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NONCOLLISION SINGULARITIES IN THE FOUR-BODY PROBLEM

BY

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ABSTRACT. It is shown that if there is a singularity in a solution of the four-body problem which is not a collision then the motion of the bodies near the singularity is nearly one-dimensional. This is established by grouping the bodies into natural clusters and showing the angular momentum of each cluster with respect to its center of mass tends to zero near the singularity. This is related to Sperling's proof of von Zeipel's theorem.

Introduction. A long-standing open problem in celestial mechanics is to describe the singularities which may occur in solutions to the equations of the n-body problem. Several pieces of this problem have been resolved. For example, it is known that in the two cases, $n = 2$ or $n = 3$, any singularity results from a collision between two or more particles. A collision is said to be any singularity in which all particles tend to limiting positions at the singularity. This idea of a collision singularity includes the fact that two or more of the limiting positions must be equal, for if not, the solution would not have a singularity at the prescribed time.

It is not known in general if all singularities have the property that the positions of the particles tend to limits at the singularity. Classical and current results will be described in the following paragraphs. One of the first tools developed to treat the general question of describing singularities of the n-body problem is a lemma of P. Painlevé **[1]** (Lemma 1 of this paper). Painlevé showed that the potential energy tends to infinity at any singularity of the n-body problem. One may apply this result to show that all singularities of the 3-body problem are collisions.

A somewhat different theorem was proposed and partially proved by H. von Zeipel **[2]**. The proof was completed in 1968 by Hans Sperling **[3]**. This result is that the maximum distance between particles tends to infinity at a noncollision singularity. A very important tool in the study of noncollision singularities which is used in the proof of von Zeipel's result is the idea of grouping the n particles into natural clusters. This idea of clustering is used extensively in this paper, but the method used to form the clusters differs from the "counting argument" used by Sperling. This clustering process and

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the behavior of these clusters is studied rigorously in this paper, so that a relatively detailed picture of a noncollision singularity in the 4-body problem is given.

The present work considers solutions of the equations of the four-body problem which may contain binary collisions. Theorem 2 asserts that such a solution which is a noncollision singularity must collapse to a line at the time of the singularity, i.e. the motion becomes essentially one dimensional near a noncollision singularity. This means that all of the bad behavior of the positions of the four particles may be confined to a certain direction in space. This is proved by grouping the masses into natural clusters and proving that the angular momentum of any cluster tends to 0 at a noncollision singularity; this is Theorem 1.

These results suggest that all singularities are due to collisions. A joint paper of R. McGehee and J. Mather [4] strongly indicates that noncollision singularities may in fact occur. They construct a function which solves the equations of the 4-body problem, transformed to regularize binary collisions, and which is a noncollision singularity. The present work, which was done in ignorance of the work of McGehee and Mather, suggests that their construction could only succeed in this linear case.

Classically speaking, this function is not a noncollision singularity. In fact, it contains infinitely many binary collisions. On the other hand, it may be possible to find a solution of the 4-body problem which is close enough to the construction of McGehee and Mather to be a noncollision singularity. It is worth mentioning that McGehee and Mather constructed their solution by taking the particles to lie on a line.

I. Basic Treatment of Singularities in the n-Body Problem

§1 includes a general discussion of the n-body problem with emphasis on the behavior of singularities. §§2 and 3 are the beginning of a careful description of the behavior of a noncollision singularity in the 4-body problem, which includes a detailed explanation of the clustering process (Lemmas 3, 4, 5 and 7). The material in §1 is completely standard, and some of the ideas underlying the work in §§2 and 3 are due to Saari [5].

1. In this section the n-body problem is stated precisely as a system of equations are given without proof. The section ends with two basic lemmas. Both Lemma 1 and Lemma 2 crudely describe the behavior of the potential near the time of a singularity in the equations of the n-body problem.
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Notation. Let R_i be the elements of R^3 and m_i be positive numbers for $i = 1, \ldots, n$. Let $R_{ij} = R_i - R_j$ and $r_{ij} = |R_{ij}|$. We shall use Σ for a summation over i, $i = 1, \ldots, n$, and Σ' for a summation over j for fixed i and $j = 1, \ldots, n$ but $j \neq i$.

EQUATION. The n-body problem is described by the equations

$$R_i^{\cdot\cdot} = -\sum{}' m_j R_{ij}/r_{ij}^3, \qquad i = 1, 2, \ldots, n.$$

CLASSICAL DEFINITION OF SINGULARITY. The right members of the equations of the n-body problem are analytic functions of R_i, $i = 1, \ldots, n$, as long as the distances r_{ij} are bounded away from 0. Cauchy's theorem for ordinary differential equations states that given $R_i(t_0) = R_{0i}$, $R_i^{\cdot}(t_0) = V_{0i}$, $i = 1, \ldots, n$, $\min_{1 \leq i < j \leq n} r_{ij}(t_0) = a > 0$ and $\max_{1 \leq i \leq n} |V_{0i}| = b$, then there exists a solution $R_i(t)$, $R_i^{\cdot}(t)$ with $R_i(t_0) = R_{0i}$, $R_i^{\cdot}(t_0) = V_{0i}$. Moreover, this solution is unique and analytic for $|t - t_0| < \delta$, and δ depends on a, b, m_i. A point t^* is a point of singularity of a solution if $R_i(t)$, $R_i^{\cdot}(t)$ analytic for t in some interval $(t^* - \varepsilon, t^*)$ and R_i, $R_i^{\cdot}$ may not be continued analytically past t^*.

COLLISIONS. If t^* is a point of singularity of a solution, R_i, $R_i^{\cdot}$ of the n-body problem and the limit from the left of $R_i(t)$, as t approaches t^*, $i = 1, \ldots, n$, exists, then the singularity is said to be a collision. There are many examples of collision singularities. General collision or total collapse is a solution with $\lim_{t \to t^*} R_i(t) = 0$. An example of some initial conditions which produce such a collision is a planar solution with equal masses and

$$R_i(t_0) = (r_0 \cos(2\pi(i/n)), r_0 \sin(2\pi(i/n)));$$
$$R_i^{\cdot}(t_0) = 0, \qquad i = 1, \ldots, n.$$

BINARY COLLISIONS. A singularity which is due to the coincidence of one or more separate pairs of masses is called a binary collision. In the classical sense, binary collisions are singularities. On the other hand, it is well known that such singularities may be removed by suitable transformations of the space and time variables. Thus a solution which "ends" in a binary collision at time, t^*, may be extended through t^* so that the branches of the solution to the left and right of t^* match in every possible sense. For a full treatment of these facts, one should consult the works of Sundman and more recently, Moser and Siegel **[6]**.

For the remainder of this paper, we assume that any solution which would end with a binary collision has been extended. It is important to note that the set of t at which binary collisions occur on a given orbit is discrete. Naturally the functions, R_i, $R_i^{\cdot}$, are analytic except at the times of the collisions and the functions R_i are continuous through the collisions.

NONCOLLISION SINGULARITIES. A solution of the n-body problem is said to have a noncollision singularity at time t^* if one or more R_i, $i = 1, \ldots, n$, has no limit at t^*. Noncollision singularities cannot occur when $n \leq 3$ and it is a long-standing open question whether they indeed can occur with more bodies.

STANDARD FORMULAS AND CONSTANTS OF MOTION. Without loss of generality assume that the center of mass of the n bodies $= \Sigma m_i R_i = 0$ identically. Define the angular momentum of system C by $\Sigma m_i R_i \times R_i^{\cdot}$. C is a constant on any given solution. Let $U = \Sigma_{1 \leqslant i < j \leqslant n} m_i m_j / r_{ij}$ be the potential of the n-body problem and $T = \Sigma m_i R_i^{\cdot 2}/2 =$ the kinetic energy. Then it follows from the equations of motion that $T - U$ is a constant on any solution. We shall call this constant h. It is well known that the angular momentum, C, and total energy, h, are conserved as any given solution passes through a binary collision.

MOMENT OF INERTIA. We define a parameter, $I = \Sigma m_i R_i^2$. If $M^* = \Sigma m_i$, then I may be expressed in terms of the distances, r_{ij}, by $I = (1/M^*)\Sigma_{1 \leqslant i < j \leqslant n} m_i m_j r_{ij}^2$. There is a relation between the moment of inertia and the potential, namely the Lagrange-Jacobi formula,

$$I^{\cdot\cdot} = 4T - 2U = 2U + 4h = 2T + 2h.$$

REMARK. It is possible to assume with no loss of generality that solutions of the n-body problem are defined for $-1 \leqslant t < 0$ and that if a singularity occurs that it happens at $t = 0$. We use $\lim_{t \to 0}$ to denote the limit from the left.

LEMMA 1. *Suppose $t = 0$ is a singularity of the n-body problem. Then* $\lim_{t \to 0} U(t) = \infty$.

PROOF. If the lemma is false, then there exists an increasing sequence, $\{t_m\}$, with limit zero and a positive number, a, such that $U(t_m) < a$. Let $m_0 = \min m_i$ and we then have

$$\min_{1 \leqslant i < j \leqslant n} r_{ij}(t_m) \geqslant m_0^2/a = a'.$$

Also by the energy relation we have $\Sigma m_i R_i^{\cdot 2}/2 = U + h$, so

$$\max|R_i^{\cdot}(t_m)| \leqslant \sqrt{2(a + h)/m_0} = b.$$

Now apply Cauchy's existence theorem to yield existence of analytic solutions in neighborhoods $|t - t_m| < \delta$ where δ does not depend on m. This implies 0 is not a singularity of this solution since $t = 0$ will eventually fall inside one of these neighborhoods.

COROLLARY 1. *If $t = 0$ is a singularity of the n-body problem, then* $\lim_{t \to 0} I(t) = 0$, *L, or ∞.*

PROOF. From the Lagrange-Jacobi formula and the above result, $\lim_{t \to 0} I^{\cdot\cdot} = \infty$. Therefore I is positive by definition and convex near 0, so the corollary follows.

Notation. We shall use a_m, $m = 1, 2, \ldots,$ to denote positive constants.

COROLLARY 2. *If $t = 0$ is not a singularity due to a general collision of all particles, i.e.,* $\lim_{t\to 0} I(t) \neq 0$, *then there exists some positive number a_1 such that near $t = 0$,* $\max_{1 \leqslant i<j \leqslant n} r_{ij}(t) > a_1$. *This is a simple consequence of the formula for I in terms of the distances, r_{ij}. Choose $a_1 = \sqrt{L/2M^*}$.*

LEMMA 2. *Suppose $t = 0$ is any singularity of the n-body problem. Then we can find a constant, a_2, such that $U(t) \geqslant a_2 t^{-2/3}$.*

PROOF. The proof of this lemma consists of finding an easy inequality for $U^{\cdot}$ and integrating it from t to 0 [7]. Define $V = 3 \cdot n$ vector of velocities, $V = (R_1^{\cdot}, R_2^{\cdot}, \ldots, R_n^{\cdot})$. Define U' = the gradient of U with respect to its $3n$ real arguments, R_i, $i = 1, \ldots, n$. $U' = (U_{R_1}, U_{R_2}, \ldots, U_{R_n})$. Let $|V|$, $|U'|$ = Euclidean norms of the respective vectors. By the energy relation,

$$T = U + h = \tfrac{1}{2} \sum m_i (R_i^{\cdot})^2 \geqslant \tfrac{1}{2} m_0 |V|^2.$$

Therefore

$$|V| \leqslant (2T/m_0)^{1/2} \leqslant b_1 U^{1/2}$$

since $T = U + h$ and U tends to infinity. Now

$$U_{R_i} = \sum{}' m_i m_j R_{ji}/r_{ij}^3.$$

Therefore

$$|U'| \leqslant (M^{*2}/m_0^4) U^2 = b_2 U^2.$$

Since $U^{\cdot} = U'V$,

$$|U^{\cdot}| \leqslant b_1 b_2 U^{5/2} = b_3 U^{5/2}.$$

The desired inequality is

$$|(U^{-3/2})^{\cdot}| \leqslant (3/2) b_3.$$

We can integrate this inequality from t to 0 since Lemma 1 gives $U^{-5/2}(S)$ tends to 0 as S tends to 0. Therefore, $|U^{-3/2}| \leqslant (3/2) b_3 |t|$, so if $a_2 = (1/(3/2)b_3)^{2/3}$, the lemma is clear.

NONCOLLISION SINGULARITY IN THE FOUR-BODY PROBLEM. In the remainder of the paper we assume that $t = 0$ is a noncollision singularity of the 4-body problem.

2. In this section we begin the study of noncollision singularities. The section begins with a discussion of "cluster coordinates", a basic tool for the study of this subject. It may be seen from the formulas given in this discussion that the center of mass of an isolated cluster of particles will move with essentially rectilinear motion, and that the motion of particles in an isolated cluster will be most strongly influenced by the mutual attractions among the particles of the cluster.

It is then inferred from Lemmas 1 and 2 that any clustering of the four masses must be of one of the two types illustrated in Figure 1. The impact of Lemmas 3–5 is that the open time interval immediately preceding the time of the singularity may be partitioned into countably many interlocking intervals on which cluster configurations A or B are alternately in effect (see Figure 2).

