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STARLIKE, CONVEX, CLOSE-TO-CONVEX, SPIRALLIKE, AND Φ-LIKE MAPS IN A COMMUTATIVE BANACH ALGEBRA WITH IDENTITY

BY

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ABSTRACT. Let $C(X)$ be the space of continuous functions on a compact T_2-space X where each point of X is a G_δ. If F: $B \to C(X)$ is a biholomorphic (in the sense that F and F^{-1} are Fréchet differentiable) map of $B = \{f \mid \|f\| < 1\}$ onto a convex domain with $DF(0) = I$, then F is Lorch analytic (i.e., $DF(f)(g) = a_f g$ for some $a_f \in C(X)$).

Let R be a commutative Banach algebra with identity such that the Gelfand homomorphism of R into $C(\mathfrak{M})$ is an isometry. Starlike, convex, close-to-convex, spirallike and Φ-like functions are defined in $B = \{x \in R \mid \|x\| < 1\}$ for L-analytic functions in B and they are related to associated complex-valued holomorphic functions in $\Delta = \{z \in \mathbf{C} \mid |z| < 1\}$.

Introduction. In §§2–7, let R be a commutative Banach algebra over the complex numbers with identity (denoted by 1) and let $\mathfrak{M}$ be the space of maximal ideals in R. Then $\mathfrak{M}$ is a compact, T_2-space where the topology is the weakest topology on $\mathfrak{M}$ such that the Gelfand transformation $x(M)$ of x is a continuous function on $\mathfrak{M}$. Assume further that the Gelfand homomorphism of R into $C(\mathfrak{M})$ is an isometry; i.e., $\|x\| = \sup\{|x(M)| \mid M \in \mathfrak{M}\}$ for all $x \in R$. Let $B = \{x \in R \mid \|x\| < 1\}$ and $\Delta = \{z \in \mathbf{C} \mid |z| < 1\}$.

If D is an open set in R, we say F: $D \to R$ is L-analytic in D if for each $x \in D$, there is $F'(x) \in R$ such that

$$\lim_{h \to 0} \frac{\|F(x+h) - F(x) - hF'(x)\|}{\|h\|} = 0$$

[11]. Thus it is clear that L-analytic functions are Fréchet differentiable. If F: $B \to R$ is L-analytic in B, then for each $x \in B$, $F(x) = \sum_{n=0}^{\infty} a_n x^n$ where $a_n \in R$ and the series converges uniformly on $\|x\| \leq \rho < 1$ [7, Theorems 3.19.1 and 26.4.1]. If F: $D \to R$ is L-analytic in D and for each $y \in F(B)$, there is an open neighborhood V of y such that F^{-1} exists and is L-analytic

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in V, then we say that F is locally bianalytic in B. If F is univalent (one-to-one) and locally bianalytic in B, we say that F is bianalytic in B. If F is L-analytic in B, then for each $M \in \mathfrak{M}$, there is an associated holomorphic function $F_M: \Delta \to \mathbf{C}$ defined by $F_M(z) \equiv F(z1)(M)$ for all $z \in \Delta$. If $F(x) = \sum_{n=1}^{\infty} a_n x^n$ is L-analytic in B, then we write $F(x)/x$ for the L-analytic function $\sum_{n=1}^{\infty} a_n x^{n-1}$.

2. Preliminary lemmas.

LEMMA 2.1. *Let* $V: B \times I \to B$ *be* L*-analytic in* B *for each* $t \in I = [0, 1]$, $V(0, t) = 0$ *for all* $t \in I$, $V(x, 0) = x$ *for all* $x \in B$. *If* $\lim_{t \to 0^+}(x - V(x, t))/(xt) = U(x)$ *exists and is* L*-analytic in* B, *then* $\operatorname{Re} U(x)(M) > 0$ *for all* $M \in \mathfrak{M}$ *and all* $x \in B$.

PROOF. For each $t \in I$, $V(x, t)$ satisfies Schwarz' Lemma [17, Theorem A] so $\|V(x, t)\| < \|x\|$ for all $x \in B$. For all $M \in \mathfrak{M}$ and all $x \in B$,

$$|V_M(x(M), t)| = |V(x, t)(M)| \leqslant \|V(x, t)\| \leqslant \|x\|.$$

For $z \in \Delta$, the choice $x = z1$ shows that $V_M(\cdot, t)$ satisfies Schwarz' lemma. Now letting $z = x(M)$, we have $|(V(x, t)/x)(M)| = |V_M(x(M), t)/x(M)| \leqslant 1$ (where the limit value is to be taken when $x(M) = 0$) and taking the maximum over all $M \in \mathfrak{M}$, we have $\|V(x, t)/x\| \leqslant 1$. Hence,

$$\operatorname{Re} \frac{x - V(x, t)}{xt}(M) \geqslant \frac{1 - \|V(x, t)/x\|}{t} \geqslant 0 \quad \text{for all } t \in I.$$

The lemma follows.

DEFINITION 2.2. Let D be a domain in R. If $U: D \to R$ is L-analytic in D and $\operatorname{Re} U(x)(M) \geqslant 0$ for each $M \in \mathfrak{M}$ and each $x \in D$, then we say U has positive real part in D.

EXAMPLE 1. Let $X = \{1, 2, \ldots, n\}$ with the discrete topology. Then $C(X) = \mathbf{C}^n$ with the multiplication $(a_1, a_2, \ldots, a_n) \cdot (b_1, b_2, \ldots, b_n) = (a_1b_1, a_2b_2, \ldots, a_nb_n)$ and the unit ball B is the polydisk $\{(z_1, z_2, \ldots, z_n): |z_j| < 1, 1 \leqslant j \leqslant n\}$. Therefore, L-analytic functions on B are functions $F(z_1, z_2, \ldots, z_n) = (F_1(z_1), F_2(z_2), \ldots, F_n(z_n))$ where each F_j is analytic in the unit disk Δ. There are n maximal ideals $M_1, M_2, \ldots, M_n$ in $C(X)$ given by $M_j = \{(z_1, z_2, \ldots, z_n) \in \mathbf{C}^n: z_j = 0\}$. It follows that $U = (U_1, U_2, \ldots, U_n): B \to \mathbf{C}^n$ has positive real part if and only if $\operatorname{Re} U_j(z_j) > 0$ whenever $|z_j| < 1$ (where it is assumed that U is L-analytic in B). Note that if U has positive real part, then zU is in the class $\mathfrak{P}$ defined in [15].

EXAMPLE 2. Let $X = [0, 1]$ with the usual topology and $R = C(X)$. Then L-analytic functions on B are the power series $F(f) = \sum_{n=0}^{\infty} a_n f^n$ where $a_n \in C(X)$ with lim sup $\|a_n\|^{1/n} \leqslant 1$. The maximal ideals are the sets $M_x = \{f \in C(X): f(x) = 0\}$ for some x, $0 \leqslant x \leqslant 1$. Therefore, if U is L-analytic in B, U has positive real part if and only if Re $U(f)(x) \geqslant 0$ for all $f \in B$ and $x \in [0, 1]$.

EXAMPLE 3. Let $R = H^{\infty}(\Delta)$. In this case, one needs to modify Definition 2.2 to replace $\mathfrak{M}$ by $\mathfrak{M}' = \text{cl}\{M_z \in \mathfrak{M}: f \in M_z \Rightarrow f(z) = 0, |z| < 1\}$. The theory in the remainder of this paper can then be applied to this space. Thus U: $B \to H^{\infty}(\Delta)$ has positive real part if Re $U(f)(z) \geqslant 0$ for all $f \in H^{\infty}$ and $z \in \Delta$. For example, $U(f) = (1 + f)(1 - f)^{-1} = 1 + 2f + 2f^2 + \ldots$ has positive real part.

