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AN ALGEBRAIC CHARACTERIZATION OF CONNECTED SUM FACTORS OF CLOSED 3-MANIFOLDS

BY

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ABSTRACT. Let M and N be closed connected 3-manifolds. A *knot group* of M is the fundamental group of the complement of a tame simple closed curve in M. Denote the set of knot groups of M by $K(M)$. A knot group G of M is *realized* in N if G is the fundamental group of a compact submanifold of N with connected boundary.

THEOREM. *Every knot group of N is realized in M iff N is a connected sum factor of M.*

COROLLARY 1. $K(M) = K(N)$ *iff M is homeomorphic to N.*

Given M, there exists a knot group G_M of M that serves to characterize M in the following sense.

COROLLARY 2. *G_M is realized in N and G_N is realized in M iff M is homeomorphic to N.*

Our proof depends heavily on the work of Bing, Feustal, Haken, and Waldhausen in the 1960s and early 1970s. A. C. Conner announced Corollary 1 for orientable 3-manifolds in 1969 which Jaco and Myers have recently obtained using different techniques.

1. Preliminaries. We will work exclusively in the PL category. **[Hem]** is an excellent reference for definitions, notation and techniques. Manifolds are usually connected but not necessarily compact, orientable, or without boundary. ∂M denotes the boundary of a manifold M. A 2-manifold S properly embedded in a 3-manifold M or contained in ∂M is *compressible* provided (1) there exists a 2-cell D in M such that $D \cap S = \partial D$ and ∂D does not bound a 2-cell in S, or (2) S bounds a 3-cell in M. We call a 2-cell D as in (1) a *compressing 2-cell for S in M*. If S is not compressible we say S is *incompressible*.

A 3-manifold M is *P^2-irreducible* if every 2-sphere in M bounds a 3-cell in M and M contains no 2-sided projective planes. M is *∂-irreducible* if every component of ∂M is incompressible. We usually follow Waldhausen **[W_1**, p. 57] in using $U(\cdot)$ for nice regular neighborhoods. (One exception is when $U(J_{i+1})$ is defined.) If N and M are compact manifolds with connected

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boundary and $N \subseteq \text{Int } M$, we use the notation $[N, M] = M - \text{Int } N$.

Projective planes, 2-spheres, and 2-cells share the property that any 2-sided simple closed curve they contain bounds a 2-cell, which is the key to proving Lemma A. (Proof omitted.)

LEMMA A. *Suppose N_1 and N_2 are P^2-irreducible, ∂-irreducible 3-manifolds such that $N_1 \cup N_2$ is a 3-manifold, $N_1 \cap N_2 = \partial N_1 \cap \partial N_2$ is a collection of 2-manifolds, $N_1 \cap N_2$ is incompressible in both N_1 and N_2, and no component of $N_1 \cap N_2$ is a 2-cell. Then $N_1 \cup N_2$ is P^2-irreducible and ∂-irreducible.*

The next two lemmas are more general than we need but have other applications. First we give some definitions. Let B be a 3-cell and A an annulus on ∂B. Suppose α is an arc properly embedded in B with an endpoint in each component of $\partial B - A$ and such that C, the closure of $B - U(\alpha)$, is not a solid torus. We call C a *cube-with-a-knotted-hole*. If M is a 3-manifold such that $C \cap M = \partial C \cap \partial M = A$. We say the 3-manifold $M \cup C$ *is obtained from M by attaching a cube-with-a-knotted-hole to M along A.*

LEMMA B. *Suppose T is a torus or Klein bottle boundary component of a 3-manifold M. Let A be an annulus in T that is not contractible in M. If N is obtained from M by attaching a cube-with-a-knotted-hole to M along A, then the boundary component of N that intersects T is incompressible in N.*

PROOF. Let L be a cube-with-a-knotted-hole such that $L \cup M = N$ and $L \cap M = (\partial L) \cap (\partial M) = A$. Note A is an incompressible surface in N. Let S be the boundary component of N that intersects T. Let $S \cap L = A'$, an annulus with $\partial A' = \partial A$.

