

## MODELOIDS. I<sup>1</sup>

BY

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**ABSTRACT.** If  $A$  is a set and  $\hat{A}$  is the collection of finite nonrepeating sequences of its elements then a modeloid  $E$  on  $A$  is an equivalence relation on  $\hat{A}$  which preserves length, is hereditary, and is invariant under the action of permutations. The pivotal operation on modeloids is the derivative. The theory of this operation turns out to be very rich with connections leading to diverse branches of mathematics. For example, in §3 we associate an action space with a modeloid and in §5 we characterize the action spaces which are associated with the basic modeloids, i.e., those which are derivatives of themselves. What emerges is a kind of stability for the action space. We then show that action spaces with this stability can be approximated by finite actions and, subject to certain requirements, this approximation is unique (see Proposition 5.7). Algebraically, the countable basic modeloids correspond to closed subgroups of the symmetric groups. This and the study of automorphisms of modeloids let us show, without any algebra, that the only nontrivial normal subgroups of the finite ( $> 5$ ) symmetric groups are the alternating groups. The last section gives, hopefully, credence to the thesis that the essence of model theory is the study of modeloids.

**Introduction.** A modeloid is an equivalence relation on a set with a binary operation. There would not, of course, be much to say in so general a setting so we impose conditions on the relation and the operation. Parts of our theory would go through for semigroups but we shall deal exclusively with the semigroups  $A^*$  (the free monoid on  $A$ ). We then require of the equivalence relation to respect the lengths of the words, to be hereditary in the sense that if two words are equivalent then so are their initial segments of the same length and, finally, that the relation be invariant under the actions induced by the permutations of the words. This is a modeloid.

What are we interested in? This may be answered best by thinking of the equivalence relation as an analogy. That is, if two objects are equivalent we say that they are analogous. We thus have an analogy among our objects (be they words, concepts, theories, or concrete things) and when presented with an analogy we invariably ask how far does it hold. We check the extent of the validity of the analogy presented by pointing out a feature on one side of the analogy and then try to find an analogous feature on the other side. Defining

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two objects equivalent iff the analogy of them is preserved by pointing out new features, we get a new equivalence relation and a better, more precise analogy for which the question of how far it extends may be repeated and the process iterated. This is a motivation for the notion of the derivative of a modeloid and its iterations with which we are preoccupied throughout the paper.

Let us take a concrete example, one which goes to the origins of mathematics. Consider collections of pebbles. Declare two such sets to be analogous just in case they are both nonempty. Hence the only distinction perceived at this level is between the empty and the nonempty. Let us assume that "pointing out a feature" means taking a pebble out of a set of pebbles (the binary operation we talked about in the beginning). Then a set consisting of one pebble is seen to be no longer equivalent to a collection of two or more pebbles. The new analogy has three equivalence classes, one consisting of the empty set, one of the collection of sets with one pebble only and the third consists of the rest. By repeating this process we distinguish more and more obtaining eventually the natural numbers by naming the equivalence classes. The Greeks did repeat this process throughout infinity while other groups of people stopped after five iterations or so. The content of the property ( $\alpha$ ) of modeloids is that this is taken for granted.

Although equivalence relations are usually simple objects, in conjunction with the structure of  $A^*$  they are very complicated and for this reason we introduce the concept of an outline of a modeloid which has the advantage that it can, sometimes, be drawn. The outline is a tree with actions on it and we extend it by what we call the last level which becomes a topological space with the group of permutations of the natural numbers acting on it. Because of this we shall use some results and terminology of topological dynamics; the reader may consult the books [5] and [7] if need be.

As we said the iteration of the derivatives of a given modeloid eventually stabilizes at a modeloid which we call basic for the given one. The basic modeloids are those where the analogy is perfect, that is, the application of the derivative gives nothing new. In §5 we represent them using the last level of the outline and arrive at the notion of the action being smooth at a point. This notion is somewhat akin to the notion of a structurally stable point in the qualitative theory of differential equations. We also make some remarks on how to view the process in the light of getting a closure operator satisfying the exchange principle.

We also study automorphisms of modeloids getting three groups in the investigation and we watch how they vary with successive derivatives. We also have one of them as a normal subgroup of another and in the end the third group becomes a quotient of the extension. We use finite modeloids and

their automorphisms to give a proof of the fact that there is only one nontrivial normal subgroup of the group of permutations on  $n$  letters,  $n \geq 5$ .

We then make apparent some connections with model theory to justify the term modeloid. This section serves as a bridge between this paper and its follow-up which will be devoted to supporting the thesis that most of model theory is a study of modeloids rather than models.

In the last section we give some thoughts on the study of the process in general terms concerning the development of language, concepts, and knowledge. We also state some of the many open questions.

Historically, the idea of defining modeloids occurred to me while working on my paper [2]. I wished to construct a structure with certain properties and I had the idea of constructing the group of its automorphism rather than the structure itself. This was successful; the thought whether constructions like that could be used generally came up. It was, of course, discarded; however, from my experience (see [1]) and that of others Fraïssé [6] and Ehrenfeucht [4], in particular, I knew that partial isomorphisms should serve very well in a general setting. It turned out that notationally equivalence relations work better than partial automorphisms; the connection is explained in the first section.

One who has been on unfamiliar ground before can appreciate the fact that this paper had to go through many versions in order to find viable paths and useful connections with other fields. In remarks scattered through the paper we make some alternatives evident and, perhaps, some of them would serve better than what we present here. But we must leave this up to the future.

Finally, I would like to say “muito obrigado” to the “Instituto de Matemática” of the “Universidade de São Paulo” for their support during the year 1976.

**0. Preliminaries.** The notation and terminology is standard. The only possible exceptions are explained below as well as some special definitions we use throughout the paper.

We denote by  $A!$  the group of permutations of  $A$ . Hence the symbols  $n!$  and  $N!$  denote the groups of permutations of  $n = \{0, 1, \dots, n-1\}$  and of the set of natural numbers  $N$ , respectively. We often consider the group  $n!$  to be a subgroup of  $m!$  if  $n \leq m$ , and in turn  $m!$  as a subgroup of  $N!$ . The identification is done in the natural way:  $p \in n!$  is identified with  $q \in m!$  such that  $p(i) = q(i)$  for  $i < n$  and  $q(j) = j$  otherwise ( $n \leq j < m$ ).

By  $A^*$  we denote the set of finite sequences of members of  $A$ , we sometimes refer to elements of  $A$  as letters and to the elements of  $A^*$  as words. Given two words  $u, v \in A^*$ , say  $u = (a_0 \dots a_{n-1})$  which we sometimes denote by  $(a_i | i < n)$ , and  $v = (b_0 \dots b_{m-1})$ , then by  $uv$  we denote their concatenation,  $uv = (a_0 \dots a_{n-1} b_0 \dots b_{m-1})$ . By  $\Lambda$  we denote the empty

word. The length of  $u \in A^*$  is denoted by  $l(u)$ . If  $u$  is an initial segment of  $v$  we denote this by  $u \leq v$  (we use this convention sparingly). If  $p \in l(u)!$  then by  $up$  we denote the effect of the action of  $p$  on  $u$ ,  $up = (a_{p(0)} \cdots a_{p(n-1)})$  if  $u = (a_i | i < n)$ . We write the action from the right for reasons which shall be soon apparent.

By  $\hat{A}$  we denote the set of finite one-to-one sequences of members of  $A$ . We thus have  $\hat{A} \subseteq A^*$ . We replace the concatenation on  $A^*$  by the operation  $\hat{\cdot}$  which is defined by induction as follows: if  $u \in \hat{A}$  and  $x \in A$  then  $u\hat{x}$  is  $ux$  if  $x$  does not occur in  $u$  and it is  $u$  if  $x$  does occur in  $u$ . We thus have  $u\hat{v} \in \hat{A}$  if  $u, v \in \hat{A}$  and we shall at the beginning use this notation but later we shall write  $u\hat{v}$  simply as  $uv$ , with the understanding that when we work on  $\hat{A}$  we really mean  $u\hat{v}$ . There is a natural projection of  $A^*$  onto  $\hat{A}$  defined by:  $\hat{u} = a_0\hat{a}_1\hat{\cdots}\hat{a}_{n-1}$  where  $u = (a_i | i < n)$  and where we use the standard practice of denoting the sequence  $(a)$  by  $a$ . We of course have  $\widehat{uv} = \hat{u}\hat{v}$ . Note that  $n!$  is in one-to-one correspondence with  $\hat{n}$ : we simply think of  $p \in n!$  as the word  $(p(i) | i < n) \in \hat{n}$  and we often do so. Then if  $u \in \hat{n}$  and  $p \in n!$  and we pretend that  $u$  is a permutation of  $n!$ ; then the result of the action as defined above is  $up$  where this time we are referring to the product operation of the group  $n!$ .

By  $nA^*$  and  $n\hat{A}$  we denote the elements of  $A^*$  and  $\hat{A}$  resp. which have length  $n$ .

### 1. The definition.

DEFINITION. A modeloid over a set  $A$  is an equivalence relation  $E$  on  $\hat{A}$  satisfying:

- ( $\alpha$ ) if  $aEb$  then  $l(a) = l(b)$ ,
- ( $\beta$ ) if  $aEb$ ,  $c \leq a$ ,  $d \leq b$  and  $l(c) = l(d)$  then  $cEd$ ,
- ( $\gamma$ ) if  $l(a) = n$ ,  $p \in n!$ , and  $aEb$  then  $apEbp$ .

Before discussing the definition let us give a few examples some of which will be used later to illustrate new notions.

EXAMPLE 1.1. Let  $A = N$  and  $H \subseteq N!$ , meaning  $H$  is a subgroup of  $N!$ . For  $(n_0 \cdots n_k), (m_0 \cdots m_k) \in \hat{N}$  we define  $(n_0 \cdots n_k)E(m_0 \cdots m_k)$  iff there are  $p, q \in N!$  such that  $pq^{-1} \in H$ ,  $p(i) = n_i$ , and  $q(i) = m_i$  ( $i \leq k$ ).

EXAMPLE 1.2 (THE WHERE-IS-0-MODELOID). Let  $A = N$  and define  $(n_0 \cdots n_k)E(m_0 \cdots m_k)$  iff 0 is on the same place in the two sequences.

EXAMPLE 1.3. Let  $(A, <)$  be an infinite linear order. Define  $(a_0 \cdots a_k) \cdot E(b_0 \cdots b_k)$  iff the map  $a_i \rightarrow b_i$  preserves  $<$ . As in the examples above it is important that the sequences are from  $\hat{A}$ .

EXAMPLE 1.4. Let  $A$  be a subset of the plane. Define  $(a_0 \cdots a_k) \cdot E(b_0 \cdots b_k)$  iff " $a_{i_1}, a_{i_2}, a_{i_3}$  ( $i_1 < i_2 < i_3$ ) are collinear iff  $b_{i_1}, b_{i_2}, b_{i_3}$  are collinear."

**EXAMPLE 1.5 (THE TYPICAL EXAMPLE).** Let  $M = (A, R, \dots)$  be a structure (functions are treated as relation). We define  $E_M$  by:

$$(a_0 \cdots a_k)E_M(b_0 \cdots b_k) \text{ iff the function } f = \{(a_i, b_i) | i \leq k\} \text{ is a partial isomorphism.}$$

By a partial isomorphism we mean this: if  $x_1, \dots, x_m$  are in the domain of  $f$  then  $R(x_1 \cdots x_m)$  iff  $R(fx_1 \cdots fx_m)$ , for all relations of the structure.

The phrase "we treat functions as relations" signifies that if an operation leads outside of the domain of  $f$  we do not check whether the image of the operation is the operation of the image. For example if we consider  $(N, +)$  then  $(2, 5, 4)E(2, 7, 4)$  but  $(2, 5, 4)\bar{E}(2, 6, 4)$ .

The reason for calling this example typical is that every modeloid on a set  $A$  is of this form. Given a modeloid  $E$  on  $A$  define  $M_E = (A, R, \dots)$  where  $R, \dots$  is a listing of all equivalence classes of  $E$ . It should be clear that  $E_{M_E} = E$ .

In words a modeloid is an invariant (under the actions) equivalence relation which preserves lengths and is hereditary (condition  $(\beta)$ ).

Let us now discuss alternative but equivalent presentations of modeloids. Instead of using the set  $\hat{A}$  we can use the more familiar set  $A^*$  and define modeloids as equivalence relations on  $A^*$  satisfying  $(\alpha)$ ,  $(\beta)$ ,  $(\gamma)$ , and the following two additional requirements

$$(\delta) \text{ if } (a_0 \cdots a_n)E(b_0 \cdots b_n) \text{ then } a_i = a_j \text{ iff } b_i = b_j.$$

$$(\epsilon) \text{ if } (a_0 \cdots a_n)E(b_0 \cdots b_n) \text{ and } i \leq n \text{ then } (a_0 \cdots a_n a_i)E(b_0 \cdots b_n b_i).$$

**PROPOSITION 1.6.** *Modeloids on  $\hat{A}$  and  $A^*$  are in a natural correspondence determined by the projection  $\hat{\phantom{A}}$  onto  $\hat{A}$ .*

**2. The derivative.** Let us fix a (nonempty) set  $A$  and consider the collection of all modeloids on this set. As modeloids are equivalence relations, operations which apply to equivalence relations apply to modeloids as well. E.g. we have

**PROPOSITION 2.1.** (a) *Intersection of modeloids is a modeloid.*

(b) *The least (in the sense of  $\subseteq$ ) equivalence relation including two given modeloids is a modeloid.*

(c) *The inclusion relation is a partial ordering of modeloids. The least element in the partial order is  $I_A$ , the identity modeloid, the largest element is  $J_A$  where  $aJ_A b$  iff  $l(a) = l(b)$ .*

Actually only (a) is important for us because it enables us to speak of the least modeloid generated by an equivalence relation but we shall go into this later. Let us rather introduce the operation which can be defined only on  $\hat{A}$

(or on  $A^*$ ) and which ties together the structure of  $\hat{A}$  and the equivalence relation.

DEFINITION. Let  $E$  be an equivalence relation of  $A^*$ . We define its derivative,  $E'$ , as follows (for  $a, b \in A^*$ ):

$aE'b$  iff for every  $x \in A$  there is  $y \in A$  with  $axEby$  and for every  $x \in A$  there is  $y \in A$  with  $ayEbx$ .

PROPOSITION 2.2. If  $E$  is an equivalence relation on  $A^*$  then so is  $E'$ .

PROPOSITION 2.3. If  $E$  is a modeloid on  $A^*$  then so is  $E'$  and  $E' \subseteq E$ .

In the case of a modeloid on  $\hat{A}$  we define

DEFINITION. Let  $E$  be a modeloid on  $\hat{A}$ .

$aE'b$  iff for each  $x \in A$  there is a  $y \in A$  such that  $a\hat{x}Eb\hat{y}$  and for each  $x \in A$  there is a  $y \in A$  such that  $a\hat{y}Eb\hat{x}$ .

Let us now look at the examples of §1.

EXAMPLE 2.5. We show that no matter what  $H \subseteq G$  we take,  $E' = E$ . Assume  $(n_0 \cdots n_k)E(m_0 \cdots m_k)$  and let  $x = n_{k+1} \in N$  be different from  $n_0, \dots, n_k$ . Let  $p$  and  $q$  be in  $N!$  as stipulated by the definition of  $E$ . Then for some  $i > k$ ,  $p(i) = n_{k+1}$ . Let  $r \in N!$  be such that  $r(i) = k+1$ ,  $r(k+1) = i$  and  $r(j) = j$  for all other  $j$ 's. Then for  $j \leq k$ ,  $pr(j) = p(j) = n_j$  and  $qr(j) = q(j) = m_j$ . We also have  $pr(k+1) = p(i) = n_{k+1}$  and  $pr(qr)^{-1} = pq^{-1} \in H$ . So taking  $y = m_{k+1} = qr(k+1)$  we see that  $(n_0 \cdots n_{k+1})E(m_0 \cdots m_{k+1})$ , consequently  $(n_0 \cdots n_k)E'(m_0 \cdots m_k)$ . As  $E' \subseteq E$  we have  $E' = E$ .

EXAMPLE 2.6. It is easy to see that we have  $E' = E$  in this example as well. Defining a modeloid  $E_n$  on  $\hat{A}$  by  $aE_nb$  iff " $l(a) = l(b) \leq n$  or  $aEb$ " we get an example of "finding antiderivatives," because we easily check that  $(E_{n+1})' = E_n$ , that is the  $n$ th derivative of  $E_n$  is  $E_0 = E$ .

EXAMPLE 2.7. In this example we get  $E' = E$  iff  $(A, <)$  is a dense linear order without first and last element. The "if" part follows easily, the "only if" part will follow from a general theorem (but can be checked directly). In the case when  $(A, <)$  is the order of natural numbers we have  $0E1$  but not  $0E'1$  as  $0xE'10$  has no solution, so  $E' \neq E$ .

EXAMPLE 2.8. To describe  $E'$  we define for  $(a_0 \cdots a_k) \in \hat{A}$   $a \notin \{a_0, \dots, a_k\}$  a graph  $G_a$  on  $\{a_0, \dots, a_k\}$  by joining  $a_i$  and  $a_j$  iff the points  $a_i$ ,  $a_j$ , and  $a$  are collinear. Then

$(a_0 \cdots a_k)E'(b_0 \cdots b_k)$  iff the map  $a_i \rightarrow b_i$  induces a correspondence of the graphs for  $(a_0 \cdots a_k)$  onto the graphs for  $(b_0 \cdots b_k)$ .