LEMMA 2.3. *Let F: $B \to R$ be bianalytic in B. Let G: $B \times I \to R$ be L-analytic for each $t \in I$, $G(x, 0) = F(x)$, for each $x \in B$, $G(0, t) = F(0)$ for each $t \in I$, and $G(B, t) \subset F(B)$ for each $t \in I$. If $\lim_{t \to 0^+}(G(x, 0) - G(x, t))/t = xH(x)$ exists and is L-analytic, then $H(x) = F'(x)U(x)$ where U has positive real part in B.*

PROOF. We will show that $V(x, t) = F^{-1}(G(x, t))$ satisfies Lemma 2.1. Fix $x \in B$, $x \neq 0$, and expand $G(x, t)$ about x,

$$G(x, t) = F(V(x, t)) = F(x) + F'(x)(V(x, t) - x) + K(V(x, t), x)$$

where $\|K(y, x)\|/\|y - x\| \to 0$ as $\|y - x\| \to 0$. Therefore,

$$\frac{G(x, 0) - G(x, t)}{t} = F'(x)\frac{x - V(x, t)}{t} - \frac{K(V(x, t), x)}{t}.$$

If we show $K(V(x, t), x)/t \to 0$ as $t \to 0^+$, then

$$\lim_{t \to 0^+} \frac{x - V(x, t)}{xt} = [F'(x)]^{-1}H(x)$$

and the lemma follows by Lemma 2.2.

To show that $K(V(x, t), x)/t \to 0$ as $t \to 0^+$, observe that $\|(x - V(x, t))/t\|$ is bounded as $t \to 0^+$; otherwise, for some sequence $\{t_n\}$, $t_n \to 0$ and $\|(x - V(x, t_n))/t_n\| \to \infty$. In this case,

$$xH(x) = \lim_{n \to \infty} \left[F'(x)\frac{x - V(x, t_n)}{\|x - V(x, t_n)\|} - \frac{K(V(x, t_n), x)}{\|x - V(x, t_n)\|} \right] \frac{\|x - V(x, t_n)\|}{t_n}$$

so that

$$F'(x)\frac{x - V(x, t_n)}{\|x - V(x, t_n)\|} \to 0 \quad \text{as } n \to \infty.$$

But this implies that $F'(x)$ is a generalized divisor of zero which contradicts the L-analyticity of F^{-1}. Thus we have shown

$$\lim_{t \to 0^+} \frac{K(V(x, t), x)}{t} = \lim_{t \to 0^+} \frac{K(V(x, t), x)}{\|V(x, t) - x\|} \frac{\|V(x, t) - x\|}{t} = 0.$$

LEMMA 2.4. *Let U have positive real part.*

(1) *If $M \in \mathfrak{M}$, then*

$$\frac{1 - \|x\|}{1 + \|x\|} \operatorname{Re} U(0)(M) \leqslant \operatorname{Re} U(x)(M) \leqslant \frac{1 + \|x\|}{1 - \|x\|} \operatorname{Re} U(0)(M) \quad \text{for all } x \in B;$$

and so $\operatorname{Re} U(0)(M) > 0$ *if and only if* $\operatorname{Re} U(x)(M) > 0$ *for all $x \in B$.*

(2) $\operatorname{Re} U(0)(M) > 0$ *for all $M \in \mathfrak{M}$ implies $U(x)$ is nonsingular for all $x \in B$.*

PROOF. For $M \in \mathfrak{M}$ and $0 \neq x \in B$, let $\rho(\lambda) = U(\lambda x/\|x\|)(M)$ for $\lambda \in \Delta$. Since ρ is holomorphic in Δ and $\operatorname{Re} \rho(\lambda) \geqslant 0$, by the classical inequality,

$$\frac{1 - |\lambda|}{1 + |\lambda|} \operatorname{Re} \rho(0) \leqslant \operatorname{Re} \rho(\lambda) \leqslant \frac{1 + |\lambda|}{1 - |\lambda|} \operatorname{Re} \rho(0)$$

so, $\lambda = \|x\|$ yields (1). (2) follows from (1) and the fact that $\operatorname{Re} U(x)(M) > 0$ for all $M \in \mathfrak{M}$ implies $U(x) \notin M$ for any $M \in \mathfrak{M}$ and, hence, $U(x)$ is nonsingular.

DEFINITION 2.5. If U has positive real part in a domain $D \subset R$ and $\operatorname{Re} U(x)(M) > 0$ for all $M \in \mathfrak{M}$ and all $x \in D$, then we write $U \in \mathcal{P}(D)$. If $D = B$, then we write $\mathcal{P}$ for $\mathcal{P}(B)$.

LEMMA 2.6. *Let $P \in \mathcal{P}$. Then, for each $x \in B$, the initial value problem*

$$dw/dt = -wP(w), \qquad w(0) = x,$$

has a unique solution $V(t) = V(x, t)$ defined on $t \geqslant 0$. For fixed $t \geqslant 0$, $V_t(x) = V(x, t)$ is L-analytic and univalent in B and

$$\|V(x, t)\| \leqslant \|x\| \exp\left(-\frac{1 - \|x\|}{1 + \|x\|}\, \delta t\right) \tag{1}$$

for all $t \geqslant 0$ and all $x \in B$ where $\delta = \min\{\operatorname{Re} P(0)(M) | M \in \mathfrak{M}\}$.

PROOF. The proof of the existence and uniqueness of the solution is covered in [**12**]. If (1) holds, then the solution can be continued to obtain a solution for all $t \geqslant 0$. The univalence of solution follows from the uniqueness of the solution, and the L-analyticity of $V(x, t)$ in B for each $t \geqslant 0$ follows from the equilocal boundedness of the successive approximations $V_m(x, t)$ of $V(x, t)$ and Theorem 8.4.3 [**6**, p. 272].

We now show (1). For each $M \in \mathfrak{M}$, $V(t)(M)$ is the solution of the initial value problem

$$du/dt = -uP_M(u), \qquad u(0) = V(0)(M) = x(M).$$

By [**1**, Lemma 1], $|V(t)(M)| \leqslant |V(0)(M)|$ for all $t \geqslant 0$. Differentiating $|V(t)(M)|^2 = V(t)(M)\overline{V(t)(M)}$, we get

$$\frac{1}{|V(t)(M)|}\ \frac{d|V(t)(M)|}{dt} = -\operatorname{Re} P_M(V(t)(M))$$
$$\leqslant -\frac{1-|V(t)(M)|}{1+|V(t)(M)|}\operatorname{Re} P_M(0)$$
$$\leqslant -\frac{1-|V(0)(M)|}{1+|V(0)(M)|}\operatorname{Re} P_M(0) \leqslant -\frac{1-\|x\|}{1+\|x\|}\,\delta$$

and (1) follows.

3. Starlike functions. In **C**, if $f(z) = \sum_{n=1}^{\infty} a_n z^n$, $a_1 \neq 0$, is holomorphic in Δ, then f is starlike in Δ if $(1-t)f(\Delta) \subset f(\Delta)$ for all $t \in I = [0, 1]$ which is equivalent to $\operatorname{Re}(zf'(z)/f(z)) > 0$ for all $z \in \Delta$. We will define starlike functions in R and relate them to starlike function in **C**.

DEFINITION 3.1. A bianalytic map F: $B \to R$ is said to be starlike in B if $F(0) = 0$ and $(1-t)F(B) \subset F(B)$ for all $t \in I$.

