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SWEEDLER'S TWO-COCYCLES AND GENERALIZATIONS OF THEOREMS ON AMITSUR COHOMOLOGY

BY

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ABSTRACT. For any (not necessarily commutative) algebra C over a commutative ring k Sweedler defined a cohomology set, denoted here by $\mathcal{H}^2(C/k)$, which generalizes Amitsur's second cohomology group $H^2(C/k)$. In this paper, if I is a nilpotent ideal of C and $\bar{C} \equiv C/I$ is k-projective, a natural bijection $\mathcal{H}^2(C/k) \tilde{\to} \mathcal{H}^2(\bar{C}/k)$ is established. Also, when $k \subset B$ are fields and C is a commutative B-algebra, the sequence $\{1\} \to \mathcal{H}^2(B/k) \xrightarrow{\iota^*} \mathcal{H}^2(C/k) \xrightarrow{r} \mathcal{H}^2(C/B)$ is shown to be exact if the natural map $C \otimes_k C \to C \otimes_B C$ induces a surjection on units, ι^* is induced by the inclusion, and r is the "restriction" map.

Introduction. Given a commutative algebra C over a commutative ring k, Amitsur introduced a cochain complex and demonstrated that in certain cases the second cohomology group $H^2(C/k)$ of this complex is isomorphic to the Brauer group of similarity classes of central separable k-algebras split by C [1]. Sweedler generalized the notion of an Amitsur two-cocycle to noncommutative algebras C and has shown that various sets of certain equivalence classes of (generalized) two-cocycles classify various types of k-algebras [8]. In this paper we study several functorial properties of the full set, denoted here by $\mathcal{H}^2(C/k)$, of certain equivalence classes of Sweedler's two-cocycles.

Sweedler's basic definitions are recalled in §1. §2 provides a generalization of a theorem of Rosenberg and Zelinsky [6, Proposition 3.3] which states that under suitable faithfully flatness conditions the Amitsur cohomology of a commutative algebra is not changed by factoring out a nilpotent ideal. We prove that, if I is a nilpotent ideal of the (not necessarily commutative) k-algebra C and $\bar{C} \equiv C/I$ is k-projective, the natural projection $C \to \bar{C}$ induces a bijection $\mathcal{H}^2(C/k) \tilde{\to} \mathcal{H}^2(\bar{C}/k)$. In §3 we restrict our attention to commutative algebras. If $k \subseteq B$ are fields, C is a commutative B-algebra, and the natural map $C \otimes_k C \to C \otimes_B C$ induces a surjection on units, we prove there is an exact sequence

$$\{1\} \to \mathcal{H}^2(B/k) \xrightarrow{\iota^*} \mathcal{H}^2(C/k) \xrightarrow{r} \mathcal{H}^2(C/B)$$

where ι^* is induced by the inclusion $\iota: B \to C$ and r is the "restriction" map.

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This sequence is a partial generalization and synthesis of two sequences Yuan used in his study of p-algebras [**10**, Theorems 3.4, 3.5].

1. Definitions. Let C be an algebra over a commutative ring k and let unadorned $\otimes$ represent $\otimes_k$. Following Sweedler [**8**] we call an element $\delta = \sum_i x_i \otimes y_i$ in $C \otimes C$ a C-one-cocycle if $\sum_i x_i \otimes 1 \otimes y_i = \sum_{i,j} x_i \otimes y_i x_j \otimes y_j$. Two C-one-cocycles $\delta = \sum_i x_i \otimes y_i$ and $\eta = \sum_i u_i \otimes v_i$ are said to be equivalent if there is a unit z in C with $\sum_i z^{-1} x_i \otimes y_i z = \sum_i u_i \otimes v_i$. An element $\sum_i x_i \otimes y_i$ in $C \otimes C$ is called vertible if $\sum_i x_i \otimes y_i^0$ is invertible in $C \otimes C^0$. (Here C^0 is the opposite k-algebra.) We denote the set of equivalence classes of vertible C-one-cocycles by $\mathcal{H}^1(C/k)$ or by $\mathcal{H}^1(C)$ if the ground ring is clear from the context.