Case 1. $T - \text{Int } A$ is an annulus. Suppose S is compressible. Then there exists a properly embedded 2-cell E in N such that $\partial E \subseteq S$, ∂E does not bound a 2-cell in S, and E is in general position with respect to A. If $(\partial E) \cap A = \varnothing$, there exists a properly embedded 2-cell E' such that $\partial E = \partial E'$ and $E' \cap A = \varnothing$. But then A is contractible in either M or L, a contradiction. So assume $(\partial E) \cap A \neq \varnothing$. Since the closures of both components of $S - A$ are annuli, we can adjust E by an ambient isotopy of M so that the closure of each component of $\partial E - A$ is an arc with endpoints in different components of ∂A, and E is in general position with respect to A. Note that if α is an arc that is a component of $E \cap A$, then the endpoints of α must also lie in different components of ∂A. Using the incompressibility of A we can remove the simple closed curve components of $E \cap A$. Hence we can assume that $E \cap A$ consists of a finite number of arcs. Let F be a component of $E \cap L$. F is a 2-cell properly embedded in L such that the algebraic intersection number of ∂F with a component J of ∂A is a nonzero integer. Since L is ∂-irreducible, ∂F must bound a 2-cell in ∂L. Hence ∂F must have zero algebraic intersection number with J. So we must conclude that S is incompressible.

Case 2. $T - \text{Int } A$ consists of two Möbius bands Q_1 and Q_2. Suppose S is compressible. As in Case 1, there exists a properly embedded 2-cell E in N such that $\partial E \subseteq S$, ∂E does not bound a 2-cell in S, E is in general position with respect to A, and $\partial E \cap A \neq \emptyset$. We can assume each component of $(\partial E) \cap A'$ is an arc with an endpoint in each component of $\partial A'$, and each component of $\partial E \cap Q_i$ is an arc that does not separate Q_i. Since A is incompressible in N we can further assume that all the components of $E \cap A$ are arcs. Let D be a 2-cell in E such that $D \cap A = \alpha$ is an arc, $D \cap \partial E = \alpha'$ is an arc, and $\partial D = \alpha \cup \alpha'$.

If $D \subseteq L$, then $\alpha' \subseteq A'$. Recall that α' must have an endpoint in each component of $\partial A'$. Hence the existence of D violates the fact that L is not a solid torus.

If $D \subseteq M$, then $\partial D \subseteq T$. Suppose $\alpha' \subseteq Q_1$. Since α' does not separate Q_1, $\alpha \cup \alpha'$ must be 1-sided in T. But $\partial D = \alpha \cup \alpha'$ must be 2-sided in T.

We are forced to conclude again that S is incompressible.

LEMMA C. *Suppose M is a compact 3-manifold with no 2-sphere or projective plane boundary components. Then there exists a simple closed curve J contained in* Int M *such that* $M - \text{Int } U(J)$ *is P^2-irreducible and ∂-irreducible. Furthermore we may choose J so that $U(J)$ is orientable.*

PROOF. Let C be a compact collar on ∂M in M and let N be the closure of $M - C$. Suppose L is a triangulation of M with subcomplex K that triangulates N. Let G be the 1-skeleton of K'', the second derived barycentric subdivision of K. Since $M - \text{Int } U(G)$ is homeomorphic to C with a finite number of 1-handles attached, $M - G$ is P^2-irreducible and ∂-irreducible. We want to use one of Bing's techniques to find a simple closed curve J_1 that approximates G. Then we can tie a knot in J_1 to obtain J to insure that $\partial U(J)$ is incompressible in $M - \text{Int } U(J)$.

Note G has the following properties:

(1) G is a connected finite graph with no points of order one;

(2) for all vertices v of G, $G - \text{st}(v, G)$ is connected ($\text{st}(v, G)$ denotes the open star of v in G);

(3) for all vertices v of G, $G - \text{st}(v, G)$ contains either K^1, the 1-skeleton of K, or K_1, the dual 1-skeleton of K.

(1) and (3) clearly hold. (2) holds since $\text{st}(v, G)$ is one component of $G - \text{lk}(v, K'')^1$ and $\text{lk}(v, K'')^1$ is connected ($\text{lk}(v, K'')$ denotes the link of v in K''. Recall that $\text{lk}(v, K'')^1$ is the 1-skeleton of $\text{lk}(v, K'')$, a 2-sphere or 2-cell.) Note any subdivision of a graph with properties (1)–(3) also has properties (1)–(3).

