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A 3-LOCAL CHARACTERIZATION OF $L_7(2)$

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ABSTRACT. Recent work of Gorenstein and Lyons on finite simple groups has led to standard form problems for odd primes. The present paper classifies certain simple groups which have a standard 3-component of type $L_5(2)$.

Introduction. D. Gorenstein and R. Lyons [8] have recently shown that any "minimal unknown" simple group G of characteristic 2 type with $e(G) > 4$ must satisfy one of three specific conditions. In [3], we consider a special case of one of those conditions. Here, we obtain characterizations of groups with $e(G) = 3$ which satisfy hypotheses analogous to those in that case.

We are concerned with the following hypothesis.

$\mathcal{H}_n$: G is a group, b is an element of G of order 3, and $J = O^{3'}(E(C(b)))$. Furthermore, the following conditions hold.

(a) $J/Z(J) \cong L_n(2)$;

(b) $C(J)$ has cyclic Sylow 3-subgroups;

(c) $\langle b \rangle$ is not strongly closed in $C(b)$; and

(d) $m_{2,3}(G) = m_3(C(b))$.

Briefly, $\mathcal{H}_n$ says that G has a standard 3-component of type $L_n(2)$ satisfying the Gorenstein-Lyons conditions. In the general case, the statement of the Gorenstein-Lyons conditions is somewhat more technical.

We also remark that by results of Schur [12] and Steinberg [13], either $J \cong L_n(2)$ or $n = 3$ or 4 and J is the unique central extension $\widehat{L_n(2)}$ of $L_n(2)$ by $\mathbf{Z}_2$.

The two main results in this paper are:

THEOREM A. *Let G satisfy $\mathcal{H}_4$ or $\mathcal{H}_5$ and assume that $F^*(G)$ is simple and that $b \notin_G J$. Then $b \notin F^*(G)$ and $F^*(G)$ has Sylow 3-subgroups of type $Z_9 \times Z_9$. In particular, G is not simple.*

THEOREM B. *Let G be a finite simple group of characteristic 2 type which satisfies $\mathcal{H}_5$. Then $G \cong L_7(2)$.*

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1. Preliminary lemmas. In this section we collect some properties of $L_4(2)$ and $L_5(2)$ which will be useful in the proofs of Theorems A and B. We also derive some elementary consequences of $\mathcal{H}_4$ and $\mathcal{H}_5$.

LEMMA 1.1. *Let $G \cong L_n(2)$, $n \geqslant 4$, and let t be an involution in the center of a Sylow 2-subgroup of G. Then the following statements hold.*

(a) *$C(t) = TL$ where $T = O_2(C(t))$ is extra-special of type 2_+^{2n-3} and $L \cong L_{n-2}(2)$;*

(b) *T contains 2 elementary abelian L-invariant subgroups U and V of rank $n-1$;*

(c) *L acts decomposably on U and on V;*

(d) *$C(t)$ acts on $\mathrm{Inv}(T)$ with the following orbits: $\{t\}$, $U \setminus \langle t\rangle$, $V \setminus \langle t\rangle$, $\mathrm{Inv}(T) \setminus [\mathrm{Inv}(U) \cup \mathrm{Inv}(V)]$;*

(e) *$t^G \cap T = U^{\#} \cup V^{\#} \cup \{t\}$;*

(f) *if $H = \mathrm{Aut}\, G$, then $T = O_2(C_H(t))$.*

PROOF. See Suzuki [**14**].

LEMMA 1.2. *Let $H = \mathrm{Aut}(L_n(2))$, $n = 4$ or 5, and set $G = H'$. Then $G \cong L_n(2)$ and the following conditions hold:*

(a) *G contains a Sylow 3-subgroup P of type E_9;*

(b) *$C_H(P) = P \times \langle \tau \rangle$ where $\tau \in H \setminus G$ is an involution, $C_G(\tau) \cong \Sigma_6$, and $H = G\langle \tau \rangle$;*

(c) *$N_G(P) = PD$ where D is dihedral of order 8;*

(d) *G has 2 classes of elements of order 3.*

Letting α and β be representatives of the two conjugacy classes of elements of order 3,

(e) *$N_H(\langle \alpha \rangle) = \langle \alpha, s\rangle \times K\langle \tau \rangle$ where $\langle \alpha, s\rangle \cong L_2(2)$, $K \cong L_2(4)$, and $\langle \alpha, s\rangle =_G C_K(\tau)$;*

(f) *$N_H(\langle \beta \rangle) = \langle \beta, t\rangle \times L\langle \tau \rangle$ where $\langle \beta, t\rangle \cong L_2(2)$, $L \cong L_{n-2}(2)$ and $\langle \beta, t\rangle =_G C_L(\tau)$; and*

(g) *$C_G(t)$ contains a Sylow 2-subgroup of G and $C_G(s)$ does not contain a Sylow 2-subgroup of G.*

(h) *$\langle \alpha \rangle$ and $\langle \beta \rangle$ are each contained in a subgroup of G of type $L_2(4)$.*

PROOF. These results all follow from routine computations.

LEMMA 1.3. *Assume that G, b, and J satisfy $\mathcal{H}_n$ where $n = 4$ or 5. Let $B \in \mathrm{Syl}_3(C(b))$ and set $B_0 = B \cap J$. Then $B = \langle b \rangle \times B_0$ and one of the following is true.*

(i) *$b \notin_G J$, $N(B)/C^*(B) \cong \Sigma_3 \wr Z_2$ and $\langle b\rangle^{N(B)} = \mathcal{E}_1(B) \setminus \mathcal{E}_1(B_0)$.*

(ii) *$b \in_G J$, $N(B)/C^*(B) \cong \Sigma_4$ and elements $b_1, b_2 \in B_0$ can be chosen so that $B_0 = \langle b_1, b_2\rangle$ and $N(B)/C(B)$ acts as the full monomial group on B with respect to the basis $\{b, b_1, b_2\}$.*

Furthermore, in case (ii), $N(B)$ *controls fusion in* B.

PROOF. $B = B_0 \times C_B(J)$ because $J \lhd C(b)$ and $\mathrm{Out}\, J$ is a $3'$-group. Setting $B_1 = C_B(J)$, we need to show that $B_1 = \langle b \rangle$. By assumption B_1 is cyclic, and B_0 is elementary abelian. Therefore B is abelian and $\mho^1(B) = \mho^1(B_1)$. If $|B_1| > 3$, then $\langle b \rangle = \Omega_1(\mho^1(B))$ is strongly closed in B, contrary to hypothesis. Therefore $B_1 = \langle b \rangle$, and $B \cong E_{27}$.

Set $N = N(B)$ and $\bar{N} = N/C^*(B)$. Then there is a natural injection $\bar{N} \to PGL(3, 3) \cong SL(3, 3)$ so we can identify $\bar{N}$ with its image in $SL(3, 3)$. Let $\tau \in N_J(B)$ be an involution which inverts B_0, so that $\langle b \rangle = C_B(\tau)$. We have $C_{\bar{N}}(\bar{\tau}) \leqslant N_{\bar{N}}(\langle b \rangle) = \overline{C(b) \cap N(B)}$, so that $C_{\bar{N}}(\bar{\tau}) \cong D_8$. Inspection of 2-local and 3-local subgroups of $SL(3, 3)$ yields that either $\bar{N} \cong E_9 \cdot D_8 \cong \Sigma_3 \wr Z_2$ or $\bar{N} \cong \Sigma_4$. In the former case $\bar{N}$ is the stabilizer of a hyperplane of B which must be B_0, whence (a) holds.