A C-two-cocycle is an element $\sigma = \sum_i a_i \otimes b_i \otimes c_i$ in $C \otimes C \otimes C$ such that

$$\sum_{i,j} a_i a_j \otimes b_j \otimes c_j b_i \otimes c_i = \sum_{i,j} a_i \otimes b_i a_j \otimes b_j \otimes c_j c_i$$

and there is an e_σ in C with $\sum_i a_i e_\sigma b_i \otimes c_i = 1 \otimes 1 = \sum_i a_i \otimes b_i e_\sigma c_i$. We shall refer to these equations as the associativity and unitary conditions, respectively, since they arose to guarantee a certain construction C^σ of Sweedler's would be a k-algebra. The C-two-cocycle $\sigma = \sum_i a_i \otimes b_i \otimes c_i$ is cohomologous to the C-two-cocycle $\tau = \sum_i r_i \otimes s_i \otimes t_i$ via $\delta = \sum_i x_i \otimes y_i$ in $C \otimes C$ if

$$\sum_i x_i e_\sigma y_i = e_\tau,$$

$$\sum_{i,j} x_i a_j \otimes b_j \otimes c_j y_i = \sum_{i,j,l} r_i x_j \otimes y_j s_i x_l \otimes y_l t_i. \tag{1.1}$$

If σ is cohomologous to τ via a vertible δ, we indicate this by $\sigma \sim^\delta \tau$ and say that σ and τ are equivalent. We denote the set of equivalence classes of C-two-cocycles by $\mathcal{H}^2(C/k)$ or by $\mathcal{H}^2(C)$ if no confusion seems likely. Given a C-two-cocycle $\sigma = \sum_i a_i \otimes b_i \otimes c_i$ and a vertible element $\delta = \sum_i x_i \otimes y_i$ with verse $\bar{\delta} = \sum_i \bar{x}_i \otimes \bar{y}_i$ (i.e. $\sum_i \bar{x}_i \otimes \bar{y}_i^0$ is the inverse of $\sum_i x_i \otimes y_i^0$) we will often wish to consider the C-two-cocycle $\sigma_1 = \sum_{i,j,l,m} \bar{x}_i a_j x_l \otimes y_l b_j x_m \otimes y_m c_j \bar{y}_i$ obtained by "altering" σ by δ. We shall say in this situation that $\sigma \sim^\delta \sigma_1$ defines the C-two-cocycle σ_1. For example, if C is commutative and $\sigma = \sum_i a_i \otimes b_i \otimes c_i$ is a C-two-cocycle, the unitary condition implies that e_σ is invertible (with inverse $\sum_i a_i b_i c_i$) and $\sigma \sim^{1 \otimes e_\sigma^{-1}} \sigma_1$ defines a C-two-cocycle σ_1 with $e_{\sigma_1} = 1$.

Any k-algebra map $C \xrightarrow{f} D$ induces a map

$$C \otimes \overset{n}{\cdots} \otimes C \xrightarrow{f^n} D \otimes \overset{n}{\cdots} \otimes D$$

given by $x_1 \otimes \cdots \otimes x_n \to f(x_1) \otimes \cdots \otimes f(x_n)$ for each positive integer n which we will denote by f if no confusion seems likely. For $i = 1, 2$, we have an induced map on cohomology $\mathcal{H}^i(C) \xrightarrow{f^*} \mathcal{H}^i(D)$.

2. A theorem of Rosenberg and Zelinsky. Throughout this section, $\mathcal{H}^2(D/k)$ will be denoted by $\mathcal{H}^2(D)$ for any k-algebra D. We prove:

THEOREM 2.1. *Let C be an algebra over a commutative ring k. Suppose I is a nilpotent ideal of C with C/I projective as a k-module, and let p: $C \to C/I$ be the natural projection. Then $\mathcal{H}^2(C) \xrightarrow{p^*} \mathcal{H}^2(C/I)$ is bijective.*

We begin by noting that Theorem 2.1 is equivalent to:

THEOREM 2.1′. *Let C be an algebra over a commutative ring k. Suppose I is an ideal of C with $I^2 = \{0\}$ and C/I is projective as a k-module. Let p: $C \to C/I$ be the natural projection. Then $\mathcal{H}^2(C) \xrightarrow{p^*} \mathcal{H}^2(C/I)$ is bijective.*

PROOF THAT 2.1′ IMPLIES 2.1. For any positive integer n we have a commutative diagram

$$\begin{array}{ccc} \mathcal{H}^2(C) & \xrightarrow{p^*} & \mathcal{H}^2(C/I) \\ & \searrow \quad \nearrow & \\ & \mathcal{H}^2(C/I^n) & \end{array}$$

with each map induced by the appropriate canonical projection. Hence if 2.1′ is known, the general result 2.1 may be shown by an induction on the index of nilpotency of I.