There are an even number of vertices of G with odd order. Hence we can pair these vertices and connect them by polygonal arcs that intersect G only in their endpoints. So we obtain a finite graph G_1 that contains G, has properties (1)–(3), and such that each point of G_1 has even order. Note that $M - G_1$ is P^2-irreducible and ∂-irreducible.

We now wish to modify G_1 to obtain a graph G_2 that satisfies (1) and (2) such that each point of G_2 has order 2 or 4, and $M - G_2$ is homeomorphic to $M - G_1$.

Let $w_1, \ldots, w_k$ be the points of G_1 that have order greater than 4. Let $D_1, \ldots, D_k$ be small regular neighborhoods of $w_1, \ldots, w_k$, respectively, in M such that D_i collapses to $G_1 \cap D_i$, $i = 1, \ldots, k$. There exist 2-cells B_i, properly embedded in D_i, such that $G_1 \cap D_i \subseteq B_i$. There exist trees $F_i \subseteq B_i$ such that B_i collapses to F_i, $F_i \cap \partial D_i = G_1 \cap \partial D_i$ consists of the endpoints of F_i, and each point of F_i in Int B_i has order 2 or 4. Let $D = \bigcup_{i=1}^{k} D_i$ and $G_2 = (G_1 - D) \cup (\bigcup_{i=1}^{k} F_i)$.

Let $z_1, \ldots, z_s$ be the points of G_2 that have order 4, and let $H_1, \ldots, H_s$ be small regular neighborhoods of $z_1, \ldots, z_s$, respectively, in Int D such that $G_2 \cap \partial H_j$ consists of 4 points. Let $H = \bigcup_{j=1}^{s} H_j$. Replace $G_2 \cap H_j$, $j = 1, \ldots, s$, by a pair of arcs α_j, α_j' properly embedded in H_j such that $(\alpha_j \cup \alpha_j') \cap \partial H_j = G_2 \cap \partial H_j$, α_j and α_j' cannot be separated in H_j by a properly embedded 2-cell, and such that

$$J_1 = (G_2 - H) \cup \left(\bigcup_{j=1}^{s} (\alpha_j \cup \alpha_j') \right)$$

is a simple closed curve. See [**B**, Figure 3 and Lemma 6] for one way to choose α_j and α_j'. In the next four paragraphs we will show $M - J_1$ is P^2-irreducible and ∂-irreducible.

$H_j - J_1$ is clearly P^2-irreducible. Using linking arguments it is easy to verify that $H_j - J_1$ is ∂-irreducible.

We claim $D_i - (J_1 \cup \text{Int } H)$ is P^2-irreducible and ∂-irreducible. $D_i - (J_1 \cup H)$, $D_i - G_2$, $D_i - G_1$ and $(\partial D_i - J_1) \times [0, 1)$ are homeomorphic. So it is sufficient to show that if E is a properly embedded 2-cell in $D_i - (J_1 \cup \text{Int } H)$ with $\partial E \subseteq \partial H_j - J_1$, then ∂E bounds a 2-cell in $\partial H_j - J_1$. By property (2), $G_2 - \text{Int } H_j$ is connected. Note E separates $M - \text{Int } H_j$. Hence $J_1 \cap \partial H_j = G_2 \cap \partial H_j$ must be contained in one component of $\partial H_j - \partial E$. Therefore ∂E bounds a 2-cell in $\partial H_j - J_1$.