Assume for the rest of the proof that $\bar{N} \cong \Sigma_4$. Then $|\bar{N} : C_{\bar{N}}(b)| = 3$, so $\langle b \rangle$ has 3 N-conjugates. $C_{\bar{N}}(b)$ has orbits of lengths 1, 2, 2, 4, and 4 on $\mathcal{E}_1(B)$ so $\langle b \rangle$ must fuse to exactly one of the orbits of length 2. Letting $\langle b_1 \rangle$ and $\langle b_2 \rangle$ be the groups in that orbit, we have $B_0 = \langle b_1, b_2 \rangle$. Since $C_N(b)/C(B)$ acts as the full monomial group on B_0 with respect to $\{b_1, b_2\}$, we conclude that $N/C(B)$ acts as the full monomial group on B with respect to $\{b, b_1, b_2\}$.

It remains to show that $N(B)$ controls fusion in B. If $P \in \mathrm{Syl}_3(N(B))$, then $P \cong Z_3 \wr Z_3$ by the above paragraph, so $B = J(P)$. Therefore B is weakly closed in P and $N(P) \leqslant N(B)$. It follows that $P \in \mathrm{Syl}_3(G)$ and that $N(B)$ controls fusion in B with respect to G.

LEMMA 1.4. *Assume that* G, b, *and* L *satisfy* $\mathcal{H}_n$, *where* $n = 4$ *or* 5. *Let* $B \in \mathrm{Syl}_3(C(b))$, *and set* $X = O_{3'}(C(b))$. *Assume that* X *has odd order and that either* $|C(B)|$ *is odd or* $n = 5$. *Then* X *is a normal Hall* $\{2, 3\}'$-*subgroup of* $C(B)$ *and* $X = O_{3'}(C(A))$ *for every group* $A < B$ *with* $b \in_G A$. *Finally, one of the following holds*:

(i) $C(b) = \langle b \rangle \times J \times X$, *or*

(ii) $C(B)$ *has even order.*

PROOF. It follows from Lemma 1.2(b) that $[C(B) : BX] \leqslant 2$. Therefore X is a normal $\{2, 3\}$-complement for $C(B)$ and $X = O_{3'}(C(a))$ for every $a \in b^G \cap C(b)$. To verify the second assertion, it suffices to assume that A is an E_9-subgroup of B containing $\langle b \rangle$. Then $C(A)$ normalizes J and $O_{3'}(C(A)/C(A) \cap C(J)) = 1$ by Lemma 1.2(b), (e), (f). Since $X = O_{3'}(C(A) \cap C(J))$, we have $X = O_{3'}(C(A))$. For the last assertion, set $C = C(b)$ and assume that $C(B)$ has odd order. Then $C_C(J) = \langle b \rangle \times X$ by transfer, so $C = N_C(J) = J \times \langle b \rangle \times X$ by Lemma 1.2(b).

LEMMA 1.5. *Let $J \cong L_5(2)$ act faithfully on $U \cong E_{2^6}$ with $C_U(J) \neq 1$. Then* $U = [U, J] \times C_U(J)$.

PROOF. Assume the contrary. Then $C_U(J) = \langle t \rangle$ has order 2 and J acts semiregularly on the set Ω of complements to $\langle t \rangle$ in U. As $|\Omega| = 32$ and J has a subgroup of order 31, it follows that J is doubly transitive on Ω. But $L_5(2)$ has no doubly transitive representations of degree 32 by **[2]**, a contradiction.

LEMMA 1.6. *Assume that $J \cong L_5(2)$, that $\beta \in J$ has order 3 and that J acts on the 2-group T so that $C_T(\beta) \leq T_0$ where T_0 is J-invariant. Then $T = T_0$.*

PROOF. It suffices to assume that $T_0 = 1$. By Lemma 1.2(h), we can choose $\gamma \in J$ of order 5 so that $\langle \beta, \gamma \rangle \cong L_2(4)$. Then T is the direct product of natural $L_2(4)$-modules by **[10]**, so $C_T(\gamma) = 1$. Since $\langle \gamma \rangle$ acts fixed point-free on a subgroup D of J of order 31, we have $[D, T] = 1$. Thus J centralizes T and $T = 1$.

2. Groups of small 3-rank. In this section, we derive two propositions about configurations which arise in the proofs of Theorems A and B.

PROPOSITION 2.1. *Assume that $G = LB$ is a finite group such that $L = F^*(G)$ is simple and $B \nleq L$ has order 3. Assume further that*

(i) $C(B) = B \times K \times O_{3'}(C(B))$ *where* $K \cong L_2(4)$.

(ii) *If* $A \in \mathrm{Syl}_3(K)$, *then* $C(A)$ *has odd order.*

(iii) *If* $P = \langle B, A \rangle \in \mathrm{Syl}_3(C(B))$ *and* $B_1 \in \mathfrak{E}_1(P) - \{A\}$, *then* $C(B_1) \cong C(B)$.

(iv) $m_{2,3}(L) = 1$.

Then $L \cong L_2(125)$, $L_2(64)$ *or* $L_3(4)$.

PROOF. Let $\mathfrak{E}_1(P) = \{A, B, B_1, B_2\}$, let $U \in \mathcal{N}_K^*(P; 2)$ and let $U \leq T \in \mathcal{N}_G^*(P; 2)$. Then $C_T(A) = 1$ so $T = C_T(B)C_T(B_1)C_T(B_2)$. Hypotheses (i) and (ii) imply that $U \in \mathrm{Syl}_2(C(B))$, so $U = C_T(B)$ and $|C_T(B_i)| \leq 4$ for $i = 1, 2$ by hypothesis (iii). Either $U \leq Z(T)$ or $1 \neq C_T(B_i) \leq Z(T)$ for $i = 1, 2$. In the latter case, we may relabel B and B_i without affecting the hypotheses of the theorem to obtain $U \leq Z(T)$.

By the Frattini argument, $N(U) = C(U) \cdot (C(B) \cap N(U)) = C_L(U)(C(B) \cap N(U))$. Setting $C = C_L(U)$, we have $3 \nmid |C|$ because $C_C(B)$ has $3'$-order. Therefore $T \in \mathrm{Syl}_2(C)$ and in fact $T \in \mathrm{Syl}_2(N(U))$. By the preceding paragraph, $|T| = 4^n$ for $n = 1, 2$, or 3. We consider each possibility in turn.

Case 1. $n = 1$. Then $U = T \in \mathrm{Syl}_2(L)$, so $L \cong L_2(q)$ for some $q \equiv 3$ or 5 (mod 8) by Walter **[15]**. By elementary properties of $\mathrm{Aut}(L_2(q))$, $C_L(B) \cong L_2(q^{1/3})$. Therefore $q = 125$, and the proposition holds.

