THE SPACES OF FUNCTIONS OF
FINITE UPPER p-VARIATION

BY

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ABSTRACT. Let Y be a Banach space, $1 \leqslant p < \infty$, and U_p be the semi-normed space of Y-valued Bochner measurable functions of a real variable which have finite upper p-variation. Let $\bar{U}_p$ be the space of U_p-equivalence classes. An averaging operator is defined with the aid of the theory of helixes in Banach spaces, which enables us to show that the spaces $\bar{U}_p$ are Banach spaces, to characterize their members, and to show that they are isometrically isomorphic to Banach spaces of Y-valued measures with bounded p-variation.

1. Introduction. Let $1 \leqslant p < \infty$ and $U_p(\mathbf{R}: Y)$ be the space of functions f on $\mathbf{R}$ to a Banach space Y such that f is Bochner measurable (i.e. f is Borel measurable and the range of f is separable), and

$$\|f\|_p \underset{\mathrm{d}}{=} \varlimsup_{h \to 0+} \frac{1}{2h} \left\{ \int_{-\infty}^{\infty} |f(t + h) - f(t - h)|^p \, dt \right\}^{1/p} < \infty. \quad (1.1)$$

Let

$$[f] \underset{\mathrm{d}}{=} \left\{ g : g \in U_p(\mathbf{R}; Y) \,\&\, \|f - g\|_p = 0 \right\}$$

and

$$\bar{U}_p(\mathbf{R}; Y) \underset{\mathrm{d}}{=} \left\{ [f] : f \in U_p(\mathbf{R}; Y) \right\}.$$

Let $L_p(\mathbf{R}; Y)$, $1 \leqslant p < \infty$, be the usual Lebesgue-Bochner L_p space of functions on $\mathbf{R}$ to Y. The purpose of this paper is to study the spaces $\bar{U}_p(\mathbf{R}; Y)$. We shall show they are Banach spaces for $1 < p < \infty$, then characterize their members, and finally show that they are isometrically isomorphic to certain spaces of Y-valued measures considered by S. Leader [7] and J. J. Uhl, Jr. [11].

As with the work of J. K. Lee [8], our proof of completeness hinges on the theory of helixes in Banach spaces, cf. P. Masani [9], specifically on the theory of the averaging operator A_p which takes a function f in $U_p(\mathbf{R}; Y)$ into

the average vector of the associated helix in $L_p(\mathbf{R}; Y)$. The completeness of $\tilde{U}_p(\mathbf{R}; Y)$ emerges as a corollary of our Main Theorem 4.7 which asserts that $\dot{A}_p([f]) =_d [A_p(f)]$ defines a one-one bounded linear operator on the Banach space $\tilde{U}_p(\mathbf{R}; Y)$ into itself, and that the range of A_p is $L_p(\mathbf{R}; Y) \cap U_p(\mathbf{R}; Y)$.

Another important corollary of our Main Theorem 4.7 is the fact that $U_p(\mathbf{R}; Y) \subset L_1^{\mathrm{loc}}(\mathbf{R}; Y)$, i.e. that every f in $U_p(\mathbf{R}; Y)$ is integrable on bounded intervals $[a, b]$ (Corollary 4.9). This allows us to adapt an argument of Hardy and Littlewood [5, p. 599] to show that if $f \in U_p(\mathbf{R}; Y)$, $1 \leqslant p < \infty$, then $f = \dot{f}$, a.e., where for $p = 1$, $\dot{f}$ is a unique, right continuous function of bounded variation on $\mathbf{R}$ to Y, and where for $1 < p < \infty$, $\dot{f}$ is a unique locally absolutely continuous function on $\mathbf{R}$ to Y (Corollaries 4.10 and 4.11). In case the Banach space Y has the Radon-Nikodym Property, and therefore $\dot{f}'$ exists, we show that $\dot{f}' \in L_p(\mathbf{R}; Y)$, $1 \leqslant p < \infty$ (Theorem 7.3).

To describe our next result, let $\mathcal{R}$ be the ring generated by the bounded intervals $(a, b] \subset \mathbf{R}$, and for each $R \in \mathcal{R}$, Π_R be the class of partitions of R into a finite number of members of $\mathcal{R}$. Define for all functions ξ on $\mathcal{R}$ to Y and $\forall R \in \mathcal{R}$,

$$|\xi|_p(R) =_d \sup_{\pi \in \Pi_R} \left\{ \sum_{E \in \pi} \frac{|\xi(E)|^p}{|\mathrm{Leb}(E)|^{p-1}} \right\}^{1/p} \leqslant \infty. \qquad (1.2)$$

Let $\mathcal{U}_p(\mathcal{R}; Y)$ be the set of functions ξ on $\mathcal{R}$ to Y which are finitely additive, absolutely continuous with respect to Lebesgue measure, and having finite "p-variation", i.e. such that

$$|||\xi|||_p =_d \sup_{R \in \mathcal{R}} |\xi|_p(R) < \infty. \qquad (1.3)$$

It follows as a special case of a general theorem of J. J. Uhl, Jr. [11, Theorem 16] (who follows S. Leader [7, Theorem 13]) that $\mathcal{U}_p(\mathcal{R}; Y)$ is a Banach space under the norm $||| \cdot |||_p$, $1 \leqslant p < \infty$. We shall show that for $1 < p < \infty$, $\tilde{U}_p(\mathbf{R}; Y)$ is isometrically isomorphic to $\mathcal{U}_p(\mathcal{R}; Y)$, and that $\tilde{U}_1(\mathbf{R}; Y)$ is isometrically isomorphic to the space $\mathrm{BVCA}(\mathcal{R}; Y)$ of countably additive measures of bounded variation on $\mathcal{R}$ to Y. The isometry in question is the correspondence $[f] \to \xi_f$ where $\dot{f}$ is the associate of f alluded to above, and ξ_f is the unique finitely additive measure on $\mathcal{R}$ defined by

$$\xi_f(a, b] =_d \dot{f}(b) - \dot{f}(a). \qquad (1.4)$$

For Banach spaces Y with the Radon-Nikodym Property, the Radon-Nikodym Theorem shows that for $1 < p < \infty$, $\tilde{U}_p(\mathbf{R}; Y)$ is in fact isometrically isomorphic to $L_p(\mathbf{R}; Y)$ (Theorem 7.3).

We thus show that unlike the spaces $V_p(\mathbf{R})$ of measurable functions on $\mathbf{R}$ to

C such that

$$\varlimsup_{h \to 0+} \left\{ \frac{1}{2h} \int_{-\infty}^{\infty} |f(t+h) - f(t-h)|^p \, dt \right\}^{1/p} < \infty,$$

considered by J. K. Lee [8], the spaces $\tilde{U}_p(\mathbf{R}; Y)$ are not new Banach spaces. We should recall that for $p = 1$, $U_1(\mathbf{R}; \mathbf{C}) = V_1(\mathbf{R})$, a case which Lee did not consider.

