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A SIMULTANEOUS LIFTING THEOREM FOR BLOCK DIAGONAL OPERATORS

BY

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ABSTRACT. Stampfli has shown that for a given $T \in B(H)$ there exists a $K \in C(H)$ so that $\sigma(T+K) = \sigma_w(T)$. An analogous result holds for the essential numerical range $W_e(T)$. A compact operator K is said to preserve the Weyl spectrum and essential numerical range of an operator $T \in B(H)$ if $\sigma(T+K) = \sigma_w(T)$ and $\overline{W(T+K)} = W_e(T)$.

THEOREM. *For each block diagonal operator T, there exists a compact operator K which preserves the Weyl spectrum and essential numerical range of T.*

The perturbed operator $T+K$ is *not*, in general, block diagonal. An example is given of a block diagonal operator T for which there can be *no* block diagonal perturbation which preserves the Weyl spectrum and essential numerical range of T.

Let $B(H)$ and $C(H)$ denote, respectively, the algebras of bounded and compact linear operators on a complex, separable Hilbert space H. Then $C(H)$ is a closed ideal in $B(H)$ and $B(H)/C(H)$, the Calkin algebra, is a C^*-algebra with identity when endowed with the quotient norm. One general problem associated with this algebra is the following: if a coset $T + C(H)$ has a certain property in $B(H)/C(H)$, is there a representative $T + K$ of the coset having the same property in $B(H)$? Much progress on this question has already been made (see, for instance, [1], [2], [5], [6], [9], [10], [11]). In particular, Stampfli [11] has shown that there exists $C \in C(H)$ such that the spectrum of $T + C$ and the Weyl spectrum of T are equal. In [6], it was proved that there is a $C \in C(H)$ such that the closure of the numerical range of $T + C$ agrees with the essential numerical range of T.

The results in the present paper were motivated by the following question: Given $T \in B(H)$, does there exist a $C \in C(H)$ such that $T + C$ simultaneously preserves the Weyl spectrum and essential numerical range of T. This problem appears to be quite hard and is still unresolved. Our main result is

THEOREM 3.7. *For each block diagonal operator $T \in B(H)$, there exists a compact operator C such that $T + C$ simultaneously preserves the Weyl spectrum and essential numerical range of T.*

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As will be seen later, the operator $T + C$ of the above theorem is not, in general, block diagonal.

In §1, preliminaries are given along with an example which shows that the simultaneous preservation problem is quite different from the spectrum and numerical range problems separately. In §2, the existence of a compact quasinilpotent operator with numerical range in an arbitrary sector is established. This result is crucial for the proof of the main theorem. In §3, the main theorem is proved.

At this time, we wish to express our gratitude to F. J. Narcowich for many valuable suggestions on material appearing in §2.

1. Preliminaries. We list here several important definitions and terms relevant to lifting problems. As usual, $\sigma(T)$ will denote the spectrum of an operator T and $\sigma_p(T)$ will symbolize the isolated eigenvalues of finite multiplicity of T. The *Weyl spectrum* of T, $\sigma_w(T)$, is given by $\lambda \in \sigma_w(T)$ if and only if $T - \lambda$ is not a Fredholm operator with zero index. Following Stampfli and Williams **[12]**, we define the *numerical range* of an operator T to be $W(T) = \{\phi(T)\colon \phi \text{ is a state on } B(H)\}$ and define the *essential numerical range* of T to be $W_e(T) \equiv \{\bigcap W(T + K)\colon K \in C(H)\}$. Note that $W(T)$ is always closed in contrast to the set $A = \{\langle Tx, x\rangle\colon \|x\| = 1\}$, which corresponds to the usual numerical range. In fact, $W(T)$ is the closure of A. An operator $T + C$ is said to *preserve* the Weyl spectrum (essential numerical range) of an operator T if $\sigma(T + C) = \sigma_w(T)$ ($W(T + C) = W_e(T)$). An operator $T + C$ is said to *simultaneously preserve* the Weyl spectrum and essential numerical range of T if $T + C$ preserves both the Weyl spectrum and essential numerical range of T.

The following assumptions will be made throughout the remainder of this paper:

(i) each block of the block diagonal operator is a matrix of finite but arbitrary order, and

(ii) each block is in upper triangular form.

The fact that for a given block diagonal T there is a compact operator C which preserves either the Weyl essential spectrum or essential numerical range of T is easily derived. In fact, suppose T is a block diagonal operator and one wishes to find a compact operator K such that $\sigma(T + K) = \sigma_w(T)$. Let $\{\lambda_k\}_{k=1}^{\infty}$ be the at most countable number of eigenvalues of finite multiplicity of T and let $\{\mu_k\}_{k=1}^{\infty}$ be any sequence in $\sigma_w(T)$ satisfying the condition

$$\lim_{k\to\infty} |\mu_k - \lambda_k| \to 0.$$

Then the operator $T + K$ determined by replacing λ_k by μ_k for all k along the diagonal of T is the required operator (recall that the blocks of T are upper triangular).