Case 2. $n = 2$. Then T is elementary abelian of order 16 because $C_T(A) = 1$. Set $N_1 = N(T) \cap N(P)$. Then $A \lhd N_1$ and $|C_{N_1}(A)|$ is odd by hypothesis, so $|N_1 : P| \leqslant 2$. Inspecting the subgroups of $L_4(2)$, we then have $|N(T) : C(T)|_2 \leqslant 2$. Let $S \in \mathrm{Syl}_2(N(T))$. Then either $S = T$ or $S \cong E_4 \wr Z_2$. In the latter case, $T = J(S)$, so $S \in \mathrm{Syl}_2(L)$ in either case. But no simple group has Sylow 2-subgroup of type $E_4 \wr Z_2$ by Corollary 6 of **[6]**, so $S = T$. But then **[15]** forces $L \cong L_2(16)$, a contradiction as B must act as a group of outer automorphisms of L. Thus Case 2 does not occur.

Case 3. $n = 3$. We argue that $T \in \mathrm{Syl}_2(L)$. It suffices to show that $N(T)$ is 3-nilpotent since $T \in И^*(P; 2)$. Set $N = N(T)$. By hypothesis (i), $N_N(B) \leqslant C(B) \cap N(U)$ has a normal 3-complement. Similarly, $N_N(B_i)$ is 3-nilpotent for $i = 1, 2$ because $C_T(B_i) \neq 1$. This implies that $\mathrm{Aut}_N(P)$ is a 3-group, so $N_N(P)$ is 3-nilpotent. If $P \leqslant Q \in \mathrm{Syl}_3(N)$, then $Q \cap L$ is cyclic by hypothesis (iv). It follows that $P = \Omega_1(Q)$. Since $A = \Omega_1(Q \cap L)$, $N_N(A) \cap L$ is 3-constrained and $N_N(A) = O_{3'}(N_N(A))(N_N(A) \cap N_N(P))$ by the Frattini argument. Therefore $N_N(A)$ is 3-nilpotent and N is 3-nilpotent by the Frobenius transfer theorem.

If T is abelian, then $G \cong L_2(64)$ by Walter **[15]**. Otherwise T is of type $L_3(4)$ [7, p. 16] in which case $L \cong L_3(4)$ by Collins **[1]**. The proof is complete.

LEMMA 2.2. *Let R be a solvable group with a normal subgroup S of index* 2 *such that $O_{3'}(R) \leqslant S$. Assume that $T \cong Z_3$ is a Sylow* 3*-subgroup of S and that $x \in \mathrm{Inv}(R) \setminus \mathrm{Inv}(S)$. Then $x \in_R N(T)$.*

PROOF. Let R be a counterexample of minimal order and let N be a minimal normal subgroup of R. Then N is an elementary abelian p-group for some $p \neq 3$ as R is solvable and $O_3(R) = 1$ by assumption. Setting $\bar{R} = R/N$ and applying induction, $\bar{x} \in_{\bar{R}} N(\bar{T})$. That is $x \in_R N \cdot N(T)$, so $R = N \cdot N(T)$ by choice of R. Furthermore $p = 2$, as otherwise $N(T)$ contains a Sylow 2-subgroup of R. Let $x = nh$ for $n \in N$ and $h \in N(T) \setminus N_S(T)$. Our choice of R implies that $R = \langle T, x\rangle = NT\langle h\rangle$. Evidently, $h^2 \in N$, so $R/N \cong \Sigma_3$. Therefore $N \cong E_4$ and $R \cong \Sigma_4$ is not a counterexample.

PROPOSITION 2.3. *Let G be a finite group with an elementary abelian Sylow* 3*-subgroup $P = \langle A, B\rangle$ of order* 9. *Assume that the following conditions are satisfied*:

(i) $E(C(B)) = K \cong L_2(4)$ *with* $A \leqslant K$.

(ii) $O_{3'}(C(B))$ *has odd order.*

(iii) *One of the following holds*:

(α) $N(A) \leqslant N(B)$ *and* $C(P)$ *has odd order.*

(β) $N(A) \leqslant N(B)$ *and* $O_{3'}(C(B)) = 1$.

(γ) $E(C(A)) \cong L_3(2)$ *and* $O_{3'}(C(A)) = O_{3'}(C(B))$.

Then $G = O_{3'}(G)N(K)$.

PROOF. We first observe that A and B are strongly closed in P with respect to G. In fact, it is evident from hypothesis (iii) that $A \not\sim_G B$. Also, $C(B)$ has 3 orbits on $\mathcal{E}_1(P)$. As $N(P)/P$ is a 3′-group, it then follows that A and B are each normal in $N(P)$. But $N(P)$ controls fusion in P, so the assertion is proved.

Let G be a counterexample of minimal order. Then $O_{3'}(G) = 1$. It follows easily from assumption (i) that $O_3(G) = 1$. Thus $F^*(G) = E(G)$ is the direct product of simple groups. Set $E = E(G)$.

We argue that $P \leqslant E$. Set $Q = P \cap E$ and assume that $Q < P$. Then $Q \neq 1$ because $3 \mid |E|$. Since $Q \lhd N(P)$, we have that $Q = A$ or $Q = B$. We shall use a 3-local characterization to obtain a contradiction. The claim is that either $|C_E(Q)| = 6$ or $|C_E(Q)|$ is odd. In fact, by transfer, $C_E(Q) \leqslant Q \cdot O_{3'}(C(Q))$, hence $|C_E(Q)|$ is odd if $Q = B$ or if $Q = A$ and either (iii)(α) or (iii)(γ) holds. On the other hand, if $Q = A$ and (iii)(β) holds, then $|C_E(Q)| = 6$. So the claim is true in all cases. Therefore $E \cong L_2(q)$, $L_3(q)$ or $U_3(q)$ for appropriate q by results of **[4]**, **[5]** and **[11]**. In any case, Out(E) is solvable, so $K \leqslant E$ which gives $Q = A$. If (iii)(γ) holds, then a similar argument shows that $Q = B$ which is absurd. If (iii)(β) holds, then $E \cong L_2(p)$ for some $p \in \{5, 7, 11, 13\}$ and Out(E) is a 3′ group. But $C(E) = 1$ and $P \leqslant N(E)$ then yield a contradiction. Therefore (iii)(α) must hold. Let τ be an involution in $N_K(A)$. Then $\tau \in E$, and $A = [C_E(A), \tau]$. It follows that $E \cong L_2(4)$, $L_3(2)$ or $U_3(4)$. But none of these groups admit an outer automorphism of order 3 in contradiction to $B \leqslant N(E)$ and $C(E) = 1$. This completes the argument that $P \leqslant E$.

We now show that E is simple. If not, then $E = E_1 \times E_2 \times \cdots \times E_n$, with $2 \leqslant n$ and E_i simple, $1 \leqslant i \leqslant n$. As $C(P)$ is 3-solvable by hypotheses (i) and (ii), $E = E_1 \times E_2$ with $P \cap E_i \neq 1$, $i = 1, 2$. As $P \cap E_i$ is inverted in E_i, $i = 1, 2$, it follows from $N_{E_i}(B) \lhd C_E(B)$, $i = 1, 2$, together with $O^{3'}(C(B)) = C_E(B) = B \times K$ that, without loss, we may set $P \cap E_1 = A$ and $P \cap E_2 = B$. But then $E_1 = K$ and therefore $E_1 \lhd G$ by hypothesis (iii) which contradicts our choice of G. We conclude that E is simple. Therefore $E = G$ by choice of G.

