TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 253, September 1979

NATURAL ENDOMORPHISMS OF BURNSIDE RINGS

BY

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ABSTRACT. The Burnside ring $\mathfrak{B}(G)$ of a finite group G consists of formal differences of finite G-sets. $\mathfrak{B}$ is a contravariant functor from finite groups to commutative rings. We study the natural endomorphisms of this functor, of its extension $\mathbf{Q} \otimes \mathfrak{B}$ to rational scalars, and of its restriction $\mathfrak{B} \restriction \mathbf{Ab}$ to abelian groups. Such endomorphisms are canonically associated to certain operators that assign to each group one of its conjugacy classes of subgroups. Using these operators along with a carefully constructed system of linear congruences defining the image of $\mathfrak{B}(G)$ under its canonical embedding in a power of $\mathbf{Z}$, we exhibit a multitude of natural endomorphisms of $\mathfrak{B}$, we show that only two of them map G-sets to G-sets, and we completely describe all natural endomorphisms of $\mathfrak{B} \restriction \mathbf{Ab}$.

Finite sets on which a finite group G acts (G-sets) can be added and multiplied in an obvious way, so their formal differences (virtual G-sets) form a ring, the Burnside ring of G. This paper is concerned with natural endomorphisms of Burnside rings, i.e., operators θ mapping virtual G-sets to virtual G-sets for all G and respecting addition, multiplication, and "restriction of scalars" along homomorphisms of groups. (Precise definitions are given in §1.) If we require θ to send G-sets to G-sets (rather than to virtual G-sets), then, as we prove in §6, the only θ other than the identity is the one that sends every G-set to the same set with G acting trivially. But, in the absence of this positivity requirement, the monoid of natural endomorphisms is uncountable (Corollary 4c), noncommutative (Corollary 4d), and apparently quite chaotic (see the end of §4). In contrast, we show in §5 that the natural endomorphisms of Burnside rings of *abelian* groups admit a uniform description and commute with each other.

In the course of obtaining these results, we describe (in §3) natural endomorphisms θ of rational Burnside algebras in terms of certain operators $\bar{\theta}$ that assign to each group a subgroup, and we give (in §4) a sufficient condition on $\bar{\theta}$ for θ to be an endomorphism of Burnside rings. Although this sufficient condition is not known to be necessary, it covers all known examples.

§1 contains definitions and well-known preliminary results, including a

Received by the editors September 16, 1975.

AMS (MOS) subject classifications (1970). Primary 20B05; Secondary 20C99, 20J99.

Key words and phrases. Permutation group, Burnside ring, marks, natural transformation.

representation of the Burnside ring of G as a subring of $\mathbf{Z}^n$, where $\mathbf{Z}$ is the ring of integers and n is the number of conjugacy classes of subgroups of G. This subring is definable by a collection of congruences, and we show in §2 that these congruences may be put into a particularly simple form. This result is the main structural property of Burnside rings needed in the later sections.

I am grateful to A. Dress for his helpful comments on this work, especially for pointing out that my proofs of Corollaries 4a and 4b could be used to establish some reasonably general results. These results eventually combined to produce Theorem 4.

1. Definitions. We shall be concerned with finite groups G acting (from the left) on finite sets X. When such an action is specified, we call X a G-set. Both the operation in G and the action on X are usually written as multiplication. G_x is the isotropy group $\{g \in G | gx = x\}$ of a point $x \in X$; note that $G_{gx} = gG_xg^{-1}$. A homomorphism of G-sets is a function f that commutes with the action of G, $f(g \cdot x) = g \cdot f(x)$. If α: $G_1 \to G_2$ is a homomorphism of groups and X is a G_2-set, then the G_1-set $\alpha^*(X)$ is defined to have the same underlying set as X with the action $g \cdot x = \alpha(g) \cdot x$, where the multiplication by $\alpha(g)$ is the given action of G_2.

For any two G-sets X and Y, there are obvious actions of G on the disjoint union $X + Y$ and (componentwise) on the product $X \times Y$. If we identify isomorphic G-sets, then these operations make the collection of (finite) G-sets a commutative semiring with 1 (which means that all the axioms for a commutative ring with 1 hold except that additive inverses do not exist). The well-known process of freely adjoining additive inverses extends this semiring to a commutative ring with 1, called the *Burnside ring* $\mathfrak{B}(G)$ of the group G. The members of $\mathfrak{B}(G)$ are formal differences of G-sets and are called *virtual G-sets*.

The additive structure of $\mathfrak{B}(G)$ is easy to describe because of the following three facts. First, every G-set is uniquely expressible as a sum of transitive G-sets (its orbits). Second, every transitive G-set is isomorphic to G/U, where U is a subgroup of G, G/U is the set of cosets aU ($a \in G$), and G acts on G/U by $g \cdot (aU) = (ga)U$. Third, G/U and G/V are isomorphic G-sets if and only if U and V are conjugate in G (see [3]). Let

$$\mathfrak{S}(G) = \{U_1, \ldots, U_n\}$$

be a system of representatives of the conjugacy classes of subgroups of G. Then we see that each $X \in \mathfrak{B}(G)$ has a unique representation in the form

$$X = \sum_{k=1}^{n} \xi(X)_k \cdot G/U_k$$

with coefficients $\xi(X)_k \in \mathbf{Z}$. The map ξ: $\mathfrak{B}(G) \to \mathbf{Z}^n$ that sends X to the n-tuple of coefficients is an additive isomorphism. The actual G-sets are those

X for which all the components of $\xi(X)$ are nonnegative. Clearly, ξ extends to an additive isomorphism (still called ξ) from the *rational Burnside algebra* $\mathfrak{B}(G) \otimes \mathbf{Q}$ onto $\mathbf{Q}^n$.

Unfortunately, the operation on $\mathbf{Z}^n$ that corresponds, via ξ, to multiplication in $\mathfrak{B}(G)$ admits no simple description. So we shall find it more convenient to use a different coordinatization of $\mathfrak{B}(G)$ with better multiplicative behavior. We define the *mark* of a subgroup U of G in a G-set X to be the number of points in X fixed by U,

$$\langle U, X\rangle = |\{x \in X \mid U \subseteq G_x\}|.$$

(The terminology is from Burnside [2], the notation from Dress [3].) For each U, the map $\langle U, -\rangle$ from G-sets to natural numbers clearly preserves both addition and multiplication, so it extends to a ring-homomorphism (still called $\langle U, -\rangle$) from $\mathfrak{B}(G)$ to $\mathbf{Z}$. Conjugate U's define the same $\langle U, -\rangle$, so we may confine our attention to $U \in \mathfrak{S}(G)$. The various ring-homomorphisms $\langle U, -\rangle$ together constitute a ring-homomorphism μ: $\mathfrak{B}(G) \to \mathbf{Z}^n$,

$$\mu(X)_k = \langle U_k, X\rangle,$$

where multiplication in $\mathbf{Z}^n$ is defined componentwise. This μ is one-to-one [3, Lemma 1], though not surjective (unless G is trivial). Its extension to rational scalars is an isomorphism between $\mathfrak{B}(G) \otimes \mathbf{Q}$ and $\mathbf{Q}^n$. Compared to ξ, μ has the advantage that it preserves multiplication and the disadvantage that the images of $\mathfrak{B}(G)$ and of the actual G-sets admit no simple description.