Therefore we need only prove Theorem 2.1′ and accordingly in the remainder of this section we will assume $I^2 = \{0\}$. With this assumption I is a C/I-bimodule and we may view C as a Hochschild extension of C/I by I **[4]**. Thus we have $C = C/I \oplus_\psi I$ as k-algebras where ψ: $C/I \otimes C/I \to I$ is a Hochschild two-cocycle. Recall that this means $C = C/I \oplus I$ as k-modules and the multiplication $*_\psi$ in $C/I \oplus_\psi I$ is given by $(a, x) *_\psi (b, y) = (ab, ay + xb + \psi(a \otimes b))$ for a,b in C/I, x,y in I. The next theorem considers the special case of Theorem 2.1′ where ψ is a trivial Hochschild two-cocycle.

THEOREM 2.2. *Let k, C, I and p be as in Theorem 2.1′. Also assume $C = C/I \oplus_\psi I$ with ψ a trivial Hochschild two-cocycle. Then $\mathcal{H}^2(C) \xrightarrow{p^*} \mathcal{H}^2(C/I)$ is bijective.*

PROOF. Since ψ is trivial there is an algebra map ι: $C/I \to C$ with $p\iota$ the identity of C/I. Denote the composite ιp by q. Then q: $C \to C$ is an algebra endomorphism satisfying $q^2 = q$ and $\ker(q) = I$. It thus follows from Theorem 4.7 of **[5]** that any C-two-cocycle σ is equivalent to its projection $q(\sigma)$ modulo I, which establishes the theorem.

We are now ready to prove Theorem 2.1′. The idea of the proof is to mimic the argument used to prove Theorem 4.7 of **[5]** while making allowance for the fact that C/I is not, in general, a subalgebra of $C/I \oplus_\psi I$.

p^* is injective. Let σ,τ be C-two-cocycles with $p(\sigma)$ and $p(\tau)$ equivalent C/I-two-cocycles, i.e. there is an element δ in $C \otimes C$ with $p(\sigma)$ equivalent to $p(\tau)$ via $p(\delta)$. Since I is nilpotent, $p(\delta)$ vertible implies that δ is vertible. Thus $\sigma \sim^{\delta} \sigma_1$ defines a C-two-cocycle σ_1 equivalent to σ with $p(\sigma_1) = p(\tau)$.

Hence we may assume

$$\sigma = \tau + \sum_i f_i \otimes g_i \otimes h_i \tag{2.3}$$

with $\Sigma_i f_i \otimes g_i \otimes h_i$ in $\mathcal{I} = I \otimes C \otimes C + C \otimes I \otimes C + C \otimes C \otimes I$; that is, $\sigma \equiv \tau$ (modulo $\mathcal{I}$). We argue by induction on n that if $\sigma \equiv \tau$ (modulo $\mathcal{I}^n$), there exist C-two-cocycles σ_1, τ_1 with σ_1 equivalent to σ, τ_1 equivalent to τ, and $\sigma_1 \equiv \tau_1$ (modulo $\mathcal{I}^{n+1}$). This will be sufficient to establish the injectivity of p^* because $\mathcal{I}^4 = \{0\}$.

Let $\tau = \Sigma_i\, a_i \otimes b_i \otimes c_i$ in (2.3) and define

$$\delta = \sum_i p(a_i e_\sigma) b_i \otimes c_i + \sum_i p(f_i e_\sigma) g_i \otimes h_i.$$