In [5, p. 599] Hardy and Littlewood proved that if the condition (1.1) holds for f on the circle group C ($= \mathbf{R} \mod 2\pi$) with $\int_0^{2\pi}$ replacing $\int_{-\infty}^{\infty}$ and $f \in L_1(C)$, then $f = \dot{f}$, a.e., where for $p = 1$, $\dot{f}$ has bounded variation on $[0, 2\pi]$, and where for $1 < p < \infty$, $\dot{f}$ is absolutely continuous on $[0, 2\pi]$ and such that $\dot{f}' \in L_p(C)$. Our Corollaries 4.10, 4.11 and Theorem 7.3 extend this result from the domain C to the domain $\mathbf{R}$ and from the range $\mathbf{C}$ to the range Y. But these results also improve the Hardy-Littlewood theorem by showing the redundancy of their hypothesis that $f \in L_1(C)$. This improvement is nontrivial, since it is not apparent that the symbol $\int_a^b f(t) \, dt$ is meaningful for a function f in $U_p(\mathbf{R}; Y)$. It is here that our Corollary 4.9, derived from helix theory, is required. We should also point out that our use of helix theory is quite different from Lee's use of this theory. Lee's proof of completeness of $V_p(\mathbf{R})$ hinges on the fact that $\alpha_f / \sqrt{2}$, where α_f is the average vector of the helix in $L_p(\mathbf{R})$ associated with f in $V_p(\mathbf{R})$, is in the equivalence class of f ($1 < p < \infty$). This result fails for $U_p(\mathbf{R}; \mathbf{C})$, $1 \leqslant p < \infty$, and with it Lee's technique, cf. Example 4.6. We find however that $|\alpha_f / \sqrt{2}|_p \leqslant \|f\|_p$, $1 \leqslant p < \infty$, and this fact allows us to get a Cauchy sequence in $L_p(\mathbf{R}; Y)$, viz. $(\alpha_{f_n} / \sqrt{2})_{n=1}^{\infty}$, from a Cauchy sequence $([f_n])_{n=1}^{\infty}$ in $\tilde{U}_p(\mathbf{R}; Y)$. We then show that its limit in $L_p(\mathbf{R}; Y)$ is $\alpha_f / \sqrt{2}$, for some f in $U_p(\mathbf{R}; Y)$, and that $[f_n]$ converges to $[f]$ in $\tilde{U}_p(\mathbf{R}; Y)$.

As a consequence of our completeness result, Theorem 5.1, we prove in §8 a special form of a theorem of Butzer and Berens [2, p. 160].

Methods similar to those used in this paper can be developed for the space $U_p(C; Y)$ of Bochner measurable functions f on the circle group C to Y such that

$$\|f\|_p \underset{\mathrm{d}}{=} \varlimsup_{h \to 0+} \frac{1}{4\pi h} \left\{ \int_0^{2\pi} |f(t+h) - f(t-h)|^p \, dt \right\}^{1/p} < \infty.$$

Results analogous to those obtained for $U_p(\mathbf{R}; Y)$ hold for $U_p(C; Y)$. But we shall not prove this in this paper.

2. Helixes in Banach spaces. We will use definitions and notations of [9].

2.1 DEFINITION. Cf. [9, p. 4]. Let X be a Banach space. We say that $x(\cdot)$ is a *helix in X* iff $x(\cdot)$ is a continuous function on $\mathbf{R}$ to X such that $\forall a, b, t \in \mathbf{R}$

$$x(b + t) - x(a + t) = U(t)\{x(b) - x(a)\}$$

where $\{U(t): t \in \mathbf{R}\}$ is a strongly continuous group of isometries on X onto X.

2.2 TRIVIALITY. Let $x(\cdot)$ be a helix in a Banach space X, and $\phi(\cdot) =_d |x(\cdot) - x(0)|_X$ be its chordal length function. Then

(a) $\phi(\cdot)$ is a nonnegative, subadditive, even, continuous function on $\mathbf{R}$ to $\mathbf{R}$ such that $\phi(0) = 0$.

(b) $\sup_{t \neq 0} |t|^{-1} \phi(t) = \lim_{s \to 0} \phi(s)/|s| < + \infty$.

(c) $\forall t \in \mathbf{R}$, $\phi(t) \leqslant |t| \lim_{s \to 0} \phi(s)/|s| < + \infty$.

PROOF. (a) See [9, Triv. 2.8]. Also $\forall s, t \in \mathbf{R}$,

$$\phi(s + t) \leqslant |x(s + t) - x(s)|_X + |x(s) - x(0)|_X = \phi(t) + \phi(s).$$

For (b) see [6, Theorem 7.11.1]. (c) follows from (b). $\square$

Although not used in the sequel the following theorem should be noted as a preliminary to our Main Theorem 4.7.

2.3 THEOREM. *Let X be a Banach space, $\{U(t): t \geqslant 0\}$ be a strongly continuous semigroup of isometries on X onto X, and D be its infinitesimal generator. Define $\forall x \in X$,*

$$T(x) =_d \int_0^\infty e^{-t}(x - U(t)x)\, dt,$$

where the integral is Lebesgue-Bochner in X. Then

(a) *T is a bounded linear operator on X into X with operator norm $|T| \leqslant 2$.*

(b) *$I - T = (I - D)^{-1}$ is a one-one contraction on X onto $\mathfrak{D}_D$.*[1]

(c) *$T = - D(I - T)$ has domain X and range $\mathfrak{R}_D$.*

PROOF. (a) is obvious from definition of T.

(b) Obviously $I - T = \int_0^\infty e^{-t} U(t)\, dt$. By the theory of the resolvent of the infinitesimal generator (cf. [6, §11.5]), $\int_0^\infty e^{-t} U(t)\, dt = (I - D)^{-1}$ is a one-one contraction on X onto $\mathfrak{D}_D$.

(c) By (a),

$$I = (I - D)(I - T) = (I - T) - D(I - T).$$

Thus $T = - D(I - T)$. Since $I - T$ is onto $\mathfrak{D}_D$, T is onto $\mathfrak{R}_D$. $\square$

3. **The seminorm $\| \cdot \|_p$, and the helix associated with a function in $U_p(\mathbf{R}; Y)$.** Let $1 \leqslant p < \infty$ and Y be an arbitrary Banach space over $\mathbf{F}$ (= $\mathbf{R}$ or $\mathbf{C}$). Let $f(\cdot)$ be a function on $\mathbf{R}$ to Y which is Bochner measurable (i.e. is Borel measurable and has a separable range). Let $|f|_p$ denote the usual $L_p(\mathbf{R}; Y)$ norm, and $\|f\|_p$ be as in (1.1). It is easily seen that $U_p(\mathbf{R}; Y)$ is a vector space over $\mathbf{F}$, and that $\| \cdot \|_p$ is a seminorm on it. Hence $\tilde{U}_p(\mathbf{R}; Y)$, the space of equivalence classes $[f]$, is a normed linear space. For each $h \in \mathbf{R}$ let τ_h be

[1] $\mathfrak{D}_S$ and $\mathfrak{R}_S$ denote the domain and range of an operator S.

translation through h, i.e. $\forall t \in \mathbf{R}$, $(\tau_h f)(t) = f(t + h)$. Let $U_t = \mathrm{Rstr}_{L_p(\mathbf{R}; Y)} \tau_t$, $t \in \mathbf{R}$.[2] For each helix, $x_{(\cdot)}$, in $L_p(\mathbf{R}; Y)$ let $\mathbb{S}_x =_{\mathrm{d}} \mathbb{S}\{x_b - x_a : a, b \in \mathbf{R}\} \subseteq L_p(\mathbf{R}; Y)$.[3] The proofs of the following two results are straightforward, and are left to the reader.