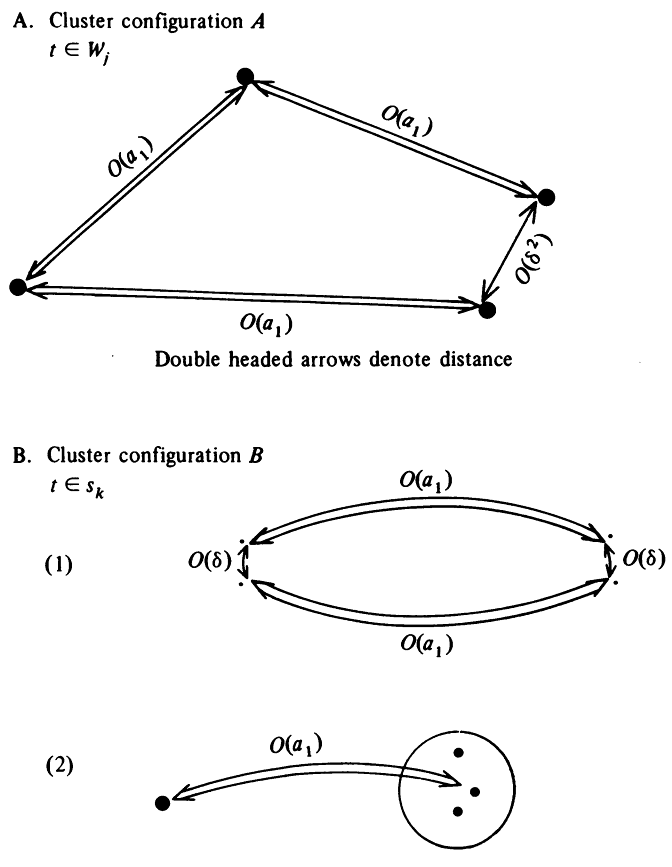


FIGURE 1

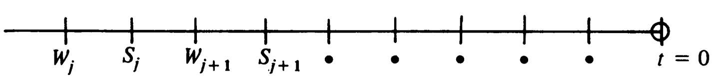


FIGURE 2

Clusters. In the study of singularities of the n-body problem it is sometimes useful to segregate the n masses into clusters. This is done in the following way. Let G_s, $s = 1, \ldots, P$, $P \leqslant n$, be disjoint subsets of the set $N = \{1, 2, \ldots, n-1, n\}$ with $\bigcup_{s=1}^{P} G_s = N$. The masses whose indices belong to the set G_s are said to be in the sth cluster.

FORMULAS. Let

$$M_s = \sum_{i \in G_s} m_i = \text{the total mass of the } s\text{th cluster.}$$

Let

$$C_s = \frac{1}{M_s} \sum_{i \in G_s} m_i R_i.$$

C_s is the center of mass of the sth cluster. We have cluster formulas for I and C. The first one is

$$I = \sum_{s=1}^{P} \sum_{i \in G_s} m_i (R_i - C_s)^2 + \sum_{s=1}^{P} M_s C_s^2.$$

The cluster formula for angular momentum is given by

$$C = \sum_{s=1}^{P} \sum_{i \in G_s} m_i (R_i - C_s) \times (R_i - C_s)^{\cdot} + \sum_{s=1}^{P} M_s C_s \times \dot{C}_s.$$

CHOOSING NATURAL CLUSTERS. As stated the selection of clusters may be arbitrary, but normally clusters are chosen so that particles which are close together are in the same cluster. Suppose the distances, r_{ij}, fall into two categories: (1) greater than a, (2) less than $a/2$, for some a. This induces a relation on N, i related to j if $r_{ij} < a/2$. By the triangle inequality, this is an equivalence relation and we can therefore use it to partition N into subsets G_s. This assures that if $i \in G_s$ and $j \notin G_s$, then $r_{ij} > a$.

ESTIMATE FOR $\ddot{C}_s$. Suppose $0 < r = \min\{r_{ij}, i \in G_s, j \notin G_s\}$. We have

$$M_s \ddot{C}_s = \sum_{i \in G_s} \sum_{\substack{j \in G_s \\ j \neq i}} \frac{m_i m_j R_{ji}}{r_{ij}^3} + \sum_{i \in G_s} \sum_{j \notin G_s} \frac{m_i m_j R_{ji}}{r_{ij}^3}$$

$$= \sum_{i \in G_s} \sum_{j \notin G_s} \frac{m_i m_j R_{ji}}{r_{ij}^3}.$$

This follows because

$$\sum_{i \in G_s} \sum_{\substack{j \in G_s \\ j \neq i}} F(i,j) = 0 \quad \text{if } F(i,j) = -F(j,i).$$

Therefore,

$$|M_s \ddot{C}_s| \leqslant M^{*2}/r^2.$$

Technical note. Normally, a group of particles is said to form a cluster if the mutual distances of these particles are all less than a preassigned positive number. In this paper we will define clusters with respect to a sliding parameter, δ, a function of time which tends to 0 at the time of the singularity. The advantage in doing this is that trivial clusters, i.e. clusters in which small velocity changes occur, are avoided. In other words, by a careful choice of the function δ, it will be possible to show that:

(a) When a particle or cluster is isolated from all other particles by a distance at least δ, then the velocity of the particle or cluster is essentially constant (Lemmas 3 and 4).

(b) In any cluster of size δ or less, immense changes of velocity must occur. Lemma 5 asserts that two successive clusters of size δ or less must consist of different particles. The fact that the clusters do not immediately recur implies that a large change in velocity, and therefore a strong interaction, must occur during the time in which the particles are clustered. It can be seen in the proof of Lemma 5 that this nonrecurrence of clusters is a trivial consequence of the choice of the parameter δ.

CONSTRUCTION OF CLUSTERING INTERVALS. Let δ be a function of time defined for $t < 0$ by $\delta(t) = |t|^{1/3}$. We restrict attention to t such that $a_2\delta^2(t) \leqslant \delta(t)/2 \leqslant a_1/16$. By Lemma 1, Corollary 2 and Lemma 2 we know that for t near 0,

$$\min_{i \neq j} r_{ij} < M^{*2}\delta^2/a_2 = a_2^*\delta^2 \quad \text{and} \quad \max r_{ij} > a_1.$$

Define E to be the set of t such that second smallest of $r_{ij}(t) \geqslant \delta/2$. On connected components of E pick out I, J the indices of the unique pair whose distance, r_{IJ}, is the minimum of all the distances, r_{ij}. Let $G_1 = (I, J)$, G_2 and G_3 each contain precisely one element. Let C_s be defined from G_s, $s = 1, 2, 3$. Let $C_{qs} = C_q - C_s$, $q = 1, 2, 3$, $s = 1, 2, 3$ and let $c_{qs} = |C_{qs}|$. Define W to be the set $\{t \in E | \min_{s \neq q} c_{sq} \geqslant \delta(t)\}$.

In the construction thus far we have only used the assumption that the solution is not a total collapse. On the other hand, we have little information about the set W.

LEMMA 3. *Let t and τ be elements of a connected component of W. Also let t_1 be left endpoint of this connected component. Then $|C_s^{\cdot}(t) - C_s^{\cdot}(\tau)| < a_3\delta(t_1)$.*

PROOF.

$$|C_s^{\cdot}(t) - C_s^{\cdot}(\tau)| \leqslant \int_\tau^t |C_s^{\cdot\cdot}(\eta)|\, d\eta \leqslant \left(\frac{M^{*2}}{m_0}\right) \int \frac{1}{r^2(\eta)}\, d\eta$$

where

$$r(\eta) = \min_{\substack{i \in G_s \\ j \notin G_s}} r_{ij}(\eta) > \delta(\eta) - a_2^*\delta^2(\eta) > \delta(\eta)/2.$$

Then

$$|C_s^{\cdot}(t) - C_s^{\cdot}(\tau)| \leqslant \frac{4M^{*2}}{m_0} \int \frac{1}{\delta^2(\eta)}\, d\eta < a_3\delta(\tau) < a_3\delta(t_1)$$

since $t_1 \leqslant \tau$.

COROLLARY. *Since $t = 0$ is a noncollision singularity of the four-body problem, $t = 0$ must be a limit point of W complement.*

PROOF. If this were not the case then the clusters G_1, G_2, G_3 in effect on the last component of W would have velocities $C_1^{\cdot}$, $C_2^{\cdot}$, $C_3^{\cdot}$ which one could integrate up to the origin. This would then give limiting values for the position vectors R_1, R_2, R_3, R_4 since two of the clusters are trivial and the other consists of two particles separated by distance less than $a_2^*\delta^2$, which tends to 0.

LEMMA 4. *Let S_1 be a component of W complement. Then on the interval S_1 there is a clustering arrangement consisting of two clusters H, H' with $\min_{i \in H; j \in H'} r_{ij} > a_1/2$ and $\max_{i,j \in H; i,j \in H'} r_{ij} < 2\delta$.*

Notation. Let H, H' be the clustering on a component of W complement. Let

$$M = \sum_{i \in H} m_i, \qquad M' = \sum_{i \in H'} m_i,$$

$$D = \frac{1}{M} \sum_{i \in H} m_i R_i, \qquad D' = \frac{1}{M'} \sum_{i \in H'} m_i R_i.$$

COROLLARY 1. *Let $\tau < t$ be two points in S_1. Then $|D^{\cdot}(t) - D^{\cdot}(\tau)| < a_4(t - \tau)$ and $|D'^{\cdot}(t) - D'^{\cdot}(\tau)| \leqslant a_4(t - \tau)$.*

PROOF. Let $a_4 = 4(M^{*2}/m_0)/a_1^2$. (See proof of Lemma 3.)

COROLLARY 2. *The point $t = 0$ is a limit point of W.*

PROOF. The preceding corollary insures that $D^{\cdot}$ and $D'^{\cdot}$ are integrable over components of W complement. If W is bounded away from 0, then $D'^{\cdot}$ and $D^{\cdot}$ are integrable and thus D and D' have limits as t tends to 0. Since i, j elements of H or i, j elements of H' imply $r_{ij} < 2\delta$, then R_i must also have a limit at 0.

Notation for Lemma 5. Let S_0, W_1, S_1 be components of W complement, W, and W complement which are adjacent, with S_0 to the left of W_1 to the left of S_1. Let H_0, H_0' be the clusters on S_0, G_1, G_2, G_3 on W_1, and H_1, H_1' on S_1.

REMARK. One of the clusters on S_0 must be the union of two of the three clusters G_1, G_2, G_3 and one of the clusters of S_1 must be the union of two of the clusters G_1, G_2, G_3. That is, Lemmas 3 and 4 tell us that the clusters H, H' break up and reform infinitely often near a noncollision singularity. We are now tacitly assuming that the entire system, S_0, W_1, S_1 is in some arbitrarily small fixed neighborhood of $t = 0$. This is possible by the corollaries to Lemmas 3 and 4.

LEMMA 5. *Suppose* $H_0 = G_{s_0} \cup G_{p_0}$ *and* $H_1 = G_{s_1} \cup G_{p_1}$ *for integers* $1 \leqslant s_k < p_k \leqslant 3$ *for* $k = 0, 1$. *Then* $\{s_0, p_0\} \neq \{s_1, p_1\}$.

PROOF. Suppose $\{s_0, p_0\} = \{s_1, p_1\}$. Since s_k was chosen less than p_k, the above supposition implies $s_0 = s_1$ and $p_0 = p_1$. Then let $s = s_1$, $p = p_1$. Let t_1, t_2 be the left and right endpoints of W_1, respectively. Define $f(t)$ for t in a neighborhood of W_1 by $f(t) = c_{sp}(t) - \delta(t)$. Notice that by construction f must have a maximum on W_1 even if W_1 is only a point, since for t immediately to the left of t_1, $f(t) < 0$, and for t immediately to the right of t_2, $f(t) < 0$, and for $t \in [t_1, t_2]$, $f(t) \geqslant 0$. Therefore there is a point, t^*, element of W_1 such that $f^{\cdot\cdot}(t^*) \leqslant 0$.

CALCULATION. Let $R(t)$ be a C^2 map of some interval into R^n. Let $r(t) = |R(t)|$ and suppose $r(t) \neq 0$. Then

$$\begin{aligned} r^{\cdot\cdot}(t) &= (R(t)R^{\cdot}(t)/r(t))^{\cdot} = R(t)R^{\cdot\cdot}(t)/r(t) \\ &\quad + \big(R^{\cdot 2}(t)/r(t) - (R(t)R^{\cdot}(t))^2/r^3(t)\big) \\ &\geqslant R(t)R^{\cdot\cdot}(t)/r(t) \geqslant -|R^{\cdot\cdot}(t)|. \end{aligned}$$

Since $W \subset E^{\cdot}$, the functions C_s, $s = 1, 2, 3$, are of class C^2 in a neighborhood of W. Apply this to the functions C_{sp} and c_{sp} in the following calculations:

$$f^{\cdot\cdot}(t^*) = c_{sp}^{\cdot\cdot}(t^*) - \delta^{\cdot\cdot}(t^*) \geqslant -|C_{sp}^{\cdot\cdot}(t^*)| + 2/(9\delta^5(t^*)).$$

However, since $t^* \in W_1$, c_{12}, c_{13}, and c_{23} satisfy $c_{sp} \geqslant \delta$ and

$$|C_{sp}^{\cdot\cdot}| = |C_s^{\cdot\cdot} - C_p^{\cdot\cdot}| \leqslant |C_s^{\cdot\cdot}| + |C_p^{\cdot\cdot}| < (8M^{*2}/m_0)/\delta^2;$$

therefore

$$f^{\cdot\cdot}(t^*) \geqslant (1/\delta^5(t^*))\big(\tfrac{2}{9} - (8M^{*2}/m_0)\delta^3(t^*)\big) > 0$$

as $\delta(t)$ tends to 0 as t tends to 0. Therefore f is strictly convex on W_1 and this is the contradiction.

COROLLARY (1). $t_2 - t_1 \geqslant (a_1/(2\max_{t \in W_1; s \in \{1,2,3\}}|C_s^{\cdot}|))$.

This is clear since one of the centers, C_s, must move in some fashion from one location to another during W_1 and these locations are separated by at least $a_1/2$.

COROLLARY (2). 0 *is the only limit point of the boundary of* W.

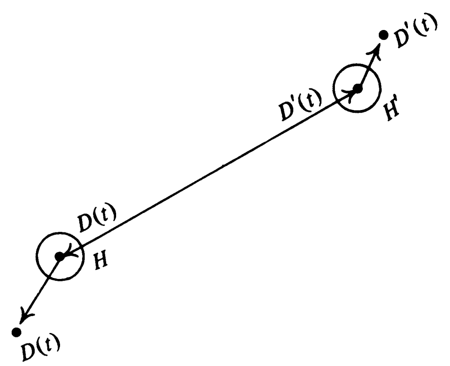
PROOF. From Lemma 5 and the construction of the clusters G_s, it is clear that any limit point of the boundary of W must be a point in time at which some of the coordinate functions R_i do not have limits. This contradicts the assumption that the solution R_i, $R_i^{\cdot}$, $i = 1, 2, 3, 4$, has no singularity for $t < 0$.