Since

$$\{p \otimes 1\}\Big(\sum_i p(a_i e_\sigma) b_i \otimes c_i\Big) = 1 \otimes 1$$

and I is nilpotent, $\Sigma_i\, p(a_i e_\sigma) b_i \otimes c_i$ is vertible. Denote its verse by $\Sigma_i\, x_i \otimes y_i$. $\sigma_1 \sim^{\delta} \sigma$ and $\tau_1 \sim^{\Sigma_i p(a_i e_\sigma) b_i \otimes c_i} \tau$ define C-two-cocycles σ_1, τ_1 with

$$\begin{aligned}\sigma_1 \equiv \tau_1 &- \sum_{i,j} a_j p(f_i e_\sigma) g_i \otimes h_i b_j \otimes c_j + \sum_i f_i \otimes g_i \otimes h_i \\ &+ \sum_{i,j} p(f_i e_\sigma) g_i a_j \otimes b_j \otimes c_j h_i - \sum_{i,j} a_j \otimes b_j p(f_i e_\sigma) g_i \otimes h_i c_j \end{aligned}$$

$$(\text{modulo } \mathcal{I}^{n+1}) \tag{2.4}$$

where we have used

$$\sum_i p(a_i e_\sigma) b_i \otimes c_i \equiv 1 \otimes 1 \equiv \sum_i x_i \otimes y_i \quad (\text{modulo } C \otimes I + I \otimes C).$$

It follows from the associativity relation for σ that

$$\begin{aligned}&\sum_{i,j} p(a_i f_j e_\sigma) g_j \otimes h_j b_i \otimes c_i + \sum_{i,j} p(f_i a_j e_\sigma) b_j \otimes c_j g_i \otimes h_i \\ &\quad\equiv \sum_{i,j} p(a_i e_\sigma) b_i f_j \otimes g_j \otimes h_j c_i + \sum_{i,j} p(f_i e_\sigma) g_i a_j \otimes b_j \otimes c_j h_i \\ &\quad\equiv \sum_j f_j \otimes g_j \otimes h_j + \sum_{i,j} p(f_i e_\sigma) g_i a_j \otimes b_j \otimes c_j h_i \quad (\text{modulo } \mathcal{I}^{n+1}),\end{aligned}$$

so we may rewrite (2.4) as

$$\sigma_1 \equiv \tau_1 + \sum_{i,j} p(f_i a_j e_\sigma) b_j \otimes c_j g_i \otimes h_i$$
$$- \sum_{i,j} a_j \otimes b_j p(f_i e_\sigma) g_i \otimes h_i c_j \quad (\text{modulo } \mathcal{J}^{n+1})$$

since $\Sigma_i f_i \otimes g_i \otimes h_i$ is in $\mathcal{J}$ and $p(I) = \{0\}$. That is, we have shown

$$\sigma_1 \equiv \tau_1 + \sum_i f_i' \otimes g_i' \otimes h_i' \quad (\text{modulo } \mathcal{J}^{n+1})$$

with $\Sigma_i f_i' \otimes g_i' \otimes h_i'$ in $C \otimes I \otimes C + C \otimes C \otimes I$.

Suppose $\tau_1 = \Sigma_i a_i' \otimes b_i' \otimes c_i'$. If we now repeat the above procedure but "push down on the right", that is alter by

$$\sum_i a_i' \otimes b_i' p(e_{\sigma_1} c_i') + \sum_i f_i' \otimes g_i' p(e_{\sigma_1} h_i'),$$

we obtain C-two-cocycles σ_2, τ_2 with σ_2 equivalent to σ_1, τ_2 equivalent to τ_1, and

$$\sigma_2 \equiv \tau_2 + \sum_i u_i \otimes v_i \otimes w_i \quad (\text{modulo } \mathcal{J}^{n+1}) \tag{2.5}$$

with v_i in I for all i.

If $\tau_2 = \Sigma_i r_i \otimes s_i \otimes t_i$, the associativity relation for σ_2 yields

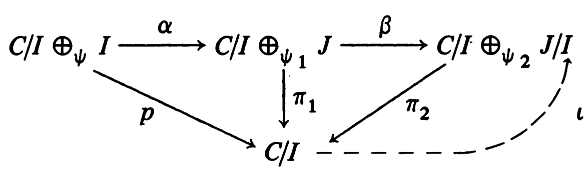
$$\sum_{i,j} r_i u_j \otimes v_j \otimes w_j s_i \otimes t_i + \sum_{i,j} u_i r_j \otimes s_j \otimes t_j v_i \otimes w_i$$
$$\equiv \sum_{i,j} r_i \otimes s_i u_j \otimes v_j \otimes w_j t_i + \sum_{i,j} u_i \otimes v_i r_j \otimes s_j \otimes t_j w_i \quad (\text{modulo } \mathcal{J}^{n+1}).$$