THEOREM 3.2. *Let $F(x) = \sum_{n=1}^{\infty} a_n x^n$ be locally bianalytic in B. Then F is starlike in B if and only if $F_M(z) = \sum_{n=1}^{\infty} a_n(M) z^n$ is starlike in Δ for all $M \in \mathfrak{M}$.*

PROOF. Assume F is starlike in B and set $G(x, t) = (1-t)F(x)$. Lemma 2.3 applies with $xH(x) = F(x)$ so that $F(x) = xF'(x)U(x)$ where U has positive real part. However, $U(0) = 1$ by equating coefficients, so by Lemma 2.4, $U \in \mathcal{P}$. Setting $x = ze$, we conclude

$$\operatorname{Re}\frac{\sum_{n=1}^{\infty} a_n(M) z^{n-1}}{\sum_{n=1}^{\infty} n a_n(M) z^{n-1}} > 0 \quad \text{for } z \in \Delta$$

and, hence, F_M is starlike for each $M \in \mathfrak{M}$.

Conversely, if F_M is starlike for every $M \in \mathfrak{M}$, then for fixed $x \in B$, the function $V(x, t) = F^{-1}(e^{-t}F(x))$, defined near $t = 0$, satisfies the initial value problem

$$\frac{\partial V(x, t)}{\partial t} = -\left[\frac{F(V(x, t))}{V(x, t)F'(V(x, t))}\right]V(x, t), \qquad V(x, 0) = x.$$

Set $P(w) = F(w)/wF'(w)$ for all $w \in B$. By hypothesis, $P \in \mathcal{P}$ and so by Lemma 2.6, $V(x, t)$ is the unique solution of the initial value problem

$$dw/dt = -wP(w), \qquad w(0) = x.$$

Then $\|V(x, t)\| \leqslant \|x\| < 1$ and $F(V(x, t)) = e^{-t}F(x)$, $t \geqslant 0$. This implies that $(1-t)F(B) \subset F(B)$, $0 \leqslant t \leqslant 1$. To see the univalence of F in B, let $x_1, x_2 \in B$ such that $F(x_1) = F(x_2)$. Suppose $V_{x_i}(t) = V(x_i, t)$ is the unique solution of

$$dw/dt = -wP(w), \qquad w(0) = x_i,$$

and let $W_{x_i}(t) = F(V_{x_i}(t))$, $i = 1, 2$. For small $t \geqslant 0$, $W_{x_i}(t)$ satisfies the initial value problem

$$dw/dt = -w, \qquad w(0) = F(x_i),$$

which has a unique solution $W_{x_i}(t) = F(x_i)e^{-t}$ for $t \geqslant 0$. Since $F(x_1) = F(x_2)$, $W_{x_1}(t) = W_{x_2}(t)$ for $t \geqslant 0$. Since $W_{x_i}(t) \to 0$ as $t \to +\infty$, and since F has a local inverse in an open neighborhood of 0, $V_{x_1}(t) = V_{x_2}(t)$ for all $t \geqslant M > 0$. Then $V_{x_1}(t) = V_{x_2}(t)$ for all $t \geqslant 0$; in particular, $x_1 = V_{x_1}(0) = V_{x_2}(0) = x_2$ and F is univalent in B.

EXAMPLE 1. Let F: $B \to R$ be given by $F(x) = x(1 - ax)^{-2}$ where $\|a\| \leqslant 1$. Let $M \in \mathfrak{M}$ and set $a(M) = \alpha$. Then $|\alpha| \leqslant 1$ and $F_M(z) = z/(1 - \alpha z)^2$, which is known to be starlike. Therefore F is starlike. If $X = \{1, 2, \ldots, n\}$ so that $C(X) = \mathbf{C}^n = R$, F has the form

$$F(z_1, z_2, \ldots, z_n) = \left(z_1/(1 - a_1 z)^2, \ldots, z_n/(1 - a_n z)^2\right)$$

where $|a_j| \leqslant 1$, $1 \leqslant j \leqslant n$.

If $R = C[0, 1]$,

$$F(f)(x) = f(x)/(1 - a(x)f(x))^2, \qquad 0 \leqslant x \leqslant 1.$$

If $R = H^\infty(\Delta)$,

$$F(f)(z) = f(z)/(1 - a(z)f(z))^2, \qquad |z| < 1.$$

EXAMPLE 2. Other choices for F: $B \to R$ that will make F starlike are

$$F(x) = x + ax^2, \qquad a \in R, \|a\| \leqslant \tfrac{1}{2},$$

$$F(x) = x \prod_{j=1}^{\infty} (1 - a_j x)^{-\alpha_j}$$

where each $\alpha_j \geqslant 0$ and $\sum_{j=1}^{\infty} \alpha_j \leqslant 2$ with $\|a_j\| \leqslant 1$ for each j.

4. Convex functions. In $\mathbf{C}$, if $f(z) = \sum_{n=1}^{\infty} a_n z^n$, $a_1 \neq 0$, is holomorphic in Δ, then f is convex in Δ if $f(\Delta)$ is a convex domain. This is equivalent to $\operatorname{Re}(1 + zf''(z)/f'(z)) > 0$ for all $z \in \Delta$. We will define convex functions in R and relate them to convex functions in $\mathbf{C}$.

DEFINITION 4.1. A bianalytic map F: $B \to R$ is said to be convex in B if $F(B)$ is a convex domain.

THEOREM 4.2. *Let $F(x) = \sum_{n=1}^{\infty} a_n x^n$ be locally bianalytic in B. F is convex in B if and only if $F_M(z) = \sum_{n=1}^{\infty} a_n(M) z^n$ is convex in Δ for each $M \in \mathfrak{M}$. Thus the Alexander relation (F is convex in B if and only if G is starlike in B where $G(x) = xF'(x)$ for all $x \in B$) holds.*

PROOF. Assume F is convex in B. Set $G(x, t) = \frac{1}{2}(F(e^{i\sqrt{t}}x) + F(e^{-i\sqrt{t}}x))$ and apply Lemma 2.3. Expanding $F(e^{\pm i\sqrt{t}}x)$ about x, we have

$$\begin{aligned}
&\tfrac{1}{2}\left(F(e^{i\sqrt{t}}x) + F(e^{-i\sqrt{t}}x)\right)\\
&\quad = \tfrac{1}{2}\Big[F(x) + F'(x)(e^{i\sqrt{t}} - 1)x + \tfrac{1}{2}F''(x)(e^{i\sqrt{t}} - 1)^2x^2\\
&\qquad\quad + F(x) + F'(x)(e^{-i\sqrt{t}} - 1)x + \tfrac{1}{2}F''(x)(e^{-i\sqrt{t}} - 1)^2x^2 + o(t)\Big]\\
&\quad = F(x) + xF'(x)(\cos\sqrt{t} - 1)\\
&\qquad + \tfrac{1}{2}x^2F''(x)(\cos 2\sqrt{t} - 2\cos\sqrt{t} + 1) + o(t).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\lim_{t\to 0^+} \frac{G(x, 0) - G(x, t)}{t}\\
&\quad = xF'(x)\lim_{t\to 0^+} \frac{1 - \cos\sqrt{t}}{t} + x^2F''(x)\lim_{t\to 0^+} \cos\sqrt{t}\left(\frac{(1 - \cos\sqrt{t})}{t}\right)\\
&\quad = \tfrac{1}{2}\left[xF'(x) + x^2F''(x)\right].
\end{aligned}$$

Therefore $F'(x)U(x) = \frac{1}{2}[F'(x) + xF''(x)]$ where U has positive real part. Equating coefficients, we conclude that $U(0) = \frac{1}{2}1$ and so $U \in \mathcal{P}$. This means

$$\operatorname{Re}\left[1 + \frac{\sum_{n=1}^{\infty} n^2a_n(M)z^{n-1}}{\sum_{n=1}^{\infty} na_n(M)z^{n-1}}\right] > 0 \quad \text{for all } M \in \mathfrak{M}$$

so that F_M is convex in Δ.