3.1 TRIVIALITY. Let $f(\cdot)$ be Bochner measurable on $\mathbf{R}$ to Y. Then for $1 < p < \infty$,

$$\|f\|_p = \varlimsup_{h \to 0+} \frac{1}{h} |\tau_h f - f|_p \leq +\infty.$$

3.2 LEMMA. Let $1 \leq p < \infty$, $f \in U_p(\mathbf{R}; Y)$, and $x_t =_{\mathrm{d}} \tau_t f - f$, $t \in \mathbf{R}$. Then $x_{(\cdot)}$ is a helix in $L_p(\mathbf{R}; Y)$ with propagator group $\{\mathrm{Rstr}_{\mathbb{S}_x} U_t : t \in \mathbf{R}\}$, and such that $x_0 = 0$. (We shall call it the helix due to f.)

Let $1 \leq p < \infty$, $f \in U_p(\mathbf{R}; Y)$, and $x_{(\cdot)}$ be the helix due to f. Since $x_0 = 0$, the chordal length function of $x_{(\cdot)}$ is $\phi(\cdot) = |\tau_{(\cdot)} f - f|_p$. Thus combining Lemma 3.2, Trivialities 2.2 and 3.1, we have our main lemma:

3.3 MAIN LEMMA. Let $1 \leq p < \infty$ and $f \in U_p(\mathbf{R}; Y)$. Then

$$\|f\|_p = \lim_{h \to 0} \frac{1}{|h|} |\tau_h f - f|_p = \sup_{h \neq 0} \frac{1}{|h|} |\tau_h f - f|_p. \tag{a}$$

$$|\tau_t f - f|_p \leq |t| \, \|f\|_p, \qquad t \in \mathbf{R}. \tag{b}$$

3.4 THEOREM. Let $f(\cdot)$ be Bochner measurable on $\mathbf{R}$ to Y, and such that for each $h \in \mathbf{R}$, $f(t + h) = f(t)$, $t \in \mathbf{R} \setminus N_h$ where N_h is a Lebesgue-negligible set. Then $f(\cdot) = y$, a.e. (Lebesgue) for some $y \in Y$.

PROOF. R. P. Boas, Jr. [1, Lemma 2] has proved this result for Borel measurable, real-valued functions of period 1. We shall show that it also holds for any Bochner measurable f with values in a Banach space Y. By hypothesis, for each $n \in \mathbf{Z}$ there is a Lebesgue-negligible set N_n such that for $t \in \mathbf{R} \setminus N_n$, $f(t + n) = f(t)$. Let

$$N =_{\mathrm{d}} \bigcup_{k \in \mathbf{Z}} \left\{ \left(\bigcup_{n \in \mathbf{Z}} N_n \right) + k \right\} \quad \text{and} \quad g(\cdot) =_{\mathrm{d}} \chi_{\mathbf{R} \setminus N}(\cdot) f(\cdot).^4 \tag{1}$$

Clearly N is Lebesgue-negligible. Also $g(\cdot)$ is Bochner measurable and has period 1. Moreover for each $h \in \mathbf{R}$ and each $t \in \mathbf{R} \setminus (N_h \cup N)$,

$$g(t + h) = f(t + h) = f(t) = g(t) \quad \text{by (1)}. \tag{2}$$

[2] $\mathrm{Rstr}_D F$ denotes the restriction of the function F to domain D. U_t is used to remind the reader that the function on which it acts is in $L_p(\mathbf{R}; Y)$.

[3] $\mathbb{S}(B)$ denotes the closed linear subspace spanned by set B.

[4] $\chi_S(\cdot)$ denotes the indicator function of the set S.

By (2) Boas' result is applicable to $|g(\cdot)|_Y$, and so $|g(\cdot)|_Y = a$, a.e. for some $a \geqslant 0$. It follows that $g(\cdot) \in L_1^{loc}(\mathbf{R};\ Y)$, i.e. $g(\cdot)$ is locally (Lebesgue-Bochner) integrable. We now use Boas' arguments but with Bochner integrals in Y. For each $n \in \mathbf{N}$ and $0 < s < 1$,

$$
n\int_0^{1/n} g(s+t)\, dt = \frac{1}{n}\int_0^n g(s+t)\, dt = \frac{1}{n}\int_s^{n+s} g(t)\, dt
$$

$$
= \frac{1}{n}\left[\int_s^1 + \int_n^{n+s}\right] g(t)\, dt + \frac{1}{n}\int_1^n g(t)\, dt
$$

$$
= \frac{1}{n}\int_0^1 g(t)\, dt + \left(\frac{n-1}{n}\right)\int_0^1 g(t)\, dt
$$

$$
= \int_0^1 g(t)\, dt. \tag{3}
$$

By (3) and an extension of the Lebesgue's Differentiation Theorem, cf. [4, Theorem III.12.8] we have, for a.a. $s \in (0,\ 1)$

$$
g(s) = \lim_{n\to\infty} n\int_0^{1/n} g(s+t)\, dt = \int_0^1 g(t)\, dt. \tag{4}
$$

Since $g(\cdot)$ has period 1, $g(\cdot) = \int_0^1 g(t)\, dt$, a.e. on $\mathbf{R}$, and by (1) $f(\cdot) = \int_0^1 g(t)\, dt$, a.e. $\square$

3.5 LEMMA. *Let* $1 \leqslant p < \infty, f \in U_p(\mathbf{R};\ Y)$, *and* $[f]$ *be the equivalence class of* f *in* $\bar{U}_p(\mathbf{R};\ Y)$, *i.e. the set of all* $g \in U_p(\mathbf{R};\ Y)$ *such that* $\|g - f\|_p = 0$. *Then*

$$
[f] = \{\, g : \exists y \in Y \ni g(\cdot) = f(\cdot) + y, a.e.\,\}.
$$

PROOF. It is sufficient to show that

$$
[0] = \{\, g : \exists y \in Y \ni g(\cdot) = y, a.e.\,\}.
$$

Obviously if $g(\cdot) = y$, a.e., then $g \in [0]$. Next let $g \in [0]$. By Main Lemma 3.3, $0 = \|g\|_p = \sup_{h\neq 0}|\tau_h g - g|_p/|h|$. Hence $|\tau_h g - g|_p = 0$, $\forall h \in \mathbf{R}$. Thus for each $h \in \mathbf{R}$, $g(t+h) = g(t)$, a.e. It follows from Theorem 3.4 that there exists $y \in Y$ such that $g(\cdot) = y$. a.e. $\square$

4. The averaging operator.

4.1 DEFINITION. Let for each $1 \leqslant p < \infty$ and for all $f \in U_p(\mathbf{R};\ Y)$, $A_p(f) =_d \int_0^\infty e^{-t}(f - \tau_t f)\, dt$, where the integral is Lebesgue-Bochner in Banach space $L_p(\mathbf{R};\ Y)$. Then A_p is called the *averaging operator*.

