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CENTER-BY-METABELIAN GROUPS OF PRIME EXPONENT

BY

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ABSTRACT. We show that a center-by-metabelian group of prime exponent p is nilpotent of class at most p, and this result is best possible. The proof is based on techniques dealing with varieties of groups.

1. Introduction. A classic result due to Meier-Wunderli [5] states that a metabelian group of prime exponent p is nilpotent of class at most p. Thus a center-by-metabelian group of exponent p is nilpotent of class at most $p + 1$. We show below that this bound can be reduced to p. Because such groups do exist for every prime $p \geqslant 5$ [4, Satz 6], the bound is best possible.

2. Notation and terminology. Our notation is generally the same as that in [6], to which we also refer the reader for elementary results concerning varieties of groups. We, however, use capital (small) italic letters for groups (elements). The variety generated by a particular group G we denote var(G), and $\{1\}$ denotes the trivial group.

Commutators are left-normed, with $(x_1, x_2, \ldots, x_n)$ an element in G_n, the nth term of the descending central series of G. The *nth Engel law* is the varietal law (x, ny), defined inductively:

$$(x, 0y) = x, \quad (x, ny) = ((x, (n-1)y), y).$$

We abbreviate $((x, y), (u, v))$ by $(x, y; u, v)$ and $((x, y; u, v), w)$ by $(x, y; u, v; w)$. Thus G is metabelian if it satisfies the law $(x, y; u, v)$; and center-by-metabelian, if it satisfies the law $(x, y; u, v; w)$. We call $(x, y; u, v)$ a *double commutator*.

H is a *factor group* of G if $H \cong G/N$ for some $N < G$. A is a factor of G if $A \cong H/K$, where $\{1\} \leqslant K < H \leqslant G$. A finite group is *critical* if it is not contained in the variety generated by its proper factors. A group is *basic* if it is critical and generates a join-irreducible variety. The p-group G is *regular* if for every $a, b \in G$, $(a, b)^p = a^p b^p c^p$, where c is a commutator word in a and b. A variety is regular if every finite group in it is regular.

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3. The main result. Let H be a center-by-metabelian group of prime exponent p. Then var(H) has exponent p, it is regular and it is center-by-metabelian. To show that H is nilpotent of class at most p, we first show that every basic group in var(H) has that property. From here on, therefore, let G denote a basic center-by-metabelian group of exponent p, $p \geqslant 5$, and assume that G is nilpotent of the maximum possible class, $p + 1$. Also, let F be a finitely generated relatively free group that generates var(G).

LEMMA 3.1. *In* var(G), 2-*generator groups have class at most* p. *Equivalently,* 2-*variable words of weight* $p + 1$ *are trivial.*

PROOF. We first note that $F'' \leqslant Z(F)$ because var(G) is center-by-metabelian. F/F'' is a finite metabelian p-group such that every finite group in var(F/F'') is regular. By [**8**, Theorem 1.4], every 2-generator subgroup of F/F'' has class less than p. In F, therefore, every commutator of weight p in 2 variables is contained in F''. Since $F'' \leqslant Z(F)$, every commutator of weight $p + 1$ in 2 variables is trivial. Q.E.D.

PROPOSITION 3.2. var(G) *satisfies the* $(p-1)$*th Engel law.*

PROOF. Let x and y be elements of a free set of generators of F. Then $(x, (p-1)y) \in F_{p+1}$ because F has exponent p [**3**, 18.4.13]. Thus in F, $(x, (p-1)y)$ can be written as a product of simple commutators in x and y, each of weight $p + 1$. By the lemma, each commutator in this product is trivial. Therefore F and, hence, var(G) satisfy the law $(x, (p-1)y) = 1$. Q.E.D.

PROPOSITION 3.3. *Let a, b and y be elements of a free set S of generators of F. Then in F, every simple commutator of the form (a, ry, b, sy) is trivial, where* $r + s = p - 1$.

PROOF. Since F satisfies the $(p-1)$th Engel law, both $(a, b, (p-1)y)$ and $(a, (p-1)y, b)$ are trivial.