The transformation $\mu\xi^{-1}$: $\mathbf{Z}^n \to \mathbf{Z}^n$ between the two coordinatizations ξ and μ of $\mathfrak{B}(G)$ is given by the $n \times n$ matrix M in which the rows and columns are labeled by the groups $U \in \mathfrak{S}(G)$ and the (U, V) entry is

$$\langle U, G/V\rangle = |\{g \in G \mid U \subseteq gVg^{-1}\}|/|V|. \tag{1}$$

This matrix is called the *table of marks* of G. For example, if G is the symmetric group S_3 on three objects and we take $\mathfrak{S}(G)$ to be $\{(e), ((12)), ((123)), S_3\}$ in this order, then

$$M = \begin{bmatrix} 6 & 3 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Let us always agree to list the groups in $\mathfrak{S}(G)$ in such an order that no group precedes any conjugates of its own subgroups. Then the entries in the first row of M are $\langle (e), G/V\rangle = |G/V|$, the indices in G of its subgroups V. The entries in the last column are $\langle U, G/G\rangle = 1$. M is upper triangular since the (U, V) entry (1) is nonzero only when U is conjugate to a subgroup of V. The diagonal entries of M are, by (1), $|N(U)/U|$, where $N(U)$ is the normalizer of U in G. Thus, the index in $\mathbf{Z}^n$ of the range of μ (which is also the range of M) is

$$\det(M) = \prod_{U \in \mathcal{S}(G)} |N(U)/U|.$$

Note that the set whose cardinality is the numerator of the right side of (1) is a union of cosets of $N(V)$, so $|N(V)/V|$ divides $\langle U, G/V\rangle$. This means that every entry of M is divisible by the diagonal entry in the same column. Thus, we can write $M = \tilde{M}D$, where D is a diagonal matrix with the same diagonal entries as M and where $\tilde{M}$ is an upper triangular matrix whose diagonal entries are 1, so $\tilde{M}^{-1}$ has integer entries.

We close this section by introducing the canonical homomorphism from $\mathfrak{B}(G)$ to the character ring of G. If X is a G-set, let $\lambda(X)$ be the complex vector space having X as a basis, equipped with the linear action of G that extends the given action of G on X. Since disjoint union and Cartesian product of G-sets correspond, via λ, to direct sum and tensor product of linear representations, λ extends to a ring-homomorphism (still called λ) from $\mathfrak{B}(G)$ to the representation ring of G. Using the canonical basis X, one immediately sees that the character of $\lambda(X)$ is given by the marks in X of cyclic subgroups of G,

$$\chi_{\lambda(X)}(g) = \langle (g), X\rangle.$$

In general, λ is neither injective nor surjective, although it is injective when G is cyclic.

2. Congruences defining the Burnside ring. We use the "mark homomorphism" μ to identify $\mathfrak{B}(G)$ with a subring of $\mathbf{Z}^n$. Thus, we use the notations $\langle U, X\rangle$ and X_i interchangeably when U is conjugate to U_i. The additive isomorphism ξ is given by the matrix $M^{-1} = D^{-1}\tilde{M}^{-1}$. Since the range of ξ is all of $\mathbf{Z}^n$, we see that a column vector $X \in \mathbf{Z}^n$ belongs to $\mathfrak{B}(G)$ if and only if the components of $D^{-1}\tilde{M}^{-1}X$ are integers. If we use the abbreviation v_i for the ith diagonal entry $|N(U_i)/U_i|$ of D, then this condition for X to be in $\mathfrak{B}(G)$ can be written

$$\sum_{j=1}^{n} (\tilde{M}^{-1})_{ij}X_j \equiv 0 \mod v_i \qquad (i = 1, \ldots, n). \tag{2}$$

The congruences (2) serve to define $\mathfrak{B}(G)$ as a subring of $\mathbf{Z}^n$.

For example, when $G = S_3$, we have

$$\tilde{M} = \begin{bmatrix} 1 & 3 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \tilde{M}^{-1} = \begin{bmatrix} 1 & -3 & -1 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

so the defining congruences for $\mathfrak{B}(S_3)$ in $\mathbf{Z}^4$ are

$$X_1 + 3X_4 \equiv 3X_2 + X_3 \bmod 6 \quad \text{and} \quad X_3 \equiv X_4 \bmod 2.$$

In general, the first of the n congruences (2), corresponding to $U_1 = (e)$, has modulus $v_1 = |N(e)/(e)| = |G|$ and contains $X_1 = \langle (e), X\rangle$ with coefficient $(\tilde{M}^{-1})_{11} = 1$. Thus, it specifies the congruence class modulo $|G|$ of $\langle (e), X\rangle$, in terms of the other marks of X. We call such a congruence a *principal congruence* for G. None of the other congruences in (2) is principal, for X_1 does not occur in them at all, but one can obtain other principal congruences for $|G|$ from the one just described, by adding suitable multiples of the other congruences in (2). Since any principal congruence determines $\langle (e), X\rangle$ modulo $|G|$, it is clear that any two principal congruences will be equivalent in the presence of the remaining congruences in which $\langle (e), X\rangle$ is not mentioned.

We shall need a second set of defining congruences for $\mathfrak{B}(G)$; although equivalent to (2), it will be easier to work with. Temporarily fix a prime number p and a subgroup H of G. For any subgroup U of $N(H)/H$, we write U' for its inverse image in $N(H)$. For any G-set X, the subset Y of points fixed by H is permuted by $N(H)$, so we can view Y as an $(N(H)/H)$-set, or indeed as an S-set for any $S \leqslant N(H)/H$. Let us choose S to be a p-Sylow subgroup of $N(H)/H$, and let $|S| = p^k$. Any principal congruence for S will hold of Y,

$$\langle (e), Y\rangle + \cdots \equiv 0 \bmod p^k,$$

where ... represents an integral linear combination of terms $\langle U, Y\rangle$ with $(e) \neq U \leqslant S$. Clearly, $\langle U, Y\rangle = \langle U', X\rangle$, so this congruence is equivalent to

$$\langle H, X\rangle + \ldots \equiv 0 \bmod p^k, \tag{3}$$

where ... now represents an integral linear combination of terms $\langle U', X\rangle$ with $H \underset{\neq}{<} U' \leqslant S'$. We call (3) the (H, p)-congruence for $\mathfrak{B}(G)$.