REMARKS. Since the boundary of W does not accumulate at any point $t_0 < 0$, we may enumerate the components of W in ascending order. W_{k+1} is the left-most component of W which lies to the right of W_k.

Similarly, let S_k be the components of the complement of W enumerated such that S_k lies between W_k and W_{k+1}.

Notation. Let H_k be one of the two natural clusters on S_k. Let G_{1k}, G_{2k}, G_{3k} be the three clusters formed on W_k. Where the meaning is clear, the subscript k will be suppressed. Let $M_k = \Sigma_{i \in H_k} m_i$. Let $D_k = (1/M_k)\Sigma_{i \in H_k} m_i R_i =$ center of mass of H_k. Let T_{1k}, T_{2k} = left and right endpoints of W_k.

3. This section completes the basic material needed to state and prove Theorem 1. Lemma 7 gives us a crude picture of how the overall size of the system must increase without bound at the time of a noncollision singularity (see Figure 3).



$D(t)D^{\cdot}(t) \longrightarrow +\infty$ as $t \longrightarrow 0$ through S_k

FIGURE 3

LEMMA 6. *Suppose $t = 0$ is a noncollision singularity of the n-body problem, then* $\lim_{t\to 0} I^{\cdot}(t) = \infty$.

PROOF. By Lemma 1 and the fact that t is chosen to increase toward 0^-, $I^{\cdot}$ is increasing for t near 0^-. If Lemma 6 is false, $I^{\cdot}$ approaches a limit at 0^-. Then by the Lagrange-Jacobi relationship, T is integrable. So the functions $R_i^{\cdot}(t)$ are actually square integrable and thus integrable near 0. This contradicts $t = 0$ is a noncollision singularity.

LEMMA 7. *The limit of $D(t)D^{\cdot}(t) = \infty$ as t tends to* 0 *through S_k.*

PROOF. The proof of this lemma uses Lemmas 5 and 6. Let H' be the other cluster on S and M' and D' defined analogously with M and D. Note $M' = M^* - M$ and $D' = -MD/M'$.

By the cluster formula for I, we have

$$I = MD^2 + M'D'^2 + I^* + I^{*\prime}$$

where

$$I^* = \sum_{i \in H} m_i(R_i - D)^2 \quad \text{and} \quad I^{*\prime} = \sum_{i \in H'} m_i(R_i - D')^2.$$

Note

$$I^* = \left(\frac{1}{M}\right) \sum_{\substack{i,j \in H \\ i<j}} m_i m_j r_{ij}^2,$$

and similarly for $I^{*\prime}$. We seek a point $\tau_k \in W_k$ near S_k such that

$$(I^*(\tau_k) + I^{*\prime}(\tau_k))^{\cdot} < 0. \tag{1}$$

Let

$$\eta_k = \sup\left\{ t \in W_k \middle| \max_{\substack{i,j \in H \\ i,j \in H'}} r_{ij} > \frac{8M^*}{m_0} \delta(T_{2k}) \right\}.$$

η_k is well defined since by Lemma 5 the clusters G_s and G_p which coalesce to form H or H' on S_k do not comprise H_{k-1} or H'_{k-1}. Therefore the set of which η_k is the supremum cannot be empty. By the continuity of r_{ij}, $r_{ij}(\eta_k) = 8M^*\delta(T_{2k})/m_0$ for some (i, j) belonging to H or H'. This implies $I^*(\eta_k) + I^{*\prime}(\eta_k) > 64M^*\delta^2(T_{2k})$. Conversely on S_k, $i \in H$ implies $|R_i - D| < 2\delta$ and $j \in H'$ implies $|R_j - D'| < 2\delta$. Therefore

$$I^*(T_{2k}) + I^{*\prime}(T_{2k}) \leq 4M^*\delta^2(T_{2k}) < I^*(\eta_k) + I^{*\prime}(\eta_k),$$

which with the mean value theorem implies the existence of $\tau_k \in (\eta_k, T_{2k})$ for which (1) is true. To show that (1) implies the assertion of the lemma, write

$$I^{\cdot}(\tau_k) \leq (MD^2 + M'D'^2)^{\cdot} = \left(MD^2 + M'(-MD/M')^2\right)^{\cdot}$$
$$= M(1 + M/M')D^{2\cdot} = 2M(1 + M/M')DD^{\cdot}$$

which, therefore, tends to infinity as k tends to infinity.

To prove the lemma, it now suffices to prove $(DD^{\cdot})^{\cdot}$ is bounded below on $(\tau_k, T_{1(k+1)})$. To do this, write

$$(DD^{\cdot})^{\cdot} = (D^{\cdot})^2 + DD^{\cdot\cdot} \geq -|D|\,|D^{\cdot\cdot}| \geq -|D|M^{*2}m_0^{-1}/r^2$$

where $r(t) = \min_{i \in H; j \in H'} r_{ij}(t)$. Observe for $\eta_k \leq t \leq T_{1(k+1)}$,

$$r \geq |D - D'| - 16M^*m_0^{-1}\delta(T_{2k})$$

by definition of η_k. Since $MD + M'D' = 0$, $|D - D'| = |D| + |D'| > |D|$. Therefore $r > |D|/2$. By Corollary 2 of Lemma 1, one distance r_{ij} is greater than a_1, so again by definition of η_k,

$$r > a_1 - 16(M^*/m_0)\delta(T_{2k}) > a_1/2.$$

Thus $(DD^{\cdot})^{\cdot} > -4(M^{*2}/m_0 a_1)$. This proves the lemma.

Summary of Cluster Construction. We have constructed a sequence of intervals W_k, S_k which accumulate at 0. On W_k there are three clusters, G_1, G_2, G_3, all separated by at least $\delta(t)/2$. It was shown that for $t, \tau \in W_k$ that $|C_s^{\cdot}(t) - C_s^{\cdot}(\tau)| < a_3\delta(T_{1k})$. On the interval S_k we have only two clusters H, H'. We have shown that one of these clusters is $G_s \cup G_p$ for $1 \leqslant s < p \leqslant 3$. We have also shown that $\min\{r_{ij}(t) | i \in H, j \in H', t \in S_k\} > a_1/2$. Therefore for $t, \tau \in S_k$, $|D^{\cdot}(t) - D^{\cdot}(\tau)| < a_4|t - \tau|$.

We also know that one of the two clusters $\{H_{k-1}, H'_{k-1}\}$ on S_{k-1} is $G_{s'_k} \cup G_{p'_k}$ for $1 \leqslant s' < p' \leqslant 3$. We have from Lemma 5 that $\{s', p'\} \neq \{s, p\}$. Therefore

$$c_{s'p'}(T_{1k}) = \delta(T_{1k}), \qquad c_{s'p'}(T_{2k}) > a_1/2,$$
$$c_{sp}(T_{1k}) > a_1/2, \qquad c_{sp}(T_{2k}) = \delta(T_{2k}).$$

II. Theorem 1

Discussion. It would be reasonable to suppose that a clustering of particles during some time interval, for example H, H' on s_k, would represent a very strong interaction among the particles of each cluster since these particles must, by definition, be in extremely close proximity. On the other hand, the degree to which particles of a cluster interact with each other is governed by the initial velocities of the individual particles as well as by the size of the cluster. Theorem 1 is proved by showing that each interaction must be sufficiently strong to produce radical changes in the subclusters G_s and G_p during the time interval, s_k. The assertions of the theorem may be interpreted to state that the clusters H, H' must be in a sense near collisions.

Notation. Let $H = G_s \cup G_p$ (see above). Define M, D, Z from H as before and define H', M', D' as in Lemma 7. Let

$$Z(t) = \sum m_i(R_i - D) \times (R_i - D)^{\cdot}$$

summed over $i \in H$ and let

$$Z' = \sum_{i \in H'} m_i(R_i - D') \times (R_i - D')^{\cdot}.$$

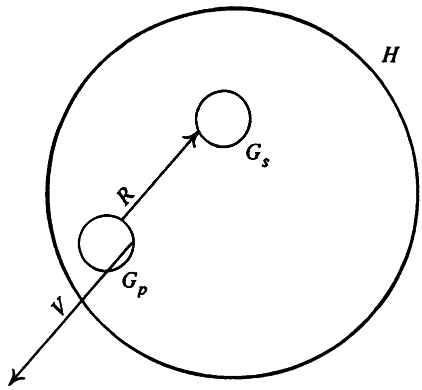
Z, Z', are defined for t near each S_k.

Theorem 1. *The limits of $Z(t)$ and $Z'(t)$ are zero as t tends to 0 through S_k.*

REMARKS. The proof of Theorem 1 will be done by contradiction, but we will not assume Theorem 1 is false yet.

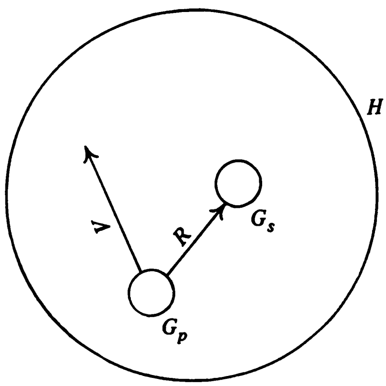
4. In this section we initiate the proof of Theorem 1. Lemma 8 asserts that, for the purpose of this analysis, the angular momentum of the clusters H, H' remains essentially constant during the time interval, S_k. Lemmas 9 and 10 reduce the problem of analyzing the angular momentum of the clusters H, H' to that of analyzing the asymptotic behavior of a vector valued function, R, and its first derivative, V. See Figure 4.

A.



$Z \longrightarrow 0$ implies $(R \times V) \longrightarrow 0$

B.



$|Z|$ bounded away from zero implies $|(R \times V)|$ bounded away from zero

FIGURE 4

LEMMA 8. *Suppose* ζ_i *satisfy*

$$\zeta_i^{\cdot\cdot} = \sum_{i \neq j} \mu_j(\zeta_j - \zeta_i)/|\zeta_j - \zeta_i|^3 + P_i$$

where P_i are bounded functions, μ_j are positive numbers, and i, j belong to some finite set of integers G. Then if $Y = \Sigma_{i \in G} \mu_i \zeta_i \times \zeta_i^{\cdot}$ we have

$$|Y^{\cdot}| < \sum_{i \in G} |\mu_i \zeta_i| \, |P_i|.$$

PROOF. Compute

$$\begin{aligned} Y^{\cdot} &= \sum_{i \in G} \mu_i \zeta_i \times \zeta_i^{\cdot\cdot} \\ &= \sum_{\substack{i,j \in G \\ i \neq j}} \mu_i \mu_j \zeta_i \times \zeta_j / |\zeta_j - \zeta_i|^3 + \sum_{i \in G} \mu_i \zeta_i \times P_i \\ &= \sum_{i \in G} \mu_i \zeta_i \times P_i, \end{aligned}$$

since the terms in the double sum are antisymmetric in i, j. Therefore $|Y^{\cdot}| < \Sigma_{i \in G} |\mu_i \zeta_i| \, |P_i|$.

COROLLARY. *For $t \in S_k$, (1) $|Z^{\cdot}(t)| \leqslant 8M^{*2}\delta(t)/a_1^2$. Also, (2) $|Z'^{\cdot}(t)| \leqslant 8M^{*2}\delta(t)/a_1^2$.*

PROOF. To get (1), apply Lemma 8, $G = H$, $\zeta_i = R_i - D$, $P_i = (\Sigma_{j \in H'} m_j R_{ji}/r_{ij}^3) - D^{\cdot\cdot}$, $\mu_i = m_i$. For (2) $G = H'$, $\zeta_i = R_i - D'$, $P_i = (\Sigma_{j \in H} m_j R_{ji}/r_{ij}^3) - D^{\cdot\cdot\prime}$, $\mu_i = m_i$.

REMARKS. We will deal with a number of velocities, $R_i^{\cdot}$, $i = 1, \ldots, 4$, $C_s^{\cdot}$, $s = 1, 2, 3$, and relative velocities, $R_{ij}^{\cdot}$, $C_{sp}^{\cdot}$, etc. All of these velocities are finite linear combinations of the four velocities $R_i^{\cdot}$. To simplify notation we find a master constant, a_5', such that each of these finite linear combinations, V, satisfies

$$|V| < a_5' T^{1/2} = a_5'(U + h)^{1/2} < a_5 U^{1/2}.$$

Notation. $d(t) = \min\{r_{ij}(t) | 1 \leqslant i < j \leqslant 4\}$. For Lemma 9, let $M_g = \Sigma_{i \in G} m_i$, $C_g = (1/M_g)\Sigma_{i \in G} m_i R_i$ and $Z_g = \Sigma_{i \in G} m_i (R_i - C_g) \times (R_i - C_g)^{\cdot}$ for G some cluster.

LEMMA 9. *Let G be a cluster with the property that $\max\{r_{ij}(t) | i, j \in G\} < a_6 d(t)$. Then $|Z_g| < M_g M^{*2} a_5 a_6 U^{-1/2}$.*

PROOF. Trivially,

$$\begin{aligned} |Z_g| &\leqslant \sum_{i \in G} m_i |R_i - C_g| \, |R_i^{\cdot} - C_g^{\cdot}| \\ &< Mg \max_{i \in G} |R_i - C_g| \max_{j \in G} |R_j^{\cdot} - C_g^{\cdot}|. \end{aligned}$$

Since C_g is a convex combination of R_i, $i \in G$, then

$$\max_{i \in G} |R_i - C_g| < \max_{i,j \in G} r_{ij} \leqslant a_6 d < a_6 M^{*2}/U.$$

Also, by the remarks in which a_5 was defined, $\max_{j\in G}|R_i^{\cdot} - C_g^{\cdot}| < a_5 U^{1/2}$. The lemma is clear.