The corresponding theorem for numerical ranges is also easy. Let $d(a, B) = \inf\{|a - b|: b \in B\}$, where $a \in \mathbf{C}$ and $B \subset \mathbf{C}$. In what follows, $d_H(A, B) \equiv \sup\{d(a, B): a \in A\}$ will designate the "one-sided" Hausdorff distance between two convex sets A and B in $\mathbf{C}$. Note that for a block diagonal operator T, $W_e(T) = \overline{\bigcap_m \bigcup_{n>m} W(T_n)}$, where T_n denotes the nth block of T. Thus $d_H(W(T_n), W_e(T)) = \varepsilon_n$ converges to zero as $n \to \infty$. Also, we may assume that $W_e(T)$ has an interior point and upon rotation and/or translation by αI that zero is an interior point. Then by taking an appropriate sequence $\alpha_n > 0$ increasing monotonically to 1, we have that $T + K \equiv \bigoplus \alpha_n T_n$ preserves the essential numerical range of T. Additionally, under the hypothesis that $W_e(T)$ contains an interior point, the α_n may be chosen so that $W(\alpha_n T_n)$ lies strictly inside $W_e(T)$ for all n. This fact is crucial for the proof of the main result.

Thus, it is now apparent that one may preserve the Weyl spectrum or essential numerical range of a block diagonal operator by means of a compact operator K in such a way that the resultant operator is also block diagonal. This is *not* the case if one wishes to simultaneously preserve the Weyl spectrum and essential numerical range of a block diagonal operator T. Consider the operator $\bigoplus_n T_n$, where T_n is an $n \times n$ matrix with constant value $1/2n$ on its diagonal, $1/n$ in each entry of its upper triangular part, and 0 in each entry of the lower triangular part. Clearly $\sigma_w(T) = 0$. Then $\operatorname{Re} T \equiv (T + T^*)/2 = P/2$ where P is a projection. Since $\operatorname{Re} T$ is positive it follows that $0 \in \partial W_e(T)$ where ∂S denotes the boundary of S. Thus if there existed a compact operator K such that $T + K$ simultaneously preserved the Weyl spectrum, the essential numerical range of T and, in addition, was block diagonal, then $T + K = \bigoplus(T_n + K_n)$ where each $T_n + K_n$ would possess the properties

(i) $\sigma(T_n + K_n) = 0$, and

(ii) $0 \in \partial W(T_n + K_n)$.

But it is well known that (i) and (ii) together imply that $(T_n + K_n) = 0$ (cf. [7]). This result holds with respect to any orthonormal basis.

Nevertheless it will be shown in §3 that for every block diagonal operator T there is a compact operator K such that $T + K$ simultaneously preserves the Weyl spectrum and essential numerical range of T. The following definition will illustrate the form that K will assume.

DEFINITION 1.1. *Let H_1 and H_2 be orthogonal subspaces of a separable Hilbert space, and let $T \in B(H_1)$. If $x \in H_1$ and B is an operator defined on the span of x and H_2, with $(Bx, x) = (Tx, x)$, then B is called an* adjunction *to T at x.*

The type of perturbation of T that will be used to establish the main result will be a direct sum of adjunctions to the diagonal elements of each T_n by compact operators having numerical ranges in decreasingly narrow wedges. The existence of such compact operators is established in §2.

2. A special compact, quasinilpotent operator. In this section the existence of a compact, quasinilpotent operator whose numerical range is contained in any given sector is established. This result will be a critical ingredient in the proof of the main theorem in §3.

A few preliminary remarks are in order. Following Kato **[8]**, we say a linear (not necessarily bounded) operator defined on a dense domain of a Hilbert space H is *accretive* if $W(T)$ is a subset of the right half-plane. If, in addition, the conditions

$$(T+\lambda)^{-1} \in B(H), \qquad \|(T+\lambda)^{-1}\| \leqslant (\operatorname{Re}\lambda)^{-1}, \qquad \operatorname{Re}\lambda > 0,$$

are satisfied, the operator is said to be *m-accretive*. We are now able to establish

THEOREM 2.1. *Let G be any sector in the right half plane which is symmetric with respect to the x-axis. Then there exists a compact quasinilpotent operator T^α satisfying $W(T^\alpha) \subset G$.*

PROOF. Let $H = L_2[0, 1]$ and define the Volterra operator $T \in B(H)$ by $Tf(x) = \int_0^x f(t)\,dt$. It is well known that T is a compact, one-to-one operator having dense range. A short computation establishes that T is m-accretive. Thus $T^{-1} = S$ exists as an unbounded inverse. By **[8**, p. 279**]**, S is also an m-accretive operator. The resolvent of S is given by $(S+\lambda)^{-1} = (1 + \lambda T)^{-1}T$ and $(S+\lambda)^{-1}$ is evidently an entire function of λ since T is a Volterra operator. Again from **[8]**, it follows that $(S+\lambda)^{-1}$ is m-accretive. Thus, for λ positive, $\|(S+\lambda)^{-1}\| \leqslant \lambda^{-1}$. On the other hand, $(S+\lambda)^{-1}$ is analytic for all $\lambda \in \mathbf{C}$. Thus the integral