THEOREM 1. *$\mathfrak{B}(G)$ consists of all vectors in $\mathbf{Z}^n$ satisfying the (H, p)-congruences for all $H \in \mathbb{S}(G)$ and all primes p.*

PROOF. We have just seen that these congruences are satisfied by every G-set and therefore, since they are linear, also by every virtual G-set. So they define an additive subgroup of $\mathbf{Z}^n$ that includes $\mathfrak{B}(G)$. We will be finished if we show that this subgroup has the same index in $\mathbf{Z}^n$ as $\mathfrak{B}(G)$ has.

If we fix a prime p and let H vary through $\mathbb{S}(G)$ in reverse order (i.e., larger groups before smaller ones), we see that each (H, p)-congruence simply restricts $\langle H, X\rangle$ to lie in one congruence class modulo p^k (where k is as above, so p^k is the largest power of p dividing $|N(H)/H|$, and where the congruence class depends on the previously chosen $\langle U, X\rangle$ for $U \underset{\neq}{>} H$). Thus, the (H, p)-congruences with one fixed p define a subgroup of $\mathbf{Z}^n$ of index $\prod_{H\in\mathbb{S}(G)} p^k$. Applying a suitable form of the Chinese remainder theorem (congruences with relatively prime moduli are independent) we find that the subgroup defined by all the (H, p)-congruences (when both p and H vary) has index

$$\prod_p \prod_{H\in\mathbb{S}(G)} p^k = \prod_{H\in\mathbb{S}(G)} \prod_p p^k = \prod_{H\in\mathbb{S}(G)} |N(H)/H|.$$

But we saw in §1 that this product, the determinant of M, is the index of $\mathfrak{B}(G)$ as well, so the proof of Theorem 1 is complete. □

REMARK. The construction of the $N(H)/H$-set Y associated to a G-set X defines a ring-homomorphism π_H^S: $\mathfrak{B}(G) \to \mathfrak{B}(S/H)$ (and similarly for rational Burnside algebras) whenever H is a normal subgroup of $S \leqslant G$. The theorem implies that

$$\mathfrak{B}(G) = \bigcap (\pi_H^S)^{-1}\mathfrak{B}(S/H) \tag{4}$$

where the intersection is over pairs $H \lhd S$ such that S/H is a p-group for some p. Indeed, the theorem says that (4) remains true if the intersection is taken over only those pairs such that S/H is Sylow in $N(H)/H$ and if $\mathfrak{B}(S/H)$ is replaced by the larger set defined by an arbitrary principal congruence for S/H. If we refrain from replacing $\mathfrak{B}(S/H)$ in this way, we can take the intersection in (4) over just those pairs such that S/H is Sylow in $N(H)/H$ and H has no proper normal subgroup of p-power index (roughly speaking, S/H is a maximal p-group subquotient of G). We omit the proof since we will not need this result.

The idea behind Theorem 1 is closely related to the result, due to Dress [3], that a congruence of the form $\langle U, X\rangle \equiv \langle V, X\rangle \bmod p$ holds in $\mathfrak{B}(G)$ if and only if the least normal subgroups of U and V with p-power index (in U and V) are conjugate in G.

In view of Theorem 1, it seems worthwhile to indicate how principal congruences for p-groups can be obtained in a rather explicit form. Let G be a p-group, and let μ_G be the Möbius function for the lattice of subgroups of G. This means that, if f maps the set of subgroups of G into $\mathbf{Z}$ (or any additive group) and if g is defined by

$$g(A) = \sum_{A\leqslant B} f(B),$$

then

$$f(A) = \sum_{A\leqslant B} \mu_G(A, B)g(B).$$

This Möbius function was calculated in [6].

THEOREM (WEISNER). *Let A and B be subgroups of a p-group G. If A is normal in B and $B/A \cong (\mathbf{Z}/p)^r$, then $\mu_G(A, B) = (-1)^r p^{r(r-1)/2}$. In all other cases, $\mu_G(A, B) = 0$.* □

For any G-set X and subgroup $A \leqslant G$, let $f(A)$ be the number of points in X with isotropy group $G_x = A$. Then the $g(A)$ defined above is precisely

$\langle A, X \rangle$, so the definition of μ_G implies that

$$f(A) = \sum_{A \leq B} \mu_G(A, B) \cdot \langle B, X \rangle.$$

But the contribution to $f(A)$ from each orbit in X is either 0 (if the orbit is G/U with U not conjugate to A) or $|N(A)/A|$ (if the orbit is G/A), so $|N(A)/A|$ divides $f(A)$. In particular, if we take $A = (e)$ so $|N(A)/A| = |G|$, and if we insert the values of μ_G from Weisner's theorem, we find that $|G|$ divides

$$\sum_{\text{all } B} \mu_G((e), B) \cdot \langle B, X \rangle = \sum_r \sum_{B \simeq (\mathbf{Z}/p)^r} (-1)^r p^{r(r-1)/2} \langle B, X \rangle$$

$$= \sum_r \sum_{\substack{B \in \mathbb{S}(G) \\ B \simeq (\mathbf{Z}/p)^r}} (-1)^r p^{r(r-1)/2} |G/N(B)| \langle B, X \rangle. \tag{5}$$

In this sum (5), $\langle (e), X \rangle$ occurs with coefficient 1, so we have produced a principal congruence for G.

3. Natural endomorphisms of rational Burnside algebras. If α: $G_1 \to G_2$ is a group homomorphism, then the transformation α^* of G_2-sets to G_1-sets defined in §1 clearly preserves addition and multiplication, so it defines a homomorphism α^*: $\mathfrak{B}(G_2) \to \mathfrak{B}(G_1)$ of Burnside rings (and similarly for rational Burnside algebras). Thus $\mathfrak{B}$ is a contravariant functor from (finite) groups to commutative rings. We wish to study the natural endomorphisms of this functor, that is, families $\theta = \{\theta_G\}$ where, for each group G, θ_G is a ring-endomorphism of $\mathfrak{B}(G)$ and, for any homomorphism α: $G_1 \to G_2$, $\alpha^* \theta_{G_2} = \theta_{G_1} \alpha^*$. Apart from the identity, only one such θ springs readily to mind, namely $\theta_G(X) = |X| \cdot 1$, that is, X with the trivial action of G (for G-sets X, extended linearly to virtual G-sets), and it is not obvious that others exist. Indeed, we shall show later that, if θ_G is required to take G-sets to G-sets (rather than to virtual G-sets) for all G, then no other θ is possible.