We shall now apply a result of G. Higman [7] to contradict the simplicity of G. Let $D \in \mathcal{E}_1(P) \setminus \{A, B\}$ and for every subgroup X of G, let $\mathcal{I}(X)$ denote the set of involutions of G which invert X. Higman's result asserts that if $t \in \mathrm{Inv}(G)$, then two of the following three sets are nonempty: $\mathcal{I}(A)^G \cap \{t\}$, $\mathcal{I}(B)^G \cap \{t\}$, $\mathcal{I}(D)^G \cap \{t\}$. In order to apply this result, we require some information about $\mathcal{I}(A)$, $\mathcal{I}(B)$ and $\mathcal{I}(D)$.

We first claim that $\mathcal{I}(D) \subseteq \mathcal{I}(P)^G$. To see this, set $H = N(D)$, $H_0 = C(D)$ and $\overline{H} = H/D$. By the first paragraph, $N_{H_0}(P) \leqslant N(B)$, so our hypothesis forces $N_{H_0}(P) = C_{H_0}(P)$. In particular, H_0 has a normal 3-

complement F. If F is not solvable, then F has a chief section which is the direct product of one or more Suzuki groups. Therefore $C_F(A) = C_F(P)$ involves $S_z(2)$. But $C(P)$ contains no elements of order 4 by hypothesis, so F is solvable and hence H is solvable. As $\bar{F} = O_{3'}(\bar{H})$, it follows from Lemma 2.2 that $\bar{x} \in_{\bar{H}} N(\bar{P})$ for every $x \in \mathcal{I}(D)$. This in turn yields $\mathcal{I}(D) \subseteq \mathcal{I}(P)^G$ as claimed.

Let $r \in \mathcal{I}(A) \cap K$ and $s \in \mathcal{I}(B) \cap C(A)$ with $[r, s] = 1$. If $C(P)$ has even order, let $t \in \mathrm{Inv}(C(P))$, otherwise, set $t = 1$. Then $\langle t \rangle \in \mathrm{Syl}_2(C(P))$ by hypothesis (ii) and we may choose t so that $\langle t, r, s \rangle$ is abelian. Observe that $K\langle s, t \rangle \cong \Sigma_5, A_5 \times Z_2$ or $\Sigma_5 \times Z_2$ and $K\langle s, t \rangle$ covers $N(B)/O_{3'}(C(B))B$. By inspection, $B \in \mathrm{Syl}_3(C(B) \cap C(r))$, hence $B \in \mathrm{Syl}_3(C(r))$. Similarly $A \in \mathrm{Syl}_3(C(s))$ and $P \in \mathrm{Syl}_3(C(t))$. Since $A \neq_G B$, we see that r, s and t belong to different G-conjugacy classes. In particular $\mathcal{I}(D)^G = \mathcal{I}(P)^G = (rs)^G \cup (rst)^G \neq \mathrm{Inv}(G)$.

As $K\langle s, t \rangle$ contains a Sylow 2-subgroup of $N(B)$, every involution of $K\langle s, t \rangle$ is K-conjugate to an element of $N(P)$. Therefore $y \in_{N(B)} N(P)$ for every $y \in \mathcal{I}(B)$. Similarly $x \in_{N(A)} N(P)$ for every $x \in \mathcal{I}(A)$. It follows that $\mathcal{I}(A)^G = r^G \cup (rt)^G \cup (rs)^G \cup (rst)^G$ and $\mathcal{I}(B)^G = s^G \cup (st)^G \cup (rs)^G \cup (rst)^G$.

If $|C(P)|$ is even, then $K\langle t \rangle \cong \Sigma_5$ by hypothesis (i), so $t =_G rt$ and $st =_G rst$. It follows that $\mathcal{I}(A)^G \cap \mathcal{I}(B)^G = \mathcal{I}(D)^G$ in any case. Let $x \in \mathrm{Inv}(G) \setminus \mathcal{I}(D)^G$. Then x belongs to at most one of $\mathcal{I}(A)^G$, $\mathcal{I}(B)^G$, $\mathcal{I}(D)^G$ contradicting Higman's result. Therefore our counterexample G does not exist.

3. Proof of Theorem A. In this section, G, b and J satisfy the hypotheses of Theorem A. That is, G is a finite group with $F^*(G)$ simple and $b \in G$ is an element of order 3 such that the following hold:

(a) J is a normal subgroup of $C(b)$ of type $L_4(2)$, $\widehat{L_4(2)}$, or $L_5(2)$;

(b) $C(J)$ has cyclic Sylow 3-subgroups;

(c) $\langle b \rangle$ is not strongly closed in $C(b)$;

(d) $m_{2,3}(G) = 3$; and

(e) $b \notin_G J$.

Choose $B \in \mathrm{Syl}_3(C(b))$ and set $B_0 = B \cap J$. Then $B_0 \lhd N(B)$ by Lemma 1.3. We set $N = N(B)$ and $\bar{N} = N/O_{3'}(N) \cdot B_0$.

LEMMA 3.1. *$\bar{N} = \langle \bar{b} \rangle \times \bar{A}\bar{D}$ or $\langle \bar{b}, \bar{t} \rangle \times \bar{A}\bar{D}$ where $\langle b, A \rangle = P \in \mathrm{Syl}_3(N)$, A is homocyclic abelian of order 3^4, $D \in \mathrm{Syl}_2(N \cap J)$ is dihedral of order 8, $\bar{A}$ is isomorphic to B_0 as a $GF(3)\bar{D}$-module and t, if it exists, is an involution which inverts B.*

PROOF. By Lemma 1.3, $\langle \bar{b} \rangle \lhd \bar{N}$ and $C_{\bar{N}}(\bar{b})/\langle \bar{b} \rangle \cong N/C^*(B) \cong \Sigma_3 \wr Z_2$. Let $P \in \mathrm{Syl}_3(N)$ so that $\bar{P} = O_3(\bar{N})$. The action of $\bar{D}$ on $\bar{P}$ implies that $\bar{P}$ is either elementary abelian or extra-special of exponent 3. In the latter case, $\mathrm{Aut}_{C(\bar{b})}(\bar{P})$ is isomorphic to a subgroup of $SL_2(3)$ whereas $\bar{D}$ acts faithfully on $\bar{P}$. Therefore $\bar{P} \cong E_{3^3}$ and we may set $\bar{P} = \langle \bar{b} \rangle \times \bar{A}$ where $\bar{A} = [\bar{P}, Z(\bar{D})]$. Clearly $\bar{D}$ acts faithfully on $\bar{A}$. Since $[b, P] = B_0$, A and B_0 are isomorphic as $GF(3)\bar{D}$-modules. Also $Z(\bar{D})$ acts regularly on $\bar{A}$ and B_0 then yields that A is homocyclic of order 3^4. Finally if b is inverted in G, hence necessarily in N, then t may be chosen as described.