For each $f \in U_p(\mathbf{R};\ Y)$, $1 \leqslant p < \infty$, $A_p(f) = \alpha_x/\sqrt{2}$ where α_x is the average vector of the helix $x_{(\cdot)}$ due to f [9, Definition 3.6]. By the theory of helixes [9, Theorem 3.15], cf. 3.2, $y_t =_d U_t(A_p(f))$, $t \in \mathbf{R}$, defines a stationary curve in $L_p(\mathbf{R};\ Y)$ and

$$
x_t = y_t - \left(y_0 - x_0 + \int_0^t y_s\, ds\right), \qquad t \in \mathbf{R}. \tag{4.2}
$$

In terms of f and $g =_d A_p(f)$, (4.2) becomes

$$\tau_t f - f = \tau_t g - g - \int_0^t U_s(g)\, ds, \qquad t \in \mathbf{R}. \tag{4.3}$$

4.4 LEMMA. *Let* $1 \leqslant p < \infty$ *and* $f \in U_p(\mathbf{R}; Y)$. *Then for a.a.* $u \in \mathbf{R}$,

$$A_p(f)(u) = \int_0^\infty e^{-t}(f(u) - f(u + t))\, dt = f(u) - \int_0^\infty e^{-t}(f(u + t))\, dt$$

where the integrals are Lebesgue-Bochner in Y.[5]

PROOF. By Lemma 3.2 and [9, Lemma 3.5],

$$F(t) \underset{d}{=} e^{-t}(f - \tau_t f)\chi_{[0,\infty)}(t), \qquad t \in \mathbf{R},$$

is Lebesgue-Bochner integrable in the Banach space $L_p(\mathbf{R}, Y)$. Clearly

$$g(u, t) \underset{d}{=} e^{-t}(f(u) - f(u + t))\chi_{[0,\infty)}(t), \qquad u, t \in \mathbf{R},$$

is Borel measurable and $g(u, t) = F(t)(u)$, a.e. with respect to Lebesgue measure over $\mathbf{R}^2$. Since $\mathscr{R}_f$ is separable and $\mathscr{R}_g \subseteq \mathfrak{S}(\mathscr{R}_f)$, $\mathscr{R}_g$ is separable and g is Bochner measurable.[6] Hence by [4, Theorem III.11.17],

$$A_p(f)(u) \underset{d}{=} \int_0^\infty e^{-t}(f - \tau_t f)\, dt(u) = \int_0^\infty e^{-t}(f(u) - f(u + t))\, dt, \quad \text{a.e.}$$

Clearly $\int_0^\infty e^{-t}f(u)\, dt = f(u)$, $u \in \mathbf{R}$, so that 4.4 holds. $\qquad\square$

4.5 LEMMA. *Let* $1 \leqslant p < \infty$ *and* $f \in U_p(\mathbf{R}; Y)$. *Then*
(a) $|A_p(f)|_p \leqslant \|f\|_p$.
(b) $\|A_p(f) - f\|_p = |A_p(f)|_p$.
(c) $\|A_p(f)\|_p \leqslant |A_p(f)|_p + \|f\|_p \leqslant 2\|f\|_p$.
(d) $\|f\|_p \leqslant |A_p(f)|_p + \|A_p(f)\|_p$.

PROOF. (a) We have

$$|A_p(f)|_p = \left| \int_0^\infty e^{-t}(\tau_t f - f)\, dt \right|_p \leqslant \int_0^\infty e^{-t}|\tau_t f - f|_p\, dt$$

$$\leqslant \|f\|_p \int_0^\infty e^{-t}t\, dt = \|f\|_p \quad \text{by Lemma 3.3(b).}$$

(b) By (4.3) using $g =_d A_p(f)$,

$$\tau_t f - f = \tau_t g - g - \int_0^t U_s(g)\, ds, \qquad t \in \mathbf{R}.$$

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[5]The existence a.e. of the last integral can be used to directly show that f is locally integrable. We obtain a stronger result in 4.9.

[6]$\mathscr{R}_F$ denotes the range of the function F.

Hence,

$$(\tau_t - 1)(g - f) = \int_0^t U_s(g)\,ds,$$

in Banach space $L_p(\mathbf{R}; Y)$. It follows that

$$\lim_{t \to 0} \frac{1}{t}(\tau_t - 1)(g - f) = g,$$

in the $L_p(\mathbf{R}; Y)$ topology. Hence,

$$\|g - f\|_p = \lim_{t \to 0} \frac{1}{|t|}|(\tau_t - 1)(g - f)|_p = |g|_p.$$

(c) follows from (b) and (a). (d) follows from (b). □

The following example gives an $f \in U_p(\mathbf{R}; Y)$ $(1 \leqslant p < \infty)$ whose equivalence class, $[f]$, is disjoint from $L_p(\mathbf{R}; Y)$.

4.6 EXAMPLE. Let $1 \leqslant p < \infty$, $y \in Y$ such that $|y|_Y = 1$, and $f(t) =_d ty\chi_{[0,1)}(t) + y\chi_{[1,\infty)}(t)$, $t \in \mathbf{R}$. Then for $0 < h < 1$,

$$|\tau_h f - f|_p^p = \frac{2}{p+1}h^{p+1} + h^p(1-h)$$

and

$$\|f\|_p^p = \overline{\lim_{h \to 0+}}\,(1/h)^p|\tau_h f - f|_p^p = 1$$

however for each $y_0 \in Y$, $f(\cdot) + y_0$ is not in $L_p(\mathbf{R}; Y)$. Thus $[f] \cap L_p(\mathbf{R}; Y) = \varnothing$.

4.7 MAIN THEOREM. *Let* $1 \leqslant p < \infty$ *and* $\tilde{A}_p([f]) =_d [A_p(f)]$ *for all* $[f] \in \tilde{U}_p(\mathbf{R}; Y)$. *Then*

(a) $\tilde{A}_p$ *is a one-one bounded linear operator on* $\tilde{U}_p(\mathbf{R}; Y)$ *into itself with operator norm* $|\tilde{A}_p| \leqslant 2$.

(b) *The range of* A_p *is* $L_p(\mathbf{R}; Y) \cap U_p(\mathbf{R}; Y)$.

PROOF. (a) It follows easily from Lemma 3.2 and Definition 4.1 that A_p is linear on $U_p(\mathbf{R}; Y)$ into $L_p(\mathbf{R}; Y)$. By Lemma 4.5(c), $\forall f \in U_p(\mathbf{R}; Y)$, $\|A_p(f)\|_p \leqslant 2\|f\|_p < +\infty$. This shows that the range $\mathfrak{R}_{A_p}$ of A_p satisfies

$$\mathfrak{R}_{A_p} \subseteq L_p(\mathbf{R}; Y) \cap U_p(\mathbf{R}; Y). \tag{1}$$

From Lemma 3.5 it is obvious that $A_p(g) = 0$, $\forall g \in [0]$. Therefore $\tilde{A}_p$ is a single valued bounded linear operator with $|\tilde{A}_p| \leqslant 2$.

Let $\mathfrak{N}$ be the null space of A_p. It suffices to show that $\mathfrak{N} = \{[0]\}$. Clearly $\{[0]\} \subseteq \mathfrak{N}$. Next let $[f] \in \mathfrak{N}$. Since $\mathfrak{R}_{A_p} \subseteq L_p(\mathbf{R}; Y)$, $A_p(f) \in [0] \cap L_p(\mathbf{R}; Y)$. Thus by Lemma 3.5, $A_p(f)(\cdot) = 0$, a.e. By Lemma 4.5(b),

$$\|f\|_p = \|f - A_p(f)\|_p = |A_p(f)|_p = 0.$$

Therefore $\mathfrak{N} \subseteq \{[0]\}$. Thus $\mathfrak{N} = \{[0]\}$ and (a) is proved.