As a first step toward the study of natural endomorphisms of $\mathfrak{B}$, we consider the easier problem of natural endomorphisms of rational Burnside algebras. Much of this material seems to be fairly well known but not in print. Since $\mathfrak{B}(G) \otimes \mathbf{Q}$ is (isomorphic via μ to) $\mathbf{Q}^n$, its endomorphisms are easily found. The primitive idempotents (= standard unit vectors) in $\mathbf{Q}^n$ must be mapped to a family of orthogonal idempotents with sum 1. So we easily see that any endomorphism θ_G of $\mathfrak{B}(G) \otimes \mathbf{Q}$ is definable by

$$\langle U, \theta_G(X) \rangle = \langle \bar{\theta}_G(U), X \rangle \tag{6}$$

for some function $\bar{\theta}_G$: $\mathbb{S}(G) \to \mathbb{S}(G)$. An arbitrary $\bar{\theta}_G$ produces an endomorphism by (6), but naturality of $\{\theta_G\}$ imposes conditions on $\{\bar{\theta}_G\}$.

THEOREM 2. *θ, defined by* (6), *is a natural endomorphism of rational Burnside algebras if and only if*

(a) *$\bar{\theta}_G(U)$ is conjugate, in G, to $\bar{\theta}_U(U)$, and*

(b) *if α: $G \twoheadrightarrow H$ is an epimorphism of groups, then $\alpha(\bar{\theta}_G(G))$ is conjugate, in H, to $\bar{\theta}_H(H)$.*

PROOF. Observe first that, for any homomorphism α: $G \to H$, any H-set X, and any $U \leqslant G$, we have

$$\langle U, \alpha^*(X)\rangle = \langle \alpha(U), X\rangle. \tag{7}$$

Now assume θ is natural and compute, with notation as above,

$$\begin{aligned}
\langle \alpha(\bar{\theta}_G(G)), X\rangle &= \langle \bar{\theta}_G(G), \alpha^*(X)\rangle && \text{by (7)}\\
&= \langle G, \theta_G\alpha^*(X)\rangle && \text{by (6)}\\
&= \langle G, \alpha^*\theta_H(X)\rangle && \text{by naturality}\\
&= \langle \alpha(G), \theta_H(X)\rangle && \text{by (7)}\\
&= \langle \bar{\theta}_H(\alpha(G)), X\rangle && \text{by (6).}
\end{aligned}$$

Therefore, by [3, p. 214], $\alpha(\bar{\theta}_G(G))$ is conjugate in H to $\bar{\theta}_H(\alpha(G))$. If we apply this result in the case where $G \leqslant H$ and α is the inclusion, we obtain (a); if we apply it in the case where α is an epimorphism, we obtain (b).

Conversely, assume (a) and (b), and compute for any $U \leqslant G$, any homomorphism α: $G \to H$, and any H-set X,

$$\begin{aligned}
\langle U, \theta_G\alpha^*(X)\rangle &= \langle \bar{\theta}_G(U), \alpha^*(X)\rangle && \text{by (6)}\\
&= \langle \bar{\theta}_U(U), \alpha^*(X)\rangle && \text{by (a)}\\
&= \langle \alpha(\bar{\theta}_U(U)), X\rangle && \text{by (7)}\\
&= \langle \bar{\theta}_{\alpha(U)}(\alpha(U)), X\rangle && \text{by (b)}\\
&= \langle \bar{\theta}_H(\alpha(U)), X\rangle && \text{by (a)}\\
&= \langle \alpha(U), \theta_H(X)\rangle && \text{by (6)}\\
&= \langle U, \alpha^*\theta_H(X)\rangle && \text{by (7).}
\end{aligned}$$

By [3, Lemma 1], we conclude $\theta_G\alpha^*(X) = \alpha^*\theta_H(X)$. □

We usually think of $\bar{\theta}_G(U)$ as being defined only up to conjugacy in G, since replacing it with a conjugate would have no effect in (6). Thus, for natural θ, part (a) of Theorem 2 permits us to drop the subscript of $\bar{\theta}$, and part (b) says that $\bar{\theta}$ commutes with epimorphisms.

If $\bar{\theta}(G)$ is normal in G, for all G, then we have the following description of $\bar{\theta}$.

THEOREM 3. *Let $\mathcal{K}$ be a class of finite groups closed under formation of homomorphic images and subdirect products (of pairs). Then every group G has a smallest normal subgroup N such that $G/N \in \mathcal{K}$. The operator $\bar{\theta}$ sending G to N commutes with epimorphisms. Conversely, any $\bar{\theta}$ that commutes with epimorphisms and sends every group to a normal subgroup is obtained in this way from such a class $\mathcal{K}$.*

PROOF. If G/N_1 and G/N_2 are in $\mathcal{K}$, then so is their subdirect product $G/(N_1 \cap N_2)$; the existence of the required N follows immediately. If α is an epimorphism from G to H, then, for any normal subgroup N of H,

$$N \supseteq \bar{\theta}(H) \Leftrightarrow H/N \in \mathcal{K} \Leftrightarrow G/\alpha^{-1}(N) \in \mathcal{K}$$
$$\Leftrightarrow \alpha^{-1}(N) \supseteq \bar{\theta}(G) \Leftrightarrow N \supseteq \alpha(\bar{\theta}(G)),$$

where the first three equivalences hold because $\mathcal{K}$ is closed under homomorphic images. Therefore, $\bar{\theta}(H) = \alpha(\bar{\theta}(G))$.