Thus it follows that, modulo $\mathcal{J}^{n+1}$,

$$\sum_{i,j} u_i r_j \otimes p(s_j) e_{\sigma_2} t_j v_i \otimes w_i \equiv \sum_{i,j} r_i \otimes p(s_i u_j) e_{\sigma_2} v_j \otimes w_j t_i \equiv 0.$$

Hence (2.5) implies $\sigma_2 \equiv \tau_2$ (modulo $\mathcal{J}^{n+1}$) which was to be shown.

p^* is surjective. Let J be the injective hull of I as a C/I-bimodule. Define ψ_1: $C/I \otimes C/I \to J$ by $\psi_1(a \otimes b) = \psi(a \otimes b)$. Then ψ_1 is a trivial Hochschild two-cocycle since J is injective [3, §IX.6] and, if we set $J^2 = \{0\}$, $C/I \oplus_{\psi_1} J$ is a k-algebra with multiplication $(a, x) *_{\psi_1} (b, y) = (ab, ay + xb + \psi_i(a \otimes b))$. We have a commutative diagram



with p, π_1, π_2 the natural projections and $\alpha(a, x) = (a, x)$, $\beta(a, y) = (a, y + I)$, $\iota(a) = (a, 0)$ for a in C/I, x in I, y in J.

Let σ be a $C/I \oplus_{\psi_1} J$-two-cocycle. Then $\beta(\sigma)$ is equivalent to $\iota\pi_2\beta(\sigma) = \iota\pi_1(\sigma)$ via δ in

$$1 \otimes 1 + J/I \otimes \{C/I \oplus_{\psi_2} J/I\} + \{C/I \oplus_{\psi_2} J/I\} \otimes J/I$$

by the proof of Theorem 2.2.

Lift δ to δ' in

$$1 \otimes 1 + J \otimes \{C/I \oplus_{\psi_1} J\} + \{C/I \oplus_{\psi_1} J\} \otimes J.$$

δ' is vertible since I is nilpotent. Let $\sigma \sim^{\delta'} \sigma'$ define the $C/I \oplus_{\psi_1} J$-two-cocycle σ'. Then

$$\iota\pi_2\beta(\sigma) \sim^{\bar{\delta}} \beta(\sigma) \sim^{\beta(\delta')} \beta(\sigma'),$$

where $\bar{\delta}$ is the verse of δ. Thus

$$\iota\pi_2\beta(\sigma) \sim^{1\otimes 1} \beta(\sigma'),$$

which implies σ' is in $\alpha(C/I \oplus_{\psi} I)$ and α^* is surjective.

We now have the commutative diagram

$$\begin{array}{ccc} \mathcal{H}^2(C/I \otimes_{\psi} I) & \overset{\alpha^*}{\rightarrow} & \mathcal{H}^2(C/I \otimes_{\psi_1} J) \\ p^* \searrow & & \swarrow \pi_1^* \\ & \mathcal{H}^2(C/I) & \end{array}$$

Since ψ_1 is a trivial Hochschild two-cocycle, π_1^* is bijective by Theorem 2.2 and hence it follows that p^* is surjective. This completes the proof of Theorem 2.1′, and hence also of Theorem 2.1.

REMARKS. (1) If we restrict our attention to commutative algebras and invertible (i.e., Amitsur) C-two-cocycles, Theorem 2.1 is included in a theorem of Rosenberg and Zelinsky [6, Proposition 3.3].

(2) One may show that the induced map $\mathcal{H}^1(C) \overset{p^*}{\rightarrow} \mathcal{H}^1(C/I)$ is bijective by a similar (and far less tedious) argument.

3. An exact sequence. The object of this section is to establish that if $k \subseteq B$ are fields and C is a commutative B-algebra there is an exact sequence of cohomology sets

$$\{1\} \rightarrow \mathcal{H}^2(B) \overset{\iota^*}{\rightarrow} \mathcal{H}^2(C) \overset{r}{\rightarrow} \mathcal{H}^2(C/B) \qquad \text{(ES)}$$

if the natural map $C \otimes C \rightarrow C \otimes_B C$ induces a surjection on the groups of units. Here ι: $B \rightarrow C$ is the inclusion and r is the "restriction" map arising from the natural projection $C \otimes C \otimes C \rightarrow C \otimes_B C \otimes_B C$. We begin with an easy extension of a theorem of Berkson [2, Theorem 1].