LEMMA 3.2. *Either G satisfies the conclusion of Theorem* A *or the following conditions hold*:

(a) $A \cong E_{3^4}$;
(b) $C(B)$ *has odd order, in particular* $|O_{3'}(C(b))|$ *is odd*;
(c) $J = O^{2'}(C(b))$; *and*
(d) P *has exponent* 3.

PROOF. By Lemma 3.1, A is homocyclic abelian of order 3^4. Assume first that $A \cong Z_9 \times Z_9$. Since $\bar{A}$ and B_0 are isomorphic as $GF(3)\bar{D}$ modules, an easy argument yields $B = \Omega_1(P)$. Therefore B is weakly closed in P with respect to G and consequently $P \in \mathrm{Syl}_3(G)$. Suppose $b =_G b^{-1}$ so that $\bar{N} = \langle \bar{b}, \bar{t} \rangle \times \bar{A}\bar{D}$ as in Lemma 3.1. Then, assuming that $t \in N(A)$, as we may, t centralizes A/B_0 and inverts B_0, an obvious contradiction. Hence $b \notin N'$. But $N(P) < N$, and P has no $Z_3 \wr Z_3$ homomorphic image. A recent transfer theorem of Yoshida [16] implies that $A = P \cap N(P)'$. Thus $A \in \mathrm{Syl}_3(F^*(G))$ because $A = [A, D]$, and G satisfies the conclusion of Theorem A.

Now assume that A is elementary abelian of rank 4. By Burnside's transfer theorem, $C(B)$ has a normal 3-complement X. Clearly $A < N(X)$. Since $m_{2,3}(G) = 3$, we have $|X|_2 = |C(B)|_2 = 1$, so (b) holds. By Lemma 1.4, this implies that $J = O^{2'}(C(b))$. Finally, P has exponent 3 since $P = \Omega_1(P)$ and P has class 2. This completes the proof of Lemma 3.2.

Now assume until a contradiction is reached that G does not satisfy the conclusion of Theorem A. Therefore conditions (a), (b), (c) and (d) of Lemma 3.2 hold.

LEMMA 3.3. *Choose $A^* \in \mathrm{Syl}_3(C(A))$ so that $b \in N(A^*)$. Then A^* is abelian and $A^*\langle b \rangle \in \mathrm{Syl}_3(G)$.*

PROOF. Set $N_1 = N(A)$. By Lemma 3.1, we may assume that $C_{N_1}(b) = O_{3'}(C_{N_1}(b))(\langle b \rangle \times B_0 D)$. Set $Y = C_{N_1}(Z(D))$. Then the regular action of $Z(D)$ on B_0 implies that $\langle b \rangle$ is a Sylow 3-subgroup of Y. As $m_{2,3}(G) = 3$, $C(A)$ has odd order, and $Z(D)$ must act regularly on some Sylow 3-subgroup of $C(A)$. Therefore A^* is abelian. As $Z(D)$ inverts A, $C(A)Z(D) \lhd N_1$;

hence, by the Frattini argument, we have that $N_1 = C(A)Y$. Thus $A^*\langle b\rangle \in \mathrm{Syl}_3(N_1)$. Now $Z(D)$ normalizes $\langle C(A), b\rangle$, so $Z(D)$ normalizes a Sylow 3-subgroup of $\langle C(A), b\rangle$ containing $\langle b\rangle$. Without loss, we may then assume that $Z(D)$ normalizes $\langle A^*, b\rangle$. If $Y^* = C_{N(A^*)}(Z(D))$, then we may argue as before to conclude that $N(A^*) = C(A^*)Y^*$ where $\langle b\rangle \in \mathrm{Syl}_3(Y^*)$. But $C(A^*) \leqslant C(A)$ yields $A^*\langle b\rangle \in \mathrm{Syl}_3(N(A^*))$ and as A^* is characteristic in $A^*\langle b\rangle$, we conclude that $A^*\langle b\rangle \in \mathrm{Syl}_3(G)$.

LEMMA 3.4. *Choose* $a \in B_0$ *so that* $K = E(C_J(a)) \cong L_2(4)$. *Then* $C_A(K) \cap N(\langle a, b\rangle) = U \cong E_9$, $b \in N(U)$ *and* $O_{3',E}(C(U))/O_{3'}(C(U)) \cong L_3(4)$.

PROOF. Set $V = N_A(\langle a, b\rangle)$. Then $V \cong E_{3^3}$ as $[A, b] = B_0$ and $A \cong E_{3^4}$. V normalizes $K = E(C(\langle a, b\rangle))$, so $U = C_V(K) \cong E_9$. Clearly $[b, U] < [b, V] = \langle a\rangle$, so $b \in N(U)\setminus C(U)$.

Set $M = N(U)$, $C = C(U)$ and $\tilde{M} = M/O_{3'}(C)$. For the proof of this lemma, we are interested only in the structure of $\tilde{M}$; therefore we shall abuse notation and identify elements and subgroups of M with their images in $\tilde{M}$. If $R = O_3(C)$, then $C_R(b) = \langle a\rangle$. Since $[a, K] = 1$, the $P \times Q$ lemma applied to $\langle b\rangle \times K$ acting on R shows that $[R, K] = 1$. This implies that $E = E(C) \neq 1$. By Lemma 3.3, $A^* \in \mathrm{Syl}_3(C)$; so $O_3(E) = 1$, and the components of E are all simple. As $m(R) \geqslant 2$ and $m_{2,3}(G) = 3$, E has at most 2 components, each of which must be normalized by b. Since $m_3(C_E(b)) = 1$, E is simple, and $C_E(b) = O_{3'}(C_E(b)) \times K$.

We argue that $E\langle b\rangle$ satisfies the hypotheses of Proposition 2.1. Since $U \leqslant C(E)$, $m_{2,3}(E) = 1$. Furthermore, if $\langle a^*\rangle \in \mathrm{Syl}_3(K)$, then $\langle a^*\rangle \in \mathrm{Syl}_3(C_E(b))$. As $N_A(\langle b, a^*\rangle)$ acts transitively on $\mathcal{E}_1(\langle b, a^*\rangle) \setminus \{\langle a^*\rangle\}$, $C_E(\beta) \cong C_E(b)$ for $\beta \in \langle b, a^*\rangle - \langle a^*\rangle$. It remains to verify that $C^* = C_E(a^*)$ has odd order. Since $A \leqslant N(C^*)$, $O_{3'}(C^*)$ has odd order. Also $C(b) \cap O_{3',E}(C^*)$ is a $3'$-group, so C^* is 3-constrained. In particular, if R^* is a $\langle b\rangle$-invariant Sylow 3-subgroup of C^*, then R^* is abelian, and $O_{3',3}(C^*) = O_{3'}(C)R^*$. Since $C_{R^*}(b) = \langle a^*\rangle$, $m(R^*) \leqslant 3$. Since $C^*/O_{3'}(C)R^*$ acts faithfully on $\Omega_1(R^*)$ and fixes $\langle a^*\rangle$, it follows that C^* is 3-nilpotent. This implies that $|C^*|$ is odd and hence $E\langle b\rangle$ satisfies the hypotheses of Proposition 2.1.