Suppose F_M is convex in Δ for each $M \in \mathfrak{M}$ and let $x, y \in B_r = \{x \in R \mid \|x\| < r\}$, $r < 1$. Since F_M is univalent, F is bianalytic in B. Let $V(t) = F^{-1}(tF(x) + (1 - t)F(y))$. Then for all $M \in \mathfrak{M}$,

$$F_M(V(t)(M)) = tF_M(x(M)) + (1 - t)F_M(y(M)).$$

Since F_M is convex in $|z| < r$, $|V(t)(M)| < r$. Choose $M \in \mathfrak{M}$ such that $\|V(t)\| = |V(t)(M)| < r$ and the convexity of F follows.

EXAMPLE. (i) $x(1 - ax)^{-1}$ when $\|a\| \leqslant 1$ is convex.

(ii) $\log[(1 + x)(1 - x)^{-1}]$ is convex.

5. Close-to-convex functions. In **C**, a holomorphic function f: $\Delta \to \mathbf{C}$ is said to be close-to-convex in Δ if there is a convex function g: $\Delta \to \mathbf{C}$ such that $\operatorname{Re}(f'(z)/g'(z)) > 0$ for all $z \in \Delta$. In **[9]**, it is shown that every close-to-convex function is univalent. We define close-to-convex functions in R and show that every close-to-convex function in B is univalent. Compare **[13]** and **[17]**.

DEFINITION 5.1. Suppose $F\colon B \to R$ is L-analytic in B. We say that F is close-to-convex if $F_M\colon \Delta \to \mathbf{C}$ is close-to-convex in Δ for all $M \in \mathfrak{M}$.

Clearly if $G\colon B \to R$ is convex in B, $U \in \mathcal{P}$, and $F'(x) = G'(x)U(x)$ for all $x \in B$, then F is close-to-convex in B.

THEOREM 5.2. *If D is a convex domain in R and $G\colon D \to R$ is such that $G' \in \mathcal{P}(D)$, then G is univalent in D.*

PROOF. Let $x_1, x_2 \in D$, $x_1 \neq x_2$. Since D is convex, $\{tx_2 + (1-t)x_1 | t \in I\} \subset D$. We have

$$\frac{d}{dt}\, G(tx_2 + (1-t)x_1) = G'(tx_2 + (1-t)x_1)(x_2 - x_1)$$

so that

$$G(x_2) - G(x_1) = (x_2 - x_1)\int_0^1 G'(tx_2 + (1-t)x_1)\, dt.$$

Let $M \in \mathfrak{M}$ be such that $\|x_2 - x_1\| = |(x_2 - x_1)(M)|$. Then

$$|(G(x_2) - G(x_1))(M)| = \|x_2 - x_1\| \left| \int_0^1 G'(tx_2 + (1-t)x_1)(M)\, dt \right|$$

$$\geqslant \|x_2 - x_1\| \int_0^1 \operatorname{Re} G'(tx_2 + (1-t)x_1)(M)\, dt > 0$$

and hence $G(x_2) \neq G(x_1)$.

THEOREM 5.3. *If F is close-to-convex in B, then F is univalent in B.*

PROOF. If there is a convex function $G\colon B \to R$ such that $F'(x) = G'(x)U(x)$ for some $U \in \mathcal{P}$, we may apply Theorem 5.2 to $F \circ G^{-1}$: $G(B) \to R$ to conclude that F is univalent.

Otherwise, let $x_1, x_2 \in B$, $x_1 \neq x_2$ and choose $M \in \mathfrak{M}$ such that $|(x_2 - x_1)(M)| = \|x_2 - x_1\|$. Since F_M is close-to-convex in Δ, there is a convex function $g\colon \Delta \to \mathbf{C}$ such that $\operatorname{Re}(F'_M(z)/g'(z)) > 0$ for all $z \in \Delta$. Define G: $B \to R$ by $G(x) = \sum_{k=1}^{\infty} (b_k 1)x^k$ where $g(z) = \sum_{k=1}^{\infty} b_k z^k$. Then G is convex (in particular, bianalytic) in B. Consider $H \equiv F \circ G^{-1}$: $G(B) \to R$ and let $y_1 = G(x_1)$ and $y_2 = G(x_2)$. As in the proof of Theorem 5.2, we have

$$F(x_2) - F(x_1) = H(y_2) - H(y_1) = \int_0^1 H'(ty_2 + (1-t)y_1)(y_2 - y_1)\, dt$$

so that

$$|(F(x_2) - F(x_1))(M)| = |(H(y_2) - H(y_1))(M)|$$

$$= \left| \int_0^1 H'(ty_2 + (1 - t)y_1)(M)\, dt \right| |(y_2 - y_1)(M)|$$

$$> \int_0^1 \operatorname{Re} H'(ty_2 + (1 - t)y_1)(M)\, dt |(y_2 - y_1)(M)|$$

$$= \int_0^1 \operatorname{Re} \frac{F'_M(G^{-1}(ty_2 + (1 - t)y_1)(M))}{g'(G^{-1}(ty_2 + (1 - t)y_1)(M))}\, dt |(y_1 - y_2)(M)| > 0$$

if $|(y_2 - y_1)(M)| \neq 0$. But

$$\|x_2 - x_1\| = |(x_2 - x_1)(M)|$$

$$= \left| \int_0^1 (G^{-1})'(ty_2 + (1 - t)y_1)(M)\, dt \right| |(y_2 - y_1)(M)|$$

so the desired result follows.

EXAMPLE. (i) $F(x) = x(1 - ax)(1 - x)^{-2}$ is close-to-convex in B where $\|a - \frac{1}{2}1\| < \frac{1}{2}$ because $F_M(z)$ is known to be close-to-convex for every $M \in \mathfrak{M}$.

(ii) Every starlike function is close-to-convex.

(iii) Every convex function is close-to-convex.

6. Spirallike functions. In C, if $f(z) = \sum_{n=1}^{\infty} a_n z^n$, $a_1 \neq 0$, is holomorphic in Δ, then f is spirallike in Δ if $\operatorname{Re}(e^{-i\alpha} z f'(z)/f(z)) > 0$ for all $z \in \Delta$ where $\alpha \in (-\pi/2, \pi/2)$. If f is spirallike in Δ, then f is univalent in Δ **[14]**. We will define spirallike functions in R and prove that they are also univalent in B.

DEFINITION 6.1. Suppose $F(x) = \sum_{n=1}^{\infty} a_n x^n$ is locally bianalytic in B. We say that F is spirallike in B if there exists $a \in R$ such that $\operatorname{Re} a(M) > 0$ for all $M \in \mathfrak{M}$ and $U \in \mathcal{P}$ such that

$$a\frac{F(x)}{x} = F'(x)U(x) \quad \text{for all } x \in B \left(\text{where } \frac{F(x)}{x} = \sum_{n=1}^{\infty} a_n x^{n-1} \right). \quad (1)$$

From (1), we see that $(F(x)/x)(M) \neq 0$ whenever $M \in \mathfrak{M}$ and so $F(x)/x$ and a are nonsingular. It is clear that (1) can be replaced by the condition $\operatorname{Re}(bxF'(x)/F(x))(M) > 0$ for all $M \in \mathfrak{M}$ where $b = a^{-1}$ and $xF'(x)/F(x)$ means $(F(x)/x)^{-1}F'(x)$.