If  $A$  is a line then  $E' = E$  but in general  $E' \neq E$ :  $(1, 0) (2, 0) (0, 1) (0, 2) E (1, 0) (2, 0) (-1, 1) (-1, 2)$  but the quadruples are not  $E'$  equivalent.

In Example 2 we have iterated the application of derivatives; let us introduce this formally.

**DEFINITION 2.9.** Let  $E$  be a modeloid on  $\hat{A}$  or on  $A^*$ .  $E^\alpha$ , where  $\alpha$  is an ordinal, is defined by induction as follows:  $E^0 = E$ ;  $E^{\alpha+1} = (E^\alpha)'$ ;  $E^\alpha = \bigcap \{E^\beta \mid \beta < \alpha\}$  for limit  $\alpha$ .

By Propositions 2.1(a) and 2.3,  $E^\alpha$  is a modeloid for each  $\alpha$ . Of course, we cannot be getting new modeloids indefinitely. If  $A$  is finite then after finitely many steps, that is for some  $n \in N$ , we find that  $E^{n+1} = (E^n)' = E^n$ . Similarly, if  $A$  is countable we have a countable  $\alpha$  with  $(E^\alpha)' = E^\alpha$  and analogous statements hold for other sets. This calls for definitions.

**DEFINITION 2.10.** The complexity of a modeloid  $E$  is the first  $\alpha$  such that  $(E^\alpha)' = E^\alpha$ . The modeloids of complexity 0, that is the modeloids  $E$  for which  $E' = E$ , are called basic modeloids.

Thus the modeloids in Examples 1 and 2 are basic. The modeloid  $E_n$  introduced in Example 2 has complexity  $n$ .

Let us turn to another aspect of the examples. In Example 2 the modeloid is defined on the basis of what happens to a single element and in Example 4 two  $n$ -tuples are equivalent if their subtriples satisfy a certain condition. The following definition captures this feature.

**DEFINITION 2.11.**  $E$  is a  $k$ -ary modeloid on  $\hat{A}$  iff for any  $(a_0 \cdots a_n)$ ,  $(b_0 \cdots b_n) \in \hat{A}$ ,  $(a_0 \cdots a_n)E(b_0 \cdots b_n)$  iff  $(a_{i_1} \cdots a_{i_k})E(b_{i_1} \cdots b_{i_k})$  for any  $0 \leq i_1 < i_2 < \cdots < i_k \leq n$ .

Thus the modeloids in Examples 2, 3, and 4 are 1-ary (or unary), 2-ary (or binary) or 3-ary (or ternary) respectively. For modeloids on  $A^*$  we have to add the condition that  $a_i = a_j$  iff  $b_i = b_j$ .

**PROPOSITION 2.12.** *All unary modeloids are basic.*

**PROOF.** We have to show that if  $E$  is unary then  $E' = E$ . We assume that  $E$  is on  $\hat{A}$ . So let  $a, b \in \hat{A}$  and  $x \in A$  be such that  $aEb$  and  $x$  does not occur in  $a$ . Let  $a = (a_1 \cdots a_n)$  and  $i_1, \dots, i_k$  be those for which  $xEa_{i_j}$  holds. This shows that there are more than  $k$  elements in the equivalence class of  $x$ . The elements  $b_{i_1}, \dots, b_{i_k}$  are also in the equivalence class of  $x$  and if  $j \notin \{i_1, \dots, i_k\}$  then  $b_j \notin x/E$ . Consequently, we can find  $y \in x/E - \{b_{i_1}, \dots, b_{i_k}\}$ . Then  $y$  does not occur in  $b$ . Because  $xEy$  and  $E$  is unary we have  $axEby$ .

The converse does not hold as can be seen on Example 1. The groups  $H$  which create unary modeloids are very special but do exist.

We also have the corollary that the derivative of a unary modeloid is unary and we might ask whether the derivative of a  $k$ -ary modeloid is  $k$ -ary or at

least  $m$ -ary for some  $m \in N$ . This is not so.

**PROPOSITION 2.13.** *There is a binary modeloid whose derivative is not  $k$ -ary for any  $k \in N$ .*

**PROOF.** We shall give an intuitive sketch. Let  $A$  be the set of nonempty open connected subsets of the plane. We shall need only countably many of them. Think of  $(a_0 a_1 \cdots a_n) \in \hat{A}$  as a Venn diagram. We define

$$(a_0 a_1 \cdots a_n)E(b_0 \cdots b_n) \text{ iff } (a_i \cap \cdots \cap a_j \neq 0 \text{ iff } b_i \cap \cdots \cap b_j \neq 0).$$

$E'$  then equalizes two such diagrams just in the case when the corresponding intersections behave (as to being empty or nonempty) in the same way. That is  $E'$  is a true isomorphism of Venn diagrams. But  $E'$  cannot be  $k$ -ary for we can certainly find two diagrams  $(a_0 \cdots a_k), (b_0 \cdots b_k)$  such that in both of them intersections of  $k$  members are nonempty (so if  $E'$  were  $k$ -ary they would be  $E'$ -equivalent) and  $a_0 \cap \cdots \cap a_n \neq 0$  while  $b_0 \cap \cdots \cap b_n = 0$ . Then:  $(a_0 \cdots a_k)\overline{E'}(b_0 \cdots b_k)$ .

There is another notion related to 'arity' which is more important for us because it is preserved under the derivative.

**DEFINITION 2.14.** A modeloid  $E$  over  $A^*$  (or over  $\hat{A}$ ) is finitary if for every  $n \in N$ ,  $nA^*/E$  is finite. That is, the words of length  $n$  fall into finitely many equivalence classes.

The modeloids of Examples 2, 3 and 4 are finitary.

**PROPOSITION 2.15.** *If  $E$  is finitary so is  $E'$  (and, consequently,  $E^n$  for every  $n < \omega$ ).*

**PROOF.** We show that  $nA^*/E'$  is finite. Let  $a \in nA^*$  and  $a' = \{ax/E \mid x \in A\}$ . We claim:  $aE'b$  iff  $l(b) = n$  and  $a' = b'$ . The implication from left to right is clear. In the other direction if  $x \in A$  is given,  $ax/E \in a' = b'$  so there is  $y \in A$  such that  $ax/E = by/E'$  i.e.  $axEby$ . If  $E$  is finitary  $a'$  is included in the finite set  $(n+1)A^*/E$  so there are only finitely many possible values for  $a'$ .

Along the lines of Proposition 2.12 we have

**PROPOSITION 2.16.** *If  $E^\omega$  is finitary then  $E^\omega$  is basic.*

**PROOF.** Let  $n \in N$  and let us watch the sequence  $E^k$  on  $nA^*$ . Because  $E^k \subseteq E^m$  if  $k \geq m$  and because  $E^\omega$  is the intersection of all  $E^k$ 's and is supposed to be finite we see that there is  $m$  such that for each  $k \geq m$ ,

$$nA^*/E^k = nA^*/E^m.$$

Let  $f(n)$  be the least such  $m$ . The function  $f$  is increasing since if  $nA^*/E^k \neq nA^*/E^{k+1}$  then the same happens for  $n' \geq n$ . By its definition  $f$  has the property that for any  $a, b \in A^*$  of length  $n$



$$aE^\omega b \text{ iff } aE^k b \text{ for some } k \geq f(n).$$

Now, we want to show  $(E^\omega)' = E^\omega$ . If  $x \in A$ ,  $aE^\omega b$ , and  $l(a) = n$  then, because  $aE^{f(n+1)+1}b$ , we have  $y \in A$  such that  $axE^{f(n+1)+1}by$ . But by what we have just established  $axE^\omega by$ . Therefore  $E^\omega$  is basic.

Turning the statement around we have that if  $E$  has complexity larger than  $\omega$  then  $E^\omega$  is not finitary and neither is the modeloid basic for  $E$ .

**3. The outline of a modeloid.** In this section we shall deal, for reasons which will be soon apparent, with modeloids on  $\hat{A}$  although we shall make remarks as to the parallel theory of modeloids on  $A^*$ .

We have seen in several arguments in the preceding section that quite a bit of information about a modeloid is contained in the equivalence classes of words of given length and in the way they fit together with equivalence classes of words of other lengths. The action of the groups may be lifted onto the equivalence classes and the result will be what we call an outline of a modeloid. We may describe this by saying that the outline is what we would see if we were looking at a modeloid from a distance. The derivative has an outline too, it is more detailed, we look from a shorter distance. The outlines of successive derivatives get more and more details until we arrive at the outline of the basic modeloid where no further resolution of the picture is possible.

Now the formal definition.

**DEFINITION 3.1.** The outline of a modeloid  $E$  on  $\hat{A}$  is the triple  $O(E) = (T_E, \leq_E, F_E)$  where:

- (i)  $T_E = \hat{A}/E$ .
- (ii)  $a/E \leq_E b/E$  iff  $l(a) \leq l(b)$  and there is an initial segment  $c$  of  $b$  of length  $l(a)$  such that  $aEc$ .
- (iii)  $F_E$  is the action: it assigns to  $p \in n!$  and  $a/E$ , where  $l(a) = n$ , the equivalence class  $ap/E$ . Instead of  $F_E(p, a/E)$  we write  $a/Ep$ .

**PROPOSITION 3.2.** (a)  $(T_E, \leq_E)$  is a tree, that is,  $\leq_E$  is a partial order with the least element and the set of predecessors of an element is linearly ordered by  $\leq_E$ .

(b) The classes  $a/E$ , where  $a \in n\hat{A}$ , form the  $n$ th level of the tree.

(c) The height of the tree is  $n + 1$  if  $|A| = n$  and it is  $\omega$  if  $A$  is infinite.

Let us now draw the tree-part of the outline of the modeloids in Examples 2 and 3. (See Figures 3.1 and 3.2.)

The tree on the left-hand side has  $n + 1$  elements on the  $n$ th level and its branches are as shown. The other tree has  $n!$  elements on the  $n$ th level.

Regarding the action let us make a definition applicable to any tree.

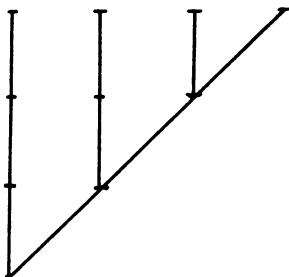


FIGURE 3.1

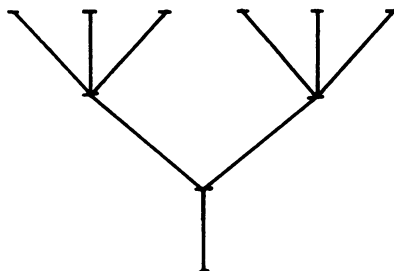


FIGURE 3.2

**DEFINITION 3.3.**  $nT$  is the  $n$ th level of a tree  $T = (T, \leq_T)$ . If  $t \in nT$  we also say that the length of  $t$  is  $n$  and write  $l(t) = n$ . If  $X \subseteq T$  we define the trace of  $X$  on the  $n$ th level as the set

$$\text{Tr}(X, n) = \{t \in nT \mid t \leq x \text{ for some } x \in X\}.$$

We denote  $\text{Tr}(\{t\}, n)$  by  $\text{Tr}(t, n)$ .

**PROPOSITION 3.4.** (a) *The actions described in 3.1(iii) are well defined.*

(b) *The actions have the following two properties.*

(P1) *If  $p_i = q_i$  for  $i < n$  where  $p, q \in l(t)!$  then  $\text{Tr}(tp, n) = \text{Tr}(tq, n)$ .*

(P2) *If  $r, s \in T_E$  there are  $t \in T_E$  and  $p \in l(t)!$  such that  $r \leq_E t$  and  $s \leq_E tp$ .*

**REMARK 3.5.** Both properties are formulated so that they make sense for any tree of height  $\leq \omega$ . (P1) relates to the fact that the identification of the group  $n!$  as a subgroup of  $m!$  for  $m \geq n$  is consistent with the actions. (P2) is a kind of coherency property. It turns out (see below) that these properties are characteristic of the outlines of modeloids; hence we define

**DEFINITION 3.6.** An outline is a tree  $(T, \leq_T)$  of height  $\leq \omega$  together with a function which assigns to each level  $nT$  of the tree an action of the group  $n!$  onto the level and the actions satisfy properties (P1) and (P2) of Proposition 3.4(b). When referring to an outline we shall usually mention only the tree-part of it.

**PROPOSITION 3.7.** *Let  $(T, \leq_T)$  be an outline. The height of the tree is equal to the length of any of its branches. (A branch of a tree is a maximal linearly ordered subset of  $T$ .)*

**PROOF.** Let  $b$  be a branch of  $T$ . If  $b$  is infinite or is equal to the height of the tree there is nothing to prove. Otherwise  $b$  has a maximal element, say  $r$ , and there is  $s \in T$  whose level is higher than that of  $r$ . By (P2) there must be  $t \in T$  with  $r \leq t$  and a permutation  $p$  such that  $s \leq tp$ . It is obvious that  $t$  is therefore strictly above  $r$ ,  $r <_T t$  so  $b$  is not maximal.

**PROPOSITION 3.8.** *Let  $(T, \leq_T)$  be an outline.*

(a) *if  $p \subseteq q$  and  $s \leq_T t$  then  $sp \leq_T tq$ .*

(b) *if  $pi = i$  for  $i < n$  then  $\text{Tr}(t, n) = \text{Tr}(tp, n)$ .*

(c) *if  $s, t \in T$ ,  $\text{Tr}(t, n) = \text{Tr}(s, n)$  then  $\text{Tr}(tp_1, n) = \text{Tr}(sp_2, n)$  provided the permutations  $p_1$  and  $p_2$  both include a permutation  $p \in n!$ .*

**PROOF.** All properties follow easily from (P1). The only thing which should be said is that we tacitly assume that whenever we apply  $p$  to  $t$  we have  $p \in l(t)!$ . Also  $p \subseteq q$  means that the permutation  $p \in n!$ ,  $q \in m!$  with  $n \leq m$ . If we write a permutation in  $n!$  as a word of length  $n$ ,  $(p_0, p_1, \dots, p_n)$ , then (a) says that if  $p$  is an initial segment of  $q$  and  $s$  is an initial segment of  $t$  then  $sp$  is an initial segment of  $tq$ .

In the case when  $A$  is finite the outline of any modeloid on  $\hat{A}$  is, by Proposition 3.7, of height  $|A| + 1$ . We therefore have the last level in the outline and the actions on lower levels may be computed from the action on the last level. This is a convenience because it enables us to separate the outline into a tree and an action on the set of its branches. We shall now proceed to show how to achieve the same effect for an arbitrary outline.

**Construction 3.9.** Let  $(T, \leq_T)$  be an outline. We shall assume that its height is infinite although the construction, with small changes, goes through for a tree of finite height.

The last level of  $T$  is defined to be

$$NT = \{b | b \text{ is a branch of } T\}.$$

If  $T$  had height  $n \in N$  then  $\omega T$  essentially is and we define it to be the last level of  $T$ , i.e.  $(n + 1)T$ . Note that if  $b \in NT$  then  $\text{Tr}(b, n)$  consists of a single element which we denote by  $nb$ . Hence  $nb \in nT$ .

We shall define an action of  $N!$  onto  $NT$  from the actions of  $n!$  onto  $nT$ . Let us take  $p \in n!$  where  $n \in N$  and  $b \in NT$ . We define  $bp$  to be the (unique) branch of  $T$  extending the set  $\{(kb)p | k \geq n\}$ . The set in question is linearly ordered by  $\leq_T$  by property (P1), or more to the point, by 3.8(a) and our conventions: if  $k \leq k'$  then  $kb \leq_T k'b$ ; the permutation  $p$ , considered in  $k!$ , is included in its natural extension  $p'$  in  $k'!$ , hence  $(kb)p \leq_T (k'b)p'$ .

Given arbitrary  $p \in N!$  and  $b \in NT$  we take  $n \in N$  and  $q \in k'$  such that  $pi = qi$  for  $i < n$ .  $q$  and  $k$  are not uniquely determined; the only condition on  $k$  is that  $k \geq pi$  for  $i < n$ . However, if take some other  $q' \in k'!$  satisfying  $q'i = pi$  we get, by (P1),  $n(bq) = n(bq')$ . We denote the element thus constructed by  $nc$ . If  $n \leq n'$  then  $nc \leq_T n'c$  again by (P1). The elements  $\{nc | n \in N\}$  thus determine a branch  $c$  which we define to be the result of the action of  $p$  onto  $b$ ,  $c = bp$ .

The last level of the tree has a natural topology whose basis consists of  $\{b \in NT | nb = t\}$  where  $t \in nT$ .

**PROPOSITION 3.10.** *The topology of  $NT$  defined in 3.9 is Hausdorff and its basis consists of clopen sets. The action is jointly continuous, i.e. the map  $(t, p) \rightarrow tp$  from  $NT \times N!$  into  $NT$  is continuous. The actions on the levels of  $T$  may be recovered from the action on  $NT$ .*

**EXAMPLE 3.11.** Let  $E$  be a modeloid on  $\hat{N}$  defined by

$$(n_1 \cdots n_k)E(m_1 \cdots m_k) \text{ iff } n_i \equiv m_i \pmod{2} \ (i = 1, \dots, k).$$

The tree part of the outline of  $E$  is isomorphic to the binary tree and hence the last level of the outline is homeomorphic to the Cantor space. A point  $b$  on the last level can be considered as a sequence 0's and 1's. If  $p \in N!$  then the coordinates of  $bp$  are those of  $b$  permuted by  $p$ .