We now show $M - (J_1 \cup \text{Int } D)$ is P^2-irreducible and ∂-irreducible. Recall $N - G_1$, which is homeomorphic to $M - (J_1 \cup D)$, is P^2-irreducible and ∂-irreducible. So we just need to show $\partial D_i - J_1$ is incompressible in $M - (J_1 \cup \text{Int } D)$. Let E be a properly embedded 2-cell in $M - (J_1 \cup \text{Int } D)$ with $\partial E \subseteq \partial D_i - J_1$. We claim E separates $M - (J_1 \cup \text{Int } D)$. Let E' be a 2-cell in ∂D_i with $\partial E' = \partial E$. The 2-sphere $E \cup E'$ is contained in either $M - K_1$ or $M - K^1$ by property (3) for G_1. Since both $M - K_1$ and $M - K^1$ are irreducible, $E \cup E'$ separates M. Hence E separates $M - (J_1 \cup \text{Int } D)$. Using property (2) for G_1, we see $G_1 - \text{Int } D_i$ is contained in one component of $M - (J_1 \cup \text{Int } D \cup E)$. Hence $G_1 \cap \partial D_i = J_1 \cap \partial D_i$ cannot meet both components of $\partial D_i - \partial E$. Therefore ∂E bounds a 2-cell in $\partial D_i - J_1$.

The results of the preceding three paragraphs imply that each component of $(\partial H \cup \partial D) - J_1$ is a separating incompressible surface in $M - J_1$, and the closures of the components of $(M - J_1) - ((\partial H \cap \partial D) - J_1)$ are P^2-irreducible and ∂-irreducible. By Lemma A, $M - J_1$ is P^2-irreducible and ∂-irreducible.

We claim J_1 does not pierce any 2-sphere in M. Suppose S is a 2-sphere in M with $S \cap J_1 = \{p\}$. We can assume $p \notin D$ and that $S \cap \partial D = \varnothing$. So $S \cap (J_1 \cup D) = \{p\} = S \cap G_1$. Hence S is contained in an irreducible submanifold of M, either $M - K_1$ or $M - K^1$. We conclude J_1 does not pierce S.

$\partial U(J_1)$ may not be incompressible in $M - \text{Int } U(J_1)$. So we need to "tie a knot" in J_1. More precisely, let P be a 3-cell in $U(J_1)$ such that $P \cap \partial U(J_1) = A$ is an annulus, $\text{cl}(U(J_1) - P) = P'$ is a 3-cell, and $P \cap J_1 = \alpha$ is an arc with an endpoint in each component of $\partial P - A$. Let α' be the arc $P' \cap J_1$. Let β be a properly embedded arc in P such that $\partial\beta = \partial\alpha$ and $L = \text{cl}(P - U(\beta))$ is a cube-with-a-knotted-hole. ($U(\beta)$ is a standard regular neighborhood of β in P.) Let $J_2 = \beta \cup \alpha'$. We say J_2 is obtained from J_1 by tieing a knot in J_1. Note $U(\beta) \cup P'$ is a regular neighborhood of J_2 in M. Hence $M - \text{Int } U(J_2)$ is homeomorphic to $M - \text{Int } U(J_1)$ with the cube-with-a-knotted-hole L attached along A. Since J_1 does not pierce any 2-sphere in M, A must not be contractible in $M - \text{Int } U(J_1)$. Applying Lemma B we see $M - \text{Int } U(J_2)$ is ∂-irreducible. $M - \text{Int } U(J_2)$ is clearly P^2-irreducible. Hence J_2 is our required J.

Suppose $U(J_2)$ is nonorientable. We wish to find a simple closed curve J such that $U(J)$ is orientable and $M - \text{Int } U(J)$ is P^2-irreducible and ∂-irreducible. Let Q be a Möbius band, with center line J_2, that is properly embedded as a 2-sided subset of $U(J_2)$. Let $J_3 = \partial Q$. Note $U(J_3)$ is orientable and J_3 is not contractible in $M - J_2$.

We claim Int Q is incompressible in $M - J_3$. Let E be a 2-cell in $M - J_3$ such that $E \cap Q = \partial E$. ∂E must be 2-sided in Q. So we can assume $\partial E \cap J_2 = \varnothing$. Now ∂E is not parallel to J_3 in Q since ∂E is contractible in $M - J_2$. Hence ∂E must bound a 2-cell in Q.

Since $M - Q$ and $M - J_2$ are homeomorphic, $M - Q$ is P^2-irreducible and ∂-irreducible. It follows that $M - J_3$ is P^2-irreducible and ∂-irreducible. J_3 cannot pierce a 2-sphere in M since J_3 bounds Q. Let J be obtained from

J_3 by tieing a knot in J_3. $U(J)$ is orientable. As before, $M - \text{Int } U(J)$ is P^2-irreducible and ∂-irreducible.