(b) Let $g \in L_p(\mathbf{R}; Y) \cap U_p(\mathbf{R}; Y)$. In view of (1) we have only to show that there exists an $f \in U_p(\mathbf{R}; Y)$ such that $A_p(f) = g$. Since $g \in L_p(\mathbf{R}; Y)$, g is locally Lebesgue-Bochner integrable in Y. Let $G(u) =_d \int_0^u g(s) \, ds$, $u \in \mathbf{R}$. Then

$$\forall u, t \in \mathbf{R}, \qquad G(u + t) - G(u) = \int_u^{u+t} g(s) \, ds = \int_0^t g(u + s) \, ds. \quad (2)$$

Hence by Hölder inequality [4, Lemma III.3.2] and Tonelli's Theorem [4, Theorem III.11.14],

$$|\tau_t G - G|_p^p \leqslant \int_{-\infty}^{\infty} \left\{ \int_0^t 1 \cdot |g(u + s)| \, ds \right\}^p du$$

$$\leqslant \int_{-\infty}^{\infty} \left\{ \left(\int_0^t ds \right)^{(p-1)/p} \left(\int_0^t |g(u + s)|^p \, ds \right)^{1/p} \right\}^p du$$

$$\leqslant (t)^{p-1} \int_{-\infty}^{\infty} \left\{ \int_0^t |g(u + s)|^p \, ds \right\} du$$

$$\leqslant (t)^{p-1} \int_0^t \left\{ \int_{-\infty}^{\infty} |g(u + s)|^p \, du \right\} ds$$

$$\leqslant t^p \cdot |g|_p^p \quad \text{for all } t > 0.$$

Therefore by Triviality 3.1,

$$\|G\|_p = \varlimsup_{t \to 0+} \frac{1}{t} |\tau_t G - G|_p \leqslant |g|_p < +\infty. \quad (3)$$

Thus $G \in U_p(\mathbf{R}; Y)$ and $f =_d g - G \in U_p(\mathbf{R}; Y)$.

We will show that $A_p(f) = g$. By Lemma 4.4,

$$A_p(g)(u) = g(u) - \int_0^{\infty} e^{-t} g(u + t) \, dt, \quad \text{a.e.} \quad (4)$$

By Lemma 4.4, (2), the Dirichlet's formula,

$$A_p(G)(u) = (-1) \int_0^{\infty} e^{-t} \left\{ \int_0^t g(u + s) \, ds \right\} dt$$

$$= (-1) \int_0^t \left\{ \int_s^{\infty} e^{-t} \, dt \right\} g(u + s) \, ds$$

$$= (-1) \int_0^{\infty} e^{-s} g(u + s) \, ds, \quad \text{a.e.} \quad (5)$$

Therefore by (4) and (5), $A_p(f) = A_p(g) - A_p(G) = g$.

From the last theorem we shall deduce three corollaries. The first of them establishes the local integrability of functions in $U_p(\mathbf{R}; Y)$ for $1 < p < \infty$.

The second corollary essentially identifies the elements of $U_p(\mathbf{R}; Y)$ for $1 < p < \infty$ with the distribution functions of the measures considered by J. J. Uhl, Jr. [11, p. 28]. The third makes a similar identification of the functions in $U_1(\mathbf{R}; Y)$ with right continuous functions of bounded variation on $\mathbf{R}$ to Y. First we will recall the concepts of local integrability, local absolute continuity and bounded variation.

4.8 DEFINITIONS. (a) $L_p^{loc}(\mathbf{R}; Y)$ will denote the class of Bochner measurable functions f on $\mathbf{R}$ to Y such that for each compact subset K of $\mathbf{R}$, $\chi_K f \in L_p(\mathbf{R}; Y)$.

(b) $AC^{loc}(\mathbf{R}; Y)$ will denote the class of functions f on $\mathbf{R}$ to Y which are *locally absolutely continuous*, i.e. for each finite closed interval $[a, b]$ and for each $\varepsilon > 0$ there exists a $\delta > 0$ such that for all finite collections of disjoint intervals $(a_i, b_i]$ contained in $[a, b]$,

$$\sum_{i=1}^{n} (b_i - a_i) < \delta \Rightarrow \sum_{i=1}^{n} |f(b_i) - f(a_i)|_Y < \varepsilon.$$

(c) Let f be a function on $\mathbf{R}$ to Y. Let B be a Borel set in $\mathbf{R}$. Let Π'_B denote the collection of all finite sequences $\pi = (b_k)_{k=0}^n$ in B such that $b_0 \leqslant b_1 \leqslant b_2 \leqslant \cdots \leqslant b_n$. Let $\nu_f(B)$ denote the variation of f over B, i.e.

$$\nu_f(B) \underset{d}{=} \sup_{\pi \in \Pi'_B} \sum_{k=1}^{n} |f(b_k) - f(b_{k-1})|_Y$$

$BV(\mathbf{R}; Y)$ will denote the class of functions f on $\mathbf{R}$ to Y which have *bounded variation* on $\mathbf{R}$, i.e. $\nu_f(\mathbf{R}) < +\infty$.

4.9 COROLLARY. *For* $1 \leqslant p < \infty$,

$$U_p(\mathbf{R}; Y) \subseteq L_p^{loc}(\mathbf{R}; Y) \subseteq L_1^{loc}(\mathbf{R}; Y).$$

PROOF. Let $f \in U_p(\mathbf{R}; Y)$. Let $g \underset{d}{=} A_p(f)$, and $G(u) \underset{d}{=} \int_0^u g(s)\, ds$, $u \in \mathbf{R}$. Then by arguments of part (b) for the proof of Theorem 4.7, $g - G \in U_p(\mathbf{R}; Y)$ and $A_p(g - G) = g$. By Theorem 4.7(a) and Lemma 3.5, $f(\cdot) = g(\cdot) - G(\cdot) + y$, a.e. for some $y \in Y$. By Theorem 4.7(b), $g \in L_p(\mathbf{R}; Y) \subseteq L_p^{loc}(\mathbf{R}; Y)$. Since $G(\cdot)$ is continuous, $G(\cdot) \in L_p^{loc}(\mathbf{R}; Y)$. Therefore $f(\cdot) = g(\cdot) - G(\cdot) + y \in L_p^{loc}(\mathbf{R}; Y)$. $\square$

4.10 COROLLARY. *Let* $1 < p < \infty$, $p' = p/(p - 1)$, *and* $f \in U_p(\mathbf{R}; Y)$. *Then* $f(\cdot) = \dot{f}(\cdot)$, *a.e., for a unique* $\dot{f} \in AC^{loc}(\mathbf{R}; Y)$ *moreover* $\dot{f}$ *satisfies the following inequality for* $a \leqslant b$,

$$|\dot{f}(b) - \dot{f}(a)|_Y \leqslant (b - a)^{1/p'} \lim_{h \to 0+} \frac{1}{h} \left\{ \int_a^b |f(t + h) - f(t)|^p\, dt \right\}^{1/p}.$$