$$T^\alpha = S^{-\alpha} = \frac{\sin \pi\alpha}{\pi} \int_0^\infty \lambda^{-\alpha}(S+\lambda)^{-1}d\lambda, \qquad 0 < \alpha < 1,$$

is a well defined bounded operator, and moreover T^α is accretive. It is also compact. To see this note that for $0 < \alpha < 1$,

$$\begin{aligned} T^\alpha &= \frac{\sin \pi\alpha}{\pi} \int_0^R \lambda^{-\alpha}(S+\lambda)^{-1}d\lambda + \frac{\sin \pi\alpha}{\pi} \int_R^\infty \lambda^{-\alpha}(S+\lambda)^{-1}\,d\lambda \\ &= T\left(\frac{\sin \pi\alpha}{\pi} \int_0^R \lambda^{-\alpha}(1+\lambda T)^{-1}\,d\lambda\right) + \frac{\sin \pi\alpha}{\pi} \int_R^\infty \lambda^{-\alpha}(S+\lambda)^{-1}\,d\lambda. \end{aligned}$$

Due to the facts that T is compact, $\lambda^{-\alpha}$ is integrable on the interval $(0, \infty)$ for $0 < \alpha < 1$, $(1 + \lambda T)^{-1}$ is uniformly bounded for $\lambda \in [0, R]$ and $\|(S+\lambda)^{-1}\|$

is bounded by λ^{-1} for positive λ, the first term on the right-hand side of the above equality integrates to a compact operator and the second term, which is bounded in norm by the number $\int_R^\infty \lambda^{-\alpha}\lambda^{-1}\,d\lambda$, goes to zero as $R \to \infty$. Hence T^α is compact.

We now proceed to "locate" the numerical range of T^α. The operators

$$T^\alpha = S^{-\alpha} = \frac{\sin \pi\alpha}{\pi}\int_0^\infty (S+\lambda)^{-1}\lambda^{-\alpha}\,d\lambda, \qquad 0 < \alpha < 1,$$

are termed the αth roots of the operator S^{-1} by Kato [8, p. 286]. The proof that the numerical range of T^α lives in the sector $|\theta| \leqslant (\pi/2)(1-\alpha)$ for $0 < \alpha < 1$ is essentially the same proof given by Kato for the special case $\alpha = \frac{1}{2}$. Following the argument in Lemma 3.40 of [8, p. 282] we integrate over the ray $\lambda = \rho e^{-i\beta}$, $|\beta| < \pi/2$, $0 < \rho < \infty$, to obtain

$$\begin{aligned}\langle T^\alpha f, f\rangle = \langle S^{-\alpha} f, f\rangle &= \frac{\sin \pi\alpha}{\pi}\int_0^\infty \lambda^{-\alpha}\langle (S+\lambda)^{-1} f, f\rangle\,d\lambda \\ &= \frac{\sin \pi\alpha}{\pi}\int_0^\infty e^{i\beta}\left(e^{i\beta}\rho + S\right)^{-1} e^{-i\beta\alpha}\rho^{-\alpha}\,d\rho \\ &= \frac{\sin \pi\alpha}{\pi}\, e^{i\beta(1-\alpha)}\int_0^\infty \rho^{-\alpha}\left(e^{i\beta}\rho + S\right)^{-1}\,d\rho.\end{aligned}$$

In [8, Lemma 3.40] Kato assumes his operator to be *strictly* m-accretive. But as he remarks later, the "strict" assumption is not needed since one may perturb the path of integrability and then take limits. Thus

$$\langle T^\alpha f, f\rangle = e^{i\beta(1-\alpha)}\langle G_{\alpha,\beta} f, f\rangle$$

where

$$G_{\alpha,\beta} \equiv \frac{\sin \pi\alpha}{\pi}\int_0^\infty \rho^{-\alpha}\left(e^{i\beta}\rho + S\right)^{-1}\,d\rho.$$

We have $\operatorname{Re} G_{\alpha,\beta} \geqslant 0$ and

$$e^{-i\beta(1-\alpha)}\langle T^\alpha f, f\rangle = \langle G_{\alpha,\beta} f, f\rangle.$$

Thus the complex number $\langle T^\alpha f, f\rangle$ may be multiplied by any scalar on the circle between $e^{-i(\pi/2)(1-\alpha)}$ and $e^{i(\pi/2)(1-\alpha)}$ while still remaining in the right half-plane which forces $\arg\langle T^\alpha f, f\rangle$ to lie between $-(\pi/2)\alpha$ and $(\pi/2)\alpha$. This completes the proof of the theorem.

3. The main theorem. In this section, a proof of Theorem 3.7 is given. Throughout this section, T will denote a block diagonal operator $T \equiv \{T_n\}$ for which $W(T_n)$ is *strictly* inside $W_e(T)$ for all n (hereafter denoted by $W(T_n) < W_e(T)$). As observed earlier, this assumption can be made without loss of generality.