THEOREM 6.2. *Every spirallike function in B is univalent in B. Furthermore, if F is spirallike in B, then F_M is spirallike in Δ for each $M \in \mathfrak{M}$.*

PROOF. Since $F'(x)$ is nonsingular for each $x \in B$, F is locally bianalytic in B. For fixed $x \in B$ and t near zero, set $V(x, t) = F^{-1}(e^{-at}F(x))$. Then $V(x, t)$ is a solution of the initial value problem

$$dw/dt = -wP(w), \qquad w(0) = x,$$

where $P(w) = aF(w)/wF'(w)$. By (1), $P \in \mathcal{P}$ and so by Lemma 2.6, $V(x, t)$ is the unique solution for $t \geqslant 0$ and $V(x, t) \to 0$ as $t \to \infty$. Let $x_1, x_2 \in B$ such that $F(x_1) = F(x_2)$ and let $V_{x_i}(t) = V(x_i, t)$ be the unique solution of the initial value problem

$$dw/dt = -wP(w), \qquad w(0) = x_i, i = 1, 2.$$

Let $W_{x_i}(t) = F(V_{x_i}(t))$ for all $t \geqslant 0$, $i = 1, 2$. For small $t > 0$, $W_{x_i}(t)$ satisfies the initial value problem

$$dw/dt = -aw, \qquad w(0) = F(x_i),$$

which has a unique solution $W_{x_i}(t) = F(x_i)e^{-at}$ for $t > 0$. Since $F(x_1) = F(x_2)$, $W_{x_1}(t) = W_{x_2}(t)$ for all $t > 0$. Since $W_{x_i}(t) \to 0$ as $t \to +\infty$, we conclude that $x_1 = x_2$ as in Theorem 3.2.

That F_M is spirallike in Δ follows from the equation $a(M)[F(x)/x](M) = F'(x)(M)U(x)(M)$, and writing $x = z1$ gives $a(M)[F_M(z)/z] = F'_M(z)U_M(z)$ where Re $U_M(z) > 0$. Write $a(M) = |a(M)|e^{i\alpha}$ where $\alpha \in (-\pi/2, \pi/2)$.

EXAMPLE. Let

$$F(x) = x(1 - ax)^{-(1+b)} = \sum_{n=1}^{\infty} \frac{(b + 1)(b + 2 \cdot 1) \cdots (b + n \cdot 1)}{n} a^n x^n$$

where $\|a\| < 1$, $\|b\| < 1$ and $-1 < \operatorname{Re} b(M) < 1$ for all $M \in \mathfrak{M}$. For example, one might take $b = \rho e^{i\alpha} \cdot 1$ where α is real, $0 < |\alpha| < \pi$ and $0 < \rho < 1$. Then

$$(F(x)/x) \cdot (F'(x))^{-1} \equiv (1 - ax)(1 + abx)^{-1},$$

and setting $U(x) = (1 + b)(F(x)/x)(F'(x))^{-1}$ yields

$$\operatorname{Re} U(x)(M) = \operatorname{Re}\Big[(1 + b(M))(1 - a(M)x(M))(1 + a(M)b(M)x(M))^{-1}\Big].$$

With $z = a(M)x(M)$ and $\beta = b(M)$,

$$U(x)(M) = (1 + \beta)(1 - z)/(1 + \beta z),$$

and it is easy to show Re $U(x)(M) > 0$.

7. Φ-like functions. See [1] for the definitions of a Φ-like function and a Φ-like domain in **C**.

DEFINITION 7.1. Let $F: B \to R$ be a locally bianalytic function in B, and $F(0) = 0$. If $\Phi: F(B) \to R$ is L-analytic, then we say F is Φ-like in B if there is $U \in \mathcal{P}$ such that $\Phi(F(x)) = xF'(x)U(x)$ for all $x \in B$.

Since $F(0) = 0$, $\Phi(0) = 0$. Letting $\alpha \to 0$ in $\Phi(F(\alpha x))/\alpha = xF'(\alpha x)U(\alpha x)$, we have $\Phi'(0)x = xU(0)$ for all $x \in B$. Setting $x = \alpha 1$, where $0 < |\alpha| < 1$, we see that $\Phi'(0) = U(0)$.

DEFINITION 7.2. Let D be a domain in R which contains 0 and let $P \in \mathcal{P}(D)$. If, for each $\alpha \in D$, the initial value problem

$$dw/dt = -\Phi(w), \qquad w(0) = \alpha,$$

where $\Phi(w) = wP(w)$, has a unique solution $w = W(t) \in D$ for all $t \geq 0$ and $W(t) \to 0$ as $t \to +\infty$, then D is said to be Φ-like.

Note that starlike, convex, close-to-convex and spirallike functions are Φ-like for appropriate choices of Φ.

THEOREM 7.3. *If F is Φ-like in B for $\Phi(v) = vP(v)$ where $P \in \mathcal{P}(F(B))$, then F is univalent in B and $F(B)$ is Φ-like.*

PROOF. The proof follows along the lines of the proof of Theorem 1 in [5].

THEOREM 7.4. *If F: $B \to R$ is bianalytic in B with $F(0) = 0$ and $F(B)$ is Φ-like, then F is Φ-like in B.*

Since our proof uses Lemma 2.3, and therefore is shorter than Theorem 2 [5], we will give our proof.

PROOF. Since $F(B)$ is Φ-like, for each $x \in B$, let $W_x(t)$ be the unique solution of $dw/dt = -\Phi(w)$, $w(0) = F(x)$ where $\Phi(w) = wP(w)$, $P \in \mathcal{P}(F(B))$. Since F is bianalytic in B, set $V_x(t) = F^{-1}(W_x(t))$ for all $t \geq 0$. Then $V_x(0) = x$ and

$$F'(V_x(t))V_x'(t) = W_x'(t) = -W_x(t)P(W_x(t)) = -F(V_x(t))P\big(F(V_x(t))\big)$$

for all $t \geq 0$. Letting $t = 0$, we have $-F'(x)V_x'(0) = \Phi(F(x))$. To show that $-V_x'(0) = xU(x)$ for some $U \in \mathcal{P}$, let $G(x, t) = W_x(t) = F(V_x(t))$ in Lemma 3.2. Then

$$\begin{aligned}\lim_{t\to 0^+} \frac{G(x,0) - G(x,t)}{t} &= \lim_{t\to 0^+} \frac{F(V_x(0)) - F(V_x(t))}{t} \\ &= -F'(x)V_x'(0) = xH(x)\end{aligned}$$

is L-analytic in B and so $H(x) = F'(x)U(x)$ where U has positive real part. Hence $xU(x) = -V_x'(0)$. Since $xF'(x)U(x) = \Phi(F(x))$, we have $U(0) = \Phi'(0) = P(0)$, and so $\operatorname{Re} U(0)(M) = \operatorname{Re} P(0)(M) > 0$ for all $M \in \mathfrak{M}$ and, by Lemma 2.4, $U \in \mathcal{P}$.