COROLLARY 1. *Suppose there exist a subsequence k_q, a positive number C^* and points $T_q \in S_{k_q}$ such that $|Z(T_q)| > C^*$. Then the minimum distance, r_{IJ}, is unique on S_{k_q}.*

PROOF. By the corollary to Lemma 8, $|Z^{\cdot}|$ tends to 0 as t approaches 0 through S_k, and since length of S_k also approaches 0, it is certainly true that for large q, $|Z(t)| > |Z(t_q)|/2 > C^*/2$, $t \in S_{k_q}$. Suppose $T_q' \in S_{k_q}$ and there are two distances equal to $d(T_q')$ = minimum distances at T_q'. Then it must be true that $i, j \in H$ or $i, j \in H'$ implies $r_{ij}(T_q') < 2d(T_q')$; therefore, by Lemma 9, $Z(T_q')$, $Z'(T_q')$ tend to 0. This is a contradiction.

Notation. $p' = 6 - s - p$. So $\{1, 2, 3\} = \{s, p, p'\}$.

COROLLARY 2. *Limit of $Z(t) = 0$ as t tends to 0 through S_k.*

PROOF. We will prove that $Z'(T_{2k})$ tends to 0. This plus the corollary of Lemma 8 will prove Corollary 2. We have $H' = G_{p'}$. At T_{2k}, G_1, G_2, G_3, each contains a single particle or two particles whose distance is the minimum. Lemma 9 then assures $Z'(T_{2k})$ tends to 0.

REMARKS. Before we proceed to Lemma 10, we should recall the cluster formula for angular momentum and that its proof does not require the center of mass of the system to be fixed at the origin. This allows us to use the formula in a slightly more general setting where the system which we wish to consider is a cluster which is the union of smaller clusters. Precisely, we view H as a system composed of two clusters, G_s, G_p, and we write

$$Z = M_s(C_s - D) \times (C_s - D)^{\cdot} + M_p(C_p - D) \times (C_p - D)^{\cdot} + Z_{gs} + Z_{gp}$$

where Z_{gs} is the angular momentum of G_s with respect to its center of mass, C_s, and likewise for Z_{gp}.

Notation. Let $R(t)$ be defined for t in a neighborhood of S_k by $R(t) = C_s(t) - C_p(t)$. Let $V(t) = R^{\cdot}(t)$, $r(t) = |R(t)|$, and $v(t) = |V(t)|$.

REMARK. The next lemma consists of two parts with separate hypotheses and conclusions. These ideas are grouped into one lemma because the methods of proof are nearly identical.

LEMMA 10. (1) *Suppose the limit of $Z(t) = 0$ as t tends to 0 through S_k. Then define $T_k = \inf\{t \in S_k | r(t) < 1/v(t)\}$. If this set is empty, then let $T_k = T_{1(k+1)}$, the right endpoint of S_k. Then the minimum distance, r_{IJ}, is unique on $T_{2k} < t < T_k$ and the limit of $|R \times V| = 0$ as t tends to 0 through $[T_{2k}, T_k]$.*

(2) *Suppose hypothesis of (1) is false. Then there is a positive number, c, and a subsequence, k_q, such that for $t \in S_{k_q}$, $|R \times V| > c$.*

PROOF. Let $\alpha = (1/(1 + M_s/M_p))$. Then $(C_s - D) = \alpha R$. Let $\beta = (-1/(1 + M_p/M_s))$. Then $(C_p - D) = \beta R$. Now if $\gamma = M_s\alpha^2 + M_p\beta^2$ then

$$\gamma R \times V = M_s(C_s - D) \times (C_s - D)^{\cdot} + M_p(C_p - D) \times (C_p - D)^{\cdot}.$$

It is clear that we may find positive numbers, a_7, a_8, such that $a_7 < \gamma < a_8$. Therefore

$$a_7|R \times V| < |M_s(C_s - D) \times (C_s - D)^{\cdot} + M_p(C_p - D) \times (C_p - D)^{\cdot}| < a_8|R \times V|. \tag{a}$$

(1) By supposition,

$$r(t) \geqslant 1/v(t) > 1/\left(a_5U^{1/2}(t)\right) > (1/a_5)d^{1/2}(t)/M^*,$$

for $t \in [T_{2k}, T_k]$. By Lemma 4 all distances r_{ij}, $i, j \in H$, are restricted to be less than 2δ, which tends to 0. Thus the previous inequality assures that $r(t)$ is much greater than $d(t)$. Therefore the minimum distance, r_{IJ}, is unique on $[T_{2k}, T_k]$. By our choice of labeling, on W_k, $G_1 = \{I, J\}$, $r_{IJ} = d$. It therefore follows that G_1 must be $\{I, J\}$ on $[T_{2k}, T_k]$ and thus by Lemma 9, Z_{g_1} tends to 0 as t tends to 0 through $[T_{2k}, T_k]$. Z_{g_2}, Z_{g_3} are identically 0 since G_2 and G_3 contain one element. Therefore from this, the modified cluster formula for angular momentum and (a), (1) is clear.

To prove (2) in the same way, all that is necessary is to use Corollary 1 of Lemma 9 to show that the minimum distance is unique on S_{k_q} for some appropriately chosen subsequence, k_q. Assume hypothesis of (1) is false; therefore there is a subsequence, k_q, and a positive number, c^*, and points $t_q \in S_{k_q}$ such that $|Z(t_q)| > c^*$. Corollary 1 of Lemma 9 tells us that the minimum distance is unique on S_{k_q}. As before, this minimum distance $= r_{IJ}$ and $G_1 = \{I, J\}$. Therefore Z_{g_1}, Z_{g_2}, Z_{g_3} tend to 0 on S_{k_q}. Corollary to Lemma 8 assures for $t \in S_{k_q}$, $|Z(t)| > |Z(t_q)/2| > c^*/2$. This, (a), and the modified cluster formula for angular momentum yield the assertion of (2) if $c = c^*/2a_8$.

5. In this section we begin with the assumption that Theorem 1 is false. Lemma 11 shows that the function $R(t)$ on the interval S_k satisfies a system of ordinary differential equations of the form of the two-body problem with a singular perturbation. Lemmas 12 and 13 show that the asymptotic solution of this system amounts to $R \approx r_k + v_k t$, for $t \in S_k$. Lemmas 14, 15, and their corollaries then show that all cluster velocities, c_i, $i = 1, 2, 3$, undergo virtually no relative change on the time interval S_k.

SUMMARY. Suppose Theorem 1 is false. Note that we need only assume that H have positive definite angular momentum on account of Corollary 2 of Lemma 9. This assumption is in effect through the remainder of the proof of Theorem 1.

We now catalogue what we know in light of this assumption. We have a subsequence, k_q, and a positive number, c, such that $t \in S_{k_q}$ implies $|R \times V| > c$. The clustering, G_1, G_2, G_3, is natural on the interval $[T_{1k_q}, T_{2k_q+1}]$, i.e., the minimum distance is unique, and the clusters are separated on this interval. Quantitatively, on $W_{k_q} \cup W_{k_q+1}$, $\min(c_{12}, c_{13}, c_{23}) \geqslant \delta$ and on S_{k_q}, $\min(c_{sp'}, c_{pp'}) > a_1/2$. Since $|R \times V| > c$, we have

$$r > c/v > c/\left(a_5 U^{1/2}\right) > (c/a_5 M^*)d^{1/2} = a_9 d^{1/2}.$$

Notation. Let $S_q = S_{k_q}$ and t_q and t_q' be the left and right endpoints of S_q, not to be confused with their previous usage in Lemmas 9 and 10. Also set $(\tau_q, \tau_q') = (T_{1k_q}, T_{2(k_q+1)})$, respectively. Assume by proper labeling that $c_{sp'}(\tau_q) = \delta(\tau_q)$ (to do this we must drop the convention that G_1 contains two elements), i.e., H_{k_q-1} or $H'_{k_q-1} = G_s \cup G_{p'}$. By construction of the intervals S_k and the assumption of nonzero angular momentum, we know for t in S_q, $c_{sp}(t) = r(t) < \delta(t)$ with equality at t_q and t_q'.

We now seek to use this information to obtain an equation for $R(t)$ on S_q and then to asymptotically solve the equation. Let $g(R) = -R/r^3$ for $r > 0$.

LEMMA 11. *For $t \in S_q$, $R(t)$ satisfies $R^{\cdot\cdot} = \mu_q g(R) + f(t)/r$ where μ_q is a positive constant and $|f(t)|$ is bounded.*

PROOF. Let B_1, B_2, be two generic bounded vector valued functions on S_q and write

$$C_s^{\cdot\cdot} = \frac{1}{M_s} \sum_{\substack{i \in G_s \\ j \in G_p}} \frac{m_i m_j (R_{ji})}{r_{ij}^3} + B_1(t)$$

where B_1 covers the contributions to $C_s^{\cdot\cdot}$ from the particles in $G_{p'}$. Also,

$$C_p^{\cdot\cdot} = \frac{1}{M_p} \sum_{\substack{i \in G_s \\ j \in G_p}} \frac{m_i m_j R_{ij}}{r_{ij}^3} + B_2.$$

Combining the two equations yields

$$\begin{aligned} R^{\cdot\cdot} &= C_s^{\cdot\cdot} - C_p^{\cdot\cdot} \\ &= \left(\frac{1}{M_s} + \frac{1}{M_p}\right) \sum_{\substack{i \in G_s \\ j \in G_p}} \frac{m_i m_j R_{ji}}{r_{ij}^3} + B_1 - B_2 \\ &= \frac{M_s + M_p}{M_s M_p} \sum_{\substack{i \in G_s \\ j \in G_p}} \frac{m_i m_j R_{ji}}{r_{ij}^3} + B_1 - B_2. \end{aligned}$$

There are at most two pairs of indices, ij, which satisfy $i \in G_s, j \in G_p$ since only one of the G clusters contains more than one element and it contains two elements.

Let $\eta_1 = R_{ij}$ and $\mu^1 = m_i m_j$ for one choice of the indices ij, and for the other choice $\eta_2 = R_{ij}$ and $\mu^2 = m_i m_j$. Then the equation for R reads

$$R^{\cdot\cdot} = \big((M_s + M_p)/M_s M_p\big)\big(\mu^1 g(\eta_1) + \mu^2 g(\eta_2)\big) + B_1 - B_2.$$

Cases. If both G_s and G_p contain a single element then $\eta_1 = \eta_2 = R$, and the above equation for R is better than the one advertised in the statement of the lemma. If one of the clusters, G_s or G_p, contains two elements, they must be separated by the minimum distance, $d(t)$, and C_s or C_p, respectively, must lie on a line between these two particles. Therefore,

$$|\eta_K - R| < d(t) < \big(1/a_9^2\big)r^2(t), \qquad K = 1, 2.$$

Let $g'(R)$ be the 3×3 matrix whose rows are the gradients of the components of g. It is a simple calculation to verify $|g'(R)| < a'_{10}/r^3$. Let $f_K = g(\eta_K) - g(R)$, $K = 1, 2$. Then by the mean value theorem

$$\begin{aligned} |f_K| &\leqslant \max_{0<\gamma<1} |g'(R + \gamma(\eta_K - R))|\, |\eta_K - R| \\ &\leqslant \Big(a'_{10}/\big(r - (1/a_9^2)r^2\big)^3\Big)\big(1/a_9^2\big)r^2 < a''_{10}/r. \end{aligned}$$

The last step is justified since

$$|R - \gamma(\eta_K - R)| > r - \gamma|\eta_K - R| > r - \big(1/a_9^2\big)r^2 > r/2.$$

Now rewrite equation for R in terms of f_K,

$$R^{\cdot\cdot} = \frac{M_s + M_p}{M_s M_p}\left((\mu^1 + \mu^2)g(R) + \sum_{K=1}^{2} \mu^K f_K\right) + B_1 - B_2.$$

The proof is now complete. Let

$$\mu_q = \big((M_s + M_p)/M_s M_p\big)(\mu^1 + \mu^2); \quad a_{10} = (M^*/m_0)^2 a''_{10};$$

$$f(t) = r(M_s + M_p)(M_s M_p)^{-1} \sum_{K=1}^{2} \mu^K f_K.$$

Definition. Let $J(t) = \frac{1}{2} r^2(t)$.

$$\begin{aligned} J^{\cdot} = RV J^{\cdot\cdot} &= v^2 + RV^{\cdot} \geqslant v^2 - \mu_q/r - |f| \\ &\geqslant v^2 - v/c - |f|. \end{aligned}$$

Therefore, since $v > c/r > c/\delta$ which tends to infinity, $J^{\cdot\cdot} \geqslant v^2/2$.

Lemma 12. *We can write for $t \in S_q$, $v(t) = v(t_q) + v^*(t)$, where $v^*(t)/v(t_q)$ tends to* 0 *as t tends to* 0 *through S_q.*

PROOF. By definition of J and the calculation of its derivatives, it follows that for large q, J is convex on S_q. Therefore $J^{\cdot}$ may change sign no more than once on S_q. Let t_q^* be this point if such a point occurs in S_q. If no such point exists, $t_q^* = t_q'$. Since $r^{\cdot}(t_q) \leqslant \delta^{\cdot}(t_q) < 0$, $J^{\cdot} < 0$ for $t_q \leqslant t < t_q^*$. Write the equation for R in the form $V^{\cdot} = \mu_q g(R) + f(t)/r$. Dot both sides with V to yield

$$VV^{\cdot} = \tfrac{1}{2}v^{2\cdot} = -\mu_q J^{\cdot}/\left(r^2(2J)^{1/2}\right) + f(t)V/r.$$

Thus

$$\begin{aligned}\left|\tfrac{1}{2}v^{2\cdot}\right| &= \left|-\mu_q J^{\cdot}/\left(2^{1/2}r^2J^{1/2}\right) + f(t)V/r\right| \\ &\leqslant \left|\mu_q J^{\cdot}/\left(2^{1/2}r^2J^{1/2}\right)\right| + |f(t)V/r| \\ &\leqslant \left|\mu_q J^{\cdot}/\left(2^{1/2}r^2J^{1/2}\right)\right| + a_{10}v/r \\ &\leqslant \left(\mu_q/c^2\right)v^2|J^{\cdot}|/\left(2^{1/2}J^{1/2}\right) + (a_{10}/c)v^2.\end{aligned}$$

Divide the inequality by $\frac{1}{2}v^2$ and get

$$\left|\tfrac{1}{2}v^{2\cdot}\right|/\left(\tfrac{1}{2}v^2\right) = \left|\log^{\cdot}\left(\tfrac{1}{2}v^2\right)\right| \leqslant \mu_q|J^{\cdot}|\left(2^{1/2}/c^2J^{1/2}\right) + (2a_{10}/c).$$

Integrate both sides of the inequality to obtain

$$\begin{aligned}&\left|\log\left(v^2(t)/v^2(t_q)\right)\right| \\ &\quad \leqslant \sqrt{2}\,\left(\mu_q/c^2\right)\left(2J^{1/2}(t_q) - 4J^{1/2}(t_q^*) + 2J^{1/2}(t_q')\right) + 2(a_{10}/c)(t - t_q) \\ &\quad < a_{11}\delta(t_q)\end{aligned}$$

where we use $r(t) < \delta(t)$, $t \in S_q$, the definition of J, δ is decreasing, and $(t - t_q) < |t_q| = \delta^3(t_q)$. This proves the lemma.