As was noted in §1, one cannot hope in general to obtain a block diagonal perturbation of a block diagonal operator T which preserves both the Weyl spectrum and essential numerical range of T. Actually the example of §1 typifies the general problem. Namely, it can be shown that a block diagonal operator whose Weyl spectrum is at a positive distance from the boundary of the essential numerical range may be perturbed to form an operator $T + K$ which remains block diagonal and which preserves both the Weyl spectrum and essential numerical range of T. This fact will become apparent in the course of the proof of the main theorem.

Motivated by the above considerations, we now proceed to "split" the point spectrum $\sigma_p(T)$ of T into two parts: the first part consists of those elements of $\sigma_p(T)$ which are "close" to points of $\sigma_w(T)$ which, in turn, are "far" from the boundary of $W_e(T)$. The second part, the complement of the previous set relative to $\sigma_p(T)$, is an also possibly infinite set whose only limit points are members of $\sigma_w(T) \cap \partial W_e(T)$. The proof of the main theorem consists of moving the first part of the point spectrum to the Weyl spectrum while preserving the block diagonal structure and the essential numerical range. Then the second part of the point spectrum is moved to the Weyl spectrum while maintaining the essential numerical range although now the block diagonal structure is lost. The details proceed as follows: let $\delta_1 > \delta_2 > \ldots$ be a monotonically decreasing null sequence and let

$$C_j = \{\lambda \in \sigma_p(T): d(\lambda, \partial W_e(T)) > \delta_j\}, \qquad j = 1, \ldots.$$

Partition $\sigma_p(T)$ into S and $\sigma_p(T) - S$ as follows: let

$$\sigma_1 = \{\lambda: \lambda \in C_1, d(\lambda, \sigma_w(T)) < \delta_1/2\} \quad \text{and}$$

$$\sigma_k = \{\lambda: \lambda \in C_k - C_{k-1}, d(\lambda, \sigma_w(T)) < \delta_k/2^k\}$$

for $k = 2, 3, \ldots$. Now set $S = \bigcup_k \sigma_k$.

PROPOSITION 3.1. *$\sigma_p - S$ is a (possibly) countably infinite set whose only limit points are contained in $\sigma_w(T) \cap \partial W_e(T)$.*

PROOF. Suppose $\sigma_p - S$ contains a sequence $\{\lambda_j\}$ which converges to $\bar{\lambda} \in \sigma_w(T) \cap \operatorname{int} W_e(T)$. Then there exists a minimal integer m for which the λ_j eventually satisfy $\lambda_j \in C_m$, $\lambda_j \notin C_{m-1}$. But those λ_j eventually belong to σ_m, a contradiction.

PROPOSITION 3.2. *Let T be a block diagonal operator satisfying $W(T_n) < W_e(T)$ with associated sets S and $\sigma_p(T) - S$. Then there exists a block diagonal compact operator $K = \bigoplus_n K_n$ where K_n and T_n are supported on the same space and satisfy*

(i) $W(T_n + K_n) < W_e(T)$,

(ii) $\sigma_p(T + K) = \sigma_p(T) - S$.

Before proving the above, the following is needed.

LEMMA 3.3. *Let C be any convex set in the complex plane. Then $d(\cdot, \partial C)$ is a concave function on C. That is, for $\lambda_i \geqslant 0$ and $C_i \in C$, $i = 1, \ldots, n$, for which $\sum_{i=1}^n \lambda_i = 1$, the inequality*

$$\sum_{j=1}^{n} \lambda_j d(C_j, \partial C) \leqslant d\left(\sum_{j=1}^{n} \lambda_j C_j, \partial C\right)$$

always holds.

PROOF OF LEMMA. The proof is easy and we omit the details.

PROOF OF PROPOSITION. We first prove the result under the hypothesis that for all n and for $\lambda_{nk} \in \sigma(T_n) \cap S$, there exists a decreasing null sequence $\{\varepsilon_n\}_{n=1}^{\infty}$ for which $d(\lambda_{nk}, \sigma_w(T)) \leqslant \varepsilon_n d(\lambda_{nk}, \partial W(T))$. For each block matrix T_n with corresponding orthonormal basis $\{e_{ni}\}$, define the diagonal operator Δ_n to be

$$\Delta_n e_{nk} = \begin{cases} \mu_{nk} - \lambda_{nk}, & \lambda_{nk} \in S, \\ 0, & \lambda_{nk} \in \sigma_p(T) - S. \end{cases}$$

Given ε_n, the scalar hypothesized for T_n above, let

$$\hat{T}_n = D_n + (1 - \varepsilon_n)\mathring{T}_n, \qquad \hat{\hat{T}}_n = D_n + \Delta_n + (1 - \varepsilon_n)\mathring{T}_n,$$

where D_n and $\mathring{T}_n$ denote the diagonal and strictly upper triangular parts of T_n respectively. Since $T_n = D_n + \mathring{T}_n$, the corresponding eigenvalues λ_{nk} are strictly inside $W_e(T)$ (recall that $W(T_n) < W_e(T)$). For the $r_n \times r_n$ matrix T_n, one obtains

$$\begin{aligned} \min_{1 \leqslant j \leqslant r_n} d(\lambda_{nj}, \partial W_e(T)) &= \min_{\|x\|=1} d(\langle D_n x, x\rangle, \partial W_e(T)) \\ &= \min_{\Sigma s_j^2 = 1} d\left(\sum \lambda_{nj} s_j^2, \partial W_e(T)\right) \equiv a_n > 0. \end{aligned}$$