This completes the proof of Lemma C.

2. Proof of the main theorem and corollaries. Let M be a closed 3-manifold. Recall that $K(M)$ is the set of fundamental groups of complements of PL simple closed curves in M. $S(M)$ is the set of fundamental groups of compact sub-3-manifolds of M that have connected boundary.

MAIN THEOREM. *Let M, N be closed 3-manifolds. $S(M)$ contains $K(N)$ if and only if N is a connected sum factor of M.*

PROOF. Suppose N is a connected sum factor of M. Let α be a simple closed curve in N, $U(\alpha)$ a regular neighborhood of α in N, and B a 3-cell in $\text{Int } U(\alpha)$. Since $N - \text{Int } B$ is homeomorphic to a subset of M, $\pi_1(N - \text{Int } U(\alpha))$ belongs to $S(M)$.

Now suppose $S(M)$ contains $K(N)$. By Lemma C there exists an orientable simple closed curve J_0 contained in N such that $L_0 = N - \text{Int } U(J_0)$ is P^2-irreducible and ∂-irreducible. Let h be a positive integer such that $h - 1$ is the maximal number of disjoint, 2-sided, nonparallel, incompressible tori contained in M [**Ha**]. We wish to add structure to L_0 by doubling J_0 and then tieing a knot in the result, repeating the process $h + 1$ times. Note that h is the only information about M used in this construction. More precisely assume J_i, $U(J_i)$, and L_i have been defined.

Let J_i^+ be a simple closed curve in $\text{Int } U(J_i)$ with winding number two. Let A_i be an annulus on $\partial U(J_i^+)$ that is contractible in $U(J_i^+)$ and does not separate $\partial U(J_i^+)$. Let $L_i^+ = N - \text{Int } U(J_i^+)$. Let C_i be a cube-with-a-knotted-hole in $U(J_i^+)$ such that $C_i \cap \partial U(J_i^+) = A_i$ and $L_{i+1} = L_i^+ \cup C_i$ is obtained from L_i^+ by attaching C_i along A_i. Now $N - \text{Int } L_{i+1}$, being a solid torus, is a regular neighborhood of a simple closed curve J_{i+1}. We depart from our $U(\cdot)$ convention and define $U(J_{i+1}) = N - \text{Int } L_{i+1}$. Recall $[L_i, L_i^+] = L_i^+ - \text{Int } L_i$. We collect some useful facts in the following lemma. Figure 1 should help picture the construction.

LEMMA 1. *$[L_i, L_i^+]$ is ∂-irreducible. $[L_i^+, L_{i+1}^+]$ is not a parallelity component. L_i is P^2-irreducible and ∂-irreducible. No nontrivial loop on A_i is freely homotopic in $[L_i, L_i^+]$ to a loop on ∂L_i.*

PROOF. Suppose D is a compressing 2-cell for $[L_i, L_i^+]$. Let $U(D)$ be a regular neighborhood of D in $[L_i, L_i^+]$. If $\partial D \subseteq \partial L_i$, the closure of $U(J_i) - U(D)$ is a 3-cell in $U(J_i)$ that contains J_i^+. If $\partial D \subseteq \partial L_i^+$, then $U(J_i^+) \cup U(D)$ is a 3-cell in $U(J_i)$ containing J_i^+. In either case the winding number of J_i^+ in $U(J_i)$ would be zero, contrary to construction.