**EXAMPLE 3.12.** In the case of the modeloid from Example 2 we see that the last level is homeomorphic to the space  $N \cup \{\infty\}$  (see the diagram of the outline following 3.2). The action on  $n \in N$  is evaluation, i.e.  $np = p(n)$ , and  $\infty$  is fixed by all  $p \in N!$ ,  $\infty p = \infty$ . On this example we can also see that the last level does not determine the outline. From Example 1.2 we find that the outline of  $E_n$  is different from the outline of  $E$  yet the last levels are isomorphic with respect to both the topology and action.

**EXAMPLE 3.13.** We shall discuss the last level of the outline for an infinite linear order. On this we wish to show that we can get a grasp of the last level for even a fairly complicated outline and that the last level gives interesting examples of action spaces.

Let us start by showing that the last level in this case may be identified with  $D = \prod_{n \in N} (n + 1)$  the topology being the usual product topology. As we mentioned any infinite linear order yields the same outline. So by selecting the order of the rational numbers to simplify matters there is no loss of generality. The simplification is due to the fact that if  $b \in NT$  then there is a sequence  $b_0, \dots, b_n, \dots$  of rationals such that

$$nb = (b_0 \cdots b_{n-1})/E.$$

Given such  $b \in NT$  we define  $f_b \in D$  by

$$f_b(n) = i \text{ iff } b_n \text{ is the } i\text{th element in } \{b_0, \dots, b_{n-1}\}.$$

E.g. if  $b_n = n$  then  $f_b(n) = n$  for each  $n$  and if  $b_n = 1/(n + 1)$  then  $f_b(n) = 0$  for each  $n$ .

In general,  $f_b \in D$  and it is clear that the map is one-to-one. It will illustrate matters further if we define its inverse (hence showing that  $b \rightarrow f_b$  is onto). Let  $f \in D$  be given. Assume we have markers  $m_0, m_1, \dots$ . We construct a linear order on them by placing  $m_n$  on the  $f(n)$ th place among the markers  $\{m_0, \dots, m_{n-1}\}$ . The branch  $b$  for which  $f = f_b$  is determined by  $nb = (m_0 \cdots m_{n-1})/E$ . It is clear that the topologies are homeomorphic under this map.

Now about the action. Intuitively it can be described by taking  $f \in D$  and  $p \in N!$ , constructing the linear order out of the markers as prescribed by  $f$ , renaming them according to  $p$  and then  $fp$  is that  $g \in D$  which gives rise to the renamed order.

It is clear that  $fp \in D$ . We shall not show that this correctly transfers the action from  $NT$  onto  $D$ ; it is fairly straightforward. But we mention that it would be a laborious exercise to prove directly from the definition given above that that map  $(f, p) \rightarrow fp$  is indeed an action.

The space  $D$  has a natural measure on its Borel sets. It is the product measure of the measures  $\mu_n$  for  $n > 0$  defined by

$$\mu_n(\{i\}) = 1/n \quad (i < n).$$

By a theorem of Ryll-Nardzewski [10] and 3.10 the orbits of the points under  $N!$  are Borel sets. They correspond to the isomorphism types of countable linear orders. The set of functions in  $D$  which "construct" the order of the rationals is an orbit and it has  $\mu$ -measure 1. The action is therefore ergodic. We shall, perhaps, return to these remarks elsewhere.

In extending the action from the outline to its last level we have used only (P1). What is the significance of (P2)? Before we answer that we have to remind the reader of some terminology from topological dynamics. Given a space  $X$  and a group  $G$  acting on it, the action is called point transitive if for some  $x \in X$

$$xG = \{xg \mid g \in G\} \text{ is dense in } X.$$

It is called transitive when for two nonempty open subsets  $U$  and  $V$  of  $X$  there is  $g \in G$  such that  $Ug \cap V \neq \emptyset$ . Finally, a universally transitive action is one under which for any  $x, y \in X$  there is  $g \in G$  such that  $xg = y$ .

**PROPOSITION 3.14.** *Let  $(T, \leq_T)$  be an outline.*

- (a) *The action of  $N!$  onto  $NT$  defined in 3.9 is transitive.*
- (b) *If  $T$  is countable the action is point transitive.*
- (c) *If  $T$  is finite the action is universally transitive.*

**PROOF.** Let us prove (a). Two nonempty open sets  $U, V$  may be replaced by two basic open sets  $U_r (= \{b \in NT \mid r \in b\})$  and  $U_s$ . Using (P2) we find  $t \in T$  such that  $r \leq_T t$  and  $s \leq_T tp$  for some  $p \in l(t)!$ . Let  $b \in NT$  be a branch containing  $t$ .  $b \in U_r$  and by the definition of the action  $s \in bp$ . Hence  $U_s \cap U_r p \neq \emptyset$ .

For (b) we enumerate the points of  $T$  thus:  $t_0, t_1, t_2, \dots, t_n, \dots$ . Using (P2) we construct by induction a sequence  $b_0, b_1, \dots, b_n, \dots$  of elements of  $T$  and permutations  $p_n \in l(b_n)!$  such that

$$b_0 \leq_T b_1 \leq_T \dots \leq_T b_n \leq_T \dots \quad \text{and} \quad t_n \leq_T b_n p_n.$$

The branch  $b$  determined by the sequence of  $b_n$ 's has the property that its orbit is dense in  $NT$ .

Item (c) follows from (a) because if  $T$  is finite its last level has the discrete topology.

This result used (P2) only; in conjunction with (P1) we can reverse the implications in (b) and (c).

**PROPOSITION 3.15.** (a) *If  $b \in NT$  then for any  $n \in N$ ,  $\text{Tr}(bN!, n)$  is countable.*

(b) *If the action of  $N!$  onto  $NT$  is point transitive then  $T$  is countable.*

(c) *If the action is universally transitive and  $|NT| \geq 2$  then  $T$  is finite.*

**PROOF.** The proofs of (a) and (b) are easy.

Regarding (c) let us remark that the assumption  $|NT| \geq 2$  is necessary because the outline of the modeloid on  $\hat{N}$  which identifies words of the same length is isomorphic to the order of natural numbers hence its last level has only one element and is (trivially) universally transitive. We show first of all that if  $NT$  is universally transitive then  $nT$  is finite for each  $n \in N$ . If not then for some  $n \in N$ ,  $nT$  is infinite and we shall for simplicity, assume  $n = 1$ . Using (P1) and (P2) we may find  $b \in NT$  such that  $\text{Tr}(bN!, 1) \neq t_0$  where  $t_0 \in 1T$  is a fixed element. Assuming we have  $b_n \in nT$  such that  $\text{Tr}(b_n n!, 1) \neq t_0$  we choose  $t$  in

$$1T - (\{t_0\} \cup \text{Tr}(b_n n!, 1))$$

and use (P2) to get  $s \geq_T b_n$  such that for some permutation  $p$   $t \leq_T sp$ . Say  $p(0) = k$ . We know  $k \geq n$ . Now we take  $q \in l(s)!$  such that  $q(k) = n \& q(n) = k$  and  $q(i) = i$  elsewhere. Then by 3.8(b)  $b_n \leq_T sq$  and we define  $b_{n+1} = \text{Tr}(sq, n+1)$ . It is clear that this gives us the required branch  $b$ . Now the action cannot be universally transitive because for no  $p \in N!$  is  $bp = b'$  where  $b' \in NT$  is such that  $\text{Tr}(b', 1) = t_0$ .

Now by the assumption,  $|nT| \geq 2$  for some  $n \in N$ . We again assume  $n = 1$ . Let  $b \in NT$  and look at  $p_k \in N!$  which switches 0 and  $k$  and fixes everything else. For some  $t \in 1T$ ,  $\text{Tr}(bp_k, 1) = t$  for infinitely many  $k$ 's. Now using the argument employed above we can find a branch  $c \in NT$  such that  $\text{Tr}(cn!, 1) = t$ . As  $|nT| \geq 2$  the action cannot be universally transitive so (c) is proved.

**REMARK 3.16.** If  $T$  is countable  $NT$  is a complete separable metric space so 3.14(b) follows from 3.14(a) and a classical result which says that in a transitive action on such space the set of transitive points is a dense  $G_\delta$  set.

Let us mention one more fact about the outlines which points out to the importance of finitary modeloids.

**PROPOSITION 3.17.** *Let  $T$  be an outline and  $X \subseteq NT$ . Then  $X$  is compact iff  $X$  is closed and for each  $n \in N$   $\text{Tr}(X, n)$  is finite. Hence  $NT$  is compact iff  $nT$  is finite for each  $n \in N$ . A modeloid  $E$  is finitary iff the last level of its outline is compact.*

**PROOF.** Imitate the proof of the fact that a subset of  $R^n$  is compact iff it is bounded and closed.

**4. Weak isomorphism and existence.** We now turn to a justification of the notion of outlines. That is, we shall show that for every (countable) outline we have a modeloid with that outline. The key to the existence of such a modeloid is 3.14(b) which we proved directly but traced in 3.16 to the Baire Category Theorem.

As we also wish to discuss the question of uniqueness we need to define the notion of isomorphism of two modeloids. There are several ways of doing this because a modeloid, being planted on a set with an intrinsic structure, is not a mere equivalence relation. When the semigroup structure as well as the equivalence relation is preserved we shall have isomorphisms. When the semigroup structure is preserved only marginally we talk of weak isomorphisms. We do not discuss at all still weaker concepts. Formally:

**DEFINITION 4.1.** Modeloids  $E_1$  and  $E_2$  on  $\hat{A}_1$  and  $\hat{A}_2$  respectively are called isomorphic if there is a one-to-one map  $f$  of  $A_1$  onto  $A_2$  such that for any  $(a_1 \cdots a_k), (b_1 \cdots b_k) \in \hat{A}_1$  we have:

$$(a_1 \cdots a_k)E_1(b_1 \cdots b_k) \text{ iff } (fa_1 \cdots fa_k)E_2(fb_1 \cdots fb_k).$$

We say that they are weakly isomorphic if their outlines are isomorphic, i.e. if there is a one-to-one map of one outline onto the other which preserves the partial order and the actions.

Weak isomorphism is a definitely weaker notion than isomorphism, because any two infinite (countable or not) modeloids of linear orders have the same outline.

**PROPOSITION 4.2.** *Let  $(T, \leq_T, F)$  be an outline and  $b \in NT$ . The equivalence relation  $E_b$  defined below is a modeloid. In case the orbit of  $b$ ,  $bN!$ , is dense in  $NT$  the outline of  $E_b$  is isomorphic to  $(T, \leq_T, F)$ . If  $E$  is a countable modeloid whose outline is  $(T, \leq_T, F)$  then  $E \simeq E_b$  for some  $b \in NT$ .*

**PROOF.** Let  $b \in NT$ . Denote  $nb$  by  $b_n$  and  $\{b_n | n \in N\}$  by  $B$ . Let  $v \in \hat{B}$  where  $v = (b_{i_0} \cdots b_{i_{k-1}})$ . Choose a permutation  $p \in N!$  such that  $p(j) = i_j$  for  $j < k$ . Let  $f(v) = \text{Tr}(bp, k)$ . Note that by (P1) the definition of  $f(v)$  does not depend on the choice of  $p$ . For  $v, w \in \hat{B}$  we define

$$vE_b w \text{ iff } f(v) = f(w).$$

We claim that  $E_b$  is a modeloid on  $\hat{B}$ . That it is an equivalence relation

preserving length is clear.  $(\beta)$  is also satisfied. To prove  $(\gamma)$  we assume  $vEw$  and let  $q \in I(v)!$ . We have  $p_1, p_2 \in N!$  such that

$$f(v) = \text{Tr}(bp_1, k) = \text{Tr}(bp_2, k) = f(w).$$

Then, by (P1),  $\text{Tr}(bp_1q, k) = (bp_2q, k)$ . As  $\text{Tr}(bp_1q, k) = f(vq)$  and similarly for  $w$  we get  $vqEwq$ .

The function  $f$ , by its definition, can be lifted to  $\hat{B}/E$ . Because  $(\beta)$  is true of  $E_b$  we have

$$v/E \leq_E w/E \text{ iff } f(v/E) \leq_T f(w/E).$$

Thus  $f$  is an embedding of the outline of  $E_b$  into the given outline. When it happens that  $bN!$  is dense in  $NT$  we see that  $f$  is onto  $T$ , hence it is the required isomorphism.

If  $E$  is a modeloid over a countable set  $A = \{a_n | n \in N\}$  let  $b$  be the branch determined by  $\{a_0 \cdots a_n/E | n \in N\}$ . Then we have  $E \simeq E_b$ .

**COROLLARY 4.3.** *Every countable outline is an outline of a countable modeloid.*

**PROOF.** By 4.2 and 3.14(b).

**PROPOSITION 4.4.** *If  $(T, \leq_T, F)$  is an outline and  $b, b' \in NT$  are in the same orbit, then  $E_b$  is isomorphic to  $E_{b'}$ . The converse does not hold.*

**REMARK 4.5.** The fact that the converse in 4.4 does not hold depends on automorphisms of outlines. In fact, the equivalence relation  $b \sim b'$  iff  $E_b \simeq E_{b'}$  can be determined from the orbits of  $NT$  and the automorphism classes of  $NT$  at least for  $b$  such that  $bN!$  is dense in  $NT$ . Assume, for such  $b$ , that  $E_b \simeq E_{b'}$ . Then for some  $p \in N!$  we have that the function  $g$  defined by  $g(nb) = p(n)b'$  is an isomorphism of  $E_b$  and  $E_{b'}$ .

Let  $f$  be the function defined in 4.2 for  $b$  and let  $f'$  be defined similarly for  $b'$ . Then  $f' \circ \bar{g} \circ f^{-1}$  (where  $\bar{g}$  stands for the natural extension of  $g$  onto the outline of  $E$ ) is an automorphism of  $T$ . We can extend it (uniquely) to an automorphism  $F$  of  $NT$ . Then  $b'p = F(b)$ . We can describe this in words by saying that  $b \sim b'$  iff an automorphism of the outline carries  $b$  into the orbit of  $b'$ .

The automorphism group of an outline is finite when we deal with an outline of a  $k$ -ary (for some  $k \in N$ ) and finitary modeloid. We do not know whether we need to assume any kind of arity, to prove this.

Let us now turn to the question whether weak isomorphism may imply strong isomorphism and under what conditions is this implication valid.



**PROPOSITION 4.6.** *Let us have two countable modeloids which are weakly isomorphic. They are isomorphic if*

- (a) *they are basic or*
- (b) *they are finite.*

**PROOF.** Let  $E$  and  $F$  be countable modeloids on  $A$  and  $B$  respectively, let  $T_E$  and  $T_F$  be their outlines and let  $f: T_E \rightarrow T_F$  be an isomorphism of the outlines. By 3.2(c),  $|A| = |B|$ .

To prove (a) we choose and fix enumerations of  $A$  and  $B$ . By a back-and-forth argument we proceed inductively to find  $\{a_n | n \in N\} = A$ ,  $\{b_n | n \in N\} = B$  so that:

- (i)  $f((a_0 \cdots a_n)/E) = (b_0 \cdots b_n)/F$  and
- (ii) if  $n$  is even  $a_n$  is the first (in the fixed enumeration of  $A$ ) different from  $a_i$  for  $i < n$
- (iii) if  $n$  is odd  $b_n$  is the first different from  $b_i$  for  $i < n$ .

Assume  $a_i$  and  $b_i$  are constructed for  $i < n$  and  $n$  is even. Then we choose  $a_n$  according to (ii). We look at  $f((a_0 \cdots a_n)/E) = t$  and choose  $b_n$  so that  $t = (b_0 \cdots b_n)/F$ . This can be done because  $F$  is basic:  $t = (c_0 \cdots c_n)/F$  and by inductive assumption  $(c_0 \cdots c_{n-1})E(b_0 \cdots b_{n-1})$ , hence there is  $b_n$  such that  $(c_0 \cdots c_n)E(b_0 \cdots b_n)$ . We treat the case when  $n$  is odd similarly, this time using the fact that  $E$  is basic. By (i)–(iii) the map  $g: A \rightarrow B$  defined by  $g(a_n) = b_n$  is an isomorphism of  $E$  and  $F$ .

If we assume that  $E$  and  $F$  are finite (in fact it is enough to assume that only one is finite) we choose a branch  $b$  on the last, say the  $n$ th, level of the outline of  $E$ . Then there is a sequence  $(a_1 \cdots a_n) \in \hat{A}$  such that  $b = (a_1 \cdots a_n)/E$ . Similarly, there is a sequence  $(b_1 \cdots b_n) \in \hat{B}$  such that  $f(b) = (b_1 \cdots b_n)/F$ . Note that  $A = \{a_1, \dots, a_n\}$  and  $B = \{b_1, \dots, b_n\}$ . The map sending  $a_i$  to  $b_i$  is defined on  $A$  and is onto  $B$ . It is an isomorphism of  $E$  and  $F$  because the action on the  $n$ th level is universally transitive (by 3.14(c)) and  $f(bp) = f(b)p$ .

**COROLLARY 4.7.** *Let  $(T, \leq_T, F)$  be an outline. There is at most one countable basic modeloid with this outline. If  $T$  is finite there is exactly one modeloid with this outline.*

**PROOF.** This follows from 4.6. Of course, phrases like “there is exactly one modeloid” mean that there is exactly one isomorphism type of modeloids, etc.

Corollary 4.7 contains an obvious question about the outlines of basic modeloids. We shall answer it in the next section.