8. Convex F-holomorphic functions in $C(X)$. If D is an open set in the Banach space $\mathfrak{B}$ and F: $D \to \mathfrak{B}$, then F is said to be F-holomorphic in D if for each $x \in D$, there is a bounded linear map $DF(x)$: $\mathfrak{B} \to \mathfrak{B}$ such that $\lim_{h\to 0} \|F(x+h) - F(x) - DF(x)(h)\|/\|h\| = 0$. If F: $D \to \mathfrak{B}$ is F-holomorphic in D and for each $y \in F(B)$, there is an open neighborhood V of y

such that $F^{-1}: V \to D$ is F-holomorphic in V, then we say F is locally biholomorphic in B. If F is univalent and locally biholomorphic in B, we say that F is biholomorphic in B. If F is F-holomorphic in D, then for each $x_0 \in D$ there is a disk in D with center at x_0 such that $F(x) = \sum_{n=0}^{\infty} 1/n!D^nF(x_0)((x - x_0)^n)$ where $D^nF(x_0) \in L_n(\mathfrak{B}, \mathfrak{B})$, space of all continuous symmetric n-linear maps of $\mathfrak{B}^n$ into $\mathfrak{B}$ and the series converges uniformly in this disk. If $F: B \to \mathfrak{B}$ is F-holomorphic in B, then $F(x) = \sum_{n=0}^{\infty} 1/n!D^nF(0)(x^n)$ for all $x \in B$ [7, Theorem 3.16.2]. If $\mathfrak{B}$ is a commutative Banach algebra with identity, it is clear that every L-analytic function is F-holomorphic; however, not every F-holomorphic function is L-analytic [7, p. 115]. We will prove that F-holomorphic implies L-analytic in the special case in which F is a biholomorphic map of the unit ball of $C(X)$ onto a convex domain in $C(X)$.

Assume X is a compact T_2-space such that each point of X is a G_δ; i.e., each $x \in X$ is the intersection of a countable number of open neighborhoods of x. Let $C(X)$ be the Banach algebra of complex valued continuous functions on X (with sup norm and pointwise multiplication).

THEOREM 8.1. *Let $C(X)$ be as above. If $F: B \to C(X)$ is a convex biholomorphic function in B such that $DF(0) = I$, then F is L-analytic in B and hence bianalytic in B.*

Without loss of generality, we assume $F(0) = 0$. The proof will be given in the following six lemmas.

LEMMA 8.2. *If $k, u \in C(X)$, $k \equiv 0$ on an open neighborhood of $x_0 \in X$, $u(x_0) = 1$ and $|u(x)| < 1$ if $x \neq x_0$ (such a peaking function u exists since every point of X is G_δ), then when $|\alpha| < 1$, we have*

$$[DF(\alpha u)]^{-1}(D^nF(\alpha u)(k^n))(x_0) = 0 \quad \text{for } n = 2, 3, \ldots .$$

PROOF. Assume $k \equiv 0$ on the open neighborhood N of x_0. Then N^c, complement of N, is compact, so that for fixed α, $0 < |\alpha| < 1$, we can choose $r > 0$ (say $r = |\alpha|(1 - m)/(\|k\| + 1)$ where $m = \sup\{|u(x)| \,|x \in N^c\} < 1$) so that

$$\|\alpha u + \beta k\| = |(\alpha u + \beta k)(x_0)| = |\alpha| \quad \text{for all } \beta \in \mathbf{C}, |\beta| < r.$$

Define $l \in C(X)^*$ by $l(f) = |\alpha| f(x_0)/\alpha$ for all $f \in C(X)$. Then $l(\alpha u) = |\alpha| = \|\alpha u\|$ and $\|l\| = 1$. Since F is convex biholomorphic in B, we know by Theorem 4 [**16**, p. 583] there is a function $w: B \times B \to C(X)$ such that w is F-holomorphic in each variable, $w(\alpha u, \alpha u) = 0$, Re $l(w(\alpha u, \alpha u + \beta k)) > 0$ if $|\beta| < r$, and

$$F(\alpha u) - F(\alpha u + \beta k) = DF(\alpha u)(w(\alpha u, \alpha u + \beta k)).$$

Expanding $F(\alpha u + \beta k)$ about αu, we have

$$F(\alpha u + \beta k) = F(\alpha u) + \sum_{n=1}^{\infty} \frac{\beta^n}{n!} D^n F(\alpha u)(k^n)$$

so that

$$w(\alpha u, \alpha u + \beta k) = -\beta k - \sum_{n=2}^{\infty} \frac{\beta^n}{n!} \left[DF(\alpha u)\right]^{-1} D^n F(\alpha u)(k^n).$$

Applying l, we have

$$v(\beta) \equiv lw(\alpha u, \alpha u + \beta k) = -\sum_{n=1}^{\infty} \frac{\beta^n}{n!} l\Big(\left[DF(\alpha u)\right]^{-1} D^n F(\alpha u)(k^n)\Big)$$

is a holomorphic function of β for $|\beta| < r$ and $\operatorname{Re} v(\beta) \geq 0$. Since $v(0) = 0$, $v(\beta) \equiv 0$ and Lemma 8.2 follows.

LEMMA 8.3. *If k, $u \in C(X)$, $k(x_0) = 0$ and u is as in Lemma 8.2, then $[DF(\alpha u)]^{-1}(D^n F(\alpha u)(k^n))(x_0) = 0$ for $n = 2, 3, \ldots$.*

We will prove this without utilizing the G_δ property. Let A be an index set of the open neighborhoods of x_0 and let A_α be the set of functions $u_\delta k$ where $u_\delta \in C(X)$, $u_\delta = 1$ on U_α^c, complement of U_α, support of $u_\delta \subset \overline{U_\delta^c}$, $0 \leq u_\delta \leq 1$, where U_δ is an open neighborhood of x_0 and $\overline{U}_\delta \subset U_\alpha$. $A_\alpha \neq \varnothing$ by Urysohn's Lemma. Let $\mathfrak{U} = \{A_\alpha | \alpha \in A\}$. It is routine to show that $\mathfrak{U}$ is a filterbase in $C(X)$ which converges (in the norm topology) to k [2, p. 211]. Apply Lemma 8.2 to each $u_\delta k$ and the result follows.

LEMMA 8.4. *If k, u are as in Lemma 8.3, we have*

$$\left[DF(\alpha u)\right]^{-1} D^2 F(\alpha u)(u, k)(x_0) = 0.$$

PROOF. First assume k is as in Lemma 8.2. Set $(1 + t^2)^{1/2} g = \alpha(1 + it)u + \beta k$ where $|\beta| = t^{1/2}$, and t is sufficiently small and positive so that

$$\|g\| = \left| \frac{\alpha(1 + it)u(x_0) + \beta k(x_0)}{\sqrt{1 + t^2}} \right| \leq |\alpha| < 1.$$

Let l be the same as in the proof of Lemma 8.2. Since F is convex biholomorphic in B, we know again by Theorem 4 [16] that $F(\alpha u) - F(g) = DF(\alpha u)(w(\alpha u, g))$ where $\operatorname{Re} l(w(\alpha u, g)) \geq 0$. Expanding $F(g)$ about αu, we have

$$\begin{aligned} F(g) &= F(\alpha u) + \sum_{n=1}^{\infty} \frac{1}{n!} D^n(\alpha u)(g - \alpha u)^n \\ &= F(\alpha u) + DF(\alpha u)(g - \alpha u) \\ &\quad + D^2 F(\alpha u)\left(\frac{i + \alpha u}{\sqrt{1 + t^2}}, \frac{\beta k}{\sqrt{1 + t^2}} \right) + O(t^2) \end{aligned}$$

since $(1+t^2)^{-1/2}-1=O(t^2)$ and $|\beta|=t^{1/2}$. By Lemma 8.2 we then have

$$w(\alpha u, g) = -(g-\alpha u) - \frac{i\alpha u t\beta}{1+t^2}\left[DF(\alpha u)\right]^{-1}D^2F(\alpha u)(u,k) + O(t^2)$$

so that

$$\frac{|\alpha|}{\alpha}\, w(\alpha u, g)(x_0) = \frac{-it|\alpha|}{\sqrt{1+t^2}} + \frac{-it|\alpha|\beta}{1+t^2}\left[DF(\alpha u)\right]^{-1}D^2F(\alpha u)(u,k)(x_0) + O(t^2).$$