We have just established that $v(t)$ remains essentially constant with respect to its starting value on the intervals S_q for large q. We now seek to show this for the function $V(t)$.

USE OF LEMMA 12.

$$\tfrac{3}{4}\,v(t_q) \leqslant v(t) \leqslant \tfrac{5}{4}\,v(t_q). \tag{a}$$

We make use of the fact that $v(t_q)$ tends to infinity and (a) so that we also have

$$J^{\cdot\cdot} = v^2 + RV^{\cdot} \geqslant v^2 - \mu_q/r - a_{10} \geqslant v^2 - 2\mu v/c > v^2(t_q)/2.$$

So

$$J^{\cdot\cdot} > v^2(t_q)/2. \tag{b}$$

LEMMA 13. *The length of* $S_q < a_{12}\delta(t_q)/v(t_q)$.

PROOF. Let

$$J^*(t) = \big(v^2(t_q)/4\big)(t - t_q)^2 - \big(\delta(t_q)v(t_q)\big)(t - t_q) + \delta^2(t_q)/2,$$

defined for $t \in S_q$. $J(t_q) = J^*(t_q)$, since $r(t_q) = \delta(t_q)$. $J^{*\cdot}(t_q) < J^{\cdot}(t_q)$ since

$$J^{*\cdot}(t_q) = -\delta(t_q)v(t_q) < R(t_q)V(t_q) = J^{\cdot}(t_q).$$

$J^{*\cdot\cdot}(t) < J^{\cdot\cdot}(t)$ for $t \in S_q$ by definition of J^* and (b). The preceding inequalities imply $J^*(t) < J(t)$, $t \in S_q$. However, suppose length of $S_q >$ $5\delta(t_q)/v(t_q)$. This would mean that there is a point of S_q, T_q'', with $t_q'' - t_q =$ $5\delta(t_q)/v(t_q)$ such that

$$J^*(t_q'') = 25\delta^2(t_q)/4 - 5\delta^2(t_q) + \delta^2(t_q)/2 > \delta^2(t_q)/2 > J(t_q'')$$

since

$$J(t_q'') = r^2(t_q'')/2 \leqslant \delta^2(t_q'')/2 < \delta^2(t_q)/2.$$

This is a contradiction. Therefore length of $S_q < 5\delta(t_q)/v(t_q)$.

COROLLARY. *For* $t \in S_q$, $|V(t) - V(t_q')| < a_{13}\delta(t_q)v(t_q')$.

PROOF.

$$|V(t) - V(t_q')| \leqslant \int_t^{t_q'} |V^{\cdot}(s)|\, ds$$

and

$$|V^{\cdot}(s)| \leqslant \mu_q/r^2(s) + a_{10}/r(s) < 2\mu^*/r^2(s)$$

where μ^* = supremum over q of μ_q. Also from Lemma 10(2), we have $rv \geqslant |R \times V| \geqslant c$ so $|V^{\cdot}| < 2\mu^* v^2/c^2$.

$$\begin{aligned}
|V(t) - V(t_q')| &< c^{-2}\int_t^{t_q'} 2\mu^* v^2(s)\, ds \\
&< \frac{50}{9}\mu^* v^2(t_q')\ (\text{length of } S_q)/c^2 \\
&\leqslant \tfrac{50}{9}\ \mu^* a_{12} v^2(t_q')\delta(t_q)/\big(c^2 v(t_q)\big) \\
&< \tfrac{250}{36}\ \mu^* a_{12}\delta(t_q)v(t_q')/c^2,
\end{aligned}$$

where we use condition (a) to get $v(s) < 5v(t_q)/4 < (5/4)(4/3)v(t_q')$.

REMARK. The preceding corollary is a crucial step in showing that the solution which we are now considering is not a noncollision singularity. The corollary shows that V is asymptotically constant with respect to its absolute value at t_q'. We show with little additional work that $C_{sp'}^{\cdot}$, $C_{pp'}^{\cdot}$ are also asymptotically constant with respect to $v(t_q')$ but this kind of estimate is worthless unless $V(t_q')$, $|C_{sp'}^{\cdot}|$ and $|C_{pp'}^{\cdot}|$ are mutually comparable.

Notation. Let $W_q = W_{k_q}$ and $W_q' = W_{k_q+1}$. Let $V_1 = C_1^{\cdot}$, $V_2 = C_2^{\cdot}$, $V_3 =$ $C_3^{\cdot}$, $V_{12} = C_{12}^{\cdot}$, $V_{13} = C_{13}^{\cdot}$, $V_{23} = C_{23}^{\cdot}$, and lower case v's denote absolute value. Let $V' = V_{p'} - V_s$ and $v' = |V'|$, defined for $t \in W_q \cup S_q \cup W'_q$.

Let ε_{0q} be a positive sequence with limit 0 such that for $t \in S_q$, $|V(t) - V(t'_q)| < \varepsilon_{0q} v(t'_q)$. This inequality is a restatement of the corollary of Lemma 13.

LEMMA 14. *We can find a sequence, ε_{1q}, with limit equal to zero such that $\tau_q \leqslant t \leqslant \tau'_q$ implies*

$$|V_{ij}(t) - V_{ij}(t'_q)| \leqslant \varepsilon_{1q} v(t'_q), \qquad 1 \leqslant i < j \leqslant 3.$$

REMARK. The proof of this lemma consists of nothing more than using the corollary to Lemma 13 and the known good behavior of V_1, V_2, V_3, on W_k.

PROOF. We have the algebraic identities $V_s - V_p = V$, $M_s V_s + M_p V_p = -M_{p'} V_{p'}$. We can write these equations in matrix forms, $A(V_s, V_p) = (V, V_{p'})$ where A is a 6×6 matrix. The matrix A depends on the cluster masses and therefore depends on q, but as we have seen before, A has only finitely many possible values. All of these possible matrices are invertible. Let a_{14} = maximum over all possibilities of norm (A^{-1}). Then if we write $(V, V_{p'}) = (V(t'_q), V_{p'}(t'_q)) + W^*$ where W^* is a map of (τ_q, τ'_q) into R^6 satisfying the preceding equation, then $|V_s(t) - V_s(t'_q)| < a_{14}|W^*(t)|$ and $|V_p(t) - V_p(t'_q)| < a_{14}|W^*(t)|$. Let $W^* = (W_1^*, W_2^*)$ where W_1^* is the first three components of W^*. By the definition of ε_{0q} for $t \in S_q$, $|W_1^*(t)| < \varepsilon_{0q} v(t'_q)$. Since G_s and G_p are distinct clusters on W_q and W'_q, by Lemma 3 the function $v(t)$ is well behaved on $W_q \cup W'_q$; $|W_1^*(t)| \leqslant 2a_3\delta(\tau_q) + \varepsilon_{0q} v(t'_q)$. The same is true of $V_{p'}$ since $G_{p'}$ is one of the three clusters on $W_q \cup W'_q$ and equals H' on S_q. Therefore for $\tau_q \leqslant t \leqslant \tau'_q$,

$$|W_2^*(t)| \leqslant a_3\delta(\tau_q) + a_4(t'_q - t_q) + a_3\delta(t'_q) \leqslant 3a_3\delta(\tau_q) < 1.$$

Since $v(t'_q)$ tends to infinity, we can certainly find a sequence, ε'_q with limit 0 such that for $\tau_q < t \leqslant \tau'_q$, $|W^*(t)| < (1/a_{14})\varepsilon'_q v(t'_q)$. A possible choice of ε'_q is $2a_{14}\varepsilon_{0q}$. We have just shown that $V_{p'}$ is much better behaved than either velocity V_s or V_p so by setting $\varepsilon_{1q} = 2\varepsilon'_q$ we can satisfy the statement of this lemma.

LEMMA 15. *There exists a positive number, a_{15}, such that $v'(\tau_q) > (1 + a_{15})v(\tau_q)$.*

PROOF.

$$\begin{aligned} V' &= (1/M_{p'})(-M_s V_s - M_p V_p) - V_s \\ &= -\left((1 + M_s/M_{p'})V + (1 + M_s/M_{p'} + M_p/M_{p'})V_p\right) \\ &= -(1 + \alpha_1)V - (1 + \beta_1)V_p. \end{aligned}$$

The preceding are algebraic identities valid on $W_q \cup S_q \cup W'_q$, α_1 and β_1 depend on M_1, M_2, M_3, the masses of the three clusters, and therefore depend on q, but since there are only finitely many possibilities for α_1 and β_1 and they are all positive, then α_1 and β_1 are bounded away from 0 and infinity.

Let $\alpha^* =$ smallest possible value of $\alpha_1 > m_0/M^*$. Then

$$\begin{aligned} v'^2 &= (1+\alpha_1)^2 v^2 + 2(1+\alpha_1)(1+\beta_1)VV_p + (1+\beta_1)^2 v_p^2 \\ &\geqslant (1+2\alpha^*)v^2 + \beta_2 VV_p + v_p^2 \end{aligned}$$

where $\beta_2 = 2(1+\alpha_1)(1+\beta_1)$. If $a_{15} = \sqrt{1+\alpha^*} - 1$ then the lemma would be proved if we could show $\alpha^* v^2(\tau_q) + \beta_2 VV_p + v_p^2 > 0$. The only term which is not automatically positive is VV_p. We will show that $V(\tau_q) = \gamma_1 C_p(\tau_q)$ + a small error term. If we then recall that one of the two clusters of $S_{k_q - 1}$ must be G_p, and use Lemma 7 to throw away most of the term VV_p, as it will be positive, what is left will be the error term times $-\beta_2 v(\tau_q)$ which we can overcome with v^2 and v_p^2.

Recall $V = \dot{C}_{sp} = \dot{C}_s - \dot{C}_p$. Then

$$C_{sp}(t_q) = C_{sp}(\tau_q) + \int_{W_q} V(\nu)\, d(\nu).$$

Let $V^* = V(t) - V(\tau_q)$. From Lemma 3 $|V^*| = v^* < 2a_3\delta(\tau_q)$. Then

$$C_{sp}(\tau_q) = C_{sp}(t_q) - \int_{\tau_q}^{t_q} V^*(\nu)\, d\nu - V(\tau_q)(t_q - \tau_q).$$

Let $\theta_q = C_{sp}(t_q) - \int_{\tau_q}^{t_q} V^*(\nu)\, d\nu$. We estimate

$$|\theta_q| < |C_{sp}(t_q) + (t_q - \tau_q)(2a_3\delta(\tau_q))| < 2\delta(\tau_q)$$

since $(t_q - \tau_q) < -\tau_q = \delta^3(\tau_q)$ and $|C_{sp}(t_q)| = \delta(t_q) \leqslant \delta(\tau_q)$. Then $C_{sp}(\tau_q) = -V(\tau_q)(t_q - \tau_q) + \theta_q$. This is very nearly the expression which we need for $C_p(\tau_q)$, but $C_{sp}(\tau_q)$ is $-$const. $C_p(\tau_q)$ + small error term. Let

$$D_q^* = (1/(M_s + M_{p'}))(M_s C_s(\tau_q) + M_{p'} C_{p'}(\tau_q)).$$

The quantity D_q^* is the center of mass of one of the clusters $H_{k_q - 1}$, $H'_{k_q - 1}$ which is the union of G_s and $G_{p'}$ at time τ_q. What we will use is $C_s(\tau_q) = D_q^* + \theta'_q$ where $|\theta'_q| < c_{sp'}(\tau_q) = \delta(\tau_q)$; this is possible because D_q^* is a convex combination of $C_s(\tau_q)$ and $C_{p'}(\tau_q)$. Then

$$C_{sp}(\tau_q) = D_q^* - C_p(\tau_q) + \theta'_q = -(1 + M_p/(M_s + M_{p'}))C_p(\tau_q) + \theta'_q.$$

Let $\beta_3 = (1/(1 + M_p/(M_s + M_{p'})))$. Then β_3 is positive and

$$C_p(\tau_q) = -\beta_3 C_{sp}(\tau_q) - \beta_3\theta'_q = \beta_3\big(V(\tau_q)(t_q - \tau_q) - \theta_q\big) - \beta_3\theta'_q.$$

Since $\beta_3 < 1$,

$$C_p(\tau_q) = \beta_3 V(\tau_q)(t_q - \tau_q) + \theta''_q \quad \text{and} \quad |\theta''_q| < a_{16}\delta(\tau_q).$$

Divide both sides of this equation to express

$$V(\tau_q) = (1/\beta_3)C_p(\tau_q)(t_q - \tau_q) + V_q$$

where

$$V_q = -\big(1/\big(\beta_3(t_q - \tau_q)\big)\big)(\theta_q'')$$

and since $\beta_3 > 1/(1 + (M^*/m_0))$, we have

$$|V_q| = v_q < a_{16}'\delta(\tau_q)/\,(t_q - \tau_q).$$

We need only one more fact about $V(\tau_q)$ to complete the proof. That is that

$$\begin{aligned} v(\tau_q) &> \sup_{\nu \in W_q} v(\nu) - 2a_3\delta(\tau_q) \\ &\geqslant |C_{sp}(\tau_q) - C_{sp}(t_q)|/\,(t_q - \tau_q) - 2a_3\delta(\tau_q) \\ &> a_1/\,\big(3(t_q - \tau_q)\big). \end{aligned}$$

This is true because $|C_{sp}(\tau_q)| > a_1/2$ and $C_{sp}(t_q) = \delta(t_q)$.