We conclude that $E \cong L_2(125)$, $L_2(64)$ or $L_3(4)$. If $E \cong L_2(125)$ or $L_2(64)$, let $a^* \in S \in \mathrm{Syl}_3(E)$ where S is $\langle b\rangle$-invariant. Then S is cyclic of order 9 and $[S, b] = \langle a^*\rangle$. But this implies that $S \leqslant N(B)$ contradicting Lemma 3.2. Hence $E \cong L_3(4)$ as required.

LEMMA 3.5. *A contains* E_9 *subgroups* U_1 *an and* U_2 *satisfying*

(i) $b \in N(U_1) \cap N(U_2)$;

(ii) *If* $L_i = O_{3',E}(C(U_i))$, *then* $L_i/O_{3'}(L_i) \cong L_3(4)$ *and* $U_i \leqslant L_{3-i}$ *for* $i = 1, 2$; *and*

(iii) $C_{L_i}(b) = O_{3'}(C_{L_i}(b)) \times K_i$ *where* $O^{2'}(C_{L_i}(b)) = K_i \cong A_5$, $i = 1, 2$.

PROOF. Choose a and U as in Lemma 3.4 and set $U_1 = U$ and $L_1 = O_{3',E}(C(U_1))$. Let $U_2 = A \cap L_1$ so that $U_2 \in \mathrm{Syl}_3(L_1)$ and $A = U_1 \times U_2$. By properties of $L_3(4)$, U_2 is inverted by $\sigma \in \mathrm{Inv}(C(\langle U_1, b\rangle))$. Recall from Lemmas 3.1 and 3.2 that a Sylow 2-subgroup D of $C(b) \cap N(A)$ is dihedral of order 8 and $Z(D) = \langle z\rangle$ inverts A. Assume that $\sigma \in D$, as we may. Then for some $d \in \mathrm{Inv}(D)$, $\sigma^d = \sigma z$. As $U_2 = [A, \sigma]$ and z inverts A, $U_2^d = [A, \sigma^d] = [A, \sigma z] \leqslant U_1$; hence, $U_2^d = U_1$. Since $N(U_1)$ satisfies (iii) and d interchanges U_1 and U_2 under conjugation, the result follows.

We are now in position to obtain our final contradiction. Let $L_i^* = \langle L_i, b\rangle$ and set $\overline{L_i^*} = L_i^*/O_{3'}(L_i)$. Then $N_{\overline{L_i^*}}(\overline{U}_j) = \overline{U}_j\overline{Q}_j\langle\overline{b}\rangle$, $i \neq j$, where $\overline{Q}_j \cong Q_8$. As $C_{\overline{L}_i}(\overline{b}) = \overline{K}_i \cong A_5$ and all involutions of $\overline{L}_i$ are conjugate, we may assume that $\overline{b}$ normalizes $\overline{Q}_j$. Now $O_{3'}(L_i)$ is a $\{2, 3\}'$-group, so we may assume that $N_{L_i}(U_j)$ contains a Q_8-subgroup Q_j and that b normalizes Q_j, $1 \leqslant i \neq j \leqslant 2$.

Let $M_i = L_iQ_i\langle b\rangle$ and set $\overline{M}_i = M_i/O_{3'}(L_i) = \overline{L}_i\overline{Q}_i\langle\overline{b}\rangle$. Since $\overline{Q}_i$ normalizes $\overline{L}_i$ and $[\overline{Q}_i, \overline{U}_j] = \overline{1}$, $i \neq j$, it follows from $|C_{\mathrm{Aut}(\overline{L}_i)}(\overline{U}_j)| = 2 \cdot 3^2$ that $[\overline{Q}_i : C_{\overline{Q}_i}(\overline{L}_i)] \leqslant 2$. But $\overline{b}$ acts regularly on $\overline{Q}_i/Z(\overline{Q}_i)$ then yields $\overline{Q}_i\overline{L}_i = \overline{Q}_i \times \overline{L}_i$. Therefore $C_{\overline{Q}_i\overline{L}_i}(\overline{b}) = Z(\overline{Q}_i) \times \overline{K}_i \cong Z_2 \times L_2(4)$ and this in turn implies that $O^{2'}(C(b))$ contains a subgroup isomorphic to $Z_2 \times L_2(4)$. But by Lemma 3.2, $O^{2'}(C(b)) = J \cong L_4(2)$ or $L_5(2)$ and no involution of J centralizes an $L_2(4)$-subgroup. With this contradiction, the proof of Theorem A is complete.

4. Proof of Theorem B. In this section, G, b, and J satisfy the hypotheses of Theorem B. Thus G is a finite simple group of characteristic 2 type such that:

(a) $b \in G$ has order 3;

(b) J is a normal subgroup of $C(b)$ of type $L_5(2)$;

(c) $\langle b\rangle$ is not strongly closed in $C(b)$; and

(d) $C(J)$ has cyclic Sylow 3-subgroups.

By Theorem A, we may assume that $b \in_G J$. As before, we choose $B \in \mathrm{Syl}_3(C(b))$ and set $B_0 = B \cap J$. By Lemma 1.3, $B_0 = \langle b_1, b_2\rangle$ where $\langle b\rangle^{N(B)} = \{\langle b\rangle, \langle b_1\rangle, \langle b_2\rangle\}$. We shall show that $G \cong L_7(2)$ by constructing the centralizer of a central involution. In order to do this, we show in Lemma 4.2 that the 3-fusion pattern in G is the same as that in $L_7(2)$. First, we show that $O_{3'}(C(b))$ has odd order.

LEMMA 4.1. *$O_{3'}(C(b))$ has odd order.*

PROOF. Set $X = O_{3'}(C(b))$ and let $V \in \mathrm{Syl}_2(X)$. Since G has characteristic 2 type, it suffices to show that $\langle b\rangle$ centralizes $V^* = O_2(N(V))$.

By Lemma 1.2, $X = O_{3'}(C(\langle b, b^*\rangle))$ for every $b^* \in J$ of order 3. Since $b^{N(B)} = \langle b\rangle^\# \cup (b^{N(B)} \cap J)$, it follows that $X = O_{3'}(C(b^*))$ for all $b^* \in b^{N(B)}$. Therefore $X \lhd N(B)$, so that $N(B) = X \cdot N_{N(B)}(V)$ by the Frattini argument. This implies that $b =_{N(V)} b_1$, so $C_{V^*}(b) = V = C_{V^*}(b_1)$ Since

$J \leqslant N(V)$, we have that $V^* = V$ by Lemma 1.6. Thus $\langle b \rangle$ centralizes V^*, as required.

Now set $\beta = b_1 b_2$ and $\gamma = b_1^2 b_2$, so that $B_0 = \langle \beta, \gamma \rangle$. Also set $H^* = C(\beta)$ and $H = O^3(H^*)$. We use Proposition 2.3 to determine the structure of H.

LEMMA 4.2. *$H^* = \langle \beta \rangle \times H$ and H has the following properties*:

(i) $\langle b, \gamma \rangle \in \mathrm{Syl}_3(H)$.