**REMARK.** To illustrate the situation of Corollary 4.7 note that the outline of linear orders (Example 3.12) does admit a basic modeloid, namely the modeloid of the linear order of the rationals. According to Corollary 4.7 this modeloid is the unique countable basic modeloid with that outline.

Now we wish to put our intuitive ideas about outlines on firm footing. In the beginning of §3 we had a picture of the outlines of successive derivatives of a modeloid as giving more and more information about the modeloid itself. That this is indeed so is the content of the next theorem where, as it turns out, we show that in fact the knowledge of the final product and its relation to what we start with suffices to determine the modeloid.

**DEFINITION 4.8.** Let  $E$  be a modeloid and  $E^b$  be its basic modeloid, i.e.  $E^b = E^\alpha$  where  $\alpha$  is the first such that  $E^\alpha$  is basic. The resolvent of  $E$  is the triple  $(O(E), O(E^b), \pi)$  where  $O(E)$  and  $O(E^b)$  are the outlines of  $E$  and  $E^b$  respectively and  $\pi$  is the projection of  $O(E^b)$  onto  $O(E)$  defined by  $\pi(w/E^b) = w/E$ . We denote the resolvent of  $E$  by  $R_E$ .

**PROPOSITION 4.9.** *Every countable modeloid is determined by its resolvent.*

**PROOF.** Let  $E$  and  $F$  be modeloids over countable sets  $A$  and  $B$  and assume that their resolvents  $R_E$  and  $R_F$  are isomorphic. This means that there are maps  $f$  and  $g$  such that:

- (i)  $f$  is an isomorphism of  $O(E)$  and  $O(F)$ ,
- (ii)  $g$  is an isomorphism of  $O(E^b)$  and  $O(F^b)$ ,
- (iii) for every  $v \in \hat{A}$   $f(\pi_E(v/E^b)) = \pi_F(g(v/E^b))$ .

By 4.6(a) there is an isomorphism  $h$  of  $E^b$  and  $F^b$  and, by condition (i) of the proof of 4.6, we can choose it so that for every  $v \in \hat{A}$   $g(v/E^b) = h(v)/F^b$  where  $h(v) = (ha_0, \dots, ha_n)$  if  $v = (a_0 \cdots a_n)$ . The following commutative diagram summarizes the relationships and shows that  $h$  may serve as an isomorphism between  $E$  and  $F$ :

$$\begin{array}{ccccc}
 \hat{A}/E & \xleftarrow{\pi_E} & \hat{A}/E^b & \xleftarrow{\quad} & A \\
 \downarrow f & & \downarrow g & & \downarrow h \\
 \hat{B}/F & \xleftarrow{\pi_F} & \hat{B}/F^b & \xleftarrow{\quad} & \hat{B}
 \end{array}$$

Proposition 4.9 says that if we are given two outlines one of them being a projection of the other then there is at most one modeloid such that its outline and the outline of its basic modeloid fit the prescribed outlines. We have to say at most one modeloid because not every outline is an outline of a basic

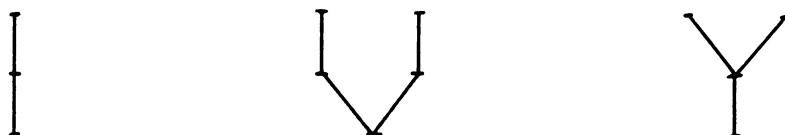
modeloid. Thus in order to describe resolvents intrinsically we need to characterize the outlines of basic modeloids which we accomplish in the next basis modeloid.

**5. Basic modeloids and representation.** We have seen on a number of occasions already that the basic modeloids play a prominent role in our study. In this section we determine their outlines and represent them using subgroups of  $N!$  As it turns out Example 1.1 is a typical example of a modeloid.

**DEFINITION 5.1.** An outline is basic if there is a basic modeloid with that outline. An outline is absolutely basic if every modeloid with that outline is basic.

**REMARK 5.2.** Among finite outlines the notions of basic and absolutely coincide. The reason is 4.6 which implies that there is only one modeloid for a given finite outline. An absolutely basic outline also has only one modeloid, now because of 4.7. An example of an absolutely basic outline is  $(N, <, F)$  where the actions are, of course, trivial. We shall see later that e.g. the binary tree is also an absolutely basic outline.

The problem of characterizing basic outlines may be illustrated on outlines of finite modeloids. Take the case of modeloids on  $\{a, b\}$ . There are just three of them so we have three outlines which are schematically depicted below



The action is completely determined as it must, by 3.14(c), be universally transitive. The first two outlines are basic, the third is not (its derivative is the second outline). The failure of its not being basic stems from the fact that the branches are rather near to each other yet we need a “radical” action to bring one onto the other. Similar difficulty is present in other outlines.

We formalize the concept which emerged for any action of a metric group onto a metric space.

**DEFINITION 5.3.** Let  $(X, d)$  be a metric space and  $(G, m)$  be a metric group acting on  $X$ . We say that the action is smooth at  $x \in X$  if for every  $\epsilon > 0$

$$\{y \in X | d(x, y) < \epsilon\} \subseteq \overline{\{xg | m(e, g) < \epsilon\}}$$

where  $e \in G$  is the identity. We say that an invariant  $Y \subseteq X$  is smooth if the action of  $G$  onto  $Y$  is smooth at every point of  $Y$ .

**REMARK.** The definition depends on the metrics. A topological definition might be phrased so that an action is smooth at  $x \in X$  if it is smooth at  $x$  for some metrization of the topologies.

The notion is very natural. It says, speaking loosely, that the action is smooth at  $x$  whenever every  $y$  close to  $x$  can be approximated by motions of  $x$  close to the identity (so there is no "cliff" between  $x$  and  $y$  around which we need to go in order to get to  $y$ ).

In our case where we deal with  $NT$  and  $N!$  (or  $nT$  and  $n!$  if the outline is finite) we have natural metrics on the spaces defined by:

$$d(b, c) = 1/(n + 1) = m(p, q)$$

where  $b, c \in NT$ ,  $b \neq c$  and  $n$  is the first  $m$  such that  $mb \neq mc$ ;  $p, q \in N!$  and  $n$  is the first  $m$  such that  $p(m) \neq q(m)$ . When talking of smooth action of  $N!$  onto  $NT$  we shall mean it with respect to these metrics.

**PROPOSITION 5.4.** *All the transitions (i.e. the maps  $x \rightarrow xp$ ) are uniformly continuous with respect to the metric  $d$ .*

**PROPOSITION 5.5.** *Let  $T$  be an outline,  $b \in NT$ . The following are equivalent:*

- (i)  $E_b$  is basic,
- (ii) the orbit of  $b$  is smooth.

**PROOF:** Assume (i). The construction of  $E_b$  may be found in 4.2. In order to show that the orbit of  $b$  is smooth it is enough to take  $q \in N!$  such that  $\text{Tr}(b, k) = \text{Tr}(bq, k)$  for some  $k \in N$ , and then for a given  $m > k$  find  $p \in N!$  such that  $\text{Tr}(bp, m) = \text{Tr}(bq, m)$ . This can be done by induction on  $m$ . We show it for  $m = k + 1$ ; the general case is entirely similar. By the definition of  $E_b$  we have

$$(b_0 \cdots b_{k-1})E_b(b_{q(0)} \cdots b_{q(k-1)}).$$

Because  $E_b$  is basic there is  $j \geq k$  such that

$$(b_0 \cdots b_{k-1}b_j)E_b(b_{q(0)} \cdots b_{q(k)}).$$

Hence taking  $p \in N!$  such that  $p(i) = i$  for  $i < k$  and  $p(k) = j$  we get  $\text{Tr}(bp, k + 1) = \text{Tr}(bq, k + 1)$ .

Now assume (ii) and let

$$(b_{n_0} \cdots b_{n_{k-1}})E_b(b_{m_0} \cdots b_{m_{k-1}}).$$

This means that there are  $p, q \in N!$  such that  $p(i) = n_i$  and  $q(i) = m_i$  for  $i < k$  and  $\text{Tr}(bp, k) = \text{Tr}(bq, k)$ . Let  $x \notin \{b_{n_0}, \dots, b_{n_{k-1}}\}$  be given. Then  $x = b_{n_k}$  and we may assume that  $p(k) = n_k$  as this will not change the trace of  $b$  on  $kT$ . Now as the orbit of  $b$  is smooth and  $bp$  is at most  $1/(k + 1)$  from  $bq$  there is  $q' \in N!$  such that  $q'(i) = i$  for  $i < k$  and

$$\text{Tr}(bp, k + 1) = \text{Tr}(bqq', k + 1). \quad (1)$$

Let  $y = b_{qq'(k)}$ . Considering the fact that there are permutations extending the order of indices,  $(n_0, \dots, n_{k-1}, n_k)$  and  $(m_0, \dots, m_{k-1}, qq'(k))$ , namely  $p$  and  $qq'$  and satisfying (1) we find that, by the definition of  $E_b$ ,

$$(b_{n_0} \cdots b_{n_{k-1}} x) E_b (b_{m_0} \cdots b_{m_{k-1}} y).$$

Hence  $E_b$  is basic.

**PROPOSITION 5.6.** *A countable outline  $T$  is basic iff there is a dense and smooth orbit of  $NT$  under the action of  $N!$ . It is absolutely basic iff every dense orbit is smooth.*

**EXAMPLE.** Let us illustrate 5.5 on the outline of the mod 2-modeloid which we discussed in Example 3.11 and where we found the outline to be isomorphic to the binary tree. The action on the last level (the Cantor space) is defined by  $(i_0, i_1, \dots)p = (i_{p(0)}, i_{p(1)}, \dots)$  so it is easy to see that there is only one dense orbit which contains sequences with infinitely many 0's and 1's. The action at this orbit is smooth so the binary tree is an absolutely basic outline.

One can check easily that the outline drawn in Figure 3.1 is absolutely basic as well.

The outline of linear orders is basic but not absolutely basic.

We shall now establish a fact on uniqueness of smooth orbits under certain kind of actions. The argument highlights some of the special properties of smooth orbits. It also gives a way of constructing outlines from some actions on metric spaces.

**PROPOSITION 5.7.** *Let  $N!$  act on a complete separable metric space  $(X, d)$  such that the diameter of  $xF_k$  is  $< 1/k$  where  $F_k \subseteq N!$  is the group of permutations which fix  $i$  for  $i < k$ . Then any two dense and smooth orbits of  $X$  are isomorphic, i.e. there is a homeomorphism of  $X$  which preserves the action ( $f(x)p = f(xp)$ ) and carries one of the orbits onto the other.*

**PROOF.** Let  $B \subseteq X$  be a smooth and dense orbit (there is nothing to prove if one does not exist).

*Claim 1.* If  $x \in B$  then  $\overline{xF_k}$  is open. We show that if  $y \in \overline{xF_k}$  then

$$U_k(y) = \{z | d(y, z) < 1/k\} \subseteq \overline{xF_k}.$$

Let  $z \in U_k(y)$  and  $\varepsilon > 0$  be given. Because  $B$  is dense in  $X$  we can find  $p \in N!$  and  $q \in F_k$  such that  $d(xp, z)$  and  $d(xq, y)$  are as small as we wish. We wish them to be so small that  $d(xp, xq) < 1/k$  and  $d(xp, z) < \varepsilon$ . Owing to the fact that the action on  $B$  is smooth we can find  $q' \in F_k$  such that  $d(xp, xqq')$  is small enough to make  $d(z, xqq') < \varepsilon$ . Since  $qq' \in F_k$  and  $\varepsilon$  was arbitrary this means that  $z \in \overline{xF_k}$ .

*Claim 2.* If  $x, y \in B$  and  $d(x, y) < 1/k$  then  $\overline{xF_k} = \overline{yF_k}$ .

Because of the smoothness  $y$  is in  $\overline{xF_k}$ . By the proof of Claim 1,  $U_k(y) \subseteq \overline{xF_k}$  and by our assumption on the diameter of  $yF_k$  we have  $yF_k \subseteq U_k(y)$ . So  $yF_k \subseteq \overline{xF_k}$ .

*Claim 3.* For  $x \in B$  and any  $p, q \in N!$  the sets  $\overline{xpF_k}$  and  $\overline{xqF_k}$  are either equal or are disjoint.

If they are not disjoint let  $y$  be in both of them. By Claim 1 even  $U_{2k}(y)$  is included in both of them. There exist, therefore,  $p'$  and  $q' \in F_k$  such that  $xpp'$  and  $xqq'$  are in  $U_{2k}(y)$ . Then  $d(xpp', xqq') < 1/k$  so, by Claim 2,

$$\overline{xpF_k} = \overline{xpp'F_k} = \overline{xqq'F_k} = \overline{xqF_k}.$$

We shall now define an outline  $(T, \leq_T, F)$  whose last level will turn out to be  $X$ . For  $k \in N$  and  $p \in N!$  let  $t_k^p = (k, \overline{xpF_k})$ . Define  $t_k^p \leq_T t_m^q$  iff  $k \leq m$  and  $xpF_k \supseteq xqF_m$ . It is easy to see that  $\leq_T$  is a partial order. To see that it is a tree let  $t_{k'}^{p'}, t_k^p \leq_T t_m^q$ . If, say  $k' \leq k$ , then

$$\overline{xqF_m} \subseteq \overline{xpF_k} \subseteq \overline{xp'F_{k'}}$$

because of Claim 3 and the fact that  $F_{k'} \supseteq F_k$ . Intuitively, by Claim 3,  $xpF_{k'}$  form a partition of  $X$  and whenever we take a smaller group  $F_k$  the partition corresponding to  $F_k$  will be a refinement. Thus  $(T, \leq_T)$  is a tree and its  $k$ th level consists of the nodes  $t_k^p$  ( $p \in N!$ ). The  $k$ th level is countable as follows from separability of  $X$  and Claims 1 and 3. The whole tree is, consequently, countable.

Now fix  $z \in B$ . Given  $x \in X$  let

$$f(x) = \{(k, \overline{zpF_k}) | x \in \overline{zpF_k}\}.$$

By Claim 3,  $f(x)$  determines a branch of  $T$ ;  $f$  is a map from  $X$  into  $NT$ . The map is one-to-one because the diameters of  $zpF_k$  tend to 0. It is onto  $NT$  because given  $b \in NT$  then

$$\bigcap_{k \in N} \{ \overline{zpF_k} | (k, \overline{zpF_k}) \in b \}$$

consists of a single point  $x \in X$ , as it is an intersection of closed sets with diameters tending to 0 and  $X$  is complete. For this point we have  $f(x) = b$ . We now transfer the action of  $N!$  onto  $X$  to an action of  $N!$  onto  $NT$  by putting  $f(x)p = f(xp)$ .

The function  $f$  is continuous (in fact uniformly with respect to the standard metric on  $NT$ ) since

$$d(x, y) < 1/k \quad \text{iff} \quad \text{Tr}(f(x), k) = \text{Tr}(f(y), k). \quad (1)$$

Assuming  $d(x, y) < 1/k$  we find  $p, q \in N!$  such that  $zp$  and  $zq$  are so close to  $x$  and  $y$  respectively that  $d(zp, zq) < 1/k$ . From Claim 1 we find that  $x \in \overline{zpF_k}$  and  $y \in \overline{zqF_k}$  and by Claim 2 we get the right-hand side of (1). The other implications follow from the definition of  $f$ .

It follows from (1) that  $f(B)$  is a smooth and dense orbit of  $NT$ . Since it is dense,  $(T, \leq, F)$  with the action determined by 3.10 from the last level is a countable outline (for (P2) see 3.14). If we take  $b = f(z)$  then the modeloid  $E_b$  is basic by 5.5.

If we had another smooth and dense orbit in  $X$  we would get a branch  $b' \in NT$  whose orbit would be dense and smooth in  $NT$  so, by 5.5 the modeloid  $E_{b'}$  would be isomorphic to  $E_b$ . We now refer to 4.5 where we showed that  $b$  and  $b'$  are interchangeable by a homeomorphism of  $NT$  which preserves the action.

Proposition 5.7 is interesting from the point of view of "stratification" of actions on spaces. By a stratification, we mean, roughly, a way of attaching to the space a tree with actions on its levels so that passing to the last level results in the action space we started with. If a stratification exists, then we have a good idea about the action space because we have "finite" approximations to it. Proposition 5.7 shows that if the action is smooth, then a stratification exists *and* it is unique. With modeloids defined on semigroups, a more general result of similar nature could be proved. The result also tells us what are the last levels of outlines of basic modeloids. For this, we have a better description in terms of closed subgroups of  $N!$ .

Let us start by reminding the reader of Example 1.1. There we defined a modeloid from a group  $H \subseteq N!$  as follows

$$(n_0 \cdots n_{k-1})E_H(m_0 \cdots m_{k-1}) \text{ iff } \exists p, q \in N! \text{ such that } p(i) = n_i, q(i) = m_i \text{ for } i < k \text{ and } pq^{-1} \in H.$$

Let us note first of all that we need only to consider closed groups.