For any fixed unit vector $x = \Sigma s_j^2 e_{nj}$, we have

$$\begin{aligned} |\langle \Delta_n x, x\rangle| &= \left|\sum (\mu_{nj} - \lambda_{nj}) s_j^2\right| \leqslant \left|\sum d(\lambda_{nj}, \sigma_w(T)) s_j^2\right| \\ &= \varepsilon_n \sum d(\lambda_{nj}, \partial W_e(T)) s_j^2 \\ &\leqslant \varepsilon_n \sum d\left(\lambda_{nj} s_j^2, \partial W_e(T)\right) = \varepsilon_n a_n, \end{aligned}$$

the last inequality following from Lemma 3.3. Thus $d_H(W(T_n), \partial W(T_n + \Delta_n)) \leqslant \varepsilon_n a_n$. But since $d_H(W(D_n), \partial W_e(T)) \geqslant a_n$, clearly $d_H(W(\hat{\hat{T}}_n), \partial W_e(T)) \geqslant \varepsilon_n a_n$, whence $W(\hat{\hat{T}}_n) < W(T) = W_e(T)$. Since $\varepsilon_n \to 0$, $\oplus\, \hat{\hat{T}}_n$ is a compact perturbation of T with the required properties.

To complete the proof, it remains to show that such ε_n in fact exist. Toward this end, let τ_1 denote the finite set $\{\lambda \in \sigma_1: d(\lambda, \sigma_w(T)) > \delta_1/4\}$. Each such $\lambda \in \tau_1$ corresponds to an eigenvalue of some matrix (o matrices) T_n. Let n_1 be the largest such n and assign the number $\frac{1}{2}$ to $T_1, \ldots, T_{n_1}$. Similarly let τ_2 denote the set $\{\lambda \in \sigma_1 \cup \sigma_2 - \tau_1: d(\lambda, \sigma_w(T)) > \delta_1/8\}$. Again each such λ corresponds to an eigenvalue of some T_n. Let n_2 be the maximum such n and assign the number $\frac{1}{4}$ to the matrices $T_{n+1}, \ldots, T_{n_2}$. Continuing in this fashion, note that there is at most a finite number of λ's in $\bigcup_{i=1}^{m} \sigma_i - \bigcup_{i=1}^{m-1} \tau_i$ satisfying $d(\lambda, \sigma_w(T)) > \delta_m/2^{m+1}$. Associate, as before, with each such λ a corresponding T_n. Let n_m be the maximum of such m and assign $1/2^m$ to the matrices $T_{n_{m-1}+1}, \ldots, T_{n_m}$. Thus to each T_n there exists a scalar of the form $1/2^{j_n}$. Set $\varepsilon_n = 1/2^{j_n}$, and the proof is complete.

In view of Proposition 3.2, to complete Theorem 3.7, it suffices to prove

PROPOSITION 3.4. *Let T be a block diagonal operator satisfying $W(T_n) < W_e(T)$ for all n and for which there is a positive null sequence $\{\alpha_n\}$ such that $d(\lambda_{ni}, \sigma_w(T) \cap W_e(T)) \leq \alpha_n$ for $i = 1, 2, \ldots, r_n$, and $n = 1, 2, \ldots$ where the λ_{ni} denote the eigenvalues of the $r_n \times r_n$ upper triangular matrix T_n. Then there exists a compact operator K with the properties*

(3.1.i) $W(T + K) = W_e(T)$, *and*

(3.1.ii) $\sigma(T + K) = \sigma_w(T)$.

PROOF. As was observed in §1, there are no nontrivial finite rank operators A satisfying $0 = \sigma(A)$ and Re $W(A) \geq 0$. We will appeal to the results of §2 to obtain infinite rank compact quasinilpotent operators having numerical range in a prescribed sector. Moreover the domains of these operators will be subspaces on which T is "roughly", in a sense made clear below, a multiple of the identity. This allows us to compute the spectrum and numerical range of the operator $T + K$. The details now follow. If $\lambda_{ni} \notin \sigma_w(T)$, let μ_{ni} denote a nearest point of $\sigma_w(T) \cap \partial W(T)$ to λ_{ni}. For each n let I_n denote the set of subscripts i for these λ_{ni}. The collection of these points $\{\mu_{ni}\}$ constitutes a set of *strong normality* of T in the sense of Stampfli.

Let D be the direct sum of diagonal operators $D_{ni} = \mu_{ni} I$, for all n and i defined on mutually orthogonal infinite dimensional Hilbert spaces H_{ni}. By [**11**, Theorem 2], T is unitarily equivalent to $T \oplus D + K$ where K is a compact operator. Following the proof of [**11**, Theorem 2], it is easily seen that $W_e(T \oplus D) = W_e(T)$ and $\sigma_w(T \oplus D) = \sigma_w(T)$. For each $n = 1, 2, \ldots$ define $D_n = \bigoplus_{i \in I_n} D_{ni}$. We wish to perturb each operator $T_n \oplus D_n$ by a compact operator K_n with $\|K_n\| \to 0$ as $n \to \infty$ so that

(i) $\sigma(T_n \oplus D_n + K_n) \subset \sigma_w(T)$, and

(ii) $W(T_n \oplus D_n + K_n) \subset W(T)$.