As a metabelian group, F/F'' satisfies the Witt identity. Thus

$$(a, b, y)(b, y, a)(y, a, b) \in F''.$$

Since $F'' \leqslant Z(F)$, we get

$$((a, b, y)(b, y, a)(y, a, b), (p-2)y) = 1.$$

Now

$$((a, b, y)(b, y, a)(y, a, b), y) = (a, b, y, y)(b, y, a, y)(y, a, b, y)g_7,$$

where $g_7 \in F_7$.

Proceeding by induction, we obtain

$$\begin{aligned} 1 &= ((a, b, y)(b, y, a)(y, a, b), (p-2)y) \\ &= (a, b, y, (p-2)y)(b, y, a, (p-2)y)(y, a, b, (p-2)y)\, g_{p+4}, \end{aligned}$$

where $g_{p+4} \in F_{p+4}$. Since $F_{p+4} = \{1\}$ and $(a, b, (p-1)y) = 1$, we obtain the following equation:

$$(b, y, a, (p-2)y) = (y, a, b, (p-2)y)^{-1} = (a, y, b, (p-2)y). \quad \text{(A)}$$

Since 2-variable commutators of weight $p + 1$ are trivial in F,

$$(x, y, x, (p-2)y) = 1,$$

where $x \in S$. Replacing x by ab and expanding, we get

$$(a, y, b, (p-2)y)(b, y, a, (p-a)y) = 1.$$

Equation (A) then implies that $(a, y, b, (p-2)y)^2 = 1$. Since $p > 2$, we have $(a, y, b, (p-2)y) = 1$.

Thus if $r = 0$, 1 or $p-1$, $(a, ry, b, sy) = 1$ in F, where $r + s = p - 1$. Continuing by induction, assume $1 \leqslant r < p - 2$ and $(a, ry, b, sy) = 1$. Now

$$((a, ry), y, b)(y, b, (a, ry))(b, (a, ry), y) \in F''$$

by the Witt identity, and since $F'' \leqslant Z(F)$,

$$(((a, ry)y, b)(y, b, (a, ry))(b, (a, ry), y), (s-1)y) = 1.$$

Reasoning as above, and expanding, this yields

$$(a, ry, y, b, (s-1)y)(y, b, (a, ry), (s-1)y)(b, (a, ry), y, (s-1)y) = 1. \quad \text{(B)}$$

Since $(y, b, (a, ry)) \in F'' \leqslant Z(F)$, $(y, b, (a, ry), (s-1)y) = 1$. Also

$$(b, (a, ry), y, (s-1)y) = ((a, ry), b, y, (s-1)y)^{-1} = (a, ry, b, sy)^{-1},$$

which is trivial by the induction hypothesis. Thus (B) becomes

$$(a, (r+1)y, b, (s-1)y) = 1,$$

and this induction step completes the proof. Q.E.D.

COROLLARY 3.4. *In F, $(a, ry; b, sy) = 1$, where r and s are positive integers such that $r + s = p - 1$. Hence $(a, ry; b, sy)$ is a law in* var(G).

PROOF. Since $(a, ry; b, sy)$ is a double commutator of weight $p + 1$, it can be written as a product of simple left-normed commutators of weight $p + 1$, each of which contains exactly one a, one b, and $p - 1$ y's. By the proposition, each of these is trivial. Q.E.D.

LEMMA 3.5. *Let $\{a, b, x, y, y_1, y_2, \ldots, y_{p-3}\}$ be a free set of generators of F. Then the following laws hold in F:*

$$(a, b; x, y, y_1, y_2, \ldots, y_{p-3}) = (a, b; x, y, y_{\sigma(1)}, y_{\sigma(2)}, \ldots, y_{\sigma(p-3)}), \quad (1)$$

where σ *is any permutation of* $\{1, 2, \ldots, p-3\}$.

$$\begin{aligned}(a, b; x, y, y_1, \ldots, y_{p-3})^{-1} &= (b, a; x, y, y_1, \ldots, y_{p-3}) \\ &= (a, b; y, x, y_1, \ldots, y_{p-3}) = (a^{-1}, b; x, y, y_1, \ldots, y_{p-3}) \\ &= \cdots = (a, b; x, y, y_1, \ldots, y_{p-3}^{-1}) = (x, y, y_1, \ldots, y_{p-3}; a, b). \quad (2)\end{aligned}$$

$$\begin{aligned}(a, b; x, y, y_1, y_2, \ldots, y_{p-3})(a, b; y, y_1, x, y_2, \ldots, y_{p-3}) \\ \times (a, b; y_1, x, y, y_2, \ldots, y_{p-3}) = 1. \quad (3)\end{aligned}$$

Let $d_1, d_2 \in F'$. *Then* $(d_1, a, d_2) = (d_1; d_2, a^{-1})$. (4)

PROOF. The following are well-known properties of metabelian groups:

(i) $(x, y, y_1, y_2, \ldots, y_{p-3}) = (x, y, y_{\sigma(1)}, y_{\sigma(2)}, \ldots, y_{\sigma(p-3)})$, where σ is any permutation of $\{1, 2, \ldots, p-3\}$.