Conversely, if $\bar{\theta}$ commutes with epimorphisms and gives normal subgroups, define $\mathcal{K}$ to be the class of groups G for which $\bar{\theta}(G) = (e)$. Closure of $\mathcal{K}$ under homomorphic images, respectively subdirect products, follows immediately from the fact that $\bar{\theta}$ commutes with the homomorphism, respectively the projections. For any G and any normal subgroup N, we have

$$G/N \in \mathcal{K} \Leftrightarrow \bar{\theta}(G/N) = (e) \Leftrightarrow \bar{\theta}(G) \subseteq N,$$

because $\bar{\theta}$ commutes with the epimorphism $G \twoheadrightarrow G/N$. So $\bar{\theta}(G)$ is the smallest normal subgroup of G with quotient in $\mathcal{K}$. □

The finite groups in any variety of groups form a class $\mathcal{K}$ satisfying the hypothesis of Theorem 3. The associated $\bar{\theta}$ sends each group to the verbal subgroup corresponding to the defining equations of the variety. For example, the variety of abelian groups yields $\bar{\theta}(G)$ = commutator subgroup of G. There are classes $\mathcal{K}$ satisfying the hypotheses of Theorem 3 but not equal to the finite part of any variety, for example the class of p-groups for any fixed prime p, or the class of solvable groups. It is proved in **[1]** that any class satisfying our hypothesis and closed under subgroups is just the union of the finite parts of an increasing sequence of varieties. But our hypothesis does not imply closure under subgroups; for example, the class of homomorphic images of subdirect powers of S_3 does not contain the alternating group A_3.

There are also operators $\bar{\theta}$ that commute with epimorphisms but do not always give normal subgroups. An example is $\bar{\theta}(G)$ = a p-Sylow subgroup of G, for any fixed p. More such operators can be formed by composing one example with operators obtained from Theorem 3, for example the p-Sylow subgroup of the smallest normal N for which G/N is solvable. I know of no good description of these operators.

4. Natural endomorphisms of Burnside rings. In this section, we present our main existence theorem for natural endomorphisms of Burnside rings. Clearly, any such endomorphism θ extends to a unique natural endomorphism of rational Burnside algebras and is therefore given, as in the preceding section, by

$$\langle U, \theta_G(X)\rangle = \langle \bar{\theta}(U), X\rangle, \tag{6}$$

where $\bar{\theta}$ commutes with epimorphisms. For example, the θ that sends every G-set X to the G-set $|X| \cdot 1$, consisting of X with the trivial G-action, corresponds to the $\bar{\theta}$ that sends every group to the trivial group (e). Unfortunately, the more interesting $\bar{\theta}$'s do not generally give endomorphisms of $\mathfrak{B}(G)$; instead, they send elements of $\mathfrak{B}(G)$ to vectors in $\mathbf{Z}^n$ that are not in $\mathfrak{B}(G)$. For example, the "commutator subgroup" operator $\bar{\theta}$ corresponds to a θ that sends the transitive 3-element S_3-set (represented by the vector (3, 1, 0, 0)) to the vector (3, 3, 3, 0) which fails to satisfy the second of the defining congruences (2) for $\mathfrak{B}(S_3)$, $X_3 \equiv X_4 \bmod 2$.

We call an endomorphism of $\mathfrak{B}(G) \otimes \mathbf{Q}$ *integral* if it maps $\mathfrak{B}(G)$ into itself, i.e., if it is (the canonical extension of) an endomorphism of $\mathfrak{B}(G)$. We call it *p-integral*, for a prime p, if it maps $\mathfrak{B}(G)$ to vectors that satisfy the (H, p)-congruences (3) for this fixed p and for all $H \in \mathbb{S}(G)$. Thus, Theorem 1 tells us that an endomorphism is p-integral for all p if and only if it is integral. We seek conditions on $\bar{\theta}$ which imply that the endomorphisms θ_G defined by (6) are integral for all G. The following theorem gives such a condition, general enough to cover all known examples.

THEOREM 4. *Let θ be a natural endomorphism of rational Burnside algebras, and let p be a prime. Assume that, for any group G and any normal subgroup H of index p in G, $\bar{\theta}(H) = H \cap U$ for some conjugate U of $\bar{\theta}(G)$ in G. Then, for all G, θ_G is p-integral.*

PROOF. We must show that, for any G-set X and any subgroup H of G, $\theta_G(X)$ satisfies the (H, p)-congruence (3). In view of (6), this reduces to showing that X satisfies the congruence obtained from the (H, p)-congruence by changing each term $\langle U', X\rangle$ to $\langle \bar{\theta}(U'), X\rangle$. This new congruence will be referred to as the *desired congruence* throughout this proof. The proof splits into two cases according to what $\bar{\theta}$ does to the cyclic group C of order p.

Case 1. $\bar{\theta}(C) = (e)$. Whenever H is normal in G with index p, the projection $G \twoheadrightarrow G/H \cong C$ must send $\bar{\theta}(G)$ to $\bar{\theta}(C) = (e)$, as $\bar{\theta}$ commutes with epimorphisms. So $\bar{\theta}(G) \subseteq H$. As H is normal, all conjugates of $\bar{\theta}(G)$ lie inside H, and the hypothesis of the theorem therefore implies that $\bar{\theta}(H)$ and $\bar{\theta}(G)$ are conjugate in G. The same holds whenever H is normal in G with index a power of p, by induction on the index using a composition series between G and H.

Looking at the way the (H, p)-congruence was obtained in §2, we see that, in every term $\langle U', X\rangle$, U' is such that H is normal in U' and $|U'/H|$ is a power of p (for $U' \leqslant S' \leqslant N(H)$ and $S'/H \cong S$ is a p-group). By the preceding paragraph, $\bar{\theta}(U')$ is conjugate (in U') to $\bar{\theta}(H)$, so all the terms $\langle\bar{\theta}(U'), X\rangle$ in the desired congruence are the same. We need only check, therefore, that the sum of the coefficients of the (H, p)-congruence is divisible by the modulus, p^k, of that congruence. But this is clear, since the (H, p)-congruence is satisfied by the element 1 of $\mathfrak{B}(G)$.

Case 2. $\bar{\theta}(C) = C$. We prove three preliminary facts.

(a) For all p-groups P, $\bar{\theta}(P) = P$. We proceed by induction on $|P|$, the result being trivial for $|P| = 1$. For nontrivial p-groups P, there is a normal subgroup H of index p. The projection $P \twoheadrightarrow P/H \cong C$ sends $\bar{\theta}(P)$ onto $\bar{\theta}(C) = C$, so H and $\bar{\theta}(P)$ together generate P. The same is true if we replace $\bar{\theta}(P)$ by a conjugate U, because H is normal. On the other hand, the induction hypothesis and the hypothesis of the theorem give $H = \bar{\theta}(H) = H \cap U$, so $H \subseteq U$, for some such U. Therefore, $U = P$ and $\bar{\theta}(P) = P$, as claimed.

(b) For any group G, let G^* be the smallest normal subgroup whose index is a power of p. (It exists; see the proof of Theorem 3.) Then G^* and $\bar{\theta}(G)$ together generate G. This follows immediately from the fact that $\bar{\theta}$ commutes with the epimorphism $G \twoheadrightarrow G/G^*$ and the fact that $\bar{\theta}(G/G^*) = G/G^*$ by (a).