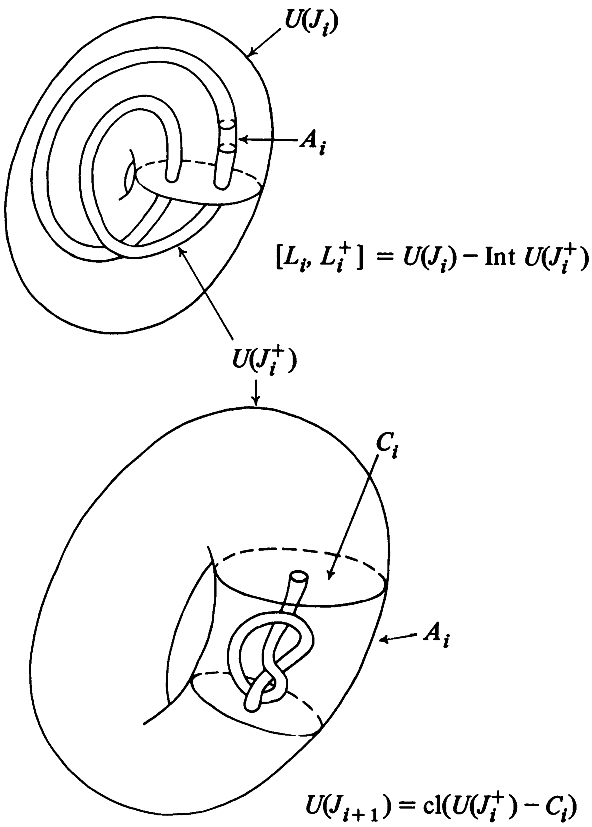

FIGURE 1

Int $[L_i^+, L_{i+1}^+]$ contains an incompressible torus (a parallel copy of ∂C_i in Int C_i) that does not separate ∂L_i^+ from ∂L_{i+1}^+ in $[L_i^+, L_{i+1}^+]$. So by the appendix to **[Ha]**, $[L_i^+, L_{i+1}^+]$ cannot be a parallelity component.

Using Lemmas A and B with the above facts we see L_i is P^2-irreducible and ∂-irreducible.

Suppose l^+ is a nontrivial loop on A_i that is freely homotopic in $[L_i, L_i^+]$ to a loop l on ∂L_i. Since A_i is incompressible l must be nontrivial on ∂L_i. By $[\mathbf{W_2}]$ there exists an annulus A in $[L_i, L_i^+]$ with one boundary component α^+ on A_i, the other α on ∂L_i, and both are nontrivial. Note α bounds a 2-cell in $U(J_i)$. So if we consider $U(J_i)$ as embedded in E^3, α does not link J_{i+1} mod 2 but α^+ does link J_{i+1} mod 2. This contradiction completes the proof of Lemma 1.

LEMMA 2. *M contains a compact P^2-irreducible, ∂-irreducible 3-manifold K such that ∂K is a torus or a Klein bottle and K is homotopy equivalent to L_{h+1}.*

PROOF. By hypothesis there is a compact 3-manifold K' in M such that $\pi_1(K') \cong \pi_1(L_{h+1})$ and $\partial K'$ is connected. Note $\pi_1(L_{h+1})$ has no elements of finite order [E, Theorem 3.2], is not a nontrivial free product [Hem, Theorem 7.1], and is not free abelian. $\partial K'$ is incompressible since otherwise $\pi_1(K')$ would be free abelian or a nontrivial free product. If K' contained a 2-sided projective plane, $\pi_1(K')$ would have elements of order two. Hence $K' = K \# H$ where H is a homotopy 3-sphere and K is P^2-irreducible with ∂K connected and incompressible. We can assume K is contained in M. Since K and L_{h+1} are $K(\pi, 1)$'s, K is homotopy equivalent to L_{h+1}. Hence

$$0 = X(\partial L_{h+1}) = 2X(L_{h+1}) = 2X(K) = X(\partial K)$$

which implies ∂K is a torus on a Klein bottle. ($X(\cdot)$ denotes the Euler characteristic.) Lemma 2 is completed.

Suppose $f\colon K \to \operatorname{Int} L_{h+1}$ is a homotopy equivalence. We can assume $f^{-1}(A_h)$ and $f^{-1}(\partial L_h)$ are collections of 2-sided properly embedded incompressible surfaces in K [Hei] with a minimum number of components. Both collections are nonempty since $f_*\colon \pi_1(K) \to \pi_1(L_{h+1})$ is an isomorphism and $\pi_1(L_{h+1})$ splits as nontrivial free products with amalgamation along both $\pi_1(A_h)$ and $\pi_1(\partial L_h)$. Suppose S is a component of either $f^{-1}(A_h)$ or $f^{-1}(\partial L_h)$ that has boundary.