(ii) $(a, b)^{-1} = (b, a)$.

(iii) $(x, y, y_1)(y, y_1, x)(y_1, x, y) = 1$ (the Witt identity).

With these, parts (1), (2) and (3) of the lemma follow easily since F is center-by-metabelian, and all double commutators are of maximal weight. Part (4) is Lemma 6.2 of [**2**]. Q.E.D.

PROPOSITION 3.6. *In* F, *commutators of weight* 2 *commute with commutators of weight* $p-1$; *i.e.*, $(F_2, F_{p-1}) = 1$.

PROOF. Let $\{a, b, x, y, y_1, y_2, \ldots, y_{p-2}\}$ be a free set of generators of F. It suffices to show that $(a, b; x, y_1, y_2, \ldots, y_{p-2})$ is a law in F.

When working with laws in F, we shall use the notation "$a \mapsto b$" to mean that each occurrence of a is replaced by b. Then "$a \leftrightarrow b$" will mean $a \mapsto b$ and $b \mapsto a$ simultaneously.

Also, we shall use the term "separation" to denote the following procedure: Assume $1 = f_1 f_2 \cdots f_r$ is a law in F, where $r < p$ and each f_i is a product of nontrivial commutators of maximal weight, each one containing exactly i occurrences of x. Then each f_i is a law in F, $i = 1, 2, \ldots, r$. (See, e.g., [**1**, Corollary 1.1] for a more detailed description of this.)

We now proceed to the proof of the proposition. In the law

$$1 = (a, y; b, (p-2)y),$$

which is valid by (3.4), let $y \mapsto yy_1$. After some expanding, we get

$$\begin{aligned}1 = (a, y; b, y, (p-3)yy_1)(a, y_1; b, y, (p-3)yy_1) \\ \times (a, y; b, y_1, (p-3)yy_1)(a, y_1; b, y_1, (p-3)yy_1).\end{aligned}$$

Using separation, we obtain from this a law written as the product of double commutators, in which exactly one y_1 occurs in each double commutator. We thus have

$$1 = (a, y;\, b, y, y_1, (p-4)y)^{p-3}(a, y_1;\, b, y, (p-3)y)(a, y;\, b, y_1, (p-3)y).$$

Since F has exponent p, this becomes

$$1 = (a, y;\, y, b, y_1, (p-4)y)^3(a, y_1;\, b, (p-2)y)(a, y;\, b, y_1, y, (p-4)y). \tag{A}$$

Using 3.5(3) and 3.5(2), we get that

$$\begin{aligned}(a, y;\, b, y_1, y, (p-4)y)\\ &= (a, y;\, y, y_1, b, (p-4)y)(a, y;\, b, y, y_1, (p-4)y).\end{aligned}$$

Inserting this in equation (A) yields

$$\begin{aligned}1 &= (a, y;\, y, b, y_1, (p-4)y)^3(a, y_1;\, b, (p-2)y)\\ &\quad\times (a, y;\, y, y_1, b, (p-4)y)\,(a, y;\, b, y, y_1, (p-4)y)\\ &= (a, y;\, y, b, y_1, (p-4)y)^2(a, y_1;\, b, (p-2)y)(a, y;\, y, y_1, b, (p-4)y)\\ &= (a, y;\, y, b, (p-4)y, y_1)^2(a, y_1;\, b, (p-3)y, y)\\ &\quad\times (a, y;\, y, y_1, b, (p-4)y)\\ &= (a, y, y_1^{-1};\, y, b, (p-4)y)^2(a, y_1, y^{-1};\, b, (p-3)y)\\ &\quad\times (a, y;\, y, y_1, b, (p-4)y).\end{aligned}$$