Then $T + C \equiv \bigoplus_n (T_n + D_n + K_n)$ will be the desired perturbation of T.

We now proceed to construct such a compact K_n. Recall that, for fixed n,

T_n is an upper triangular matrix with respect to an orthonormal basis e_{ni}, $i = 1, \ldots, r_n$, having eigenvalues λ_{ni}, $i = 1, \ldots, r_n$. For each $i \in I_n$ let B_{ni} be the resultant operator of a translation and rotation of $T_n + D_{ni}$ for which

(3.2.i) D_{ni} is transformed to the zero operator;

(3.2.ii) $W(B_{ni})$ lies in a half-plane which contains a wedge symmetric with respect to the positive real axis;

(3.2.iii) $(B_{ni}e_{ni}, e_{ni}) = a_{ni} + ib_{ni}$ where $\sqrt{a_{ni}^2 + b_{ni}^2} = |\lambda_{ni} - \mu_{ni}|$, $a_{ni} > 0$ and $|a_{ni}| \gg |b_{ni}|$. In fact, $|b_{ni}|$ could be chosen to be zero since λ_{ni} is assumed to be an interior point of $W(T_n)$. However we assume $|a_{ni}| \gg |b_{ni}| > 0$ for the reason stated in (3.4.i) below.

Let $\dot{V}_{ni}$ be a Volterra (i.e. compact and quasinilpotent) operator defined on $\text{sp}\{e_{ni}, H_{ni}\}$ which is an adjunction (cf. §1) to B_{ni} and which, by Theorem 2.1, satisfies the following

(3.3.i) $\dot{V}_{ni}$ is m-accretive

(3.3.ii) $W(\dot{V}_{ni})$ is contained in the sector (in the right half plane) of angle $\tan^{-1}\gamma_n$ (γ_n specified later).

Since $\dot{V}_{ni}$ is an adjunction to B_{ni} we can assume

(3.4.i) the real part of $\dot{V}_{ni}$, $\dot{V}_{Rni}$, satisfies $\dot{V}_{Rni}e_{ni} = a_{ni}e_{ni}$ (it is for this reason we have chosen $|b_{ni}| > 0$. Otherwise e_{ni} would be an eigenvalue of $\dot{V}_{ni}$ which contradicts the assumptions on $\dot{V}_{ni}$). If need be, pick $\dot{V}_{ni}$ to satisfy $\dot{V}_{Rni}e_{ni} = \|\dot{V}_{Rni}\|e_{ni}$ and then pick the operator B_{ni} to satisfy (3.2.iii) above where now $a_{ni} = \|\dot{V}_{Rni}\|$.

(3.4.ii) the imaginary part of $\dot{V}_{ni}$, $\dot{V}_{Ini}$, satisfies $\dot{V}_{Ini}e_{ni} = b_{ni}e_{ni} + v_{ni}$, $v_{ni} \in H_{ni}$.

Moreover $\dot{V}_{ni}$ can be chosen so that $W(\dot{V}_{ni}) \subset W(B_{ni})$ by making γ_n small enough. The constants γ_n (and hence upper bounds for the $|b_{ni}|$) will be specified later. Letting $\{P_{e_{ni}}\}$ denote the rank one orthogonal projection onto the one dimensional space $\text{sp}\{e_{ni}\}$ define $V_{ni} = \dot{V}_{ni} - P_{e_{ni}}\dot{V}_{ni}P_{e_{ni}}$. Thus V_{ni} has the same matrix representation as $\dot{V}_{ni}$ except that $(V_{ni}e_{ni}, e_{ni}) = 0$.

The operator $W_{ni}(\dot{W}_{ni})$ is defined to be the inverse rotation and translation of $V_{ni}(\dot{V}_{ni})$. Observe that for each i there is a possibly different rotation and translation. Each W_{ni} corresponds to an adjunction of a 1-dimensional piece of T. Although W_{ni} is infinite dimensional it corresponds to "fixing up" only a 1-dimensional piece of T. The perturbation of T_n to be examined is $\overline{T}_n = T_n + \bigoplus_{i \in I_n} W_{ni}$. If γ_n is small enough, since $\|V_{Rni}\| = a_{ni}$, it follows from the fact that the numerical radius is an equivalent norm [12] that $\|V_{ni}\| < 3|\lambda_{ni} - \mu_{ni}|$. For $i \neq j$, the supports of V_{ni} and V_{nj} are orthogonal. So $\overline{T}_n$ is a compact perturbation of $T_n \oplus D_n$ by a compact operator of norm less than $3\max\{|\mu_{ni} - \lambda_{ni}|\colon i \in I_n\} \equiv \alpha_n$. Hence $\overline{T} = \bigoplus_{n=1}^{\infty} \overline{T}_n$ is a compact perturbation of $T \oplus D$, since $\alpha_n \to 0$.