Since $\operatorname{Re} l(w(\alpha u, g)) \geqslant 0$, we conclude that

$$\operatorname{Re} \frac{-i|\alpha|t\beta}{1+t^2}\left[DF(\alpha u)\right]^{-1}D^2F(\alpha u)(u,k)(x_0) \geqslant 0.$$

Since $\arg\beta$ is arbitrary, we have

$$\left[DF(\alpha u)\right]^{-1}D^2F(\alpha u)(u,k)(x_0) = 0 \quad \text{if } k \equiv 0$$

in an open neighborhood of x_0. Now assume $k \in C(X)$ and $k(x_0)=0$. Apply the argument in Lemma 8.3 to obtain the conclusion.

LEMMA 8.5. *If k, u are as in Lemma* 8.3, *then*

$$\left[DF(\alpha u)\right]^{-1}D^nF(\alpha u)(u^l, k^{n-l})(x_0) = 0$$

when $0 \leqslant l < n$, $n = 2, 3, 4, \ldots$.

PROOF. When $l=0$ and $n \geqslant 2$, the result is Lemma 8.3. When $l=1$ and $n=2$, the result is Lemma 8.4. Assume the result is true for fixed l and n and prove it for $l+1$ and $n+1$. Let $G=F^{-1}$. Then, for all $f \in B$, we have $G \circ F(f) = f$, $DG(F(f)) \circ DF(f) = I$, identity map from $C(X)$ into $C(X)$, and

$$D^2G(F(f))(DF(f)(g), DF(f)(h)) + DG(F(f))(D^2F(f)(g,h)) = 0,$$

zero map from $C(X)$ into $C(X)$, for all $f \in B$ and all $g, h \in C(X)$. Hence

$$\begin{aligned} D^2G(F(f))(g,h) &= -DG(F(f))\Big[D^2F(f)\Big(\left[DF(f)\right]^{-1}(g), \left[DF(f)\right]^{-1}(h)\Big)\Big] \\ &= -\left[DF(f)\right]^{-1}\Big[D^2F(f)\Big(\left[DF(f)\right]^{-1}(g), \left[DF(f)\right]^{-1}(h)\Big)\Big]. \end{aligned}$$

Define $H\colon B \to \mathfrak{L}_n(C(X), C(X))$, space of all continuous symmetric n-linear maps of $C(X)^n$ into $C(X)$, by

$$H(f) = \left[DF(f)\right]^{-1}D^nF(f) = DG(F(f))D^nF(f) \quad \text{for all } f \in B.$$

By the induction assumption $H(\alpha u)(u^l, k^{n-l})(x_0) = 0$ and $H(\alpha u + \varepsilon u)(u^l, k^{n-l})(x_0) = 0$ for small $|\varepsilon| > 0$. Hence

$$DH(\alpha u)(u)(u^l, k^{n-l})(x_0)$$
$$= \lim_{\varepsilon \to 0} \frac{H(\alpha u + \varepsilon u)(u^l, k^{n-l})(x_0) - H(\alpha u)(u^l, k^{n-l})(x_0)}{\varepsilon} = 0.$$

On the other hand,

$$DH(f)(u) = D^2G(F(f))(D^nF(f), DF(f)(u))$$
$$+ DG(F(f))(D(D^nF(f))(u))$$
$$= -[DF(f)]^{-1}[D^2F(f)(H(f), u)] + [DF(f)]^{-1}(D(D^nF(f))(u)).$$

Since $H(\alpha u)(u^l, k^{n-l}) \in C(X)$ and vanishes at x_0, the first term is zero when evaluated at $f = \alpha u$ and (u^l, k^{n-l}) and x_0 by Lemma 8.4. Hence

$$0 = DH(\alpha u)(u)(u^l, k^{n-l})(x_0)$$
$$= [DF(\alpha u)]^{-1}(D(D^nF(\alpha u)(u^l, k^{n+1-(l+1)}))(u))(x_0)$$

which is the result for $l + 1$ and $n + 1$.

LEMMA 8.6. *Let $u \in C(X)$ such that $u(x_0) = 1$ and $0 < u(x) < 1$ if $x \neq x_0$. If $f \in B$, then $F(f)(x_0) = F(uf)(x_0)$.*

PROOF. If $f(x_0) = 0$, then $(uf)(x_0) = 0$ and

$$F(f)(x_0) = \sum_{n=1}^{\infty} \frac{1}{n!} D^nF(0)(f^n)(x_0) = f(x_0)$$

since f plays the role of k in Lemma 8.3. Similarly, $F(uf)(x_0) = (uf)(x_0)$.

Assume $f(x_0) \neq 0$. Let $N = \{x \in X | u(x) < |f(x_0)|/2\|f\|\}$ and set $v(x) = u(x)$ if $x \in N$ and $v(x) = \min(u(x), |f(x_0)|/|f(x)|)$ if $x \in N^c$. Then $v \in C(X)$ and $|(vf)(x)| < \frac{1}{2}|f(x_0)|$ if $x \in N$ and $|(vf)(x)| = \min(u(x)|f(x)|, |f(x_0)|)$ if $x \in N^c$. Therefore $vf/f(x_0)$ plays the role of u in Lemma 8.5. Setting $\alpha = 0$, in Lemma 8.5, we have, for all nonnegative integers $l < n$,

$$D^nF(0)\left((vf/f(v_0))^l, ((1 - v)f)^{n-l}\right)(x_0) = 0$$

and hence

$$D^nF(0)\left((vf)^l, ((1 - v)f)^{n-l}\right)(x_0) = 0.$$

Then

$$F(f)(x_0) = F(vf + (1 - v)f)(x_0)$$

$$= \sum_{n=1}^{\infty} \frac{1}{n!} D^n F(0)\big((vf + (1 - v)f)^n\big)(x_0)$$

$$= \sum_{n=1}^{\infty} \sum_{l=0}^{n} \frac{1}{l!\,(n - l)!} D^n F(0)\big((vf)^l, ((1 - v)f)^{n-l}\big)(x_0)$$

$$= \sum_{n=1}^{\infty} \frac{1}{n!} D^n F(0)\big((vf)^n\big)(x_0) = F(vf)(x_0).$$

If we use uf, instead of f, the same v works for uf and we have $F(uf)(x_0) = F(vuf)(x_0)$. Since vu can be used instead of v for f, we have $F(f)(x_0) = F(vuf)(x_0)$ and hence $F(f)(x_0) = F(uf)(x_0)$.