Using 3.5(2) we rewrite this as

$$1 = (a, y, y_1;\, b, y, (p-4)y)^2(y_1, a, y;\, b, (p-3)y)(a, y;\, y, y_1, b, (p-4)y). \tag{B}$$

Using the Witt identity, we have

$$(y_1, a, y;\, b, (p-3)y) = (y, a, y_1;\, b, (p-3)y)(y_1, y, a;\, b, (p-3)y),$$

from which equation (B) becomes

$$1 = (a, y, y_1;\, b, (p-3)y)(y_1, y, a;\, b, (p-3)y)(a, y;\, y, y_1, b, (p-4)y).$$

Taking inverses and rearranging yields

$$1 = (a, y;\, y_1, y, b, (p-4)y)(y, a, y_1;\, b, (p-3)y)(y, y_1, a;\, b, (p-3)y). \tag{C}$$

Now from (3.4), $1 = (a, y;\, b, (p-3)y)$. Letting $y \mapsto xy$, we get, after some expanding,

$$\begin{aligned}1 &= (a, x;\, b, x, (p-3)xy)(a, y;\, b, y, (p-3)xy)\\ &\quad\times (a, x;\, b, y, (p-3)xy)(a, y;\, b, x, (p-3)xy).\end{aligned}$$

Using separation, we obtain from this a law in which each double commutator contains exactly one x. We get

$$1 = (a, y; b, y, x, (p-4)y)^{p-3}(a, x; b, y, (p-3)y)(a, y; b, x, (p-3)y)$$
$$= (a, y; b, y, (p-4)y, x)^{-3}(a, x; b, (p-3)y, y)(a, y; b, x, (p-3)y)$$
$$= \left(a, y, x^{-1}; b, (p-3)y\right)^{-3}\left(a, x, y^{-1}; b, (p-3)y\right)(a, y; b, x, (p-3)y),$$

or

$$1 = (a, y, x; b, (p-3)y)^3(x, a, y; b, (p-3)y)(a, y; b, x, (p-3)y). \quad \text{(D)}$$

Using the Witt identity again, we have

$$(x, a, y; b, (p-3)y) = (y, a, x; b, (p-3)y)(x, y, a; b, (p-3)y).$$

Substituting this into (D) yields

$$1 = (a, y, x; b, (p-3)y)^2(x, y, a; b, (p-3)y)(a, y; b, x, (p-3)y).$$

Letting $x \leftrightarrow b$, we get

$$1 = (a, y, b; x, (p-3)y)^2(b, y, a; x, (p-3)y)(a, y; x, b, (p-3)y).$$

Multiplying these last two laws yields

$$1 = (a, y, x; b, (p-3)y)^2(a, y, b; x, (p-3)y)^2 \times (x, y, a; b, (p-3)y)(b, y, a; x, (p-3)y). \quad \text{(E)}$$

We now note the following:

$$\begin{aligned}(b, y, a; x, (p-3)y) &= \left(b, y; x, (p-3)y, a^{-1}\right)\\ &= \left(b, y; x, y, a^{-1}, (p-4)y\right)\\ &= \left(b, y, y^{-1}; x, y, a^{-1}, (p-5)y\right) = \cdots\\ &= \left(b, y, (p-4)y^{-1}; x, y, a^{-1}\right)\\ &= \left(b, y, (p-4)y; x, y, a^{-1}\right)^{(-1)^{p-4}}\\ &= (b, (p-3)y; x, y, a).\end{aligned}$$

Using this in equation (E) yields

$$1 = \left[(a, y, x; b, (p-3)y)(a, y, b; x, (p-3)y)\right]^2.$$

Because p is odd, we get

$$1 = (a, y, x; b, (p-3)y)(a, y, b; x, (p-3)y). \quad \text{(I)}$$

From equation (C), with $a \leftrightarrow b$ and $y_1 \mapsto x$, we obtain

$$1 = \big(b, y;\, x, y, a, (p-4)y\big)\big(y, b, x;\, a, (p-3)y\big)\big(y, x, b;\, a(p-3)y\big)$$
$$= \big(b, y, a^{-1};\, x, (p-3)y\big)\big(y, b, x;\, a, (p-3)y\big)\big(y, x, b;\, a, (p-3)y\big).$$