The figures below illustrate $T_n \oplus D_{ni}$ and $T_n + W_{ni}$ for an arbitrary i.

$$\begin{pmatrix} T_n & 0 \\ 0 & D_{ni} \end{pmatrix} = \left(\begin{array}{ccc|ccc} \lambda_{n1} & & * & & & \\ & \ddots & & & 0 & \\ 0 & & \lambda_{nr_n} & & & \\ \hline & & & \mu_{ni} & & 0 \\ & 0 & & & \ddots & \\ & & & 0 & & \ddots \end{array}\right),$$

$$T_n + W_{ni} = \left(\begin{array}{ccccc|c} \lambda_{n1} & & & & * & 0 \\ & \ddots & & & & \\ & & \lambda_{ni} & & & * \text{———} * \\ & & & \ddots & & \\ 0 & & & & \lambda_{nr_n} & 0 \\ \hline & & * & & & \\ 0 & & \vert & & 0 & * \\ & & * & & & \end{array}\right).$$

Since $\overline{T}_n = T_n + \bigoplus_{i \in I_n} W_{n_i}$, a picture of $\overline{T}_n$ would have $\mathrm{card}(I_n)$ blocks adjuncted to T_n. The way is now prepared for

PROPOSITION 3.5. $\sigma(\overline{T}_n) \subseteq \sigma_w(T)$.

PROOF. If λ were an eigenvector of $\overline{T}_n$ with eigenvectors $x = re_{ni} + h_{ni}$, $h_{ni} \in H_{ni}$ it would follow that $\lambda = \mu_{ni} \in \sigma_w(T)$ because with respect to the space $\mathrm{sp}\{e_{ni}, H_{ni}\}$, $\overline{T}_n$ is the operator $\mu_{ni}(I + V)$ where V is a Volterra operator. In general suppose that λ is an eigenvalue of $\overline{T}_n$ with eigenvector $x = \sum_{i=1}^{r_n} s_i e_{ni} + \sum_{i \in I_n} t_i h_{ni}$, $h_{ni} \in H_{ni}$. Let l denote the integer for which $s_i = 0$ for $i > l$ and $s_l \neq 0$. If $l \notin I_n$ then clearly $\lambda = \lambda_{nl} \in \sigma_w(T)$. If $l \in I_n$ then $e_{nl} + t_l h_{nl}$ must be an eigenvector with eigenvalue λ for the compression of $T_{nl} + W_{nl}$ to $\mathrm{sp}\{e_{nl}, \ldots, e_{nr_n}, H_{nl}\}$. But with respect to the space $\mathrm{sp}\{e_{nl}, H_{nl}\}$ this operator has the general form $\mu_{nl}(I + V)$, where V is a Volterra operator and thus $\lambda = \mu_{nl} \in \sigma_w(T)$. Thus $\sigma(\overline{T}) \subset \sigma_w(T)$, and the proof is complete.

We now check the numerical range inclusion property for $\overline{T}$.

PROPOSITION 3.6. *$W(\overline{T}_n) \subset W_e(T)$ for all n.*

PROOF. The verification that $W(\overline{T}_n) \subset W_e(T)$ for each $n = 1, 2, \ldots$ is a relatively straightforward computation. It is here that the sequence $\{\gamma_n\}$ must be chosen correctly. Fix n and let $x = ry + \sum_{i \in I_n} t_i z_{ni}$, where $y \in H_n$, $z_{ni} \in H_{ni}$, $\|y\| = 1$, $\|z_{ni}\| = 1$ for each $i \in I_n$ and $|r|^2 + \sum_{i \in I_n} |t_i|^2 = 1$. Then

$$\left(\overline{T}_n x, x\right) = ((T_n + \oplus W_{ni})x, x)$$
$$= |r|^2(T_n y, y) + \sum_{i \in I_n} \left\{(W_{ni} ry, t_i z_{ni}) + (W_{ni} t_i z_{ni}, ry) + |t_i|^2(W_{ni} z_{ni}, z_{ni})\right\}. \tag{$*$}$$

Furthermore $y = \sum_{i \in I_n} l_i e_{ni} + sy^*$, where y^* is orthogonal to e_{ni} for each i in I_n, $\|y^*\| = 1$, and $\sum_{i \in I_n} |l_i|^2 + |s|^2 = 1$. It then follows from ($*$) and the fact that $W_{ni} y = W_{ni} l_i e_{ni}$ that