**PROPOSITION 5.8.**  $E_H = E_{\bar{H}}$  where  $\bar{H}$  is the closure of  $H$  in  $N!$ .

**PROPOSITION 5.9.** *Every countable basic modeloid is of the form  $E_H$  for some closed group  $H$  of  $N!$ .*

**PROOF.** By Example 2.5 we know that  $E_H$  is basic. Let  $E$  be a countable basic modeloid. We shall assume that  $E$  is infinite because the proof for finite modeloids is similar (however in this case we can find  $H$  in  $n!$  for some  $n \in N$  which we shall discuss in the section on finite modeloids). Let us also assume that  $E$  is on  $\hat{N}$ . Let  $T$  be the outline of  $E$ . Given  $b \in NT$  we may find  $s: N \rightarrow N$  such that

$$\text{Tr}(b, k) = (s_0 s_1 \cdots s_{k-1})/E \quad (1)$$

for every  $k \in N$ . This is done by induction. Assuming (1) we find  $s'_0, \dots, s'_k$  such that  $\text{Tr}(b, k+1) = (s'_0, \dots, s'_k)/E$ . Then using the fact that  $E$  is basic we find  $s_k$  such that  $(s'_0 \cdots s'_k)E(s_0 \cdots s_k)$ . Note that  $s$  is one-to-one. Conversely given  $s: N \rightarrow N$  which is one-to-one there is only one branch

$b \in NT$  which satisfies (1) for all  $k \in N$ . We denote the branch by  $s/E$ . The action on  $NT$  then satisfies

$$(s/E)p = sp/E \quad (2)$$

where  $sp$  is the composition of  $s$  and  $p$ ,  $sp(n) = s(p(n))$ .

Now if  $e \in N!$  is the identity,  $e(n) = n$  for all  $n \in N$ , then set  $H = \{p \in N! | p/E = e/E\}$  which is a closed subgroup of  $N!$  because, by (2), it is the group of  $p \in N!$  which fix the branch  $e/E$ .

Because  $E$  is basic  $(n_0 \cdots n_{k-1})E(m_0 \cdots m_{k-1})$  iff there are  $p, q \in N!$  such that  $p(i) = n_i$ ,  $q(i) = m_i$  for  $i < k$  and  $p/E = q/E$ . By (2)  $p/E = q/E$  iff  $(e/E)p = (e/E)q$  which happens just in the case when  $pq^{-1}$  fixes  $e/E$ , i.e.  $pq^{-1} \in H$ . So  $E = E_H$ .

Concerning uniqueness we have:

**PROPOSITION 5.10.** *Let  $H, H' \subseteq N!$  be closed subgroups. Then  $E_H = E_{H'}$  iff  $H$  and  $H'$  are conjugate.*

**EXAMPLE 5.11.** Propositions 5.9 and 5.10 imply that the isomorphism types of countable basic modeloids are in correspondence with conjugacy classes of closed subgroups of  $N!$ . In the case of the where-is-zero modeloid the corresponding conjugacy class consists of groups  $H_n = \{p \in N! | p(n) = n\}$ . In the case of the modeloid whose outline is the binary tree the conjugacy class consists of  $H_A = \{p \in N! | A, N - A \text{ infinite invariant under } p\}$ .

By 5.9 we are able to determine the outline of a basic modeloid just from the group  $H$  corresponding to it and by 5.7 all we need to do is to determine the last level. The construction of the last level is a well-known and useful construction in topological dynamics.

Let a group  $G$  act continuously on a space  $X$ . For every  $g \in G$  the  $\pi^g$  defined by  $\pi^g(x) = xg$  is in  $X^X$  which we assume to have the product topology. The enveloping semigroup of  $X$ ,  $E(X)$ , is the closure of  $\{\pi^g | g \in G\}$  in  $X^X$ . The group  $G$  then acts on  $E(X)$  by  $sg = s\pi^g$  where  $s\pi^g(x) = s(\pi^g x) = s(xg)$ . Now given a closed subgroup  $H$  of  $G$  we define an equivalence relation  $\sim_H$  on  $E(X)$  by

$$s \sim_H r \text{ iff in every open neighborhood } U \text{ and } V \text{ of } s \text{ and } r \text{ respectively, there are } \pi^g \text{ and } \pi^h \text{ such that } gh^{-1} \in H.$$

If we think of  $G$  as a subset of  $E(X)$  via the map  $g \rightarrow \pi^g$  (not necessarily one-to-one) then  $s, r \in E(X)$  are identified by  $\sim_H$  iff for any two open neighborhoods of the points there is a right coset of  $H$  which they both intersect. It is not difficult to see that  $\sim_H$  is a closed and invariant equivalence relation. We denote the space  $E(X)/\sim_H$  by  $E(X)/H$ . The action on this space is defined by  $(s/H)g = sg/H$ .



**PROPOSITION 5.12.** *The last level of the outline of  $E_H$ , where  $H \subseteq N!$  is closed, is isomorphic to  $E(N)/H$  where  $E(N)$  is the enveloping semigroup of the action by evaluation of  $N!$  onto  $N$  (i.e.  $np = p(n)$ ).*

The enveloping semigroup  $E(N)$  has a semigroup structure defined by  $pq(n) = p(q(n))$ . The group  $N!$  may be considered as a subgroup of  $E(N)$ . The action of  $N!$  onto  $E(N)/H$  may be extended to an action of  $E(N)$  onto  $E(N)/H$  by  $(s/H)p = (sp)/H$ . We just need to show that  $s \sim_H s'$  then  $sp \sim_H s'p$ . This is the same as proving that if

$$(s(0) \cdots s(k))E_H(s'(0) \cdots s'(k))$$

for each  $k$  then  $(s(p(0)) \cdots s(p(k)))E_H(s'(p(0)) \cdots s'(p(k)))$ . The last statement is true because for each  $k$  we can replace  $p$  with  $q \in N!$  agreeing with  $p$  up to  $k$ , and the statement becomes a property of modeloids.

The orbits of  $E(N)/H$ , which is the last level of the outline of  $E_H$ , need not be disjoint because  $E(N)$  is a semigroup only. In fact the orbit of  $e/E$  is equal to the last level.

**PROPOSITION 5.13.** *Let  $E$  be  $E_H$  where  $H \subseteq N!$  is closed, and let  $T$  be the outline of  $E$ . If  $b \in NT$  is in a smooth orbit then the orbit of  $b$  under the action of  $E(N)$  onto  $NT$  is closed.*

*Resumé of §5.* The process of taking successive derivatives of a modeloid may, in the light of results in this section, be termed a process of “smoothing out” or a process of stabilization if we recall that a smooth point corresponds, more or less, to a Poisson stable point in the qualitative theory of differential equations. Let us be specific and take a countable modeloid  $E$  over  $N$ . Then  $E$  is  $E_b$  for some  $b \in NT$  (by 4.2) and we may assume that  $b = \{(0, \dots, n-1)/E | n \in N\}$ . If the orbit of  $b$  is smooth then, by 5.5,  $E$  is basic, and the process is completed. If  $b$  is not a smooth point then we do take the next derivative and its effect is a remetrisation of the last level in such a way that points in the orbit near each other but needing a radical action to get closer to each other under  $F$  are pushed further apart from each other under  $E'$ . This is repeated till we achieve a smooth orbit.

Proposition 5.7 is a purely topological study of smooth orbits and its proof contains more about smooth orbits than the formulation of the statement. Specifically, see Claims 1 and 3.

In 5.8–5.10 we show that countable basic modeloids are in correspondence with conjugacy classes of closed subgroup of  $N!$ , and in 5.12 we compute the last level of the outline from the group using the concept of enveloping semigroup.

**6. Automorphisms of modeloids.** Besides the automorphisms which correspond to the two notions of isomorphisms we have one more type of automorphism which we call strong automorphisms. The three groups associated with these notions are useful for finding out properties of modeloids.

**DEFINITION 6.1.** Let  $E$  be a modeloid over  $A$ . A weak automorphism of  $E$  is an automorphism of the outline of  $E$ . The group of weak automorphisms is denoted by  $WA(E)$ .

An automorphism of  $E$  is an isomorphism of  $E$  with itself. The group of automorphisms of  $E$  is denoted by  $A(E)$ .

A strong automorphism of  $E$  is an automorphism  $f$  of  $E$  such that, for any  $(a_1 \cdots a_n) \in \hat{A}$ ,  $(a_1 \cdots a_n)E(fa_1 \cdots fa_n)$ . The group of strong automorphisms of  $E$  is denoted by  $SA(E)$ .

The basic relations between these three groups are described in:

**PROPOSITION 6.2.** *Let  $E$  be a modeloid over  $A$ . Then*

- (i)  $SA(E)$  is a normal subgroup of  $A(E)$ ,
- (ii) The map  $W: A(E) \rightarrow WA(E)$  defined by

$$W(g)((a_1 \cdots a_n)/E) = (g(a_1) \cdots g(a_n))/E$$

is a homomorphism of  $A(E)$  into  $WA(E)$  whose kernel is  $SA(E)$ .

**PROOF.** Given a permutation  $g$  of  $A$  there is a unique extension  $\bar{g}$  of  $g$  onto  $\hat{A}$  satisfying  $\bar{g}(uv) = \bar{g}(u)\bar{g}(v)$ . It is defined by:

$$\bar{g}(a_1 \cdots a_n) = (g(a_1) \cdots g(a_n)).$$

For ease in notation we shall denote the extension by  $g$  again.

To prove (i) let  $g \in SA(E)$  and  $h \in A(E)$ . Given  $u \in \hat{A}$  we have  $h^{-1}(u) \cdot Egh^{-1}(u)$  and  $uEhgh^{-1}(u)$  showing that  $hgh^{-1} \in SA(E)$  so  $SA(E)$  is a normal subgroup of  $A(E)$ .

Let  $g \in A(E)$  and let  $h = W(g)$  where  $W$  is defined as in (ii). We need to show that  $h \in WA(E)$ . Clearly  $h$  is a permutation of the outline  $T$  of  $E$ . If  $t \leq_T s$  then  $t = (a_1 \cdots a_n)/E$  and  $s = (b_1 \cdots b_m)/E$  where  $n \leq m$ . By the definition of the outline  $(a_1 \cdots a_n)E(b_1 \cdots b_n)$  so  $g(a_1 \cdots a_n) \cdot Eg(b_1 \cdots b_n)$  hence  $h(t) \leq_T h(s)$ . If  $p \in l(t)!$  then

$$h(tp) = g(a_{p(1)} \cdots a_{p(n)})/E = g(a_1 \cdots a_n)p/E = h(t)p.$$

Thus  $h \in WA(E)$ . Similar computations show that  $W$  is a homomorphism. Finally,  $W(g)$  is the identity of  $WA(E)$  iff for every  $u \in \hat{A}$ ,  $uEg(u)$  which is iff  $g \in SA(E)$ .

**COROLLARY 6.3.** *If  $W: A(E) \rightarrow WA(E)$  is onto  $WA(E)$  then*

$$A(E)/SA(E) \simeq WA(E).$$

In general  $A(E)/SA(E)$  is isomorphic to a subgroup of  $WA(E)$ .

We shall now state a condition which characterizes the modeloids for which the homomorphism is onto  $WA(E)$ . Although we shall be able to make the condition more elegant later when we shall deal with other properties of modeloids, it will now suffice for proving the “onteness” of  $W$  for finite and basic modeloids.

**PROPOSITION 6.4.** *Let  $E$  be a countable modeloid on  $A$ . The homomorphism  $W: A(E) \rightarrow WA(E)$  is onto iff whenever  $a: N \rightarrow A$  (or  $a: n \rightarrow A$ ) is a one-to-one enumeration of  $A$  and  $b \in NT$  is such that  $b = \{(a_0 \cdots a_n)/E | n \in N\}$  then for any  $h \in WA(E)$  there is a one-to-one enumeration  $c: N \rightarrow A$  (or  $c: n \rightarrow A$ ) such that  $h(b) = \{(c_0 \cdots c_n)/E | n \in N\}$ .*

**PROOF.** In stating the condition we are taking the liberty of extending the automorphism  $h$  to the last level of the outline  $T$  of  $E$ . The extension is defined by  $h(b) = \{h(t) | t \in b\}$  and it is easy to see that  $h(bp) = h(b)p$  and that  $h$  is a homomorphism of  $T$  and even better, it is an isometry:

$$\text{Tr}(b, k) = \text{Tr}(c, k) \quad \text{iff} \quad \text{Tr}(h(b), k) = \text{Tr}(h(c), k).$$

Now, assume  $W$  is onto  $WA(E)$  and let  $h \in WA(E)$ . So there is  $g \in A(E)$  such that  $h(u/E) = g(u)/E$  for every  $u \in \hat{A}$ . Assuming  $a: N \rightarrow A$  is an enumeration of  $A$  without repetitions and  $b = \{(a_0 \cdots a_{n-1})/E | n \in N\}$  we then take  $c: N \rightarrow A$  defined by  $c_i = g(a_i)$ . Clearly  $h(b) = \{(c_0 \cdots c_{n-1})/E | n \in N\}$ .

Conversely, let  $h \in WA(E)$  be given and assume that  $a, b$  and  $c$  are as in the condition. Define then  $g(a_i) = c_i$ . Let  $u = (a_{i_0} \cdots a_{i_{k-1}})$  and  $p \in N!$  be such that  $p(j) = i_j$  for  $j < k$ . Then

$$h(u/E) = h(\text{Tr}(bp, k)) = \text{Tr}(h(b), k)p = g(u)/E$$

the equalities being true since  $h \in WA(E)$ .

**PROPOSITION 6.5.** *Let  $E$  be a countable modeloid. If  $E$  is finite or basic then the homomorphism  $W$  is onto  $WA(E)$ .*

**PROOF.** If  $E$  is finite the condition of 6.4 is satisfied since for any  $b$  in the last level of the outline there is an enumeration of  $A$  whose “type”  $b$  is.

Let  $E$  be basic and infinite and let  $a: N \rightarrow A$  enumerate  $A$  without repetitions. Let us remind the reader of a fact we have used many times already. If we are given  $b_0, \dots, b_n \in A$  and  $(a_0 \cdots a_{n-1})E(b_0 \cdots b_{n-1})$  then there is a permutation  $p \in N!$  such that  $p(i) = i$  for  $i < n$  and  $\text{Tr}(ap, n+1) = (b_0 \cdots b_n)/E$ . We expressed this so that it would be clear that if  $b = \{(a_0 \cdots a_{n-1}) | n \in N\}$  and  $h \in WA(E)$  then the branch  $d = h(b)$  has

the same property

$$\begin{aligned} \text{If } \text{Tr}(d, n) < t \in (n+1)T \text{ there is } p \in N! \text{ such that } p(i) = i \\ \text{for } i < n \text{ and } \text{Tr}(dp, n+1) = t. \end{aligned} \quad (1)$$

We shall now define by induction an enumeration  $c: N \rightarrow A$  of  $A$ . We choose as  $c_0$  any element of  $h(a_0/E)$  and assume  $c_i$  has been defined for  $i < n$  so that  $h((a_i|i < n)/E) = (c_i|i < n)/E$ . We now take the first element in the enumeration  $a$  which does not appear in  $\{c_i|i < n\}$ , say it is  $x$ . Let  $t = (c_0 \cdot \dots \cdot c_{n-1}x)/E$ . The condition of (1) is fulfilled so there exists  $p \in N!$  which fixes  $i < n$  and  $\text{Tr}(dp, n+1) = t$ . Assume that  $p(n) = k$  and  $p^{-1}(i) < m$  for  $i \leq k$ . Now using the fact that  $E$  is basic we define a sequence  $d_i$  for  $i < m$  such that  $d_i = c_i$  for  $i < n$ ,  $d_n = x$ , and  $(d_i|i < m) = \text{Tr}(dp, m)$ . This being done we define  $c_i$  as  $d_{p^{-1}(i)}$  for  $i \leq k$ . Note that as  $p(i) = i$  for  $i < n$  there is no conflict with what we had previously. Also  $c_k = d_{p^{-1}(k)} = d_n = x$ . Finally

$$\text{Tr}(d, k+1) = \text{Tr}(\text{Tr}(dp, m)p^{-1}, k+1) = (c_i|i \leq k)$$

the middle term making sense because if  $i \leq k$  then  $p^{-1}(i) < m$  so, as far as  $\text{Tr}(dp, m)$  is concerned,  $p^{-1}$  is a permutation in  $m!$ . Proceeding inductively we shall achieve an enumeration  $c$  of  $A$  satisfying the condition  $\text{Tr}(d, k) = (c_i|i < k)$  of 6.4 thus proving that  $W$  is onto  $WA(E)$ .

REMARK 6.6. Since  $h$  on the last level is an isometry and  $b \in NT$  as above is in a dense and smooth orbit so is  $d = h(b)$ . However, it can be shown by an example that this fact alone does not yield the enumeration  $c$ . We do need the stronger property (1), which, fortunately, is also preserved under automorphism of  $T$ .

COROLLARY 6.7. *If  $E$  is finite or countable and basic then  $WA(E) \simeq A(E)/SA(E)$ .*

PROOF. By 6.5 and 6.3.

EXAMPLE 6.8. Corollary 6.7 enables us to compute the group  $WA(E)$  if the outline of  $E$  is basic. We can then choose a basic modeloid  $F$  with that outline and find  $A(F)$  and  $SA(F)$ . Then  $WA(E) = A(F)/SA(F)$ . In the case of the outline of infinite linear orders (see Figure 3.2) we choose the modeloid  $F$  corresponding to the order of the rationals. Then  $g \in A(F)$  iff for any two rationals  $a < b$   $g(a) < g(b)$  or for any two rationals  $a < b$   $g(a) > g(b)$ . Also  $g \in SA(F)$  iff  $a < b$  implies  $g(a) < g(b)$ . Thus even without knowing the groups we do know that  $A(F)/SA(F)$  is the cyclic group of order 2 and by the above remark this is the group of automorphisms of the outline of Figure 3.2. The “switch” of the two elements on  $2T$  of the outline determines a nontrivial automorphism of  $T$  and there are no other nontrivial automorphisms.

We shall now discuss the relations between the groups of  $E$  and the groups associated with its derivatives.