(ii) *If $K = E(C_H(b))$ and $L = E(C_H(\gamma))$, then $K \cong L_2(4)$, $L \cong L_3(2)$, $b \in L$ and $\gamma \in K$.*

(iii) *$H = C(B)E(H)$ where $E(H) = K \times L$ and $C(B)$ is 2-nilpotent with $O_{3'}(C(B)) = O_{3'}(H) = O_{3'}(C(b))$.*

PROOF. We first argue that H satisfies the hypotheses of Proposition 2.3. By Lemma 1.3, $B \in \mathrm{Syl}_3(N_{H^*}(B))$ and $\langle b, \gamma \rangle = B \cap N_{H^*}(B)'$, so Grün's theorem implies that $\langle b, \gamma \rangle \in \mathrm{Syl}_3(H)$. Now $B_0 =_G \langle b, b_1 \rangle$. Thus $L = E(C_H(\gamma)) \cong E(C(\langle b, b_1 \rangle))$ and $b \in L$. Furthermore $K = E(C_H(b)) = E(C(\langle b, \beta \rangle))$ and $\gamma \in K$. Also, by Lemma 1.4, $O_{3'}(C_H(\gamma)) = O_{3'}(C(b)) = O_{3'}(C_H(b))$ has odd order. Since either $K \cong L_2(4)$, $L \cong L_3(2)$ or $K \cong L_3(2)$, $L \cong L_2(4)$, it follows from Proposition 2.3, that $H = N_H(K)O_{3'}(H)$ or $N_H(L)O_{3'}(H)$ according to whether $K \cong L_2(4)$ or $L \cong L_2(4)$, respectively.

Assume first that $K \cong L_2(4)$. Set $H_0 = O_{3'}(H)$. Then $C_{H_0}(b) \leqslant O_{3'}(C(b)) = O_{3'}(C(B_0))$ implies that $C_{H_0}(b) = C_{H_0}(L)$. As $L \cong L_3(2)$, we have $[L, H_0] = 1$ whereupon $H_0 \leqslant O_{3'}(C(b))$. But then $[K, H_0] = 1$, and $K \lhd H$. An easy argument shows that $H_0 = O_{3'}(C(b))$ and $E(H) = K \times L$. By the Frattini argument, $H = C(B)E(H)$ proving (iii). On the other hand, if $K \cong L_3(2)$, then $C_{H_0}(\gamma) \leqslant O_{3'}(C(B_0)) = O_{3'}(C(b))$; hence, $C_{H_0}(K) = C_{H_0}(\gamma)$. Therefore $[K, H_0] = 1$ and $H_0 \leqslant O_{3'}(C(B_0))$. This in turn yields $[L, H_0] = 1$, whereupon $L \lhd H$. As before, $H_0 = O_{3'}(C(b))$, $E(H) = K \times L$ and $H = C(B)E(H)$. Thus (iii) is true in either case.

In order to complete the proof of (ii), assume that $K \cong L_3(2)$ and $L \cong L_2(4)$ for purpose of a contradiction. Let U be a B-invariant fours subgroup of L and set $V = O_2(N(U))$. Then $[B, U] = U$ gives $U \leqslant Z(V)$. Now $C_V(\beta) = V \cap H \leqslant O_2(C_H(U))$, so $[K, C_V(\beta)] \leqslant K \cap O_2(C_H(U)) = 1$. But $C_H(K) \leqslant C(B_0)$, and $C(B_0)$ has dihedral Sylow 2-subgroups. It follows that $U = C_V(\beta)$. As $\beta =_{N(B_0)} \gamma$ and $L = E(C(B_0))$, an application of the Frattini argument yields $\beta =_{N(U)} \gamma$. Hence $U = C_V(\gamma) = C_V(K)$ whereupon $[K, V] = 1$. This contradicts $V = F^*(N(U))$.

LEMMA 4.3. *$C(\beta) = \langle \beta \rangle \times O_{3'}(C(\beta)) \times K \times L$ and $C(b) = \langle b \rangle \times O_{3'}(C(b)) \times J$.*

PROOF. By Lemma 4.2(iii) and Lemma 1.4, it suffices to show that $C(B)$ has odd order. Assume not and let $\tau \in \mathrm{Inv}(C(B))$. Then $J\langle \tau \rangle \cong \mathrm{Aut}(L_5(2))$, so $O_2(C(\langle b, \tau\rangle)) = \langle\tau\rangle$ by Lemma 1.2(b). Since $b =_{N(B)} b_i$, $i = 1, 2$, and $\langle\tau\rangle \in \mathrm{Syl}_2(C(B))$, we have $O_2(C(\langle b_i, \tau\rangle)) = \langle\tau\rangle$ for $i = 1, 2$. By Lemma 1.2(e), (f), $O_2(C_K(\tau)) = O_2(C_L(\tau)) = 1$, so $O_2(C_{KL}(\tau)) = 1$. As $KL\langle\tau\rangle$ contains a Sylow 2-subgroup of $C(\beta)$ and $\beta =_{N(B)} \gamma$, we have $O_2(C(\langle\beta, \tau\rangle)) = O_2(C(\langle\gamma, \tau\rangle)) = \langle\tau\rangle$. It follows from the action of B_0 on $O_2(C(\tau))$ that $O_2(C(\tau)) = \langle\tau\rangle$, a contradiction since $C(\tau)$ is 2-constrained.

For the remainder of the section, let t be an involution in $C(B_0)$ which inverts b. Set $C = C(t)$ and $T = O_2(C)$. We shall show that C is isomorphic to the centralizer of an involution in $L_7(2)$.

LEMMA 4.4. *$J \leqslant C$, $T = C_T(b_1) \cdot C_T(b_2)$ and T is extra-special of type 2^{11}_+. Furthermore $C_T(b_1) = O_2(C(t) \cap E(C(b_1)))$.*

PROOF. By Lemmas 4.2 and 4.3, $C(\langle b_1, t\rangle)$ contains a subgroup isomorphic to $L_3(2)$ which is centralized by b. Therefore t centralizes J by Lemma 1.2(b), whence $J \leqslant C$.

Observe that $T = \langle C_T(x) : x \in B_0^\#\rangle \leqslant \langle O_2(C(\langle x, t\rangle)) : x \in B_0^\#\rangle$. Setting $D = O_2(C(\langle\beta, t\rangle))$, it follows from Lemma 4.3 that $D \leqslant C(B_0)$ and that D is dihedral of order 8. This implies that $C_T(\beta) = C_T(B_0) = D \cap T$. Since $\beta =_J \gamma$, we have that $T = C_T(b_1) \cdot C_T(b_2)$.