PROOF. Let $f \in U_p(\mathbf{R}; Y)$. Then by Corollary 4.9, $f \in L_1^{loc}(\mathbf{R}; Y)$. Now by an extension of the Lebesgue Differentiation Theorem [4, Theorem III.12.8],

there exists a Lebesgue-negligible set E such that

$$f(a) = \lim_{h \to 0+} \frac{1}{h} \int_a^{a+h} f(t) \, dt, \quad \forall a \in \mathbf{R} \setminus E. \tag{1}$$

Therefore for $a, b \in \mathbf{R} \setminus E$, $a \leqslant b$,

$$f(b) - f(a) = \lim_{h \to 0+} \frac{1}{h} \left[\int_{a+h}^{b+h} - \int_a^b \right] f(t) \, dt$$

$$= \lim_{h \to 0+} \frac{1}{h} \int_a^b f(t+h) - f(t) \, dt. \tag{2}$$

Then using the Hölder inequality $(1/p + 1/p' = 1)$,

$$|f(b) - f(a)|_Y \leqslant \varliminf_{h \to 0+} \frac{1}{h} \int_a^b 1 \cdot |f(t+h) - f(t)| \, dt$$

$$\leqslant (b-a)^{1/p'} \varliminf_{h \to 0+} \frac{1}{h} \left\{ \int_a^b |f(t+h) - f(t)|^p \, dt \right\}^{1/p}$$

$$\leqslant (b-a)^{1/p'} \|f\|_p. \tag{3}$$

Thus f is uniformly continuous on $\mathbf{R} \setminus E$. Since $\mathbf{R} \setminus E$ is dense in $\mathbf{R}$ it follows, cf. [4, p. 23], that f has a unique continuous extension $\dot{f}$ on $\mathbf{R}$. Since $\dot{f}$ is continuous and $\dot{f} = f$, a.e.,

$$\dot{f}(a) = \lim_{h \to 0+} \frac{1}{h} \int_a^{a+h} \dot{f}(t) \, dt, \quad \forall a \in \mathbf{R},$$

$$= \lim_{h \to 0+} \frac{1}{h} \int_a^{a+h} f(t) \, dt, \quad \forall a \in \mathbf{R}. \tag{1$'$}$$

This implies the following inequality in the same way that (1) implies (3): for $a, b \in \mathbf{R}$, $a \leqslant b$,

$$|\dot{f}(b) - \dot{f}(a)|_Y \leqslant (b-a)^{1/p'} \varliminf_{h \to 0+} \frac{1}{h} \left\{ \int_a^b |f(t+h) - f(t)|^p \, dt \right\}^{1/p}. \tag{3$'$}$$

Now let $\{(a_k, b_k)\}_{k=1}^n$ be a finite collection of disjoint intervals contained in a finite interval $[a, b]$. Applying (3)$'$ to each interval $(a_k, b_k]$ and summing from 1 to n we get on applying the Hölder inequality to the sum that

$$\sum_{k=1}^{n} |f(b_k) - f(a_k)|_Y$$

$$\leqslant \sum_{k=1}^{n} \left[(b_k - a_k)^{1/p'} \cdot \varlimsup_{h \to 0+} \frac{1}{h} \left\{ \int_{a_k}^{b_k} |f(t+h) - f(t)|^p \, dt \right\}^{1/p} \right]$$

$$\leqslant \left[\sum_{k=1}^{n} (b_k - a_k) \right]^{1/p'} \cdot \varlimsup_{h \to 0+} \frac{1}{h} \left\{ \sum_{k=1}^{n} \int_{a_k}^{b_k} |f(t+h) - f(t)|^p \, dt \right\}^{1/p}$$

$$\leqslant \left[\sum_{k=1}^{n} (b_k - a_k) \right]^{1/p'} \cdot \|f\|_p. \tag{4}$$

Therefore $\dot{f}$ is locally absolutely continuous. $\quad\square$

4.11 COROLLARY. *Let $p = 1$ and $f \in U_1(\mathbf{R}; Y)$. Then $f(\cdot) = \dot{f}(\cdot)$, a.e., for a unique right continuous function $\dot{f} \in \mathrm{BV}(\mathbf{R}; Y)$; moreover the total variation $v_{\dot{f}}(\mathbf{R})$ of $\dot{f}$ satisfies*

$$v_{\dot{f}}(\mathbf{R}) \leqslant \|f\|_1.$$

PROOF. Let $f \in U_1(\mathbf{R}; Y)$. Then by Corollary 4.9, $f \in L_1^{\mathrm{loc}}(\mathbf{R})$, and as in the previous proof there exists a Lebesgue-negligible set E such that for $a < b$, $a, b \in \mathbf{R} \setminus E$

$$|f(b) - f(a)|_Y \leqslant \varlimsup_{h \to 0+} \frac{1}{h} \int_a^b |f(t+h) - f(t)| \, dt. \tag{1}$$

Let $a < b$ and $(a_k)_{k=0}^n \in \Pi_{(a,b]\setminus E}$. Applying (1) to each cell $(a_{k-1}, a_k]$ and summing from 1 to n we get

$$\sum_{k=1}^{n} |f(a_k) - f(a_{k-1})|_Y \leqslant \varlimsup_{h \to 0+} \frac{1}{h} \int_{a_0}^{a_n} |f(t+h) - f(t)| \, dt \leqslant \|f\|_1. \tag{2}$$

Therefore $v_f((a, b] \setminus E) \leqslant \|f\|_1$, $\forall a < b$, cf. Definition 4.8(c). Let

$$v(s) \underset{\mathrm{d}}{=} \begin{cases} -v_f((s, 0] \setminus E), & s < 0, \\ v_f((0, s] \setminus E), & s \geqslant 0. \end{cases}$$

Then $v(\cdot)$ is monotone increasing and satisfies

$$|f(b) - f(a)|_Y \leqslant v_f((d, b] \setminus E) \leqslant v(b) - v(d) \tag{3}$$

for $a, b \in \mathbf{R} \setminus E$, $d \in \mathbf{R}$ such that $d < a < b$. Clearly $v(\cdot)$ is continuous except on a countable set N, and $\dot{v}(s) =_{\mathrm{d}} \inf\{v(t): s < t \in \mathbf{R} \setminus E\}$ is a monotone increasing, right continuous function such that $v(\cdot) \leqslant \dot{v}(\cdot)$ with $v(s) = \dot{v}(s)$, $\forall s \in \mathbf{R} \setminus N$. Let $E' = E \cup N$. Then for each $a, b \in \mathbf{R} \setminus E'$, $a < b$, we have on letting $d \to a -$ in (3),

$$|f(b) - f(a)|_Y \leqslant \dot{v}(b) - \dot{v}(a), \tag{4}$$

by the continuity of $\dot{v}$ at a. Let $c \in \mathbf{R}$ and $(t_n)_{n=1}^{\infty}$ be any sequence in $\mathbf{R} \setminus E'$ such that $t_n \to c\ +$. By (4) and the right continuity of $\dot{v}$,

$$\left| f(t_n) - f(t_m) \right|_Y \leqslant \left| \dot{v}(t_n) - \dot{v}(t_m) \right| \to 0.$$

Since Y is complete there exists a $y \in Y$ such that $\lim_{n \to \infty} f(t_n) = y$. Moreover if $(t'_m)_{m=1}^{\infty}$ is any other sequence in $\mathbf{R} \setminus E'$ such that $t'_m \to c\ +$, then it is easily seen that $y = \lim_{n \to \infty} f(t'_m)$. Thus $\mathrm{Rstr}_{\mathbf{R} \setminus E'} f(\cdot)$ has a unique continuous extension $\dot{f}$ on $\mathbf{R}$.