THEOREM 3.1. *Let k be a commutative ring and C be a k-algebra which is free as a k-module. Then $\mathcal{H}^1(C) = \{1\}$.*

PROOF. Since C is free as a k-module, we may write any vertible C-one-cocycle as $\Sigma_{i=1}^{s} v_i \otimes w_i$ with $\{v_i\}$, $\{w_i\}$ sets of linearly independent elements of C. Then the one-cocycle relation

$$\sum_i v_i \otimes 1 \otimes w_i = \sum_{i,j} v_i \otimes w_i v_j \otimes w_j$$

implies

$$\sum_{i,j} v_i \otimes (w_i v_j - \delta_{ij}) \otimes w_j = 0$$

where δ_{ij} is the Kronecker delta. Since $\{v_i\}$ is linearly independent and $\{w_i\}$ is linearly independent we have $w_i v_j = \delta_{ij}$. Suppose $s > 2$. Then

$$(w_1 \otimes v_1^0)\Big(\sum_i v_i \otimes w_i^0\Big) = 1 \otimes 1^0 = (w_2 \otimes v_2^0)\Big(\sum_i v_i \otimes w_i^0\Big).$$

Hence, since inverses are unique and we are assuming that $\Sigma_i v_i \otimes w_i^0$ is invertible, $w_1 \otimes v_1 = w_2 \otimes v_2$, a contradiction to the linear independence hypothesis. Thus $\Sigma_i v_i \otimes w_i = v_1 \otimes w_1$. Since $v_1 \otimes w_1$ is a C-one-cocycle, $v_1 w_1 = 1$. Above we showed that $w_1 v_1 = 1$. Hence $v_1 \otimes w_1 = v_1 \otimes v_1^{-1}$, which is clearly equivalent to $1 \otimes 1$.

REMARK. The proof of Theórem 3.1 is Berkson's proof slightly modified to deal with the noncommutativity of C.

A close look at the proof of Theorem 3.1 yields a fact we will later need.

LEMMA 3.2. *If C is a commutative k-algebra and a free k-module, every C-one-cocycle is invertible.*

PROOF. The vertible hypothesis in Theorem 3.1 is needed since $w_i v_j = \delta_{ij}$.

Notation. If $k \subseteq L \subseteq C$ are commutative rings and C is an L-algebra, we define $m_L\colon C \otimes C \to C \otimes_L C$ by

$$m_L\Big(\sum_i x_i \otimes y_i\Big) = \sum_i x_i \otimes_L y_i$$

and

$$m_{L,L} = \{1 \otimes m_L\} \circ \{m_L \otimes 1\}\colon C \otimes C \otimes C \to C \otimes_L C \otimes_L C.$$

THEOREM 3.3. *Let $k \subseteq L$ be fields and C be a commutative L-algebra. If σ,τ are L-two-cocycles and $\sigma \sim^{\delta} \tau$ with δ a vertible element of $C \otimes C$, then $\delta = (\alpha \otimes \alpha^{-1})\delta'$ with δ' in $L \otimes L$ and α a unit of C.*

PROOF. Since L is commutative we may without loss of generality (cf. §1) assume $e_\sigma = 1 = e_\tau$. $m_{L,L}$ applied to (1.1) shows that $m_L(\delta)$ is a one-cocycle because σ,τ are in $L \otimes L \otimes L$. By Theorem 3.1 and Lemma 3.2 $m_L(\delta) = \alpha \otimes_L \alpha^{-1}$ for some α in C and hence $\delta = \alpha \otimes \alpha^{-1} + \sum_i d_i \otimes e_i$ with $\{d_i\}$ k-linearly independent, $\{e_i\}$ k-linearly independent, and $\sum_i d_i \otimes_L e_i = 0$. Starting again with (1.1), applying the map $1 \otimes m_L$ and substituting $\alpha \otimes \alpha^{-1} + \sum_i d_i \otimes e_i$ for δ, we find that

$$\sum_i d_i \otimes 1 \otimes_L e_i = \sum_i d_i \otimes e_i \alpha \otimes_L \alpha^{-1}.$$

A straightforward application of linear algebra yields $e_i = m_i \alpha^{-1}$ for all i, with m_i in L. Similarly $d_i = n_i \alpha$ for all i, with n_i in L. Therefore

$$\delta = (\alpha \otimes \alpha^{-1})\left(1 \otimes 1 + \sum_i n_i \otimes m_i\right),$$

which was to be shown.