$$\left(\overline{T}_n x, x\right) = |r|^2\left(|s|^2 + \sum_{i \in I_n} |l_i|^2\right)(T_n y, y)$$
$$+ \sum_{i \in I_n} \left\{(W_{ni} r l_i e_{ni}, t_i z_{ni}) + (W_{ni} t_i z_{ni}, r l_i e_{ni}) + |t_i|^2(W_{ni} z_{ni}, z_{ni})\right\}$$
$$= |r|^2\, |s|^2(T_n y, y)$$
$$+ \sum_{i \in I_n} \left\{|l_i|^2\, |r|^2(T_n y, y) + (W_{ni} r l_i e_{ni}, t_i z_{ni}) + (W_{ni} t_i z_{ni}, r l_i e_{ni}) + |t_i|^2(W_{ni} z_{ni}, z_{ni})\right\}$$
$$= |r|^2\, |s|^2(T_n y, y) + \sum_{i \in I_n} \left(|l_i|^2\, |r|^2 + |t_i|^2\right)$$
$$\cdot \left\{ \frac{|l_i|^2\, |r|^2}{|l_i|^2\, |r|^2 + |t_i|^2} (Ty, y) + \frac{1}{|l_i|^2\, |r|^2 + |t_i|^2} \left[(W_{ni} r l_i e_{ni}, t z_{ni}) + (W_{ni} t_i z_{ni}, r l_i e_{ni})\right] + \frac{|t_i|^2}{|l_i|^2\, |r|^2 + |t_i|^2} (W_{ni} z_{ni}, z_{ni}) \right\}.$$

If it can be shown that each braced term in the above sum is in $W(T)$ ($= W_e(T)$) then $(\overline{T}_n x, x) \in W(T)$, because the entire expression is then a convex combination of points of $W(T)$.

So the goal is to verify that each braced term in the above sum is a point in $W(T)$. Let $a^2 = |r l_i|^2/(|r l_i|^2 + |t_i|^2)$ and $b^2 = |t_i|^2/(|r l_i|^2 + |t_i|^2)$. From the above, it is apparent that it suffices to verify that

$$|a|^2(T_n y, y) + (W_{ni} ae_{ni}, bz_{ni}) + (W_{ni} bz_{ni}, ae_{ni}) + |b|^2(W_{ni} z_{ni}, z_{ni}) \in W(T).$$

Furthermore, the a and b can be assumed to be real, with no loss in generality. One more reduction is necessary to clarify the argument. Namely let B be the resultant operator after rotating and translating T in such a way that $T_n + D_n$ is transformed into B_{ni}. Then it becomes sufficient to show that

$$a^2(B_{ni} y, y) + (V_{ni} ae_{ni}, bz_{ni}) + (V_{ni} bz_{ni}, ae_{ni}) + |b|^2(V_{ni} z_{ni}, z_{ni}) \qquad (**)$$

is in $W(B)$. The two middle terms of (**) can be rewritten as

$$\begin{aligned}
&(V_{ni} ae_{ni}, bz_{ni}) + (V_{ni} bz_{ni}, ae_{ni}) \\
&\quad = ((V_{Rni} + iV_{Ini})ae_{ni}, bz_{ni}) + ((V_{Rni} + iV_{Ini})bz_{ni}, ae_{ni}) \\
&\quad = (V_{Rni} ae_{ni}, bz_{ni}) + \left(\overline{V_{Rni} ae_{ni}, bz_{ni}}\right) + i\left[(V_{Ini} ae_{ni}, bz_{ni}) + \left(\overline{V_{Ini} ae_{ni}, bz_{ni}}\right)\right] \\
&\quad = 2\,\mathrm{Re}(V_{Rni} ae_{ni}, bz_{ni}) + 2i\,\mathrm{Re}\,(V_{Ini} ae_{ni}, bz_{ni}) \\
&\quad = 2i\,\mathrm{Re}\,(V_{Ini} ae_{ni}, bz_{ni})
\end{aligned}$$

since $V_{Rni} ae_{ni} = al_i e_{ni}$. By the generalized Hölder inequality then

$$\begin{aligned}
|2i\,\mathrm{Re}\,(V_{Ini} ae_{ni}, bz_{ni})| &\leq 2|(V_{Ini} ae_{ni}, ae_{ni})|^{1/2}|(V_{Ini} bz_{ni}, bz_{ni})|^{1/2} \\
&\leq a^2|(V_{Ini} e_{ni}, e_{ni})| + b^2|(V_{Ini} z_{ni}, z_{ni})|.
\end{aligned}$$

Since $a^2 + b^2 = 1$ the term $a^2(B_{ni} y, y) + b^2(V_{ni} z_{ni}, z_{ni})$ is in $W(T)$. By construction, $b_{ni} \leq \gamma_{ni} a_{ni}$. Thus, using the hypothesis that $W(T_n) < W_e(T)$, it is possible to select γ_n small enough that $(B_{ni} y, y) + i|(V_{Ini} e_{ni}, e_{ni})| \in W(B)$. Similarly it is also possible to select γ_n small enough that $W(V_{Rni} + 2iV_{Ini}) \subset W(B)$; so $(V_{ni} z_{ni}, z_{ni}) + i|(V_{Ini} z_{ni}, z_{ni})|$ is in $W(B)$. Thus, (**) is in $W(B)$, as required, and the proof of the proposition is complete.

Propositions 3.2 and 3.4 now yield

THEOREM 3.7. *For each block diagonal operator $T \in B(H)$, there exists a compact operator C such that $T + C$ simultaneously preserves the Weyl spectrum and essential numerical range of T.*

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