**PROPOSITION 6.9.** *Let  $E$  be a modeloid over  $A$ . Then*

- (i)  $A(E) \subseteq A(E')$ ,
- (ii)  $SA(E) = SA(E')$ ,
- (iii)  $\cup \{A(E^\alpha) | \alpha < \beta\} \subseteq A(E^\beta)$ ,
- (iv)  $SA(\cap \{E | E \in M\}) = \cap \{SA(E) | E \in M\}$  where  $M$  is a set of modeloids on  $A$ .

**COROLLARY 6.10.** *If  $\alpha < \beta$  then  $A(E^\alpha) \subseteq A(E^\beta)$  and  $SA(E^\alpha) = SA(E^\beta)$ .*

In general, there does not seem to be a rigid connection between  $WA(E)$  and  $WA(E')$ . When it is finite, then, by 6.5 and 6.8, we have  $WA(E) \subseteq WA(E') \subseteq \dots$  where  $\subseteq$  here means an isomorphic embedding.

In the preceding section we expressed every basic modeloid on  $N$  in the form  $E_H$  where  $H \subseteq N!$  is a closed group. We should be able to compute all three automorphism groups of  $E$  just from  $H$  and this is what we do in:

**PROPOSITION 6.11.** *Let  $H$  be a closed subgroup of  $N!$  and let  $E$  be  $E_H$ . Then  $SA(E) = H$ ,  $A(E)$  is the normalizer of  $H$  in  $N!$  (the largest group  $\subseteq N!$  in which  $H$  is normal), and  $WA(E) \simeq A(E)/H$ .*

Let us recall that by  $E^b$  we denote the basic modeloid for  $E$ ;  $E^b = E^\alpha$  where  $E^\alpha$  is basic.

**PROPOSITION 6.12.** *Let  $E$  be a countable modeloid on  $A$ . Then for any  $u, v \in \hat{A}$*

$$uE^bv \text{ iff for some } g \in SA(E) \ g(u) = v.$$

$E^b$  is the largest basic modeloid included in  $E$ .

**Application 6.13.** We know, by 6.9, that if  $E = E_H$  where  $H$  is a normal subgroup on  $n!$  or  $N!$  then any permutation of the set on which the modeloid  $E$  is defined is in fact an automorphism of  $E$ . One can then easily imagine that there are just a few of such modeloids and, therefore, just a few normal subgroups of  $n!$ . The last fact is, of course, well known; its proof is not trivial and involves (usually) Sylow's Theorem. What we intend to do now is to describe the modeloids  $E_H$  with  $H$  normal, thus getting a different look at the normal subgroups of  $n!$ . We shall not avoid making some computations but they will be simple and of a kind different from the ones employed in group theory. In fact, the only fact from group theory we need to know is that  $n!$  is generated by transpositions.

So let  $E$  be a basic modeloid on  $A$  such that any permutation of  $A$  is an automorphism and also assume that  $E$  is neither the modeloid which identifies everything nor the modeloid which distinguishes everything. Thus  $uEv$  for

some  $u \neq v$  and  $u\bar{E}v$  for some  $u, v \in \hat{A}$  of the same length. Note that if  $aEb$  holds iff  $a = b$  for  $a, b \in A$  then, because  $E$  is basic  $uEv$  iff  $u = v$  for  $u, v \in \hat{A}$ . Thus for some  $a, b \in A, a \neq b$  we have  $aEb$ . Now taking a permutation which fixes  $a$  and exchanges  $b$  with some  $x \neq a$  we get  $aEx$ . Thus for any  $x, y \in A$   $xEy$ . If  $|A| \leq 2$  we are done. In fact, we can go on higher with this procedure if  $A$  has enough elements, specifically,  $E$  has to identify all sequences of length  $|A| - 2$ . For this reason we shall now assume that  $A = \{i | i < n + 2\}$ , we denote  $(0, 1, \dots, n - 1)$  by  $e$  and by  $x$  and  $y$  we denote  $n$  and  $n + 1$ . We also assume that  $E = E_H$  where  $H$  is a nontrivial normal subgroup of  $(n + 2)!$ , whose unit we denote by  $e_{n+2}$ . We know from 5.9 that  $H = \{p \in (n + 2)! | pEe_{n+2}\}$  where we treat a permutation  $p \in (n + 2)!$  as the word  $(p(0), \dots, p(n + 1))$  of  $\hat{A}$ ; we look similarly at  $p \in k!$ . Let  $G = \{p \in n! | pEe\}$ . If  $p, q \in G$  then, because  $E$  is basic (see e.g. 5.9) there are  $\bar{p}, \bar{q} \in H$  which extend  $p$  and  $q$ . Then  $\bar{p}\bar{q} \in H$ , i.e.  $\bar{p}\bar{q}Ee_{n+2}$  so by property  $(\beta)$  of modeloids  $pqEe$ . Similarly if  $p \in G$  then  $p^{-1} \in G$  so  $G$  is a subgroup of  $n!$ .

*Claim 1.*  $G = n!$ .

By what we mentioned it is enough to show that every transposition belongs to  $G$ . For this it is enough to show that  $eE(1, 0, 2, \dots, n - 1)$  because if we have this then taking the permutation which switches 0 and  $i$  and 1 and  $j$  respectively and fixes everything else we get  $(ij \dots)E(ji \dots)$  which, by property  $(\gamma)$  and the definition of  $G$ , means that the transposition  $i \leftrightarrow j$  is in  $G$ . Thus, to prove the claim, we show by induction on  $k < n, k \geq 1$ , that  $(0, 1, \dots, k)E(1, 0, \dots, k)$ . If  $k = 1$  this amounts to  $0E1$  which we already checked. Assume that  $k \geq 2$  and  $uEv$  where  $u = (0, 1, \dots, k - 1)$  and  $v = (1, 0, \dots, k - 1)$ . Since  $E$  is basic for some  $z \in A$   $ukEvz$ . If  $z = k$  we are done so let  $z \neq k$ . Note that neither  $k$  nor  $z$  appears in  $u$  or  $v$ . We then have

$$uxEv \quad \text{and} \quad vyEvx. \quad (1)$$

The first equivalence follows from  $ukEvz$  by taking a permutation which sends  $k$  to  $x$  and  $z$  to  $y$ . The second follows since sending  $z$  to  $x$  we get  $ukEvx$  and sending  $z$  to  $y$  we get  $ukEv y$ , hence  $vyEukEv x$ . Now (1) implies  $uxEv x$  so sending  $x$  to  $k$  we get  $ukEv k$  which ends the proof of Claim 1.

*Claim 2.* If  $u \in \hat{A}$  has length  $n$  then  $uEup$  for any  $p \in n!$ . This means  $u$  is  $E$ -equivalent to any arrangement of its members.

To prove this we just need to take  $q \in (n + 2)!$  such that  $q(i)$  is the  $i$ th member of  $u$ , use Claim 1 to see that  $eEp$  and then the fact that  $q$  is an automorphism of  $E$ .

*Claim 3.* If  $n \geq 3$  (so  $|A| \geq 5$ ) and  $u \in \hat{A}$  is of length  $n$  then  $uEe$ .

We shall use Claim 2 liberally without referring to it. Also note that it is enough to show  $uxEe$  and  $vyxEe$  for some  $u$  and  $v$  of length  $n - 1$  and  $n - 2$

respectively and consisting of numbers  $< n$  for the rest follows by the usual permutation argument.

Let  $u = (k | k < n - 2)$ ;  $n - 2$  and  $n - 1$  do not appear in  $u$  and we denote them, for ease in notation, by  $i$  and  $j$ . Since  $uiEuj$  we have  $uxEuy$  (by  $i \leftrightarrow x$  and  $j \leftrightarrow y$ ). Thus, because  $E$  is basic we have  $z \in A$   $uxiEuyz$ , and, of course,  $z = x$  or  $z = i$  or  $z = j$ .

Case 1.  $z = x$ . Exchanging  $x$  and  $j$  we have  $ujiEujy$ , and also  $ujiEujx$ .

Case 2.  $z = i$ . We exchange  $y$  and  $j$  getting  $uixEuij$ .

Case 3.  $z = j$ . We have  $uixEujy$ . Exchanging first of all  $i$  and  $y$  we get  $uxyEe$  so in this case if  $v \in \hat{A}$  has length  $n - 2$   $vxyEe$ . Now we use the assumption that  $n \geq 3$ ; it tells us that  $u$  is nonempty so 0 occurs in  $u$ . Take the permutation of  $A$  which sends 0 to  $y$ ,  $y$  to  $i$ , and  $i$  to 0 and fixes everything else ( $0 \rightarrow y \rightarrow i \rightarrow 0$ ). Then  $uix$  becomes, after rearranging,  $vxy$  and  $ujy$  becomes  $wy$  with  $x$  not appearing in  $w$ . So  $vxyEwy$  and in conjunction with the above equivalence we get  $wyEe$ .

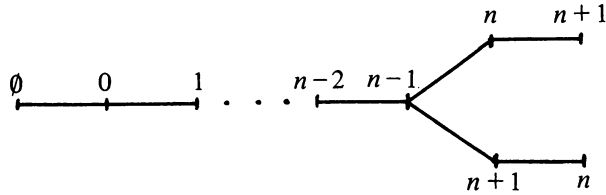
Now if  $u = vxy$  has length  $n$  and contains both  $x$  and  $y$  then it does not contain, say,  $i < j < n$ . By the first part we have  $vxEvi$  so for some  $z$   $vxyEviz$ . Since  $viz$  contains at most one of  $x$  and  $y$  we have  $eEvizEvxy$ .

Let  $u = vz$ ,  $z \in A$ , be of length  $n + 1$ . If  $vz\bar{E}ex$  then, since  $E$  is basic, and  $vEe$  by Claim 3, we must have  $vzEey$ ,  $y$  being the only remaining element available. This means that there are at most two equivalence classes of words of length  $n + 1$ . If  $exEey$  then, because  $E$  is basic,  $vEw$  iff  $l(v) = l(w) = n + 2$  so  $E$  would be trivial (alternatively: if  $exEey$  then  $exyEyx$  so  $H$  contains a transposition, hence all of them, thus  $H = (n + 2)!$ ). Thus  $ex\bar{E}ey$ . This completely describes the modeloid  $E$ . The group  $H$  has the property that  $p \in H$  iff for some transposition  $q$ ,  $qp \notin H$ : if  $p \in H$  then  $exyEp$  so  $eyxEqp$  where  $q$  transposes  $x$  and  $y$ ;  $eyx \notin H$  since  $exy\bar{E}eyx$ . This is a slightly disguised form of the definition of the alternating group.

**COROLLARY 6.14.** *If  $n \geq 5$  then the alternating group is the only nontrivial normal subgroup of  $n!$ . There is no nontrivial closed normal subgroup of  $N!$ .*

**PROOF.** In view of the above only the second statement requires argument. So let  $H \subseteq N!$  be closed and normal,  $H \neq \{e\}$ , and take  $n \in N$ . Note that we can repeat the argument in Claim 1 (we just needed two extra elements here we have infinitely many) and we get that  $n! \subseteq H$ . This, of course, means that every "finite" permutation is in  $H$  so  $H$  is dense in  $N!$  hence  $H = N!$  being closed.

Notice that we also took care of the case  $n = 3$  because we do not need Claim 3 here. If  $n = 4$  then Case 3 of Claim 3 splits into two cases getting two modeloids so two normal subgroups of  $4!$ . Finally the outlines of the modeloid  $E_H$  with nontrivial normal  $H$  look like



The even permutations leave the upper branch fixed and move the lower branch onto the upper branch.

*Resumé of §6.* The process of taking successive derivatives may be described, from the point of view of the automorphism groups, in terms of normal extensions of groups. The situation looks as follows:

$$\begin{aligned} 1 &\rightarrow SA(E) \rightarrow A(E) \xrightarrow{W} WA(E) \rightarrow 1, \\ 1 &\rightarrow SA(E) \rightarrow A(E') \xrightarrow{W} WA(E') \rightarrow 1, \\ 1 &\rightarrow SA(E) \rightarrow A(E^b) \xrightarrow{W} WA(E^b) \rightarrow 1. \end{aligned}$$

The group  $SA(E)$  is fixed, the groups  $A(E^\alpha)$  increase, and the groups  $WA(E^\alpha)$  change so that the last sequence is exact. We do now know though whether the sequence becomes exact at the same time that the action becomes smooth, i.e. when we arrive at the basic modeloid.

In 6.13 we use the previous results to show that  $n!$  has only one nontrivial normal subgroup ( $n \geq 5$ ) using the fact that any permutation is an automorphism of the modeloid corresponding to such a group. The whole argument is thus an exercise in pretending.

**7. Some connections with model theory.** In this section we sketch connections with model theory where the notion of modeloid originated. The sequel to this paper will consist of elaborations and expansions of these connections.

Let us start by reminding the reader of Example 1.5. Given a structure  $M = (A, \dots R \dots)$  we define a modeloid  $E_M$ , called the modeloid of  $M$ , by declaring  $u, v \in \hat{A}$   $E_M$ -equivalent iff the map which sends the  $i$ th letter of  $u$  onto the  $i$ th letter of  $v$  is an isomorphism with respect to the relations of the structure  $M$ . We have also seen that if a modeloid  $E$  over  $A$  is given then we can define a structure  $M$  on  $A$  such that  $E_M = E$ . We can do so, for example, by taking the equivalence classes for the relations of the structure. This may not be the most economical way of capturing the spirit of the modeloid because there are structures with countably many relations whose modeloids have continuum many equivalence classes. We shall not be concerned with this problem here, the example just serves to show that for a given modeloid  $E$  there may be many structures  $M$  such that  $E_M = E$ .

Let us amplify this on a simpler example. Assume that  $E$  is unary, hence



determined by the equivalence classes of  $A$ , and that it has three equivalence classes, say,  $P_0$ ,  $P_1$  and  $P_2$ . The canonical structure  $M$  in this case is (essentially)  $(A, P_0, P_1, P_2)$ . Another structure which gives the same modeloid is  $(A, P_0, A - P_1, P_2)$ , a structure still in the same language. But the modeloid of  $(A, P_0 \cup P_1, P_0 \cup P_2)$  is also  $E$  though the structure is in a different language.

In order to be able to relate our study of modeloids to model theory we prove a definability result which is contained, in a different set-up, in the works of Fraïssé [6], Ehrenfeucht [4], and Karp [8]. Let us mention that  $L_{\infty, \omega}$  is the language which allows arbitrary disjunctions and conjunctions of sets of formulas and quantification over individuals only. The quantifier rank (or depth) of a formula is defined by induction on its complexity: for formulas without quantifiers the quantifier rank is zero;  $qr(V\{\phi | \phi \in \emptyset\}) = \sup\{qr(\phi) | \phi \in \emptyset\}$ ;  $qr(\neg\phi) = qr(\phi)$ ; only when we apply quantifiers we raise the rank, thus  $qr((\exists v)\phi) = qr((\forall v)\phi) = qr(\phi) + 1$ . We call a structure  $M$  an  $E$ -structure if  $M$  is relational and  $E_M \simeq E$ .

**PROPOSITION 7.1 (THE CORRESPONDENCE THEOREM).** *Let  $E$  be a modeloid over  $A$ ,  $M$  be an  $E$ -structure and  $a, b \in \hat{A}$  be of the same length.*

(i) *Every equivalence class of  $E^\alpha$  is definable, in  $M$ , by a formula of  $L_{|A|, \omega}$  of quantifier rank  $\leq \alpha$ . If  $E$  is finitary and  $\alpha = n < \omega$  then the classes of  $E^n$  are defined by a finite formula of rank  $\leq n$ .*

(ii) *The equivalence  $aE^\alpha b$  holds iff  $a$  and  $b$  satisfy the same formulas of  $L_{\infty, \omega}$  of quantifier rank  $\leq \alpha$ . When  $E$  is finitary and  $\alpha = n < \omega$  then  $aE^n b$  iff  $a$  and  $b$  satisfy the same finite formulas of rank  $\leq n$ .*

(iii)  *$aE'b$  holds iff  $a$  and  $b$  satisfy the same  $L_{\infty, \omega}$ -formulas with at most one quantifier. When  $E$  is finitary we can replace " $L_{\infty, \omega}$ -formulas" by "finite formulas".*

**PROOF.** We prove (i) by induction on  $\alpha$ . If  $\alpha = 0$  take the intersection of all  $n$ -ary relations on  $A$  which contain  $a$  and which can be expressed as Boolean combinations (with identification of coordinates permitted) of the relations of  $M$ . Clearly, this intersection is an intersection of at most  $|A|$  of such relations. If we take the conjunction of these  $|A|$  definitions we get a quantifier-free formula defining the equivalence class of  $a$ .