An immediate consequence of Theorem 3.3 is

COROLLARY. *Let $k \subseteq L$ be fields and C be a commutative L-algebra. Then the inclusion ι: $L \to C$ induces an injection $\mathcal{H}^2(L) \xrightarrow{\iota^*} \mathcal{H}^2(C)$.*

The next theorem is the key result we need to prove exactness at $\mathcal{H}^2(C)$ in the purportedly exact sequence (ES) exhibited at the beginning of this section. First, however, we need a preliminary lemma.

LEMMA 3.4. *Let $k \subseteq B$ be commutative rings and C be a commutative B-algebra. Suppose $\sigma = \sum_i a_i \otimes b_i \otimes c_i$ is a C-two-cocycle with $m_{B,B}(\sigma) = 1 \otimes_B 1 \otimes_B 1$. Then under the map*

$$C \otimes C \otimes_B C \to (C \otimes C) \otimes_{C \otimes B} (C \otimes C)$$

given by $x \otimes y \otimes_B z \to (x \otimes y) \otimes_{C \otimes B} (1 \otimes z)$ the element $\{1 \otimes m_B\}(\sigma)$ maps to a one-cocycle.

PROOF. Applying $1 \otimes m_{B,B}$ to the associativity relation for σ yields

$$\sum_{i,j} a_i a_j \otimes b_j \otimes_B c_j b_i \otimes_B c_i = \sum_i a_i \otimes b_i \otimes_B 1 \otimes_B c_i \tag{3.4a}$$

since $m_{B,B}(\sigma) = 1 \otimes_B 1 \otimes_B 1$ by assumption. Under the map

$$C \otimes C \otimes_B C \otimes_B C \xrightarrow{\sim} (C \otimes C) \otimes_{C \otimes B} (C \otimes C) \otimes_{C \otimes B} (C \otimes C)$$

given by

$$w \otimes x \otimes_B y \otimes_B z \to (w \otimes x) \otimes_{C \otimes B} (1 \otimes y) \otimes_{C \otimes B} (1 \otimes z),$$

(3.4a) transforms to

$$\sum_{i,j} (a_j \otimes b_j) \otimes_{C \otimes B} (a_i \otimes c_j b_i) \otimes_{C \otimes B} (1 \otimes c_i)$$
$$= \sum_i (a_i \otimes b_i) \otimes_{C \otimes B} (1 \otimes 1) \otimes_{C \otimes B} (1 \otimes c_i).$$

This establishes the lemma.

THEOREM 3.5. *Let $k \subseteq B$ be fields and C be a commutative B-algebra. Suppose σ is a C-two-cocycle with $m_{B,B}(\sigma) = 1 \otimes_B 1 \otimes_B 1$. Then σ is equivalent to a B-two-cocycle.*

PROOF. By Lemma 3.4, $\{1 \otimes m_B\}(\sigma)$ is a one-cocycle. Since $C \otimes C$ is a free $C \otimes B$-module, Theorem 3.1 and Lemma 3.2 imply

$$\{1 \otimes m_B\}(\sigma) = \sum_{i,j} x_i \bar{x}_j \otimes y_i \otimes \bar{y}_j$$

where $\delta = \sum_i x_i \otimes y_i$ is a unit in $C \otimes C$ with inverse $\bar{\delta} = \sum_i \bar{x}_i \otimes \bar{y}_i$ and $m_C(\delta) = 1$. $\sigma_1 \sim^{\delta} \sigma$ defines a C-two-cocycle σ_1 with $\{1 \otimes m_B\}(\sigma_1) = 1 \otimes 1 \otimes_B 1$ since

$$\{m_B \circ (m_C \otimes 1)\}(\sigma) = 1 \otimes_B 1$$

implies $m_B(\bar{\delta}) = 1 \otimes_B 1$. Since B is a field, an application of $1 \otimes 1 \otimes m_B$ to the associativity relation for σ_1 demonstrates that σ_1 is in $C \otimes C \otimes B$.