Recall we wish to show

$$\alpha^* v^2(\tau_q) + \beta_2 V(\tau_q)V_p(\tau_q) + v_p^2(\tau_q) > 0.$$

Since $V(\tau_q) = (1/\beta_3)C_p(\tau_q)/(t_q - \tau_q) + V_q$,

$$\begin{aligned} \beta_2 V(\tau_q)V_p(\tau_q) &= \beta_2 V_p(\tau_q)V_q + (\beta_2/\beta_3)V_p(\tau_q)C_p(\tau_q)/\,(t_q - \tau_q) \\ &\geqslant -\beta_2^* v_p(\tau_q)v_q \end{aligned}$$

where β_2^* is the greatest possible value of β_2. To do this we have used Lemma 7 which tells us $V_p(\tau_q)C_p(\tau_q)$ tends to positive infinity so it is certainly positive for large q. Finally the polynomial in $v_p(\tau_q)$, $v_p^2(\tau_q) - \beta_2^* v_q v_p(\tau_q)$, is bounded below by $-\beta_2^{*2}v_q^2/4 \geqslant -a_{17}\delta^2(\tau_q)/(t_q - \tau_q)^2$. This bound is independent of v_p. We have just shown that $v(\tau_q) > a_1/(3(t_q - \tau_q))$ so if we take q large enough that $\alpha^* a_1^2/9 > a_{17}\delta^2(\tau_q)$, we then have $\alpha^* v^2(\tau_q) + \beta_2 V(\tau_q)V_p(\tau_q) + v_p^2(\tau_q)$ positive which proves the lemma.

COROLLARY 1. $v_{sp'}(t_q') > (1 + a_{15}/2)v(t_q')$.

PROOF.

$$\begin{aligned} v_{sp'}(t_q') &> v_{sp'}(\tau_q) - \varepsilon_{1q}v(t_q') \\ &> (1 + a_{15})\big(v(t_q') - \varepsilon_{0q}v(t_q') - \varepsilon_{1q}v(t_q')\big) - \varepsilon_{1q}v(t_q') \\ &> (1 + a_{15}/2)v(t_q'). \end{aligned}$$

COROLLARY 2. $v_{pp'}(t_q') > a_{15}v(t_q')/2$.

PROOF. Corollary 2 follows from Corollary 1 and the triangle inequality.

COROLLARY 3. *We can find a sequence* ε_{2q} *with limit* 0 *such that* $|V_{ij}(t) - V_{ij}(t_q')| < \varepsilon_{2q}v_{ij}(t_q')$ *for* $\tau_q \leqslant t \leqslant \tau_q'$ *and* $i, j = 1, 2, 3$.

PROOF. Let $\varepsilon_{2q} = 2\varepsilon_{1q}/a_{15}$. The corollary then follows from Lemma 14 and Corollaries 1 and 2 of Lemma 15.

6. In §5, Theorem 1 was assumed to be false. From this, it was deduced that the cluster velocities, V_i, $i = 1, 2, 3$, undergo little change on the interval S_q. Lemmas 17 and 18 show that the cluster velocities must undergo radical changes on each interval, S_k, in order for the clusters to ever again reform. Since the clusters must regroup in order for the solution to be a noncollision singularity, the contradiction is apparent.

LEMMA 16. *Suppose $F(t)$ is a C_1 map of a real interval into R^n. Let t_1 and t_2 be the endpoints of this interval, not necessarily numbered in left, right order. Moreover, suppose $|F(t_1)| = \delta$ and $|F^{\cdot}(t) - F^{\cdot}(t_2)| < \varepsilon|F^{\cdot}(t_2)|$. Finally suppose $\delta < \varepsilon|F^{\cdot}(t_2)|\ |t_2 - t_1|$. Then $F(t_2) = F^{\cdot}(t_2)(t_2 - t_1) + \eta$ where $|\eta| \leqslant 2\varepsilon|F^{\cdot}(t_2)(t_2 - t_1)|$.*

REMARKS. This lemma may seem entirely trivial and so it is, but we include it because it will streamline the last part of the proof of Theorem 1.

PROOF.

$$\begin{aligned} F(t_2) &= F(t_1) + \int_{t_1}^{t_2} F^{\cdot}(\nu)\, d\nu \\ &= F(t_1) + F^{\cdot}(t_2)(t_2 - t_1) + \int_{t_1}^{t_2} (F^{\cdot}(\nu) - F^{\cdot}(t_2))\, d\nu \\ &= F^{\cdot}(t_2)(t_2 - t_1) + \eta \end{aligned}$$

where $\eta = F(t_1) + \int_{t_1}^{t_2}(F^{\cdot}(\nu) - F^{\cdot}(t_2))\, d\nu$. Therefore

$$|\eta| \leqslant |F(t_1)| + \left|\int_{t_2}^{t_1} (F^{\cdot}(\nu) - F^{\cdot}(t_2))\, d\nu\right| \leqslant \delta + \varepsilon|F^{\cdot}(t_2)(t_2 - t_1)|$$

which by hypothesis is less than or equal to $2\varepsilon|F^{\cdot}(t_2)(t_2 - t_1)|$.

Notation. Let K be the unique integer, 1, 2, 3, such that $H_{k_q+1} = G_K \cup G_{p'}$. Recall H_i is the cluster on S_i composed of two of the clusters of W_i. Note that Lemma 5 justifies the assertion that one of the clusters of W'_q which is included in H_{k_q+1} is $G_{p'}$. $K = s$ or $K = p$, also by Lemma 5. Let

$$\varepsilon_q = \max\left(\varepsilon_{2q}, \frac{\delta(\tau_q)}{v_{sp'}(t'_q)(t'_q - \tau_q)}, \frac{\delta(t'_q)}{v_{Kp'}(t'_q)(\tau'_q - t'_q)}\right).$$

By the mean value theorem and Lemma 3,

$$v_{Kp'}(t'_q)(\tau'_q - t'_q) > |C_{Kp'}(t'_q) - C_{Kp'}(\tau'_q)| - 2a_3\delta(t'_q)(\tau'_q - t'_q) > a_1/3.$$

Similarly

$$v_{sp'}(t'_q)(t'_q - \tau_q) > (1 - \varepsilon_{2q})|C_{sp'}(t'_q) - C_{sp'}(\tau_q)| > a_1/3.$$

Here instead of Lemma 3 we must use Corollary 3 of Lemma 15. The above two inequalities assure that ε_q tends to zero. Further notice that the hypotheses of Lemma 16 are satisfied by $\delta = \delta(\tau_q)$, $\varepsilon = \varepsilon_q$, $t_1 = \tau_q$, $t_2 = t'_q$, $F(t) = C_{sp'}(t)$, and by $\delta = \delta(\tau'_q)$, $\varepsilon = \varepsilon_q$, $t_1 = \tau'_q$, $t_2 = t'_q$, $F(t) = C_{Kp'}(t)$.

LEMMA 17.

$$\lim_{q\to\infty} \frac{C_{sp'}(t'_q)V_{sp'}(t'_q)}{c_{sp'}(t'_q)v_{sp'}(t'_q)} = 1 \quad \textit{and} \quad \lim_{q\to\infty} \frac{C_{Kp'}(t'_q)V_{Kp'}(t'_q)}{c_{Kp'}(t'_q)v_{Kp'}(t'_q)} = -1.$$

REMARKS. This lemma rules out $K = s$, and is used in the proof of Lemma 18.

PROOF. Apply Lemma 16 to the function $C_{sp'}(t)$ on the interval (τ_q, t'_q) with $\delta = \delta(\tau_q)$ and $\varepsilon = \varepsilon_q$. Then use Lemma 16 to write

$$C_{sp'}(t'_q) = V_{sp'}(t'_q)(t'_q - \tau_q) + \eta_q$$

where

$$|\eta_q| < 2\varepsilon_q |V_{sp'}(t'_q)|(t'_q - \tau_q).$$

Now

$$\frac{V_{sp'}(t'_q)C_{sp'}(t'_q)}{\{v_{sp'}(t'_q)c_{sp'}(t'_q)\}} = \frac{(v^2_{sp'}(t'_q)(t'_q - \tau_q) + V_{sp'}(t'_q)\eta_q)}{\{v_{sp'}(t'_q)|V_{sp'}(t'_q)(t'_q - \tau_q) + \eta_q|\}} > (1 - 2\varepsilon_q)/(1 + 2\varepsilon_q)$$

which proves part 1 of the lemma since ε_q tends to 0, and $V_{sp'}C_{sp'} \leq v_{sp'}c_{sp'}$. Similarly, apply Lemma 16 to the interval (t'_q, τ'_q) with $t_1 = \tau'_q$, $t_2 = t'_q$. Let $F(t) = C_{Kp'}(t)$, $\delta = \delta(\tau'_q)$ and $\varepsilon = \varepsilon_q$. Then use Lemma 16 to express

$$C_{Kp'}(t'_q) = V_{Kp'}(t'_q)(t'_q - \tau'_q) + \eta'_q$$

with

$$|\eta'_q| \leq 2\varepsilon_q |V_{Kp'}(t'_q)|(\tau'_q - t'_q).$$

Then

$$\begin{aligned} \frac{V_{Kp'}(t'_q)C_{Kp'}(t'_q)}{v_{Kp'}(t'_q)c_{Kp'}(t'_q)} &= \frac{v^2_{Kp'}(t'_q)(t'_q - \tau'_q) + V_{Kp'}(t'_q)\eta'_q}{v_{Kp'}(t'_q)|V_{Kp'}(t'_q)(t'_q - \tau'_q) + \eta'_q|} \\ &\leq \frac{v^2_{Kp'}(t'_q)(t'_q - \tau'_q) + v_{Kp'}(t'_q)|\eta'_q|}{v^2_{Kp'}(t'_q)(\tau'_q - t'_q) + v_{Kp'}(t'_q)|\eta'_q|} \leq \frac{-1 + 2\varepsilon_q}{1 + 2\varepsilon_q}. \end{aligned}$$

This proves the second part of the lemma since

$$V_{Kp'}(t'_q)C_{Kp'}(t'_q)/(v_{Kp'}(t'_q)c_{Kp'}(t'_q)) \geq -1.$$

FINAL STEP IN PROOF OF THEOREM 1. By the remark preceding the proof of Lemma 17, we know that Lemma 17 implies $K = p$. We assume this and then show that for large q, $V_{pp'}(t'_q)C_{pp'}(t'_q)$ is positive. To do this we use part 1 of Lemma 17 and the corollary to Lemma 14. Naturally, the statement that $V_{pp'}(t'_q)C_{pp'}(t'_q) > 0$ for all large q contradicts Lemma 17.

LEMMA 18. *$V_{pp'}(t'_q)C_{pp'}(t'_q) > 0$ for all large q.*

PROOF. By definition of V, $V_{pp'} = V_{sp'} - V$. Let

$$\varphi = C_{pp'}(t'_q) - C_{sp'}(t'_q) = -C_{sp}(t'_q).$$

Therefore $|\varphi| = \delta(t'_q)$. Now we have

$$C_{pp'}(t'_q)V_{pp'}(t'_q) = V_{sp'}(t'_q)C_{sp'}(t'_q) - V(t'_q)C_{sp'}(t'_q) - V(t'_q)\varphi + V_{sp'}(t'_q)\varphi.$$

By part 1, Lemma 17, we can find a sequence, β_q, with limit 0 such that

$$V_{sp'}(t'_q)C_{sp'}(t'_q) > (1 - \beta_q)v_{sp'}(t'_q)c_{sp'}(t'_q).$$

Now write

$$\begin{aligned}
&V_{sp'}(t'_q)C_{sp'}(t'_q) - V(t'_q)C_{sp'}(t'_q) - V(t'_q)\varphi + V_{sp'}(t'_q)\varphi \\
&\quad \geqslant (1 - \beta_q)v_{sp'}(t'_q)c_{sp'}(t'_q) - v_{sp'}(t'_q)\delta(t'_q) - v(t'_q)c_{sp'}(t'_q) - \delta(t'_q)v(t'_q) \\
&\quad = (1 - \beta_q - \delta(t'_q)/c_{sp'}(t'_q))v_{sp'}(t'_q)c_{sp'}(t'_q) - (1 + \delta(t'_q)/c_{sp'}(t'_q))v(t'_q)c_{sp'}(t'_q).
\end{aligned}$$

Observe that since $c_{sp'}(t'_q) > a_1/2 - 2\delta(t'_q) > a_1/3$, we certainly have $\delta(t'_q)/c_{sp'}(t'_q)$ tends to 0. Therefore we let $\beta'_q = \delta(t'_q)/c_{sp'}(t'_q)$ and $\beta''_q = \beta_q + \beta'_q$. We can write

$$V_{pp'}(t'_q)C_{pp'}(t'_q) > (1 - \beta''_q)v_{sp'}(t'_q)c_{sp'}(t'_q) - (1 + \beta'_q)v(t'_q)c_{sp'}(t'_q).$$

By Corollary 1 to Lemma 15, $v_{sp'}(t'_q) \geqslant (1 + a_{15}/2)v(t'_q)$ so

$$\begin{aligned}
V_{pp'}(t'_q)C_{pp'}(t'_q) &\geqslant (1 - \beta''_q)(1 + a_{15}/2)v(t'_q)(c_{sp'}(t'_q)) - (1 + \beta'_q)v(t'_q)c_{sp'}(t'_q) \\
&\geqslant ((1 - \beta''_q)(1 + a_{15}/2) - (1 + \beta'_q))v(t'_q)c_{sp'}(t'_q)
\end{aligned}$$

which is positive in view of the fact that β'_q and β''_q are tending to 0.