Let us assume that we have (i) for  $\alpha$  and let  $P$  be the  $E^{\alpha+1}$  equivalence class of  $a \in \hat{A}$ ,  $P = a/E$ . Let  $\{Q_i | i \in I\}$  be a list of  $\{ax/E^\alpha | x \in A\}$ . By the inductive assumption each  $Q_i$  has a definition  $\phi_i(v, u)$  where  $\phi_i$  is a formula of  $L_{\kappa, \omega}$  of quantifier rank  $\leq \alpha$ ,  $v$  is a string of variables of length  $l(a)$  and  $u$  is a variable. Let  $\phi(v)$  be the formula

$$\bigwedge_{i \in I} (\exists u) \phi_i \wedge (\forall u) \bigvee_{i \in I} \phi_i. \quad (1)$$

It is clear that  $\phi$  has quantifier rank  $< \alpha + 1$ . It is a formula of  $L_{\kappa, \omega}$  because  $\{Q_i | i \in I\}$  forms a partition of a subset of  $(n + 1)\hat{A}$ , so  $|I| < |A|$ . Now if  $b \in n\hat{A}$  satisfies (1) then  $bE^{\alpha+1}a$ : if  $x \in A$  then, by (1), for some  $i \in I$  we have  $\phi_i(b, x)$  in  $M$ . By the definition of  $\phi_i$ 's this means that for some  $y \in A$   $\phi_i(a, y)$ . Since  $\phi_i$  defines the class of  $ay$  we get  $bx E^\alpha ay$ . If, on the other hand, we have  $y \in A$  then for some  $i \in I$   $ay/E^\alpha$  is definable by  $\phi_i$ . By (1)  $b$  satisfies in  $M$   $(\exists u)\phi_i(b, u)$ , hence for some  $x$   $bx E^\alpha ay$ . By the definition of the derivative we have  $aE^{\alpha+1}b$ .

If  $\alpha$  is limit and  $< |A|^+$  the equivalence classes of  $E^\alpha$  are intersections of the previous equivalence classes so we get the  $L_{\kappa, \omega}$ -definitions by inductive assumption. Finally we need not bother with  $\alpha \geq |A|^+$ , because by a simple cardinality argument the complexity of a modeloid  $E$  on  $A$  is less than  $|A|^+$ . Hence for some  $\alpha < |A|^+$   $E^\alpha$  is basic and there are no new equivalence classes in  $E^\beta$  for  $\beta > \alpha$ .

If  $E$  is finitary, then using an appropriate inductive assumption and 2.13 we get that the set  $I$  which figured in the proof above is finite and so is the formula (1).

The proof of (ii) goes by induction as well and it is clear that we need to worry about the successor step only. Assume, therefore,  $aE^{\alpha+1}b$  and let  $\phi(v)$  be a formula of rank  $\alpha + 1$  ( $v$  is a sequence of variables of length  $l(a)$ ). The formula  $\phi$  may be a negation or conjunction of some other formulas, and we pass over these cases easily and get to the issue, namely to a formula of the form  $(\forall u)\psi(v, u)$  where  $\psi$  has quantifier rank  $\alpha$ . We, therefore, assume that  $\phi$  has the form  $(\forall u)\psi$  and that  $M \models \phi[a]$ . Since we want to check  $M \models (\forall u)\psi(b, u)$  we choose  $x \in A$  and apply the assumption  $aE^{\alpha+1}b$  to get  $y \in A$  such that  $ay E^\alpha bx$ . Then  $M \models \psi[a, y]$  and since  $qr(\psi) < \alpha$  so does  $bx$ . Therefore  $M \models \phi[b]$ .

If  $a$  and  $b$  satisfy the same formulas of rank  $< \alpha + 1$  then by (i) they are in the same equivalence class of  $E^{\alpha+1}$  hence  $aE^{\alpha+1}b$ . The finitary case follows from these arguments by inspection.

Finally, to prove (iii) it is, by (ii), enough to show that if  $a$  and  $b$  satisfy the same formulas with at most one quantifier then  $aE'B$ . As in (i) enumerate  $\{ax/E | x \in A\}$ : say  $\{Q_i | i \in I\}$  is such enumeration. Each  $Q_i$  has a quantifier-free definition, say,  $\phi_i(v, u)$ . By an argument similar to the one used in (i) we get that if  $b \in \hat{A}$  satisfies all the formulas

$$\{(\exists u)\phi_i | i \in I\} \cup \{(\forall u)\bigvee_{i \in I} \phi_i\}$$

then  $bE'a$ . Since all the formulas have only one quantifier we are done; the finitary case follows easily. Note that we cannot proceed by induction since zero is the only ordinal  $\alpha$  satisfying  $\alpha + \alpha = \alpha$ . The number of quantifiers in a formula does not have much to do with the quantifier rank. A formula may

have a million quantifiers but its quantifier rank may be 1.

The derivative, in the light of 7.1, may be seen as a kind of quantification:

$aEb$  holds iff  $a$  and  $b$  are indistinguishable without the use of quantifiers.  $aE'b$  holds iff we cannot distinguish them even with one quantifier. This case is not completely typical. More informative is  $aE''b$  which means that  $a$  and  $b$  are indistinguishable by formulas with the property that if a quantifier is in the scope of another quantifier then it applies only to a quantifier-free formula.

This theorem enables us to elucidate the definition of basic modeloids.

**PROPOSITION 7.2.** *Let  $E$  be a finitary modeloid and let  $M$  be an  $E$ -structure. Then  $E$  is basic iff the theory of  $M$  admits elimination of quantifiers.*

**PROOF.** Let  $E$  be basic, thus  $E' = E$ . As is well known, in order to prove that  $T = \text{Th}(M)$  admits elimination of quantifiers it is enough to check that for every formula  $(\exists v)\phi(v, u)$ , where  $u$  is a string of variables and  $\phi$  is quantifier-free, we can find a quantifier-free  $\psi(u)$  such that

$$T \vdash (\exists v)\phi(v, u) \leftrightarrow \psi(u).$$

So let  $(\exists v)\phi(v, u)$  be given as above. It will be more convenient to use  $A^*$  in this instead of  $\hat{A}$  because it might happen that no  $a \in \hat{A}$  is such that  $M \models (\exists v)\phi(v, a)$ , but some  $b \in A^*$  does satisfy it. We would have to split the proof into considerations of several formulas depending on whether or not we can have  $u_0 = u_1$  etc. Thus let  $p_i$  for  $i < m$  be an enumeration of all equivalence classes  $P$  of  $E$  on  $A^*$  such that if  $a \in P$  then  $M \models (\exists v)\phi(v, a)$ . If there are no such classes we can take for  $\psi$  a contradiction. Otherwise, if  $a \in A^*$  is of the appropriate length and  $M \models (\exists v)\phi(v, a)$  then  $a \in P_i$  for some  $i < m$  (since  $E' = E$ ). By 7.1(i) each  $P_i$  is definable by a quantifier-free formula  $\psi_i$ ; let  $\psi$  be  $\psi_0 \vee \psi_1 \vee \dots \vee \psi_{m-1}$ . It is then immediate that  $M \models (\exists v)\phi \leftrightarrow \psi$ . Hence  $T \models (\exists v)\phi \leftrightarrow \psi$ .

Conversely, if  $T$  admits elimination of quantifiers we let  $a, b \in A^*$  be such that  $aEb$ . Given  $x \in A$  we have a quantifier-free definition of  $ax/E$ , say  $\phi$ . Then  $M \models (\exists v)\phi(v, u)$  (where  $l(u) = l(a)$ ) and since it is equivalent to a quantifier-free  $\psi(u)$  we have  $M \models \psi[a]$ . Because of 7.1(ii)  $M \models \psi[b]$ , hence  $M \models (\exists v)\phi(v, b)$ . So for some  $y \in A$  we have  $axEby$  which shows that  $E$  is basic.

**REMARK 7.3.** If  $E$  is basic but not finitary then we have elimination of quantifiers in the language  $L_{|A|^{+}, \omega}$  as can be easily seen from the proof.

If we are interested in infinite languages then the following example points out how to formulate the theorem in this case. Consider  $\text{Th}((N, +))$  (so-called Pressburger's arithmetic). It admits elimination of quantifiers after we adjoin to it the equivalence relation "mod  $k$ " for every  $k \in N$ . Since every finite formula contains only finitely many relations it is clear that we

eliminate the quantifiers if we prove: for every  $k \in N$  there exists  $m \in N$  such that  $E_m \subseteq E'_k$  where for  $n \in N$ ,  $E_n$  is the modeloid of the structure  $(N, +, \equiv_2, \dots, \equiv_n)$ .

Let us mention that 7.2 in conjunction with 4.6(a) proves that  $\text{Th}((Q, <))$  is the only theory of linear orders which admits elimination of quantifiers.

**COROLLARY 7.4.** *If  $M = (A, U_1, \dots, U_n)$  where all the relations are unary, or  $M$  is a structure in a finite language whose modeloid is the  $\omega$ th derivative of some other modeloid then  $\text{Th}(M)$  admits elimination of quantifiers.*

**PROOF.** By 2.9, 2.14, and 7.2.

We are now ready to discuss some basic aspects of models in terms of modeloids. We shall do so in an informal manner without formulating the results precisely as the reader may find their formulations in a standard reference book (e.g. [3] or [12]).

**7.5. Scott's sentences.** A Scott sentence of a model  $M$  is a sentence  $\phi$  of  $L_{\omega_1\omega}$  with the property that  $M \models \phi$  and any countable model satisfying  $\phi$  is isomorphic to  $M$ . Scott showed that any countable model (for a countable language) has such a sentence. The construction of such a sentence may be seen in the light of 4.9. According to that result the modeloid of a countable structure is described if we know the outline of the modeloid, the outline of the basic modeloid, and the projection of the basic outline onto the starting outline.

Thus, given a countable structure  $M$  we take its modeloid  $E$  and using 7.1 we write down in  $L_{\omega_1\omega}$  the following information:

(a)  $E^\alpha$  (where  $\alpha < \omega_1$  is the complexity of  $E$ ) is basic while  $E^\beta$  for  $\beta < \alpha$  is not.

(b) description of the outlines of  $E$  and  $E^\alpha$  and of the way they are related.

To write down (a) we just replace the definition concerning modeloids by its definitions in  $M$ . For example,  $uEv$  is a disjunction of  $\phi(u) \wedge \phi(v)$  with  $\phi$  ranging through the definitions of the equivalence classes of  $E$ . The last part of (b) involves describing when a definition of an equivalence class of  $E^\alpha$  implies a definition of a class of  $E$ .

If another countable structure  $N$  satisfies what (a) and (b) say then its modeloid  $F$  has isomorphic resolvent, so by 4.9  $E$  and  $F$  are isomorphic. This alone would not guarantee that  $M \simeq N$ . However, taking the natural isomorphism of the outlines defined by

$$a/E = \{x \in \hat{A} \mid M \models \phi(x)\} \rightarrow \{x \in \hat{B} \mid N \models \phi(x)\} = b/F$$

we shall ensure even the preservation of the names of the equivalence classes which means that the models themselves are isomorphic.

The resolvent is, therefore, a code for sentences. We say sentences although for a given language it codes essentially only one sentence. But it does not

depend on the language; the resolvent may be the same for structures in completely different languages.

**7.6. Automorphism.** Let  $M$  be a structure and  $E$  its modeloid. Then  $SA(E)$ , the strong automorphisms of  $E$ , are exactly the automorphisms of  $M$ : indeed, if  $g \in SA(E)$  then for any  $a, b \in M(a, b)E(g(a), g(b))$ . That means  $R(a, b)$  iff  $R(g(a), g(b))$  for any (binary) relation of  $M$ .

The reader familiar with Kueker's theorem on the size of the group of automorphisms of a countable structure will find that for a countable modeloid  $E$  the group  $SA(E)$  is either countable or has size continuum. However, the direct proof of this is easier since one does not have to mention  $L_{\omega, \omega}$  and only basic modeloids need to be considered. In conjunction with 7.1 we also get Kueker's description of the case when the group is countable.

**7.7. Elementary equivalence.** The most important notions in model theory are variants of elementary equivalence, like elementary substructures etc. The correspondence theorem characterizes the notion of elementary equivalence of two models in terms of weak isomorphisms or their derivatives, provided the models have finite language.

Let a structure  $M$  be given, let  $E$  be its modeloid and  $T_E$  its outline. There is then a natural way to label the nodes of  $T_E$  by formulas, or precisely, by sets of formulas. It is defined, by taking  $a/E \in T_E$  and assigning to it the set of all definitions of the equivalence class  $a/E$  (using 7.1 for the existence of such definitions). We may similarly label the outlines of  $T_{E^n}$  for any  $n \in \mathbb{N}$ . Given another structure, say  $K$ , in the language of  $M$ , which we assume to be finite, we have (denoting the modeloid of  $K$  by  $F$ ) that  $M \equiv K$  iff the following diagram is commutative

$$\begin{array}{ccccccc}
 T_E & \longleftarrow & T_{E'} & \longleftarrow & T_{E''} & \longleftarrow & \cdots & \longleftarrow & T_{E^n} & \longleftarrow & \cdots \\
 \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \\
 T_F & \longleftarrow & T_{F'} & \longleftarrow & T_{F''} & \longleftarrow & \cdots & \longleftarrow & T_{F^n} & \longleftarrow & \cdots
 \end{array}$$

where the arrows aiming left are the natural projections and the arrows aiming down are the maps induced by the labeling (i.e.  $t \in T_{E^n}$  labeled by  $s$  is mapped onto that  $s \in T_{F^n}$  which has the same label; if one does not exist the diagram is not commutative).

The finiteness assumption is needed for the implication from left to right only. Let  $a/E^n \in T_{E^n}$  have definition. Then because  $K \equiv M$  and  $\phi$  is finite,  $\phi$  defines in  $K$  an equivalence class of  $E^n$ ,  $b/F^n$ . If we have some other definition  $\psi$  of  $a/E^n$  then  $M \models \phi \leftrightarrow \psi$  so  $K \models \phi \leftrightarrow \psi$  hence  $a/E^n$  and  $b/F^n$

have the same label. It is easy to see this defines an isomorphism  $f$  of  $T_{E^n}$  and  $T_{F^n}$ ; if  $m \leq n$  and  $\pi$  is the projection of  $T_{F^n}$  onto  $T_{F^m}$  ( $\pi(b/F^n) = b/F^m$ ) then the label of  $a/F^m$  is the same as the label of  $\pi(f(a/F^n))$ , so the diagram commutes.

Let us now discuss the other directions. As a by-product we shall obtain a notion of a sentence for modeloids to which we shall devote more attention in the follow-up to the present paper. We assume now that  $E$  and  $F$  are modeloids of  $M, N$  respectively, and that the diagram commutes. We want to show that  $M \equiv N$ . This does follow easily from 7.1 but we wish to analyze the role of a sentence in the framework of modeloids. So let a sentence  $\phi$  be given. By elementary manipulations we may assume that  $\phi$  is in the prenex normal form, that is

$$\phi = (Q_0 v_0) \cdots (Q_{n-1} v_{n-1}) \psi(v_0 \cdots v_{n-1})$$

where  $Q_i$  for  $i < n$  is  $\forall$  or  $\exists$  and  $\psi$  is quantifier-free. Let

$$\text{True}(\psi) = \{a \in A^* \mid l(a) = n \text{ and } M \models \psi[a]\}.$$

By the definition of  $E_M$  we see immediately that  $\text{True}(\psi)$  is a union of equivalence classes  $a/E$  where  $a \in A^*$  and  $l(a) = n$ . We may, therefore, consider  $\text{True}(\psi)$  as a subset  $X$  of the  $n$ th level of the outline of  $E$ ;  $X \subseteq nT_E$ . What is then

$$\text{True}((Q_{n-1} v_{n-1}) \psi) = \{a \in A^* \mid l(a) = n-1 \text{ and } M \models (Q_{n-1} v_{n-1}) \psi[a]\}?$$

Assume that  $Q_{n-1}$  is  $\exists$ . It is then clear that  $\text{True}((\exists v_{n-1}) \psi)$  is a union of some equivalence classes  $a/E$ , where  $a \in A^*$  and  $l(a) = n-1$ : if  $M \models (\exists v_{n-1}) \psi[a]$  and  $bE'a$  then if  $x \in A$  is such that  $M \models \psi[ax]$  we find  $y \in A$  such that  $axEby$  hence  $M \models \psi[by]$  by 7.1 so  $M \models (\exists v_{n-1}) \psi[b]$ . So we may consider  $\text{True}((\exists v_{n-1}) \psi)$  as a subset of the outline of  $E'$ ,  $T_{E'}$ . Let  $\pi$  be the natural projection of  $T_{E'}$  onto  $T_E$ :  $\pi(a/E') = a/E$ . Then

$$\text{True}((\exists v_{n-1}) \psi) = \text{Tr}(\pi^{-1}(\text{True}(\psi)), n-1).$$

This suggests defining, for  $X \subseteq nT_E$ ,

$$\exists X = \text{Tr}(\pi^{-1}(X), n-1).$$

If  $Q_{n-1}$  is  $\forall$  then  $(\forall v_{n-1}) \psi$  is equivalent to  $\neg(\exists v_{n-1}) \neg \psi$  hence we define

$$\forall X = (n-1)T_{E'} - \text{Tr}(\pi^{-1}(nT - X), n-1).$$

With these definitions if  $X = \text{True}(\psi) \subseteq nT_E$  then

$$Q_{n-1} X = \text{True}((Q_{n-1} v_{n-1}) \psi) \subseteq T_{E'}.$$

This can of course be iterated once we know the projections of  $T_{E^n}$  onto  $T_{E^m}$  for  $m \leq n$ . This means that given any  $Q \in \{\forall, \exists\}^*$ , a word composed of  $\forall$  and  $\exists$ , whose length is  $n$ , and given  $X \subseteq mT_E$ ,  $n \leq m$ , we have defined  $QX$ ,

a subset of  $(m - n)T_{E^n}$ . In particular, in our case when  $Q = Q_0 \cdot \dots \cdot Q_{n-1}$  and  $X = \text{True}(\psi)$  then  $QX \subseteq OT_{E^n}$ . So  $QX$  is either the set  $\Lambda/E^n = \{\Lambda\}$  ( $\Lambda$  is the empty word) or the empty set. It is easy to see that  $M \models \phi$  iff  $QX = \{\Lambda\}$ .