Each component of ∂S is a nontrivial simple closed curve on ∂K. In order to see this note $(f|_S)_*\colon \pi_1(S) \to G$ is a monomorphism where G is either $\pi_1(A_h)$ or $\pi_1(\partial L_h)$. Hence $\pi_1(S)$ is free abelian. If S is an annulus the boundary components must be nontrivial. If S were a Möbius band with trivial boundary we could find a 2-sided projective plane in K. If S were a 2-cell with trivial boundary, one component of $K - \operatorname{Int} U(S)$ would be a 3-cell and we could reduce the number of components of either $f^{-1}(A_h)$ or $f^{-1}(\partial L_h)$. Our assertion follows.

Note $f^{-1}(A_h) \cap \partial K$ is nonempty.

LEMMA 3. *$f^{-1}(\partial L_h) \cap \partial K$ is empty.*

PROOF. Suppose not. Then there exist nontrivial 2-sided simple closed curves α and β on ∂K such that $f(\alpha) \subseteq A_h$ and $f(\beta) \subseteq \partial L_h$. Let $U(\alpha \cup \beta)$ be a regular neighborhood of $\alpha \cup \beta$ in ∂K. Each component of $\partial K - \operatorname{Int} U(\alpha \cup \beta)$ must have Euler characteristic zero. Since at least one such component must have two boundary simple closed curves, $\alpha \cup \beta$ must bound an annulus A' in ∂K. We can assume $\operatorname{Int} A'$ misses $f^{-1}(A_h)$ and $f^{-1}(\partial L_h)$. Hence $f(\alpha)$ is a nontrivial loop in A_h that is freely homotopic in $[L_h, L_h^+]$ to a loop in ∂L_h, which violates Lemma 1. So Lemma 3 holds.

Hence $f(\partial K) \subseteq \operatorname{Int}[L_h, L_{h+1}]$. We can modify f so that $f(\partial K) \subseteq [L_h, L_{h+1}]$ but $f(\partial K) \cap \partial L_h$ is nonempty. Let x be a point in ∂K such that $f(x)$ belongs to ∂L_h. We now wish to apply the main result of [F]. Consider the geometric splitting of $L_{h+1} = L_h \cup_{\partial L_h} [L_h, L_{h+1}]$ and $f_*\colon \pi_1(K, x) \to \pi_1(L_{h+1}, f(x))$. Under $(f_*)^{-1}$ we obtain an algebraic splitting of $\pi_1(K, x)$ that respects the peripheral structure of $\pi_1(K, x)$, i.e., $\pi_1(\partial K, x)$ under inclusion is a subgroup of $f_*^{-1}(\pi_1([L_h, L_{h+1}], f(x)))$. So applying [F] there exists a separating incompressible torus $T \subseteq \operatorname{Int} K$ (denote the closure of the component of $K - T$ that has boundary T as K_h; then the closure of the other component is $K - \operatorname{Int} K_h = [K_h, K]$; let y belong to T) and an isomorphism $d\colon \pi_1(K, y) \to \pi_1(K, x)$ such that

$$f_* d(\pi_1(T, y)) = \pi_1(\partial L_h, f(x))$$

and

$$f_* d(\pi_1(K_h, y)) = \pi_1(L_h, f(x)) \quad \text{or} \quad \pi_1([L_h, L_{h+1}], f(x)).$$

Since $\pi_1(\partial L_h)$ is a peripheral subgroup of both $\pi_1(L_h)$ and $\pi_1([L_h, L_{h+1}])$ and neither L_h nor $[L_h, L_{h+1}]$ is a product I-bundle, [W] and [**Hei**] apply to conclude K_h is homeomorphic to either L_h or $[L_h, L_{h+1}]$. The only possibility is L_h.

Let $g\colon L_h \to K_h$ be a homeomorphism. First some notation is needed. Let $K_i = g(L_i)$ and $K_i^+ = g(L_i^+)$. Now $\partial K_0^+, \ldots, \partial K_{h-1}^+$ are nonparallel disjoint 2-sided tori in M. At least one must be compressible, say ∂K_j^+, $0 < j < h - 1$. Let J be a boundary component of the annulus A_j.