By a similar argument using the analog of Lemma 3.4 for the map

$$C \otimes_B C \otimes B \tilde{\to} (C \otimes B) \otimes_{B \otimes B} (C \otimes B)$$

given by $x \otimes_B y \otimes z \to (x \otimes 1) \otimes_{B \otimes B} (y \otimes z)$, there exists a unit δ_1 in $C \otimes B$ with $m_C(\delta_1) = 1$ such that $\sigma_2 \sim^{\delta_1} \sigma_1$ defines a C-two-cocycle σ_2 with

$$\{m_B \otimes 1\}(\sigma_2) = 1 \otimes_B 1 \otimes 1.$$

Since δ_1 is in $C \otimes B$, $\{1 \otimes m_B\}(\sigma_2) = 1 \otimes 1 \otimes_B 1$.

By applying $m_B \otimes 1 \otimes 1$ to the associativity relation for σ_2 one may show that σ_2 is in $B \otimes C \otimes B$. That σ_2 is actually in $B \otimes B \otimes B$ then follows from an application of $1 \otimes m_B \otimes 1$ to the associativity relation for σ_2. Hence we are done.

We are now ready to prove:

THEOREM 3.6. *Let $k \subseteq B$ be fields and let C be a commutative B-algebra. Suppose also that the natural map m_B: $C \otimes C \to C \otimes_B C$ induces a surjection on the groups of units. Then*

$$\{1\} \to \mathcal{H}^2(B) \xrightarrow{\iota^*} \mathcal{H}^2(C) \xrightarrow{r} \mathcal{H}^2(C/B)$$

is exact.

PROOF. The injectivity of ι^* follows from the corollary to Theorem 3.3. Hence to finish the proof we must show exactness at $\mathcal{H}^2(C)$.

Let σ be a C-two-cocycle whose cohomology class is mapped to the trivial class under r, that is, $m_{B,B}(\sigma)$ is cohomologous to $1 \otimes_B 1 \otimes_B 1$ via an invertible element δ in $C \otimes_B C$. By hypothesis, we may lift δ to δ_1 in $C \otimes C$.

Then $\sigma_1 \sim^{\delta_1^{-1}} \sigma$ defines a C-two-cocycle σ_1 with $m_{B,B}(\sigma_1) = 1 \otimes_B 1 \otimes_B 1$. Thus by Theorem 3.5 there is a B-two-cocycle σ_2 with σ_2 equivalent to σ_1, and hence equivalent to σ. This completes the proof.

COROLLARY 3.7. *Let $k \subseteq B$ be fields and C be a commutative B-algebra. Then*

$$\{1\} \to \mathcal{H}^2(B) \xrightarrow{\iota^*} \mathcal{H}^2(C) \xrightarrow{r} \mathcal{H}^2(C/B)$$

is exact if either

(i) *B is a purely inseparable field extension of k, or*

(ii) *B is a separable field extension of k and C is purely inseparable over B.*

PROOF. By the theorem, it suffices to show that either (i) or (ii) implies m_B induces a surjection on units.

(i) Assume B is purely inseparable over k. The kernel of m_B is generated as a C-bimodule by $\{b \otimes 1 - 1 \otimes b | b \text{ is in } B\}$. When B is purely inseparable over k, this is clearly an ideal generated by nilpotent elements and hence lies in the Jacobson radical of $C \otimes C$. The surjectivity on units thus follows since units may be lifted modulo the radical.

(ii) Since C is a purely inseparable B-algebra (cf. **[9]**), the kernel of the multiplication map $C \otimes_B C \to C$ given by $x \otimes_B y \to xy$ is contained in the Jacobson radical of $C \otimes_B C$. Thus any unit of $C \otimes_B C$ may be written as $\alpha \otimes_B 1 + \sum_i u_i \otimes_B v_i$ with α a unit in C and $\sum_i u_i \otimes_B v_i$ nilpotent. If $\sum_i x_i \otimes y_i$ is a separability idempotent for B over k, $\alpha \otimes 1 + \sum_{i,j} u_i x_j \otimes y_j v_i$ is a unit of $C \otimes C$ which maps to $\alpha \otimes_B 1 + \sum_i u_i \otimes_B v_i$. Therefore we are done.

REMARK. If we restrict our attention in Corollary 3.7 to the groups of units, we obtain well-known results on Amitsur cohomology. The reader is referred to Yuan **[10**, Theorems 3.4, 3.5].

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