Let us now take  $K$ , its modeloid  $F$ , its outline  $T_F$ , and assume that the labeling gives a commutative diagram. This means that it commutes with the projection so if  $Y \subseteq nT_F$  is the set corresponding to  $\psi$ , the isomorphism maps  $X$  onto  $Y$  and also  $\exists X$  onto  $\exists Y$  etc. Hence  $QX = \{\Lambda\}$  iff  $QY = \{\Lambda\}$  meaning  $M \models \phi$  iff  $K \models \phi$ , i.e.  $M \equiv K$ .

This analysis shows that a complete theory essentially is a sequence of outlines  $T_0, T_1, \dots$  together with projections  $\pi_n: T_n \rightarrow T_{n-1}$  and a labeling of the elements of the outlines which is provided by the language of theory. The language allows us to agree on names of the equivalence classes formed by the successive derivatives. Because modeloids have no language associated with them and therefore no natural labeling of the outlines the role of a complete theory will be played by a sequence

$$T_0 \xleftarrow{\pi_1} T_1 \xleftarrow{\pi_2} T_2 \xleftarrow{\dots} \xleftarrow{\pi_n} T_n \xleftarrow{\dots}$$

where  $T_i$ 's are outlines and  $\pi_i$ 's projections. Of course, we shall need to describe which sequences are "consistent"; not every  $\pi_1: T_1 \rightarrow T_0$  is realized as a  $\pi: T_E \rightarrow T_E$  for some modeloid  $E$ . We call this system a partial resolvent. One of our aims is to study in how many ways a partial resolvent can be extended to a resolvent, a project intimately connected with a conjecture of Vaught.

A theory which is not complete does not fix all of the outlines and the associated projections. In fact, the length of the sequence of outlines a theory describes completely is a measure of its success in deciding questions. Our preliminary investigations show that for example set theory fixes the outlines up to the fourth, perhaps to the fifth outline. This, of course, implies that any formula with five quantifiers is decidable in set theory. The seventh (or so) outline is not fixed because of the continuum hypothesis. Of course, by fixing the outline to which the continuum hypothesis pertains we are giving information about it. Is there a "largest" outline which could be fitted at this place in a manner consistent set theory?

**7.8. Realizing and omitting of types.** Let  $M$  be a structure in a finite language,  $E$  its modeloid. What is the type of a finite sequence  $a$  from  $M$ ? It is clear by the discussion in 7.7 that the set of sequences realizing the same type as  $a$  is the equivalence class  $a/E^\omega$ . So  $E^\omega$  codes up the information about the types realized in  $M$ . If  $E$ , and hence  $M$ , is defined over a countable set  $A$ , then  $E^\omega$  is again defined over  $A$ ; thus there are only countably many types realized in  $M$ , while there may be uncountably many types of  $\text{Th}(M)$ .

As we discussed in 7.7 the theory of  $M$  from the point of modeloids is the sequence, the partial resolvent

$$T_0 \xleftarrow{\pi_1} T_1 \xleftarrow{\pi_2} T_2 \xleftarrow{\dots} \dots \xleftarrow{\pi_n} T_n \xleftarrow{\dots} \dots$$

where  $T_n$  is the outline of  $E^n$  and  $\pi_n$  is the natural projection of  $T_n$  onto  $T_{n-1}$ . This system has what is usually called an inverse limit  $\bar{T}_\omega$ . The inverse limit is again an outline. The nodes of the tree are sequences  $t = (t_n | n \in N)$  such that  $t_n \in T_n$  and  $t_n = \pi_{n+1}(t_{n+1})$  for each  $n \in N$ . We define, for two such sequences, say  $s$  and  $t$ ,  $s \leq t$  iff  $s_n \leq t_n$  for each  $n \in N$ . Also  $tp = (t_n p | n \in N)$  and from these we easily see that we have an outline. There is a natural projection  $\pi_n$  of  $\bar{T}_\omega$  onto  $T_n$  defined by  $\pi^n((t_n | n \in N)) = t_n$ . The outline  $\bar{T}_\omega$  is not, usually, the outline of  $E^\omega$ , because  $\bar{T}_\omega$  may be uncountable. But the outline of  $E^\omega$ , call it  $T_\omega$ , is a part of  $\bar{T}_\omega$ , in fact it is dense in  $\bar{T}_\omega$  in the sense that if  $n \in N$  and  $t_n \in T_n$  are given there is  $t \in T_\omega$  such  $\pi^n(t) = t_n$ . The density of  $T_\omega$  in  $\bar{T}_\omega$  is not the only condition on  $T_\omega$ . Even more important property of  $T_\omega$  is that, loosely speaking, we are well on our way toward a smooth branch. To describe this more precisely,  $T_\omega$  should have a branch  $b$  such that if  $p \in N!$  and  $\text{Tr}(\pi^n(b), k) = \text{Tr}(\pi^n(bp), k)$  then for some  $q \in N!$  such that  $q(i) = i$  for  $i < k$  we have  $\text{Tr}(\pi^{n-1}(b), k+1) = \text{Tr}(\pi^{n-1}(bpq), k+1)$ . These two conditions, i.e. density and the existence of a “relatively smooth” branch characterize the possible  $\omega$ th outlines of modeloids.

With this we are ready to look at types. There are two kinds: those  $(t_n | n \in N)$  such that for some  $n \in N$   $\pi_{n+1}^{-1}(t_n)$  is a single element  $t_{n+1}$  and  $\pi_{n+1}^1(t_{n+1})$  is a single element etc.; these are the principal types and by the density condition they must all be in  $T_\omega$ . The other kind are the elements  $t = (t_n | n \in N) \in \bar{T}_\omega$  such that for every  $n$  there are at least two  $s, s' \in T_{n+1}$  such that  $\pi_{n+1}(s) = t_n = \pi_{n+1}(s')$ . These are the nonprincipal types. Using Baire Category Theorem (or a straight construction) we can always find, for any of these types, a relatively smooth branch in  $\bar{T}_\omega$  which does not get near the type, that is the type is not in the closure of the orbit of the branch.

This is the content of the omitting of types theorem put in the framework of modeloids. This approach has the advantage that it works in the same way for the outlines of  $E^\alpha$  with  $\alpha$  limit and with modifications, for outlines of  $E^\alpha$  with  $\alpha$  a successor ordinal as we shall see later.

The  $\omega$ -categoricity theorem of Ryll-Nardzewski takes the following form in terms of modeloids: a partial resolvent  $T_0 \leftarrow T_1 \dots$  of finitary outlines determines, up to isomorphism, a unique countable modeloid iff the inverse limit of it,  $\bar{T}_\omega$ , is finitary.

THE PROOF. Assuming that the partial resolvent determines a unique countable modeloid then by the omitting of types theorem  $\bar{T}_\omega$  consists only of the principal sequences. Because we deal with finitary outlines this implies, by



a simple use of König's Theorem, that  $\bar{T}_\omega$  is finitary. Conversely, if  $\bar{T}_\omega$  is finitary then it consists of principal types only. Hence, if we take a modeloid whose partial resolvent is  $T_0 \leftarrow T_1 \leftarrow \dots$  then its  $\omega$ th outline must be  $\bar{T}_\omega$  (by the density condition discussed above;  $\bar{T}_\omega$  is the bare minimum). This means that  $E^\omega$  is finitary and by 2.14 it is basic. Therefore,  $(T_0, \bar{T}_\omega, \pi^0)$  is the resolvent of the modeloid and by 4.9 this determines the isomorphism type of the modeloid.

**7.9. Notions of submodeloids.** Let  $E$  and  $F$  be modeloids over  $A$  and  $B$  respectively, and assume that  $A \subseteq B$ . Then we call  $E$  a (weak) submodeloid of  $F$  if  $F$  restricted to  $A$  is  $E$ . In other words for  $a, b \in \hat{A}$  we have  $aEb$  iff  $aEb$ . We may of course ask whether this equivalence is preserved when we pass to the derivatives of  $E$  and  $F$ , that is whether  $aE'b$  iff  $aF'b$  for  $a, b \in \hat{A}$ . We thus define

**DEFINITION 7.10.** Assuming that  $E$  and  $F$  are as above we call  $E$  a weak  $\alpha$ -submodeloid of  $F$  if for  $\beta < \alpha$  and any  $a, b \in \hat{A}$   $aE^\beta b$  iff  $aF^\beta b$ .

This notion is too weak to be preserved under unions of chains of modeloids. The reason is that a weak submodeloid may have fewer equivalence classes than its extension. We therefore define submodeloids in such a way that the outlines remain the same. Extending the notion to include derivatives we get:

**DEFINITION 7.11.** Let  $E$  and  $F$  be modeloid over  $A$  and  $B$  respectively, and let  $A \subseteq B$ . We say that  $E$  is an  $\alpha$ -submodeloid of  $F$  if it is a weak  $\alpha$ -submodeloid of  $F$  and for any  $\beta < \alpha$  and  $b \in \hat{B}$  there is  $a \in \hat{A}$  such that  $aF^\beta b$ .

**PROPOSITION 7.12.** Let  $(E_i | i \in I)$  be a chain of modeloids such that for  $i, j \in I, i \leq j$ ,  $E_i$  is  $\alpha$ -submodeloid of  $E_j$ . Then, denoting by  $E$  the union of the modeloids  $E_i, i \in I$ , we have for every  $i \in I$ ,  $E_i$  is an  $\alpha$ -submodeloid of  $E$ .

**PROOF.** We shall suppose that  $E_i$  is on  $A_i$  and that  $A$  is the union of  $A_i$ 's. We show first of all, that if  $\beta < \alpha$  then  $E_i^\beta \subseteq E^\beta$ . This is done by induction on  $\beta < \alpha$ ; it is clear for  $\beta = 0$ , in fact,  $E_i$  is a 1-submodeloid of  $E$ ; for limit  $\beta$  it follows from the inductive hypothesis. Now let  $\beta = \gamma + 1$  and assume  $E_i^\gamma \subseteq E^\gamma$  and  $aE_i^\gamma b$ . Then, given  $x \in A$  we have that  $x \in A_j$  for some  $j \in I$  and we also have  $aE_j^\gamma b$ . So there exist  $y \in A_j$  such that  $axE_j^\gamma by$  hence, by the inductive assumption,  $axE^\gamma by$ . This, by symmetricity, implies  $aE^\beta b$ .

Having this let us check that the outlines of  $E_i^\beta$  and  $E^\beta$  are the same. Taking  $a \in \hat{A}$  we have, for some  $j \in I$ ,  $a \in \hat{A}_j$ . Then because  $E_i$  is an  $\alpha$ -submodeloid of  $E_j$  we get  $b \in A_i$  such that  $aE_j^\beta b$ . Thus, by what we proved above,  $aE^\beta b$ .

It remains to prove that  $i \in I, \beta < \alpha, a, b \in \hat{A}_i$  and  $aE^\beta b$  then  $aE_i^\beta b$ . We prove it by induction on  $\beta$ , concentrating on the successor step only;

$\beta = \gamma + 1$ . Let  $x \in A_i$  be given. Then for some  $y'$  in some  $A_j$  ( $j \geq i$ ) we have  $axE^\gamma by'$ , hence by the inductive assumption and the fact that  $ax, by' \in A_j$  we have  $axE_j^\gamma by'$ . Because  $E_i$  is an  $\alpha$ -submodeloid of  $E_j$  we find  $c \in \hat{A}_i$  and  $z \in A_i$  such that  $by'E_j^\beta cz$ . This implies  $bE_j^\beta c$  hence also  $bE_i^\beta c$  so there exist  $y \in A_i$  such that  $byE_i^\gamma cz$ . We have  $axE_j^\gamma by'E_j^\beta czE_i^\gamma by$  therefore  $axE_i^\gamma by$ , so  $aE_i^\beta b$ .

If  $\alpha = \omega$  and the modeloids in the chain are finitary this, by 7.1, is the elementary chain theorem (see [3]). For other  $\alpha$ 's the result may be translated to model theory using 7.1.

7.13. *Ultraproducts*. Let  $E_i$  be a modeloid over  $A_i$  for  $i \in I$  and let  $D$  be an ultrafilter over  $I$ . We form the ultraproduct of the  $A_i$ 's in the usual way and get the set  $A$  of the equivalence classes of functions modulo  $D$ . For  $(f_j/D | j < n)$ ,  $(g_j/D | j < n)$  we define

$$(f_j/D | j < n)E(g_j/D | j < n) \text{ iff } \{i \in I | (f_j(i) | j < n)E_i(g_j(i) | j < n)\} \in D.$$

It is easy to see that  $E$  is a modeloid, in fact  $D$  just have to be a filter. We call  $E$  the ultraproduct of  $E_i$ 's modulo  $D$ , and denote it by  $\prod E_i/D$ . The fundamental theorem on ultraproducts takes this form in the modeloid set-up.

PROPOSITION 7.14.  $(\prod E_i/D)' = \prod E_i'/D$ .

Of course, by induction on  $n \in N$ , we have the corollary that if  $E$  is  $\prod E_i/D$  then  $E^n$  is  $\prod E_i^n/D$  which together with 7.1 gives the usual form of the theorem of L6s. There are some compactness results to be deduced from this but of that we shall have more to say later.

The reader may have noticed that we have not ultraproducted the semi-group part of the modeloid. This would, of course, lead us out of the realm of standard modeloids; however the construction is interesting and merits investigation.

7.15. *Homogeneous and saturated models*. Let us recall that a countable structure  $M$  over  $A$  is homogeneous if for any  $a, b \in \hat{A}$  which realize the same type in  $M$  we can find an automorphism of  $M$  which carries  $a$  onto  $b$  coordinatewise.

PROPOSITION 7.16. *If the language of  $M$  is finite and  $M$  is countable then  $M$  is homogeneous iff its modeloid  $E_M$  is of complexity  $< \omega$ .*

PROOF. Let  $M$  be homogeneous and consider  $E_M^\omega$ . By 7.7 (or 7.1) we have  $aE_M^\omega b$  iff  $a$  and  $b$  realize the same type in  $M$ . By 6.12 we know that  $E_M^\omega$  is the basic modeloid for  $E$  iff for any  $a, b \in \hat{A}$   $aE_M^\omega b$  there is  $g \in SA(E_M)$  such that  $g(a) = b$ . By 7.6 the group  $SA(E_M)$  is equal to the group of automorphisms of  $M$ , so the result follows.

The definition of saturated models depends on addition of new constants

to the language of the structure, an expansion of the language often used in model theory. These expansions may be easily imitated in terms of modeloids. Let  $B \subseteq A$ ; there is a largest modeloid,  $E_B$ , such that if  $a, b \in B$  then  $a = b$  iff  $aE_B b$ . It is defined explicitly by:  $(a_i | i < n)E(b_i | i < n)$  iff for every  $i \in I$  such that  $a_i$  or  $b_i$  is in  $B$  we have  $a_i = b_i$ . It is easy to see that  $E_B$  is basic for every  $B \subseteq A$  and that the conjugacy class corresponding to  $E_B$  on the basis of 5.9 and 5.10 contains the group of permutations of  $A$  which fix all elements  $b \in B$ .

If  $E$  is a modeloid on  $A$  and  $B \subseteq A$  then  $E \cap E_B$  corresponds to adding to an  $E$ -structure names for the elements of  $B$ .

Therefore, if  $M$  is in a finite language and countable and  $E$  is its modeloid then  $M$  is saturated iff for every finite  $B \subseteq A$  the  $\omega$ th outline of  $E \cap E_B$  is  $\bar{T}_\omega$ , the inverse limit of the outlines of  $(E \cap E_B)^n$  (see 7.8).

Regarding the proof let us just note that on the basis of what we said above it is shorter than the definition of a saturated model.

In model theory one often adds constants to the theory under study in order to define e.g. the concept of stability and similar notions. We shall leave this to our future inquiries concerning the bond of modeloids and models.

**8. Open questions.** 1. Which modeloids have antiderivatives? Formally, describe the modeloids  $E$  over a set  $A$  for which there exists a modeloid  $F$  such that  $E$  is a proper subset of  $F$  and  $F' = E$ . Regarding this investigate the uniqueness question, i.e., what can be said about  $F_1$  and  $F_2$  if  $F'_1 = F'_2$ ?

2. Is it true that if  $E \simeq E'$  then  $E$  is basic? This is easily seen to be true in the case when  $E$  is finitary. A more specific question along these lines: consider the integers,  $Z$ , and the shift function  $f, f(n) = n + 1$ . Is there a modeloid  $E$  on  $\hat{Z}$  which is not basic such that its shift is isomorphic to  $E'$ ?

3. Question 2 is connected with a problem of defining the relation "one modeloid is more complex than another". We wish to define it along the following lines:  $E$  is more complex than  $F$  if  $F$  is isomorphic to  $E^\alpha$  for some  $\alpha$ . The basic modeloids are thus the least complex modeloids. The relation is not a partial order since it might happen that  $E \simeq F^\alpha$  and  $F \simeq E^\beta$ . So  $E \simeq E^\gamma$  for some  $\gamma > 0$  and question 2 is an initial step in the inquiry whether this can happen.

4. Proposition 6.12 says that the basic modeloid for  $E$  is the largest among the basic modeloids which are included in  $E$ . Find a characterization of  $E'$  with a similar flavor.

5. Assume that  $A(E)$  is the normalizer of  $SA(E)$ . Hence, by §6,  $A(E^\alpha) = A(E)$  for every  $\alpha$ . Is the complexity of the modeloid finite?

6. More generally, can the complexity of the modeloid  $E$  be computed from the groups  $SA(E), A(E), WA(E)$ ?

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