(c) If $G^* \leqslant H \leqslant G$, then $\bar{\theta}(H) = H \cap U$ for some U conjugate to $\bar{\theta}(G)$ in G. This follows easily from the hypothesis of the theorem (which is the special case where H is normal of index p in G) by induction on $|G/H|$, using the fact that, since G/G^* is a p-group, H is normal of index p in some subgroup of G (unless $H = G$).

Using the same notation as in the derivation of the (H, p)-congruence, consider any term $\langle U', X\rangle$ in that congruence and the corresponding term $\langle\bar{\theta}(U'), X\rangle$ in the desired congruence. Since $S'/H = S$ is a p-group, (b) tells us that $\bar{\theta}(S')$ and H together generate S', and (c) tells us that we may assume $\bar{\theta}(S') \cap H = \bar{\theta}(H)$. (It may be necessary to replace $\bar{\theta}(S')$ with a conjugate in S', but this is harmless.) Thus, $\bar{\theta}(S')/\bar{\theta}(H) \cong S'/H = S$. We know, by (c), that $\bar{\theta}(U')$ is the intersection of U' with some conjugate in S' of $\bar{\theta}(S')$; replacing U by a conjugate in S, if necessary, we assume from now on that $\bar{\theta}(U')$ is conjugate (in U') to $U' \cap \bar{\theta}(S')$. Arguing as in the derivation of the (H, p)-congruence, we let Z be the set of $\bar{\theta}(H)$-fixed points in X. It is permuted by $\bar{\theta}(S')$ as $\bar{\theta}(H) = H \cap \bar{\theta}(S')$ is normal in $\bar{\theta}(S')$. So we may view Z as a $\bar{\theta}(S')/\bar{\theta}(H)$-set and thus as an S-set. That Z satisfies the principal congruence of S implies that X satisfies a congruence which is like the (H, p)-congruence but with each term $\langle U', X\rangle$ replaced by $\langle U'', X\rangle$, where U'' is the preimage of $U \leqslant S \cong \bar{\theta}(S')/\bar{\theta}(H)$ in $\bar{\theta}(S')$. But this U'' is just

$U' \cap \bar{\theta}(S')$ which is conjugate to $\bar{\theta}(U')$. So the congruence involving $\langle U'', X \rangle$, which we know to be satisfied by X, is equivalent, term by term, to the desired congruence. □

COROLLARY 4a. *For each prime p, the $\bar{\theta}$ that sends every group to a p-Sylow subgroup defines an integral θ.* □

COROLLARY 4b. *If $\bar{\theta}$ commutes with epimorphisms and if $\bar{\theta}(G)$ depends, up to conjugacy, only on the smallest normal subgroup of G with solvable quotient, then the associated θ is integral.* □

COROLLARY 4c. *The number of natural endomorphisms of Burnside rings is the cardinality of the continuum.*

PROOF. A natural endomorphism is completely determined when $\bar{\theta}(G)$ is specified for one G in every isomorphism class. There are only countably many nonisomorphic G's, and each one has only finitely many subgroups to serve as $\bar{\theta}(G)$, so the cardinality of the continuum is an upper bound. On the other hand, for each set $\mathcal{Q}$ of (isomorphism classes of) finite simple groups, we can define $\bar{\theta}(G)$ to be the smallest normal subgroup N of G such that every composition factor of G/N is abelian or in $\mathcal{Q}$. This is well defined and commutes with epimorphisms, by Theorem 3 and the Jordan-Hölder theorem. The corresponding natural endomorphism θ of rational Burnside algebras is integral by Corollary 4b. As there are infinitely many nonisomorphic finite simple groups, the number of choices for $\mathcal{Q}$ is the cardinality of the continuum, and different $\mathcal{Q}$'s give different θ's. □

COROLLARY 4d. *The monoid of natural endomorphisms of Burnside rings is not commutative.*

PROOF. The natural endomorphisms θ and η defined by
$\bar{\theta}(G)$ = smallest normal N with G/N solvable and
$\bar{\eta}(G)$ = 2-Sylow subgroup of G
do not commute, because, for the alternating group A_5, $\bar{\eta}\bar{\theta}(A_5)$ has order 4 while $\bar{\theta}\bar{\eta}(A_5)$ is trivial. □

Corollary 4d should be contrasted with the result of End [4] that the monoid of natural endomorphisms of representation rings is commutative. Also, the proof of Corollary 4c shows that any complete description of the monoid of natural endomorphisms of Burnside rings would have to involve knowledge of all finite simple groups. In particular, questions about commutation of natural endomorphisms of the sort used in Corollary 4c lead to questions about existence of composition series in which the composition factors occur in a prescribed order. Since such questions seem entirely intractable, especially if some of the composition factors are as yet unknown simple groups, there is little hope for a full description of the monoid of natural endomorphisms of Burnside rings analogous to the work of End [4] for representation rings. There are, however, two situations in which natural endomorphisms of Burnside rings behave well enough to admit a satisfactory description. One is that the Burnside ring functor is restricted to abelian groups; the results here are quite similar to those of [4]. The other is that the endomorphisms are required to take G-sets to G-sets (rather than to virtual G-sets); this requirement turns out to be very strong. The following two sections are devoted to the study of these two situations.

5. The abelian case. In this section, we consider the Burnside ring functor restricted to the category of finite abelian groups, and we obtain a complete description of its natural endomorphisms. Since all groups in this section are abelian, we use additive rather than multiplicative notation. As before, every natural endomorphism of Burnside rings is given by

$$\langle U, \theta_G(X)\rangle = \langle \bar{\theta}(U), X\rangle \tag{6}$$

for some θ that commutes with epimorphisms. A major simplification produced by our restriction to the abelian case is that we need not worry about integrality questions as in §4.

THEOREM 5. *Let θ be a natural endomorphism of rational Burnside algebras, and let G be an abelian group. Then θ_G is integral.*

PROOF. If we view G as a G-set by left multiplication, its marks are all zero except $\langle (e), G\rangle = |G|$. Therefore, by (6), all the marks of $\theta_G(G)$ are zero or $|G|$. Since the defining congruences of $\mathfrak{B}(G)$ have moduli dividing $|G|$, it follows that $\theta_G(G) \in \mathfrak{B}(G)$. Now, if H is a normal subgroup of G, and π: $G \to G/H$ is the projection, then the G-set G/H is π^* of the (G/H)-set G/H, so, by naturality of θ,

$$\theta_G(G/H) = \theta_G\pi^*(G/H) = \pi^*\theta_{G/H}(G/H) \in \pi^*(\mathfrak{B}(G/H)) \subseteq \mathfrak{B}(G).$$

If G is abelian, then every G-set X is a finite sum of such G/H's, so $\theta_G(X) \in \mathfrak{B}(G)$. □

Thus, the classification of natural endomorphisms θ of Burnside rings of abelian groups reduces to the classification of the associated operators $\bar{\theta}$. Consider any such $\bar{\theta}$, commuting with epimorphisms.