LEMMA 4. *$g(J)$ bounds a 2-cell in $M - \operatorname{Int} K_{j+1}$.*

PROOF. We adapt the Bing-Martin proof that composite knots have property P [**B**, **M**]. Recall $L_{j+1} = C_j \cup L_j^+$. Let K_j' be a concentric copy of K_j^+ in $\operatorname{Int} K_j^+$. Since ∂K_j^+ is compressible in M, there exists a compressing 2-cell D for $\partial K_j'$ in $M - \operatorname{Int} K_j'$. We can assume D misses $g(C_j)$ (put D in general position with respect to $g(\partial C_j)$. If all the simple closed curves in $D \cap g(\partial C_j)$ are trivial on $g(\partial C_j)$ we can find the desired compressing 2-cell. Otherwise we can find a compact 3-manifold Q' that contains $g(C_j)$, is contained in $M - K_j'$ and has a 2-sphere boundary. Once again we can find the desired 2-cell.) Now put D in general position with respect to ∂K_j^+. We can assume all simple closed curves in $D \cap \partial K_j^+$ are nontrivial on ∂K_j^+ ($D \cap \partial K_j^+$ is nonempty since $K_j^+ - \operatorname{Int} K_j'$ has incompressible boundary). Let $E \subseteq D$ be a 2-cell such that $E \cap \partial K_j^+ = \partial E$. E is contained in $M - \operatorname{Int} K_{j+1}$ since D misses $g(C_j)$ and $K_j^+ - K_j'$ has incompressible boundary. Since ∂E misses $g(A_j)$ and is nontrivial in ∂K_j^+, ∂E is parallel to $g(J)$ on ∂K_{j+1}. The required 2-cell exists and the proof of Lemma 4 is complete.

Let D_1 be a 2-cell in N with $D_1 \cap L_{j+1} = \partial D_1 = J$. Let D_2 be a 2-cell in M with $D_2 \cap K_{j+1} = \partial D_2 = g(J)$. Extend $g|_{L_{j+1}}\colon L_{j+1} \to K_{j+1}$ first to $g'\colon L_{j+1} \cup D_1 \to K_{j+1} \cup D_2$ and then to regular neighborhoods $g''\colon U(L_{j+1} \cup D_1) \to U(K_{j+1} \cup D_2)$. Since $N - \operatorname{Int} U(L_{j+1} \cup D_1)$ is a 3-cell, there exists a closed 3-manifold Q such that M is a connected sum of N and Q. The proof of the main theorem is complete.

COROLLARY 1. *Suppose M and N are closed 3-manifolds. $K(M) = K(N)$ if and only if M is homeomorphic to N.*

PROOF. By the main theorem, M is a connected sum of N and a closed 3-manifold Q_1. Again N is a connected sum of M and a closed 3-manifold Q_2. So M is a connected sum of M, Q_2, and Q_1. [**Hem**, Theorem 3.21] applies to allow us to conclude Q_1 and Q_2 are trivial. Hence M and N are homeomorphic.

Suppose N is a closed 3-manifold. Let $h_N = h$ be the positive integer such that $h - 1$ is the maximal number of disjoint, 2-sided, nonparallel, incompressible tori in N [**Ha**]. Choose $L_{h+1} \subseteq N$ as in the proof of the main theorem. Denote $\pi_1(L_{h+1})$ by G_N. If M is a closed 3-manifold and G_N belongs to $S(M)$, we say G_N is *realized in M*.

COROLLARY 2. *Let M and N be closed 3-manifolds. G_M is realized in N and G_N is realized in M if and only if M is homeomorphic to N.*

PROOF. Consider the positive integers h_N and h_M as defined above. Suppose $h_M \leq h_N$. By the proof of the main theorem, N is a connected sum factor of M. So $h_N \leq h_M$. Apply the proof of the main theorem again to conclude M is a connected sum factor of N. As in Corollary 1, M is homeomorphic to N.

We conclude with the following question.

QUESTION. Does the main theorem hold for compact 3-manifolds with no 2-sphere boundary components?

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