Now since $\dot{f} = f$ on $\mathbf{R} \setminus E'$, we have by (2), $\forall (a_k)_{k=0}^n \in \Pi'_{\mathbf{R} \setminus E'}$

$$\sum_{k=1}^{n} \left| \dot{f}(a_k) - \dot{f}(a_{k-1}) \right| \leqslant \|f\|_1. \tag{5}$$

Next let $\pi = (b_k)_{k=1}^n \in \Pi'_{\mathbf{R}}$. Taking in (5) $a_k > b_k$ and letting $a_k \to b_k\ +$ we see from the right continuity of $\dot{f}$ that

$$\sum_{k=1}^{n} \left| \dot{f}(b_k) - \dot{f}(b_{k-1}) \right| \leqslant \|f\|_1. \quad \square$$

5. A direct proof of completeness of $\tilde{U}_p(\mathbf{R}; Y)$. The completeness of $\tilde{U}_p(\mathbf{R}; Y)$, $1 \leqslant p < \infty$, follows readily from our Main Theorem 4.7, and may be regarded as a corollary to it.

5.1 THEOREM. *Let $1 \leqslant p < \infty$ and Y be a Banach space. Then $\tilde{U}_p(\mathbf{R}; Y)$ is a Banach space under norm $\| \cdot \|_p$.*

PROOF. Let $([f_n])_{n=1}^{\infty}$ be a Cauchy sequence in $\tilde{U}_p(\mathbf{R}; Y)$. Set $g_n =_d A_p(f_n)$, $n \in \mathbf{N}$. By Lemma 4.5(a) we have that

$$\forall n, m \in \mathbf{N}, \qquad |g_n - g_m|_p = \left| A_p(f_n - f_m) \right|_p \leqslant \|f_n - f_m\|_p,$$

so that $(g_n)_{n=1}^{\infty}$ is a Cauchy sequence in $L_p(\mathbf{R}; Y)$. Since $L_p(\mathbf{R}; Y)$ is complete, there exists a $g \in L_p(\mathbf{R}; Y)$ such that $g_n \to g$ in $L_p(\mathbf{R}; Y)$, as $n \to +\infty$.

We shall now show that $\|g_n - g\|_p \to 0$. For this, let $\varepsilon > 0$. There exists $N(\varepsilon)$ such that,

$$m, n \geqslant N(\varepsilon) \Rightarrow \|f_n - f_m\|_p < \varepsilon/4.$$

Take $n \geqslant N(\varepsilon)$. We will show that $\|g - g_n\|_p < \varepsilon$. For each $h > 0$ there exists an $m_h \geqslant N(\varepsilon)$ such that $|g - g_{m_h}|_p < h(\varepsilon/4)$. Hence $\forall h > 0$,

$$\frac{1}{h} \left| (\tau_h - 1)(g - g_n) \right|_p \leqslant \frac{1}{h} \left| (\tau_h - 1)(g - g_{m_h}) \right|_p + \frac{1}{h} \left| (\tau_h - 1)(g_{m_h} - g_n) \right|_p$$

$$\leqslant \frac{2}{h} \left| g - g_{m_h} \right|_p + \| g_{m_h} - g_n \|_p \quad \text{by Lemma 3.3(a)}$$

$$< \frac{\varepsilon}{2} + 2 \| f_{m_h} - f_n \|_p \quad \text{by Theorem 4.7(a)}$$

$$< \varepsilon.$$

Now by Triviality 3.1, $\|g - g_n\|_p < \varepsilon$.

We have shown that $\|g_n - g\|_p \to 0$, as well as $g_n \to g$ in $L_p(\mathbf{R};\ Y)$. Thus $g \in L_p(\mathbf{R};\ Y) \cap U_p(\mathbf{R};\ Y)$. Hence by Theorem 4.7(b), $g = A_p(f)$ for some $f \in U_p(\mathbf{R};\ Y)$. Now

$$\|f - f_n\|_p \leqslant |A_p(f - f_n)|_p + \|A_p(f - f_n)\|_p \quad \text{by Lemma 4.5(d)}$$

$$\leqslant |g - g_n|_p + \|g - g_n\|_p \to 0 \quad \text{as } n \to +\infty. \quad \square$$

6. The isometric isomorphism of $\tilde{U}_p(\mathbf{R};\ Y)$ to the space $\mathfrak{U}_p(\mathfrak{R};\ Y)$ of measures. Let $\mathscr{P}$ be the prering of finite intervals $(a, b]$ where $-\infty < a < b < +\infty$. Let $\mathfrak{R}$ and $\mathfrak{A}$ be the ring and algebra generated by $\mathscr{P}$. Let μ be Lebesgue measure on $\mathfrak{A}$. Clearly $\mathfrak{R}$ is the collection of all $A \in \mathfrak{A}$ with $\mu(A) < +\infty$. For each $R \in \mathfrak{R}$ let Π_R be the class of all partitions $\pi = (E_i)$ of E into a finite number of members E_i of $\mathfrak{R}$. Note that $\mu(E_i) > 0$, $\forall E_i \in \pi \in \Pi_R$. Let $FA(\mathfrak{R};\ Y)$ denote the space of finitely additive set functions on $\mathfrak{R}$ into Banach space Y. For each $1 < p < \infty$ and each $\xi \in FA(\mathfrak{R};\ Y)$ let $|\xi|_p(\cdot)$ and $\||\xi\||_p$ be as defined in (1.2) and (1.3). We say that $\xi \in FA(\mathfrak{R};\ Y)$ is *absolutely continuous* with respect to μ, denoted by $\xi \ll \mu$, if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $R \in \mathfrak{R}$, $\mu(R) < \delta$ implies that $|\xi(R)|_Y < \varepsilon$.

6.1 DEFINITION. For $1 < p < \infty$ let $\mathfrak{U}_p(\mathfrak{R};\ Y)$ be the space of all $\xi \in FA(\mathfrak{R};\ Y)$ such that $\xi \ll \mu$ and $\||\xi\||_p < +\infty$, cf. (1.3).

J. J. Uhl's generalization [11, Theorem 16] of S. Leader's result [7, Theorem 13] asserts that $\mathfrak{U}_p(\mathfrak{R};\ Y)$, $1 < p < \infty$, is a Banach space under the norm $\||\xi\||_p$.