Temporarily fix a prime p, and let n be the largest integer for which $\bar{\theta}(\mathbf{Z}/p^n)$ is trivial. If $\bar{\theta}(\mathbf{Z}/p^n)$ is trivial for all n, formally set $n = \infty$, and note that equation (8) below holds. If n is finite, then the fact that $\bar{\theta}$ commutes with the epimorphisms $\mathbf{Z}/p^{k+1} \to \mathbf{Z}/p^k$ for all $k \geqslant n$ implies $\bar{\theta}(\mathbf{Z}/p^k) \cong \mathbf{Z}/p^{k-n}$. Therefore, for all cyclic p-groups G,

$$\bar{\theta}(G) = p^nG \underset{\text{def}}{=} \{x \in G \mid x \text{ is divisible by } p\ n \text{ times}\}. \tag{8}$$

For noncyclic abelian p-groups $G = G_1 \times \cdots \times G_r$, where the G_i are cyclic of order p^{k_i}, we use the commutativity of $\bar{\theta}$ with the projections to infer that

$$\bar{\theta}(G) \subseteq (p^n G_1) \times \cdots \times (p^n G_r) = p^n G$$

and that the projections of $\bar{\theta}(G)$ to $p^n G_i$ are surjective. In particular, $\bar{\theta}(G)$ is trivial if and only if all k_i are $\leqslant n$. Since $\bar{\theta}$ commutes with the epimorphism $G \to G/\bar{\theta}(G)$, it follows that each cyclic factor of $G/\bar{\theta}(G)$ has order $\leqslant p^n$, so $p^n G \subseteq \bar{\theta}(G)$. Therefore, (8) holds for all abelian p-groups.

Now let p vary, and write $n(p)$ for what was previously called n. Every finite abelian group G is a product of p_i-groups G_i (with distinct p_i). Applying the previous result (8) to these factors, and using the commutativity of $\bar{\theta}$ with the projections from G to the factors, we find that $\bar{\theta}(G)$ is a subdirect product of the groups $\bar{\theta}(G_i) = p_i^{n(p_i)} G_i$. Since the orders of these groups are relatively prime, the only subdirect product is the direct product, so

$$\bar{\theta}(G) = \prod_i (p_i^{n(p_i)} G_i) = \Big(\prod_p p^{n(p)}\Big) G$$
$$\underset{\text{def}}{=} \{x \in G \,|\, x \text{ is divisible by each prime } p \; n(p) \text{ times}\}. \tag{9}$$

Observe that our definition of $(\prod_p p^{n(p)})G$ agrees with the usual definition when all the $n(p)$ are finite and all but finitely many are zero, so that the product exists. In general, for any particular group G the formal infinite product is equivalent to a finite subproduct, since $n(p)$ can be replaced by the minimum of $n(p)$ and the exponent of p in $|G|$.

This calculation shows that the natrual endomorphisms of Burnside rings of abelian groups are in canonical one-to-one correspondence with the functions $n(p)$ from the set P of primes to $J = \{0, 1, 2, \ldots, \infty\}$. Composition of endomorphisms corresponds to componentwise addition of functions, with the usual convention $x + \infty = \infty$.

Following End [4], we metrize the set of natural endomorphisms by $d(\theta, \theta') = 2^{-q}$ where $q = \min\{|G|: \bar{\theta}(G) \neq \bar{\theta}'(G)\}$. The correspondence with J^P becomes a homeomorphism if we first topologize J so that ∞ is the limit of the sequence $1, 2, \ldots$ and so that all points $\neq \infty$ are isolated and then give J^P the product topology.

For comparison with the analogous results for character rings in [4], we define ψ_m to be the natural endomorphism given by $\bar{\psi}_m(G) = mG$. Then $\psi_m \circ \psi_n = \psi_{mn}$, and $\{\psi_m: m = 1, 2, \ldots\}$ is dense in the space of natural endomorphisms. Thus, this space may be described as the completion of the multiplicative monoid of positive integers with respect to the metric

$$d(m, m') = 2^{-q} \quad \text{where } q = \min\{|G|: mG \neq m'G\} = \text{the smallest prime power dividing one of } m, m' \text{ but not both.}$$

This metric is smaller than the corresponding one in [4], and the topological monoid of natural endomorphisms of Burnside rings of abelian groups is a quotient of the topological monoid of natural endomorphisms of character rings. A typical element of the latter is specified by giving, for each prime p, a p-adic integer z, while the corresponding element of the former is specified by giving, for each p, the largest n such that p^n divides z. Thus, the Burnside situation is essentially the character situation modulo p-adic units.

It should be noted that the homomorphism λ from Burnside rings to character rings respects ψ, i.e., $\lambda \circ \psi_m = \psi_m \circ \lambda$, where the first ψ is the one defined here and the second is the usual Adams operation for character rings as used in [4]. It follows immediately that λ similarly intertwines any natural endomorphism of character rings with the corresponding endomorphism of Burnside rings.

Finally, we remark that the ψ_m defined above are the "Adams operations" for a λ-ring structure [5] on the Burnside rings of abelian groups. This should be contrasted with the (unpublished) result of E. Boorman that $\mathfrak{B}(S_3)$ admits no λ-ring structure (contrary to a claim in [5]).

6. Natural endomorphisms preserving actual G-sets. Except for the identity and the endomorphism $X \to |X| \cdot 1$ (induced by the trivial homomorphism $G \to G$), the natural endomorphisms given by Theorem 4 and its corollaries seem to lack simple descriptions in terms of G-sets. We shall prove a theorem implying that these two "trivial" natural endomorphisms are the only ones that send G-sets to actual (not virtual) G-sets. To state this theorem in its full strength, we define an element X of $\mathfrak{B}(G) \otimes \mathbf{Q}$ to be *nonnegative* if, in its expansion as a linear combination of transitive G-sets, all the coefficients $\xi(X)_k$ are nonnegative.