Taking inverses yields

$$1 = \big(b, y, a;\, x, (p-3)y\big)\big(b, y, x;\, a, (p-3)y\big)\big(x, y, b;\, a, (p-3)y\big). \quad \text{(J)}$$

Now from (I), with $a \leftrightarrow b$, we get

$$1 = \big(b, y, x;\, a, (p-3)y\big)\big(b, y, a;\, x, (p-3)y\big).$$

Using this, we eliminate the first two factors in (J), getting

$$1 = \big(x, y, b;\, a, (p-3)y\big). \quad \text{(K)}$$

Now using the Witt identity one last time, we have

$$1 = \big(x, y, b;\, a, (p-3)y\big)\big(y, b, x;\, (p-3)y\big)\big(b, x, y;\, a, (p-3)y\big).$$

Therefore, by using (K) and then taking inverses, we get

$$1 = \big(b, y, x;\, a, (p-3)y\big)\big(x, b, y;\, a, (p-3)y\big).$$

From this, and from (K) with $b \leftrightarrow x$, we obtain

$$1 = \big(x, b, y;\, a, (p-3)y\big) = \big(x, b;\, a, (p-3)y, y^{-1}\big).$$

Letting $a \leftrightarrow x$ and taking inverses yields

$$1 = \big(a, b;\, x, (p-2)y\big).$$

Thus, $F/Z(F')$ satisfies the law $(x, (p-2)y)$. Since $F'' \leqslant Z(F')$ and $F_p \leqslant Z(F')$, $F/Z(F')$ is a metabelian group of class at most $p - 1$. By a well-known result [**1**, Corollary 2.1], any metabelian group of small class satisfying the law $(x, (p-2)y)$ also satisfies the law $(x, y_1, y_2, \ldots, y_{p-2})$. Therefore $F/Z(F')$ satisfies this law, and $(a, b;\, x, y_1, y_2, \ldots, y_{p-2})$ is a law in F. Q.E.D.

THEOREM 3.7. *Let G be a basic center-by-metabelian group of prime exponent p. Then G is nilpotent of class at most p.*

PROOF. Let F be a relatively free group that generates var(G), having $\{a, b, x, y_1, y_2, \ldots, y_{p-2}\}$ as a free set of generators. It is known that F is nilpotent of class at most $p + 1$; so assume F has class exactly $p + 1$. From the above proposition, we have that $1 = (a, b;\, x, y_1, y_2, \ldots, y_{p-2})$. By 3.5(4), we get $1 = (a, b, y_{p-2}^{-1};\, x, y_1, y_2, \ldots, y_{p-3})$, and taking inverses yields $1 = (a, b, y_{p-2};\, x, y_1, y_2, \ldots, y_{p-3})$.

Proceeding by induction, we see that every double commutator of weight $p + 1$ is trivial. Because F/F'' is a metabelian group of exponent p, $F_{p+1} \leqslant F''$ [**5**]. Thus the simple commutator $(a, b, x, y_1, y_2, \ldots, y_{p-2})$ can be written as a product of double commutators, each of weight $p + 1$ and containing all the generators of F. Since these are all trivial, so is F_{p+1}. Therefore, F has class at most p; and, hence, so does G. Q.E.D.

THEOREM 3.8. *Let H be a center-by-metabelian group of prime exponent p. Then H is nilpotent of class at most p.*

PROOF. Since groups of exponent 2 are abelian, and groups of exponent 3 are nilpotent of class at most 3 [**6**, Theorem 34.32], H is metabelian if $p < 5$. Thus we may assume that $p \geqslant 5$ and H is not metabelian.

Since H is center-by-metabelian but not metabelian, $Z(H) > \{1\}$ and $H'' \leqslant Z(H)$. Therefore H and, hence, var(H) are solvable. Because a solvable group of exponent p is locally finite [**7**, Theorem 7.16], var(H) is a locally finite variety. Hence var(H) is generated by its basic groups, and it remains only to show that those are nilpotent of class at most p.

Let G be a basic group in var(H). If G is abelian, it has class 1; and if G is metabelian, it has class at most p since every metabelian group of exponent p does.

If G is not metabelian, it must be a basic center-by-metabelian group of exponent p. Hence it has class at most p by 3.7. Therefore all the basic groups in var(H) have class at most p. Hence H itself is nilpotent of class at most p. Q.E.D.

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