III. THEOREM 2

Notation. For the remainder of the paper we will reserve the symbols L and L_k to designate lines. Moreover, for any point X in R^n, $|X - L| = \min\{|X - Z|\ |Z \in L\}$. We also have for X, Y in R^n, $|X - L| \leqslant |X - Y| + |Y - L|$.

THEOREM 2. *If $t = 0$ is a noncollision singularity of the four-body problem then there exists a line L through the origin such that $\lim_{t\to 0}|R_i(t) - L| = 0$ for $i = 1, 2, 3, 4$.*

REMARK. The proof of Theorem 2 will naturally use Theorem 1. It is worth noting that the only part of Theorem 1 which is critical to the proof of Theorem 2 is that the angular momentum of each cluster H and H' is bounded independent of t.

Notation. We keep much of the notation of Theorem 1. The points T_{1k} and T_{2k} are the endpoints of W_k; G_1, G_2, G_3 are the clusters on W_k with M_i the total mass of G_i and C_i the coordinate of the center of mass of G_i. Also define

C_{ij}, c_{ij} from C_i as before. $H = G_s \cup G_p$ = one of the clusters on S_k, $1 \leq s < p \leq 3$. $H' = G_{p'}$ = other clusters on S_k, $p' = 6 - s - p$. D, D' are the centers of mass of H, H', respectively. Z, Z' is the angular momentum of H, H' with respect to D, D', respectively. We also keep the labeling convention that $G_s \cup G_{p'}$ is one of the clusters H, H' on S_{k-1}. We therefore let $D_{-1} = (1/(M_s + M_p))(M_s C_s + M_{p'} C_{p'})$ and $D'_{-1} = -(M_s + M_{p'})D_{-1}/M_p$. D_{-1} and D'_{-1} are the centers of mass of the clusters H, H' on S_{k-1}. The primes on D do not necessarily agree with those on H since we do not know the clustering arrangement on W_{k-1}. $R = C_{sp}$, $V = R^{\cdot}$, $r = |R|$, and $v = |V|$. Also keep $d(t) = \min\{r_{ij}(t) | 1 \leq i < j \leq 4\}$, and define T_k as in Lemma 10, part 1.

7. In this section we show under the assumption that the angular momentum of each cluster, H_k, with respect to its center of mass is bounded, that the cluster velocities, C_i, $i = 1, 2, 3$, undergo changes which are bounded independent of K as long as the clusters H, H' are larger than $1/|C^{\cdot}_{sp}(t_{2k})|$. The preceding bound on the size of clusters is crucial to showing the convergence of the particle positions to a fixed line.

LEMMA 19. *If* $T_{2k} \leq t \leq T_k$ *then* $r^{\cdot}(t) < -v(t)/2$.

PROOF. We shall use the identity $|A|^2|B|^2 = (AB)^2 + (A \times B)^2$ for points A and B in R^3. By hypothesis $r(t)v(t) > 1$. By Theorem 1 and part 1 of Lemma 10, $R \times V$ tends to 0 as t tends to 0 through the closed intervals $[T_{2k}, T_k]$ so we can assume that $|R \times V| < \frac{1}{2}$. We then have $(RV)^2 = r^2v^2 - (R \times V)^2 \geq 3r^2v^2/4$ for t as restricted by the hypothesis. Therefore since $r^{\cdot} = RV/r$ and $r^{\cdot}(T_{2k}) \leq \delta^{\cdot}(T_{2k}) < 0$, we have more than the claim of the corollary.

LEMMA 20. *Let* $S'_k = \{t \in S_k | r(t)v(t) < 1\}$. *Then for sufficiently large* k, S'_k *is not empty.*

PROOF. Suppose S'_k empty. Then $T_k = T_{1k+1}$ and by Lemma 10. part 1, the minimum distance is unique on S_k. Therefore the clusters G_1, G_2, G_3 on W_k are also natural on W_{k+1}. Also by Lemma 5, $c_{sp}(T_{2k+1}) \geq a_1/2$. Therefore

$$v(T_{1k+1}) = |C^{\cdot}_{sp}(T_{1k+1})| \geq a_1/\left(3(T_{2k+1} - T_{1k+1})\right) > a_1/\left(3\delta^3(T_{1k+1})\right).$$

By the corollary to Theorem 1, $r^{\cdot}(T_{1k+1}) < -a_1/(6\delta^3(T_{1k+1}))$. On the other hand,

$$r^{\cdot}(T_{1k+1}) \geq \delta^{\cdot}(T_{1k+1}) = -1/\left(3\delta^2(T_{1k+1})\right).$$

Both statements cannot hold since $\delta(T_{1k+1})$ tends to 0, therefore S'_k cannot be empty for large k.

COROLLARY. $r(T_k)v(T_k) \leq 1$.

PROOF. This follows from the definition of T_k, S'_k not empty and continuity of r on the closed interval $[T_{2k}, T_k]$. Note we have not excluded the possibility that $T_k = T_{2k}$.

LEMMA 21. *If* $T_{2k} < t < T_k$, *then* $|V(t) - V(T_{2k})| < 16M^{*2}/m_0$.

PROOF. Let $r'(t) = \min\{r_{ij}(t) | i \in G_s, j \in G_p\}$. We have

$$|V(t) - V(T_{2k})| < \int_{T_{2k}}^{t} (M^{*2}/m_0 r'^2(s))\, ds.$$

Since $r > 1/v > 1/(a_5 U^{1/2}) > d^{1/2}/(a_5 M^*)$, we have $r'^2 > (r - d)^2 > r^2/2$. We now have

$$|V(t) - V(T_{2k})| < \int_{T_{2k}}^{t} (2M^{*2}/m_0 r^2(s))\, ds.$$

Assume Lemma 21 is false. Let

$$\tau = \inf\{t \in [T_{2k}, T_k] \,|\, |V(t) - V(T_{2k})| > 16M^{*2}/m_0\}.$$

The supposition that Lemma 21 is false implies the set of which τ is the infimum cannot be empty. Also if Lemma 21 is false, we may assume T_{2k} does not belong to S'_k, for if $T_{2k} \in S'_k$, then $T_k = T_{2k}$, and Lemma 21 is automatically true. By continuity $|V(\tau) - V(T_{2k})| = 16M^{*2}/m_0$. We further require k large enough so that $v(T_{2k}) > 32M^{*2}/m_0$. We can do this because $v(T_{2k}) > 1/r(T_{2k}) = 1/\delta(T_{2k})$ tends to infinity. Here we use our observation that T_{2k} does not belong to S'_k. We now have $v(t) > v(\tau)/3$. We use these facts to approximate a solution of the inequality

$$|V(\tau) - V(T_{2k})| < \int_{T_{2k}}^{\tau} (2M^{*2}/m_0 r^2(s))\, ds.$$

Multiply the right side of this inequality by $-2\dot{r}/v$ which is greater than 1 and write

$$\begin{aligned} |V(\tau) - V(T_{2k})| &< \frac{4M^{*2}}{m_0} \int_{T_{2k}}^{\tau} \frac{-\dot{r}(s)\, ds}{v(s) r^2(s)} < \frac{12M^{*2}}{m_0 v(\tau)} \int_{T_{2k}}^{\tau} \frac{-\dot{r}(s)\, ds}{r^2(s)} \\ &< 12(M^{*2}/m_0 v(\tau))(1/r(\tau)) < 12M^{*2}/m_0 \end{aligned}$$

which contradicts the definition of τ. This proves the lemma.

Notation. Let $V_i = \dot{C}_i$, $V_{ij} = \dot{C}_{ij}$. Lower case v's denote absolute value of capital V's.

COROLLARY TO LEMMA 21. *We can write for* $T_{k-1} < t < T_k$, $V_p(t) = V_p(T_{k-1}) + V_p^*(t)$ *where* $|V_p^*| = v_p^* < a_{19}$.

PROOF. For $t \in [T_{k-1}, T_{1k}]$, $v_p^*(t) < a_4(T_{1k} - T_{k-1})$ since G_p is one of the clusters on S_{k-1}. On W_k, moreover, $|V_p(t) - V_p(T_{1k})| < a_3\delta(T_{1k})$. Therefore, for $T_{k-1} \leqslant t \leqslant T_{2k}$,

$$v_p^*(t) < a_4(T_{1k} - T_{k-1}) + a_3\delta(T_{1k}) < 1.$$

Finally we use the matrix A of Lemma 14 to write $A(V_s, V_p) = (V, V_{p'})$. If we also borrow the function, $W^*(t) = (V(t), V_{p'}(t)) - (V(T_{2k}), V_{p'}(T_{2k}))$ for $t \in S_k$, Corollary 1 of Lemma 4 and Lemma 21 assert that $|W^*(t)|$ is bounded for $T_{2k} \leqslant t \leqslant T_k$. It is clear from this that

$$\begin{aligned}|V_p(t) - V_p(T_{2k})| &\leqslant |(V_s(t), V_p(t)) - (V_s(T_{2k}), V_p(T_{2k}))| \\ &< a_{14}|W^*(t)| < a'_{19}.\end{aligned}$$

Combining the estimates for the intervals $[T_{k-1}, T_{1k}]$, $[T_{1k}, T_{2k}]$, and $[T_{2k}, T_k]$, we now have for $T_{k-1} \leqslant t \leqslant T_k$,

$$v_p^*(t) < a_4(T_{1k} - T_{k-1}) + a_3\delta(T_{1k}) + a'_{19} \leqslant a_{19}.$$

8. In the previous section, it was shown that the clusters H, H' must be effectively smaller in size than $1/V(T_k)$. This bound will enable us to show that the sequence of lines L_k (see notation below), must converge extremely rapidly. This rapid convergence of the sequence L_k to a limiting line, L, will force the convergence of the particle positions R_i, $i = 1, 2, 3, 4$, to the limiting line L.

Notation. Let L_k be a line defined by the two points $D(T_k)$, $D'(T_k)$.

REMARK. L_k may be defined equivalently by the pairs of points $D(T_k)$ and 0; or $D'(T_k)$ and 0. Therefore, L_{k-1} is also defined by $C_p(T_{k-1})$ and 0.

LEMMA 22. *We can find a positive number, a_{22}, such that $|D(T_k) - L_{k-1}| < a_{22}(T_k - T_{k-1})$.*

PROOF. By Theorem 1, we can restrict our attention to k large enough that $|Z| + |Z'| < 1$ for $t \in S_{k-1}$. Now use the cluster formula for angular momentum to express

$$C = (M_s + M_{p'})D_{-1} \times \dot{D}_{-1} + M_p D'_{-1} \times \dot{D}'_{-1} + Z + Z'.$$

We then rearrange to write

$$|(M_s + M_{p'})D_{-1} \times \dot{D}_{-1} + M_p D'_{-1} \times \dot{D}'_{-1}| \leqslant |C| + 1.$$

Now let us write $D_{-1} = (-M_p/(M_s + M_{p'}))D'_{-1}$ since $M_p D'_{-1} = M_p C_p = -M_s C_s - M_{p'} C_{p'} = -(M_s + M_{p'})D_{-1}$, and use this to express

$$\begin{aligned}&(M_s + M_{p'})D_{-1} \times \dot{D}_{-1} + M_p D'_{-1} \times \dot{D}'_{-1} \\ &\qquad = (M_p M^*/(M_s + M_{p'}))D'_{-1} \times \dot{D}'_{-1}.\end{aligned}$$

Therefore

$$\begin{aligned}|D'_{-1} \times \dot{D}'_{-1}| = |C_p \times V_p| &\leqslant (M_s + M_{p'})(|C| + 1)/(M_p M^*) \\ &\leqslant (|C| + 1)/m_0 = a_{20}.\end{aligned}$$

We now will outline the remainder of the proof of the lemma. We follow $D'_{-1}(T_{k-1}) = C_p(T_{k-1})$ from T_{k-1} at which time G_p comprises part of H. We then use the estimate $|C_p(T_{k-1}) \times V_p(T_{k-1})| \leqslant a_{20}$ to make $|C_p(T_k) - L_{k-1}|$ small. We use Lemma 21 and the fact that $D(T_k)$ is a convex combination of $C_p(T_k)$, $C_s(T_k)$ to show that $|C_p(T_k) - D(T_k)|$ is small. We finally use the triangle inequality to show that $|D_k(T_k) - (L_{k-1})|$ is small which completes the proof.