THEOREM 6. *Let θ be a natural endomorphism of rational Burnside algebras. Assume that, for all G, $\theta_G(X)$ is nonnegative whenever X is. Then either θ is the identity or $\theta(X) = |X| \cdot 1$ for all X.*

PROOF. Let $\bar{\theta}$ be the operator that defines θ by (6). We must show that either $\bar{\theta}(G) = G$ for all G or $\bar{\theta}(G) = (e)$ for all G.

Temporarily fix an arbitrary G, and let

$$\mathcal{C} = \{H \leqslant G | \bar{\theta}(H) = (e)\}.$$

As $\mathcal{C}$ is nonempty (for it contains (e)), it has a maximal member K. Set

$$a = |N(K)/K| \quad \text{and} \quad b = |G/K|.$$

Expand $\theta_G(G)$ in terms of transitive G-sets

$$\theta_G(G) = \sum_{i=1}^{n} r_i \cdot G/U_i \tag{10}$$

where the r_i are nonnegative rational numbers. The marks of $\theta_G(G)$ are

$$\langle U_j, \theta_G(G)\rangle = \langle \bar{\theta}(U_j), G\rangle = \begin{cases} |G| & \text{if } U_j \in \mathcal{C}, \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

Thus, for $U_j \notin \mathcal{C}$, r_j must be zero, because otherwise the jth term on the right side of (10) would make a nonzero contribution $r_j|N(U_j)/U_j|$ to $\langle U_j, \theta_G(G)\rangle$ and this contribution cannot be cancelled, as required by (11), since all the coefficients are nonnegative. On the other hand, if k is the subscript such that U_k is conjugate to K, then, applying $\langle U_k, -\rangle$ to (10), we get $|G|$ on the left side by (11), while on the right the jth term contributes 0 if U_j does not include a conjugate of U_k (for then $\langle U_k, G/U_j\rangle = 0$), 0 if U_j properly includes a conjugate of U_k (for then $U_j \notin \mathcal{C}$ by maximality of K, so $r_j = 0$ by the preceding discussion), and $r_k a$ if $j = k$. Therefore, $r_k = |G|/a$.

Next, we apply $\langle (e), -\rangle$ to both sides of (10), obtaining $|G|$ on the left side by (11). On the right, the kth term contributes $r_k|G/K| = |G| \cdot b/a$, while all the other terms make nonnegative contributions. So $a \geqslant b$. By the definition of a and b, it follows that $a = b$ and K is a normal subgroup of G. So the kth term on the right equals the left side, and all the other terms on the right must vanish. Since $\langle (e), G/U_j\rangle = |G/U_j| \neq 0$, we infer that $r_j = 0$ for all $j \neq k$. So (10) reduces to

$$\theta_G(G) = r_k \cdot G/K = (|G|/a) \cdot G/K.$$

Inserting this result into (11), we find that $U_j \in \mathcal{C}$ if and only if $U_j \subseteq K$. (We can ignore conjugation because K is normal.) Thus, for all $H \leqslant G$,

$$\bar{\theta}(H) = (e) \text{ if and only if } H \leqslant K.$$

Apply this result with $G = A_n$, the alternating group of degree $n \geqslant 5$. Since A_n is simple, the corresponding K, call it K_n, is either A_n or (e).

If, for some $n \geqslant 5$, $K_n = A_n$, then $\bar{\theta}(A_n) = (e)$. For $m \geqslant n$, we consider A_n as a subgroup of A_m, and infer $A_n \subseteq K_m$, so $K_m = A_m$. Therefore, $\bar{\theta}(G) = (e)$ for any subgroup G of A_m for any $m \geqslant n$. But this includes all finite groups, so the theorem is proved in this case.

There remains the case that $K_n = (e)$ for all $n \geqslant 5$. By embedding an arbitrary G into a large alternating group, we obtain

$$\bar{\theta}(G) = (e) \text{ if and only if } G = (e). \tag{12}$$

We shall prove that $\bar{\theta}(G) = G$ for all G, thereby completing the proof of the theorem. Suppose the contrary, and let G be a counterexample of smallest possible order. Let X be the G-set $G/\bar{\theta}(G)$. For $H < G$, we have $\bar{\theta}(H) = H$, so $\langle H, \theta_G(X)\rangle = \langle H, X\rangle$, while $\langle G, \theta_G(X)\rangle = \langle \bar{\theta}(G), X\rangle = |N(\bar{\theta}(G))/\bar{\theta}(G)|$. Expand $\theta_G(X)$ as a linear combination of transitive G-sets

$$\theta_G(X) = \sum_{i=1}^{n} s_i \cdot G/U_i \tag{13}$$

with $s_i \geqslant 0$ by hypothesis. If we take the mark of G in both sides of (13), all but the last term on the right (with $U_n = G$) vanish, so

$$\langle G, \theta_G(X)\rangle = s_n.$$

In particular, $s_n > 0$. If we take the mark of any $H \lneq G$ in both sides of (13), we get $\langle H, X\rangle$ on the left, s_n from the last term on the right, and a nonnegative contribution from the rest of the right side. So $\langle H, X\rangle \geqslant s_n > 0$, which means that H is conjugate to a subgroup of $\bar{\theta}(G)$.

We have shown that every proper subgroup of G has a conjugate included in $\bar{\theta}(G)$. If G were not a p-group for any prime p, $\bar{\theta}(G)$ would have to include a p-Sylow subgroup of G for each p, so $\bar{\theta}(G)$ would have to be all of G. Therefore, G is a p-group for some p, and $\bar{\theta}(G)$ is a maximal proper subgroup of G. As proper subgroups of p-groups are properly included in their normalizers, $\bar{\theta}(G)$ is normal in G. Since $\bar{\theta}$ commutes with the epimorphism $G \to G/\bar{\theta}(G)$, we find that $\bar{\theta}(G/\bar{\theta}(G)) = (e)$. By (12), $\bar{\theta}(G) = G$. This contradicts the definition of G and thus completes the proof of the theorem. □

References

1. J. T. Baldwin and J. Berman, *Varieties and finite closure conditions* (preprint).

2. W. Burnside, *Theory of groups of finite order*, Cambridge Univ. Press, 1911.

3. A. Dress, *A characterization of solvable groups*, Math. Z. **110** (1969), 213–217.

4. W. End, *Operationen für den komplexen Charakterring*, Arch. Math. **23** (1972), 370–379.

5. D. Knutson, *λ-rings and the representation theory of the symmetric group*, Lecture Notes in Math., vol. 308, Springer-Verlag, New York, 1973.

6. L. Weisner, *Some properties of prime-power groups*, Trans. Amer. Math. Soc. **38** (1935), 485–492.

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