LEMMA 8.7. *If $f, g \in B$ such that $f(x_0) = g(x_0)$, then $F(f)(x_0) = F(g)(x_0)$.*

PROOF. Let $\varepsilon > 0$ be given. Since F is continuous at f, there is $\delta > 0$ such that $\|F(f) - F(h)\| < \varepsilon$ if $\|f - h\| < \delta$. Let $N_1 = \{x \in X \mid |f(x) - g(x)| < \delta\}$ and let N be an open neighborhood of x_0 such that $\overline{N} \subset N_1$. Since x_0 is a G_δ, there is $u \in C(X)$ such that $u(x_0) = 1$, $0 \leqslant u(x) < 1$, if $x \neq x_0$, and $u \equiv 0$ on N_1^c. By Urysohn's Lemma, there is $v \in C(X)$ such that $0 \leqslant v(x) \leqslant 1$ for all $x \in X$, $v \equiv 1$ on N and $v \equiv 0$ on N_1^c. Set $h = vg + (1 - v)f$. Then $\|f - h\| < \delta$ and $ug = uh$. Therefore, by Lemma 8.6,

$$|F(f)(x_0) - F(g)(x_0)| = |F(f)(x_0) - F(ug)(x_0)|$$

$$= |F(f)(x_0) - F(uh)(x_0)|$$

$$= |F(f)(x_0) - F(h)(x_0)| \leqslant \|F(f) - F(h)\| < \varepsilon.$$

Since ε can be made arbitrarily small, $F(f)(x_0) = F(g)(x_0)$.

We can now prove Theorem 8.1.

PROOF. Let $f, g \in B$ and $x_0 \in X$. For small $|\alpha|$, we have that $f + \alpha g$, $f + \alpha g(x_0)1 \in C(X)$ and agree at x_0; therefore, by Lemma 8.7, $F(f + \alpha g)(x_0) = F(f + \alpha g(x_0)1)$. Therefore,

$$DF(f)(g)(x_0) = \lim_{\alpha \to 0} \frac{F(f + \alpha g)(x_0) - F(f)(x_0)}{\alpha}$$

$$= \lim_{\alpha \to 0} \frac{F(f + \alpha g(x_0)1)(x_0) - F(f)(x_0)}{\alpha}$$

$$= DF(f)(g(x_0)1)(x_0) = g(x_0)DF(f)(1)(x_0).$$

Since x_0 is arbitrary, we have $DF(f)(g) = gDF(f)(1)$; i.e., F is L-analytic in B and $F'(f) = DF(f)(1)$.

REMARK 1. The normalization $DF(0) = I$ is necessary in Theorem 8.1 as is seen in the following example. Define F: $C[0, 1] \to C[0, 1]$ by $F(f)(x) = (x + \frac{1}{2})f(x) + (\frac{1}{2} - x)f(x + \frac{1}{2})$ if $x \in [0, \frac{1}{2})$ and $F(f)(x) = f(x)$ if $x \in [\frac{1}{2}, 1]$. Then F is a continuous linear map of $C[0, 1]$ onto $C[0, 1]$ and so F is a convex biholomorphic function in B. But F is not L-analytic and $DF(0) = F \neq I$.

Without the normalization, we have

COROLLARY. *If F is biholomorphic in B, and $F(B)$ is convex then $F = L \circ G$ where L is a univalent affine map of $C(X)$ onto $C(X)$ and G is bianalytic in B.*

PROOF. Define $L(f) = F(0) + DF(0)(f)$ for all $f \in B$ and $G = L^{-1} \circ F$ satisfies Theorem 8.1.

REMARK 2. The proof of Theorem 8.1 depends on the existence of a peaking function u at each point $x \in X$. In general, an arbitrary compact T_2-space X does not have peaking functions at each point; for example, if X is the set of all ordinals which are less than or equal to the first uncountable ordinal with the order topology, then the first uncountable ordinal is not a G_δ point. See [**10**, Exercises 1.I, 5.C, and 4.J]. It would be interesting to know if Theorem 8.1 is true for an arbitrary compact T_2-space.

Note that Theorem 8.1 contains Theorem 3 of [**15**]. To see this, take $X = \{1, 2, \ldots, n\}$ with the discrete topology. Also compare Theorem 8 of [**16**].

9. Example and a remark. We now give an example of a function which is univalent and F-holomorphic in B such that F^{-1} is not F-holomorphic in $F(B)$, $F(B)$ contains an open set, but $F(B)$ is not open. The example is in the Banach algebra $H^\infty = \{f | f: \Delta \to \mathbf{C}$ is holomorphic in Δ and $\sup\{|f(z)| \, | z \in \Delta\} < \infty\}$ [3], [8]. Define F: $B \to H^\infty$ by $F(f) = f + af^2$ for nonconstant $a \in H^\infty$ such that for some $z_0 \in \Delta$, $|a(z_0)| = \frac{1}{2}$. To show that F is univalent in B, suppose that $f, g \in B$ such that $F(f) = F(g)$. Then $(f - g)(1 + a(f + g)) = 0$ which implies that $f(z) = g(z)$ or $f(z) + g(z) = -1/a(z)$ for each $z \in \Delta$. We claim the first equation always holds. By hypothesis, there is $z_0 \in \Delta$ such that $|a(z_0)| = \frac{1}{2}$. Since $f, g \in B$, there is $\delta > 0$ such that $|f(z)| < 1 - \delta$ and $|g(z)| < 1 - \delta$ in some open neighborhood of z_0. The second equation implies $1/|a(z)| < 2 - 2\delta$ in this neighborhood, hence $|a(z_0)| > 1/(2 - 2\delta) > \frac{1}{2}$, which is a contradiction. Hence $f(z) = g(z)$ in this open neighborhood of z_0 and therefore $f = g$.

To prove that F^{-1} is not F-holomorphic in $F(B)$, observe that $F'(f) = 1 + 2af$ and so $F'(f)$ is nonsingular as long as $1 + 2a(z)f(z) \neq 0$ for $z \in \Delta$. If $|a(z_1)| > \frac{1}{2}$, let f be the constant function $-1/2a(z_1)$. Then $\|f\| < 1$ and $F'(f)$ is singular. It follows that F^{-1} is not F-holomorphic in $F(B)$.

$F(B)$ contains an open set since the equation $\varepsilon = f + af^2$ has the solution

$f = (-1 + \sqrt{1 + 4a\varepsilon})/2a = 2\varepsilon/(1 + \sqrt{1 + 4a\varepsilon})$ where $\varepsilon \in H^\infty$ and $\|\varepsilon\|$ is small.

To prove that $F(B)$ is not open, suppose $|a(z_1)| > \frac{1}{2}$ and let f be the constant function $-1/2a(z_1)$. Then $F(f) = a/4a^2(z_1) - 1/2a(z_1)$, and if $\varepsilon \in H^\infty$, $\|\varepsilon\|$ small, we have $a/4a^2(z_1) - 1/2a(z_1) + \varepsilon = f + af^2$ for some $f \in B$ if and only if $f = -1/2a(z_1) + \delta$ where $\delta \in H^\infty$ and $\varepsilon(z) = \delta(z)(1 - a(z)/a(z_1)) + a(z)\delta^2(z)$. Let ε have the property that $\varepsilon(z_1) = 0$. Then $0 = \varepsilon(z_1) = a(z_1)\delta^2(z_1)$ so we conclude $\delta(z_1) = 0$. Since $\varepsilon(z) = \delta(z)(1 - a(z)/a(z_1)) + a(z)\delta^2(z)$, it follows that ε has at least a double zero at z_1. This means that any function of the form $a/4a^2(z_1) - 1/2a(z_1) + \varepsilon$ such that $\varepsilon(z_1) = 0$ and $\varepsilon'(z_1) \neq 0$ cannot lie in $F(B)$.

This example shows that Theorem 5 [5] is false. F is univalent and F-holomorphic in B with $F(0) = 0$ and $DF(0) = I$, but $F(B)$ is not open. Hence F is not locally biholomorphic in B and therefore F is not Φ-like for any function Φ.

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