Suppose we define a coordinate system in which we express V_p in the following way. We choose this coordinate system to have three pairwise orthogonal axes, with the third axis parallel to L_{k-1}. Let u_1, u_2 be the first components of V_p. Then

$$|C_p(t) - L_{k-1}| < \int_{T_{k-1}}^{t} \left(u_1^2 + u_2^2\right)^{1/2} dt.$$

Let $\rho(t) = (u_1^2 + u_2^2)^{1/2}$. It is a computation to verify

$$\rho(T_{k-1}) = |C_p(T_{k-1}) \times V_p(T_{k-1})| / c_p(T_{k-1}).$$

We submit

$$\begin{aligned} c_p(T_{k-1}) &= |D'_{-1}(T_{k-1})| \\ &= \left(1/\left(1 + (M_s + M_{p'})/M_p\right)\right)|D'_{-1}(T_{k-1}) - D_{-1}(T_{k-1})| \\ &> m_0 a_1/3M^* = a_{21}. \end{aligned}$$

We then compute $\rho(T_{k-1}) \leqslant a_{20}/a_{21}$. Let

$$\rho^*(t) = \left[\sum_1^2 \left(u_i(t) - u_i(T_{k-1})\right)^2\right]^{1/2}.$$

It is clear that $\rho^*(t) \leqslant v_p^*(t) < a_{19}$ by corollary to Lemma 21. By the triangle inequality,

$$\rho(t) < \rho(T_{k-1}) + \rho^*(t) \leqslant a_{19} + a_{20}/a_{21} \quad \text{for } T_{k-1} \leqslant t \leqslant T_k.$$

Therefore

$$|C_p(T_k) - L_{k-1}| \leqslant \int_{T_{k-1}}^{T_k} \rho(s)\, ds \leqslant \left(a_{19} + \frac{a_{20}}{a_{21}}\right)(T_k - T_{k-1}).$$

By the corollary to Lemma 20 and the fact that $D(T_k)$ is a convex combination of $C_p(T_k)$ and $C_s(T_k)$, we have $|C_p(T_k) - D(T_k)| \leqslant r(T_k) \leqslant 1/v(T_k)$. We have, by Lemma 21, $v(T_k) > v(T_{2k}) - (16M^2/m_0)$. By the mean value theorem and Lemma 3,

$$v(T_k) > \frac{|C_{sp}(T_{1k}) - C_{sp}(T_{2k})|}{T_{2k} - T_{1k}} - 2a_3\delta(T_{1k}) - \frac{16M^{*2}}{m_0}$$

$$> \frac{a_1/3 - \delta(T_{2k}) - (2a_3\delta(T_{1k}) + 16M^{*2}/m_0)(T_{2k} - T_{1k})}{T_{2k} - T_{1k}}$$

$$> a_1/4(T_{2k} - T_{1k}).$$

Therefore, $|C_p(T_k) - D(T_k)| < 4(T_{2k} - T_{1k})/a_1$. We put the estimate for $|C_p(T_k) - L_{k-1}|$ and the one for $|C_p(T_k) - D(T_k)|$ together with the triangle inequality to give the desired result,

$$|D(T_k) - L_{k-1}| < (a_{19} + a_{20}/a_{21} + 4/a_1)(T_k - T_{k-1}) = a_{22}(T_k - T_{k-1}).$$

COROLLARY 1. $|D'(T_k) - L_{k-1}| < M^* a_{22}(T_k - T_{k-1})/m_0$.

PROOF. Since $(M_s + M_p)D = -(M_{p'}D')$, then the corollary follows.

COROLLARY 2. *Let θ_k be the angle between L_k and L_{k-1} measured so it will always be positive and less than or equal to $\pi/2$. Then* $\lim_{k\to\infty}\theta_k = 0$.

PROOF. θ_k is the angle between $D(T_k)$ and L_{k-1}, and equals the angle between $D'(T_k)$ and L_{k-1}. θ_k must tend to 0 by Lemma 22 and $|D(T_k) - D'(T_k)| > a_1/3$.

Notation. Let $\zeta_k = \pm(D(T_k) - D'(T_k))$ and $u_k = \zeta_k/|\zeta_k|$. We choose the sign of ζ_k and u_k such that $u_k u_{k+1}$ is positive for large k. Corollary 2 of Lemma 22 assures us that this is possible.

LEMMA 23. *Let $k' > k$; then* $|\zeta_{k'}| > m_0|\zeta_k|/4M^*$.

Notation. For clarity, we subscript the sets H, H'. Let H_i, H_i' be sets, respectively, H, H' on S_i.

PROOF. Merely pick k great enough that t larger than T_k assures $I^{\cdot}(t)$ positive. We also want

$$\min_{\substack{i\in H_k\\ j\in H_k'}} r_{ij}(T_k) > \tfrac{1}{2}|\zeta_k| \quad\text{and}\quad \max_{\substack{i\in H_{k'}\\ j\in H_{k'}'}} r_{ij}(T_{k'}) < 2|\zeta_{k'}|.$$

This and the formula for I in terms of the mutual distances, r_{ij}, implies

$$I(T_k) \geqslant \left(\tfrac{1}{4}m_0^2/M^*\right)|\zeta_k|^2, \qquad I(T_k') \leqslant 4M^*|\zeta_k|^2.$$

Combining the preceding two inequalities with $I(T_{k'}) \geqslant I(T_k)$, we obtain $|\zeta_{k'}| \geqslant \frac{1}{4}\,(m_0/M^*)|\zeta_k|$.

REMARKS. In Lemma 24, we will prove that the centers of mass of the clusters H and H' on S_k, D, D' tend to a line, L, at the sequence of times T_k. To show Theorem 2 from Lemma 24 is not hard. The real work comes in finding the limiting line L. We have already done most of this work in Lemmas 20–22.

LEMMA 24. *There exists a line L such that* $\lim_{k\to\infty}|D(T_k) - L| = \lim_{k\to\infty}|D'(T_k) - L| = 0$.

PROOF. It suffices to show $|D(T_k) - L|$ tends to 0 since

$$|D(T_k) - L| = (|D(T_k)|/|D'(T_k)|)|D'(T_k) - L|.$$

However, $m_0/M^* \leqslant |D|/|D'| \leqslant M^*/m_0$. We now try to find L. To find L we do the easier problem of finding a limit for the sequence of vectors u_k. By geometry, $|u_k - u_{k-1}| = 2\sin(\theta_k/2) \leqslant \theta_k$. Also, $|D(T_k) - L_{k-1}| = |D(T_k)|\sin(\theta_k)$. By Corollary 2 of Lemma 22, θ_k tends to 0. We restrict attention to k large enough that $0 \leqslant \theta_k \leqslant \pi/6$. Thus, $\theta_k \leqslant 2\sin(\theta_k)$ by the mean value theorem. Therefore, we can write

$$|u_k - u_{k-1}| \leqslant (2/|D(T_k)|)a_{22}(T_k - T_{k-1}),$$

by Lemma 22. This inequality would show that $\{u_k\}$ is a Cauchy sequence if we know $|D(T_k)|$ bounded away from 0 since $(T_k - T_{k-1})$ is summable. We have $|\zeta_k| = |D(T_k)| + |D'(T_k)|$ since $M_{p'}D' + (M_s + M_p)D = 0$. However, $|D'| \leqslant (M^*/m_0)|D|$ so $|D(T_k)| \geqslant (1/(1 + M^*/m_0))|\zeta_k|$ which is bounded away from 0 since $|\zeta_k| > a_1/3$. Therefore the sequence $\{u_k\}$ converges. Let u be the limit of this sequence and let L be the line $\{\gamma u|\gamma \in R\}$. By geometry, we have $|D(T_k) - L| < |\zeta_k|\cdot|u - u_k|$. Therefore if we can show that $|\zeta_k|\cdot|u - u_k|$ tends to 0, then we have proved this lemma. We know that

$$\begin{aligned}|u - u_k| &\leqslant \sum_{i=k+1}^{\infty}|u_i - u_{i-1}| \leqslant \sum_{i=k+1}^{\infty}\frac{2}{|D(T_i)|}a_{22}(T_i - T_{i-1})\\ &\leqslant 2\left(1 + \frac{M^*}{m_0}\right)\sum_{i=k+1}^{\infty}\frac{1}{|\zeta_i|}a_{22}(T_i - T_{i-1}).\end{aligned}$$

By Lemma 23, we can replace $(1/|\zeta_i|)$ by $4(M^*/m_0)(1/|\zeta_k|)$; so combining all this, we obtain

$$\begin{aligned}|D(T_k) - L| &\leqslant |\zeta_k|\sum_{i=k+1}^{\infty}|u_i - u_{i-1}|\\ &\leqslant \frac{8M^*}{m_0}\left(1 + \frac{M^*}{m_0}\right)a_{22}\sum_{i=k+1}^{\infty}(T_i - T_{i-1}) = a_{23}(-T_k)\end{aligned}$$

which tends to 0 as was the claim of the lemma.

LEMMA 25. *Suppose $f(x)$ is a C^1 map of some interval $I = \{x|\alpha \leqslant x \leqslant \beta\}$ into R^n. Moreover, suppose L is a line in R^n and $|f(\alpha) - L| = b_1$, $|f(\beta) - L| = b_2$ and $|f'(x) - f'(\alpha)| < b_3$ for b_1, b_2, b_3 positive numbers. Then*

$$\max_I |f(t) - L| \leqslant b_3(\beta - \alpha) + \max\{b_1, b_2 + b_3(\beta - \alpha)\}.$$

PROOF. Let $g(x) = f(\alpha) + f^{\cdot}(\alpha)(x - \alpha)$. Then $f(\alpha) = g(\alpha)$ and $|f^{\cdot} - g^{\cdot}| \leqslant b_3$. Therefore $|f - g| < b_3(\beta - \alpha)$. Since the points $g(x)$ form a line, we know that

$$\max|g(x) - L| < \max\{|g(\alpha) - L|, |g(\beta) - L|\}$$
$$\leqslant \max(b_1, b_2 + b_3(\beta - \alpha)).$$

However, since we have established that $|f - g| < b_3(\beta - \alpha)$, we can now use the triangle inequality to give

$$|f(x) - L| \leqslant \max|g(x) - L| + \max|f(x) - g(x)|$$
$$\leqslant b_3(\beta - \alpha) + \max(b_1, b_2 + b_3(\beta - \alpha)).$$

LEMMA 26. $\lim_{t\to 0}|R_i(t) - L| = 0$.

REMARK. This lemma completes the proof of Theorem 2.

PROOF. We have shown in Lemma 24 that

$$\lim_{k\to\infty} |D(T_k) - L| = \lim_{k\to\infty} |D'(T_k) - L| = 0.$$

For $i \in H$, $j \in H'$, the distances $|R_i - D|$ and $|R_j - D'|$ are less than $\max_{i,j\in H, i,j\in H'} r_{ij} < 2\delta$ for all $t \in S_k$. Therefore we have $\lim_{k\to\infty}|R_i(T_k) - L| = 0$, $i = 1, 2, 3, 4$. We seek to extend this statement to include all t near 0. We first show that limit of $|R_i(t) - L|$ is 0 as t tends to 0 through S_k. This will be done by applying Lemma 25 to the function $C_p(t)$ on the interval $[T_{k-1}, T_k]$. On S_{k-1}, G_p is one of the clusters H or H'. On S_k, G_p is contained in H. In particular, if i, j both belong to G_p then $r_{ij}(T_{k-1}) < 2\delta(T_{k-1})$ and $r_{ij}(T_k) < 2\delta(T_k)$. Therefore

$$\lim_{k\to\infty} |C_p(T_{k-1}) - L| = \lim_{k\to\infty} |C_p(T_k) - L| = 0.$$

From the corollary to Lemma 21 for $T_{k-1} \leqslant t \leqslant T_k$, $V_p(t) = V_p(T_{k-1}) + V_p^*$ where $v_p^* < a_{19}$. Now, by Lemma 25, $\lim_{k\to\infty}|C_p(t) - L| = 0$ for all $t \in [T_{k-1}, T_k]$. We will only need this for $T_{k-1} \leqslant t \leqslant T_{1k}$ and $T_{2k} \leqslant t \leqslant T_k$. Now we observe that for $T_{k-1} \leqslant t \leqslant T_{1k}$,

$$|D_{-1}(t) - L| = |C_p(t) - L|\,|D_{-1}(t)|/C_p(t) \leqslant |C_p(t) - L|(m^*/m_0).$$

Therefore the limit of $|D_{-1}(t) - L|$, as t tends to 0 through the closed intervals $[T_{k-1}, T_{1k}]$, equals the limit of $|C_p(t) - L|$ as t tends to 0 through the closed intervals $[T_{k-1}, T_{1k}]$ which equals 0. Now since $C_p = D'_{-1}$ and D_{-1} are the centers of mass of the clusters H and H' on S_{k-1}, we then have the limit of $|R_i(t) - L|$ as t tends to 0 through the closed intervals $[T_{k-1}, T_{1k}]$ equals 0.

We now wish to show the same thing for t approaching 0 through the closed intervals $[T_{2k}, T_k]$. From above, we have that the limit of $|C_p(t) - L|$, as t tends to 0 through $[T_{2k}, T]$, equals 0. We also know that for $t \in S_k$,

$|C_p(t) - D(t)| \leqslant C_{sp}(t) \leqslant S(t)$. Therefore we have the limit of $|D(t) - L|$, as t tends to 0 through $[T_{2k}, T_k]$, equals 0. Similarly,

$$|D' - L| = |D - L|\,|(D'/|D|)| \leqslant |D - L|(m^*/m_0).$$

Therefore, limit of $|D(t) - L|$, as t tends to 0 through $[T_{2k}, T_k]$, equals the limit of $|D'(t) - L|$, as t tends to 0 through $[T_{2k}, T_k]$, equals 0. We therefore have the desired statement, limit of $|R_i(t) - L|$, as t tends to 0 through S_k, equals 0. We can finish the proof by showing that the limit of $|R_i(t) - L|$ as t tends to 0 through W_k equals 0. This can be done by using the result for S_k which was just proved. That is, we have

$$\lim_{k\to\infty} |R_i(T_{1k}) - L| = \lim_{k\to\infty} |R_i(T_{2k}) - L| = 0.$$

We use Lemma 25 on the functions C_i, $i = 1, 2, 3$, in the intervals $[T_{1k}, T_{2k}]$. For $t \in W_k$, $\max_{i,j\in G_q} r_{ij}(t) < a_2S^2$ where $q = 1, 2, 3$. Therefore,

$$\lim_{k\to\infty} |C_i(T_{1k}) - L| = \lim_{k\to\infty} |C_i(T_{2k}) - L| = 0.$$

From Lemma 3 the velocities V_i satisfy $|V_i(t) - V_i(T_{1k})| < a_3S(T_{1k}) < 1$. Therefore we can apply Lemma 25 to obtain limit of $|C_i(t) - L|$, $i = 1, 2, 3$, as t tends to 0 through W_k equals 0. This immediately gives the desired result since each point R_j is less than a_2S^2 from one of the points C_i on W_k. Therefore the limit of $|R_i(t) - L|$ as t approaches 0 through $W_k = 0$. We have already shown that the limit of $|R_i(t) - L|$ as t approaches 0 through S_k equals 0. Therefore $\lim_{t\to 0}|R_i(t) - L| = 0$.

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