We now show that T is extra-special. Set $T_i = C_T(b_i)$ and $R_i = O_2(C(\langle b_i, t\rangle))$ so that $T_i = T \cap R_i$, $i = 1, 2$. By Lemma 1.2(g), $C \cap E(C(b_i)) = R_i L_i$ where $L_i \cong L_3(2)$, $b_i \in L_{3-i}$, R_i is extra-special of type 2^7_+ and $D \leqslant R_1 \cap R_2$. Setting $\bar{C} = C/\langle t\rangle$ and observing that $\bar{T} = F^*(\bar{C})$, we have $\bar{R}_i$ is abelian, hence $\bar{D} \leqslant C_{\bar{C}}(\bar{T}) = Z(\bar{T})$. Since $\bar{D} = C_{\bar{T}}(\beta)$, Lemma 1.6 implies that $\bar{T}$ is elementary abelian. As $C(\beta) \cap Z(T) = \langle t\rangle$, we have $Z(T) = \langle t\rangle$ by the same lemma. Thus T is extra-special.

By Lemma 4.3, K centralizes D, so γ acts regularly on the K-invariant section T/D. Therefore $|T/D| = 2^{4r}$ for some $r \geqslant 1$. But $|T| = |T_1T_2| = |T_1|\,|T_2|/|D| \leqslant |R_1|\,|R_2|/|D| = 2^{11}$ gives $r \leqslant 2$. On the other hand, J acts faithfully on T forcing $r > 1$. We conclude that $r = 2$ whereupon $|T| = 2^{11}$ and $T_i = R_i$, $i = 1, 2$.

Since T_1 is extra-special, $T = T_1 * C_T(T_1)$. As $C_T(T_1) \cap C(b_1) = C_T(T_1) \cap T_1 = \langle t\rangle$, we have $C_T(T_1) \leqslant T_2$. But $C_T(T_1) = [C_T(T_1), b_1] \leqslant [T_2, b_1]$ which is extra-special of type 2^5_+ so that $T = T_1 * [T_2, b_1]$ is extra-special of type 2^{11}_+. This completes the proof.

LEMMA 4.5. *TJ is isomorphic to the centralizer of a central involution of $L_7(2)$.*

PROOF. Recall from Lemma 4.4, that $C \cap E(C(b_i)) = T_iL_i$ where $T_i = C_T(b_i)$ is extra-special of type 2^7_+, $L_i \cong L_3(2)$ and $b_j \in L_i$, $1 \leqslant i \neq j \leqslant 2$. If we set $S_i = [T_{3-i}, b_i]$, then $T = T_i * S_i$, $S_i = [T, b_i]$ and S_i is extra-special of type 2^5_+, $i = 1, 2$. Also L_i may be chosen so that $L_i < J$ whereupon $J = \langle L_1, L_2\rangle$. We shall use the action of L_i on T to prove that TJ has the required isomorphism type.

As S_i does not admit the faithful action of L_i, we have $[S_i, L_i] = 1$, $i = 1, 2$. By Lemma 1.1(b), T_i contains precisely 2 L_i-invariant E_{16}-subgroups, say U_i and V_i, with $U_i \cap V_i = \langle t\rangle$. Recall from the proof of Lemma 4.4 that $D = C_T(B_0)$ is dihedral of order 8 and $D = T_1 \cap T_2$. It is easy to check that $D \cap U_i = C_{U_i}(b_{3-i})$ has order 4. Relabelling U_2 and V_2, if necessary, we may assume that $D \cap U_1 = D \cap U_2$. Set $U = U_1U_2$ and $W_i = [U_i, b_{3-i}]$. Then $U_i = W_i \times (D \cap U_i)$ implies that U has order 2^6 and that $U = U_1W_2 = U_2W_1$. Since $W_{3-i} < S_i$, $L_i < C(W_{3-i}) \cap N(U_i) < N(U)$. Therefore $J = \langle L_1, L_2\rangle < N(U)$. As $|U| = 2^6$, it follows immediately that U is elementary abelian. Similarly $V = V_1V_2$ is a J-invariant E_{2^6}-subgroup of T. Furthermore J acts trivially on $U \cap V$ then gives $T = UV$ and $U \cap V = \langle t\rangle$.

By Lemma 1.5, $U = \langle t\rangle \times U_0$ where U_0 is J-invariant. Clearly $U_0 \cap C_T(V) = U_0 \cap V = 1$, so U_0J acts faithfully on V. Since $L_7(2)$ contains a subgroup isomorphic to the split extension of V by $\mathrm{Aut}(V)$, we may embed $VU_0J = TJ$ in $L_7(2)$. But then consideration of orders verifies that TJ is isomorphic to the centralizer of a central involution of $L_7(2)$.

PROPOSITION 4.6. $C(t) = TJ$.

PROOF. As in the proof of the previous lemma, let U and V be the J-invariant E_{2^6}-subgroups of T. Then $N_C(U) = C_C(U)TJ$ because $\mathrm{Aut}_C(U) = \mathrm{Aut}_{TJ}(U)$. Furthermore, the $P \times Q$ lemma applied to $O^2(C_C(U)) \times U$ acting on T gives $O^2(C_C(U)) \leq T$ since $C_T(U) = U$. Therefore $C_C(U)$ is a 2-group and $O_2(N_C(U)) = C_C(U)T$. We shall argue that $N_C(U) = TJ$ and that $N_C(U) \lhd C$.

We claim that $t^G \cap T = U^\# \cup V^\#$. First, we show that $U^\# \cup V^\# \subseteq t^G \cap T$. In fact, TJ has 2 orbits on $U^\#$, namely $\{t\}$ and $U\setminus\langle t\rangle$. Since $D \cap U$ is a fours group and $\mathrm{Inv}(D) \subseteq t^G$ (recall that $D \leq E(C(B_0)) \cong L_3(2)$ by Lemma 4.3), we have $U^\# \subseteq t^G$. Similarly $V^\# \subseteq t^G$. As $C_T(b_1) = T_1 = O_2(C \cap E(C(b_1)))$ by Lemma 4.4, it follows from Lemma 1.1(e) that T contains an involution x with $x \in_G C(\langle b_1, bb_2\rangle)$. Certainly $x \neq_G t$ because $\langle b_1, bb_2\rangle \neq_G B_0 \in \mathrm{Syl}_3(C)$. But an easy argument shows that TJ acts transitively on $\mathrm{Inv}(T)\setminus(U^\# \cup V^\#)$. Hence $t^G \cap T = U^\# \cup V^\#$ as claimed.

If $W = U^g \leq T$, then U and V are the unique E_{2^6}-subgroups of T generated by conjugates of t implies that $W = U$ or $W = V$. Therefore $[C : N_C(U)] \leq 2$ and $N_C(U) \lhd C$. Thus $T = O_2(N_C(U))$, $U = C_C(U)$ and $N_C(U) = TJ$.

We have $C = TJ \cdot N_C(B_0)$ by the Frattini argument. Also, by Lemmas 4.3 and 4.4, $C_C(B_0)$ is 2-closed and $O_2(C_C(B_0)) \leq T$. As $|\mathrm{Aut}_C(B_0)| \leq |\mathrm{Aut}_G(B_0)| = 8$ by Lemma 1.4 and $|\mathrm{Aut}_J(B_0)| = 8$ by Lemma 1.2(c), TJ contains a Sylow

2-subgroup of $N_C(B_0)$. Therefore $C = TJ$, as required.

By Suzuki's theorem [14], we conclude from Proposition 4.6 that $G \cong L_7(2)$. This completes the proof of Theorem B.

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