Let $1 < p < \infty$ and $\xi \in FA(\mathfrak{R};\ Y)$. Then for each $E \in \mathfrak{R}$,

$$|\xi(E)|_Y \leqslant \{\mu(E)\}^{1/p'} |\xi|_p(E) \leqslant +\infty. \tag{6.2}$$

To see this note that for each $E \in \mathfrak{R}$ and $\pi \in \Pi_E$,

$$|\xi(E)|_Y \leqslant \sum_{E_i \in \pi} |\xi(E_i)|_Y = \sum_{E_i \in \pi} \frac{|\xi(E_i)|_Y}{\mu(E_i)^{1/p'}} \cdot \mu(E_i)^{1/p'}$$

$$\leqslant \left\{ \sum_{E_i \in \pi} \frac{|\xi(E_i)|_Y^p}{\mu(E_i)^{p-1}} \right\}^{1/p} \left\{ \sum_{E_i \in \pi} \mu(E_i) \right\}^{1/p'}$$

$$\leqslant |\xi|_p(E) \cdot \mu(E)^{1/p'},$$

by Hölder's inequality.

The proof of the following result is an easy consequence of (6.2) and is left to the reader.

6.3 TRIVIALITY. Let $CA(\mathfrak{R};\ Y)$ denote the set of all (strongly) countably additive Y-valued measures on $\mathfrak{R}$. Then for $1 < p < \infty$,

$$\mathfrak{U}_p(\mathfrak{R}; Y) \subseteq \mathrm{CA}(\mathfrak{R}; Y).$$

6.4 DEFINITION. Let $1 \leqslant p < \infty$. For each $f \in U_p(\mathbf{R}; Y)$ let ξ_f be the unique finitely additive measure on $\mathfrak{R}$ to Y defined by

$$\xi_f (a, b] \underset{\mathrm{d}}{=} \dot{f}(b) - \dot{f}(a), \qquad (a, b] \in \mathscr{P},$$

where $\dot{f}$ is as in Corollaries 4.10, 4.11 for $1 < p < \infty$ and $p = 1$ resp.

The following lemma, the proof of which is left to the reader, will be needed for our next two theorems.

6.5 LEMMA. *Let $v(\cdot)$ be a bounded, monotone increasing function on $\mathbf{R}$. Then* $\forall h > 0$

$$\frac{1}{h} \int_{-\infty}^{\infty} (v(s + h) - v(s)) \, ds \leqslant v(+\infty) - v(-\infty).$$

6.6 THEOREM. *Let $1 < p < \infty$. The correspondence $[f] \to \xi_f$ is an isometric isomorphism on the Banach space $\tilde{U}_p(\mathbf{R}; Y)$ onto the Banach space $\mathfrak{U}_p(\mathfrak{R}; Y)$.*

PROOF. Obviously the mapping $f \to \xi_f$ is linear. By Lemma 3.5 it is easily seen that the correspondence $[f] \to \xi_f$ is linear and one-one.

Let $f \in U_p(\mathbf{R}; Y)$, and $\xi =_{\mathrm{d}} \xi_f$. By Definition 6.4 and Corollary 4.10 for each $(a, b] \in \mathscr{P}$,

$$|\xi(a, b]|_Y \leqslant \{ \mu(a, b] \}^{1/p'} \lim_{h \to 0+} \frac{1}{h} \left\{ \int_a^b |f(t + h) - f(t)|^p \, dt \right\}^{1/p}. \quad (1)$$

Since for each $E \in \mathfrak{R}$, there exists $(J_i) \in \Pi_E$ such that $J_i \in \mathscr{P}$, it follows by the triangle inequality, (1), and Hölder's inequality by the argument used to prove (4) of the proof of 4.10 that

$$|\xi(E)|_Y \leqslant \{ \mu(E) \}^{1/p'} \lim_{h \to 0+} \frac{1}{h} \left\{ \int_E |f(t + h) - f(t)|^p \, dt \right\}^{1/p}$$

for each $E \in \mathfrak{R}$. Thus

$$\frac{|\xi(E)|_Y^p}{\mu(E)^{p-1}} \leqslant \lim_{h \to 0+} \left(\frac{1}{h} \right)^p \int_E |f(t + h) - f(t)|^p \, dt. \quad (2)$$

Now let $E \in \mathfrak{R}$ and $\pi = (E_i) \in \Pi_E$. Then applying (2) to each $E_i \in \pi$ and summing over π we have that

$$\sum_{E_i \in \pi} \frac{|\xi(E_i)|_Y^p}{\mu(E_i)^{p-1}} \leqslant \lim_{h \to 0+} \left(\frac{1}{h} \right)^p \int_E |f(t + h) - f(t)|^p \, dt \leqslant \|f\|_p^p.$$

Thus $\xi \in \mathfrak{U}_p(\mathfrak{R}; Y)$ and $|||\xi|||_p \leqslant \|f\|_p$.

Conversely let $\xi \in \mathfrak{A}_p(\mathfrak{R}; Y)$ and define

$$f_0(t) \underset{d}{=} \begin{cases} -\xi(t, 0], & t < 0, \\ \xi(0, -t], & t \geqslant 0. \end{cases}$$

Since $\xi \ll \mu$, $f_0(\cdot)$ is (locally absolutely) continuous and hence Bochner measurable. Moreover since ξ is finitely additive,

$$f_0(b) - f_0(a) = \xi(a, b], \qquad (a, b] \in \mathcal{P}. \tag{3}$$

For each $s \in \mathbf{R}$, let

$$v(s) \underset{d}{=} \sup_{\substack{E \subseteq (-\infty, s] \\ E \in \mathfrak{R}}} \left\{ |\xi|_p(E) \right\}^p.$$

Then $v(\cdot)$ is monotone increasing with $v(+\infty) = \|\|\xi\|\|_p^p$. Moreover for each $h > 0$ and $s \in \mathbf{R}$,

$$\left\{ |\xi|_p(s, s + h] \right\}^p \leqslant v(s + h) - v(s).$$

Therefore by (3) and (6.2)

$$|f_0(s + h) - f_0(s)|_Y^p = |\xi(s, s + h]|_Y^p$$

$$\leqslant h^{p-1} \left\{ |\xi|_p(s, s + h] \right\}^p \leqslant h^{p-1}(v(s + h) - v(s)).$$

Hence by Lemma 6.5,

$$\left(\frac{1}{h} \right)^p \int_{-\infty}^{\infty} |f_0(s + h) - f_0(s)|_Y^p \, ds \leqslant \frac{1}{h} \int_{-\infty}^{\infty} (v(s + h) - v(s)) \, ds$$

$$\leqslant v(+\infty) = \|\|\xi\|\|_p^p.$$

Therefore $\|f_0\|_p \leqslant \|\|\xi\|\|_p$ and $f_0 \in U_p(\mathbf{R}; Y)$. Moreover by (3) $\xi = \xi_{f_0}$ so that $\|\|\xi\|\|_p \leqslant \|f_0\|_p$ and $\|\|\xi\|\|_p = \|f_0\|_p$. $\square$

6.7 DEFINITIONS. (a) Let $\xi \in \mathrm{FA}(\mathfrak{R}; Y)$. For each $E \in \mathfrak{R}$ let

$$|\xi|(E) \underset{d}{=} \sup_{\pi \in \Pi_E} \left\{ \sum_{E_i \in \pi} |\xi(E_i)|_Y \right\},$$

and let

$$\|\xi\| \underset{d}{=} \sup_{E \in \mathfrak{R}} |\xi|_p(E).$$

(b) Let $\mathrm{BVCA}(\mathfrak{R}; Y)$ be the space of all $\xi \in \mathrm{FA}(\mathfrak{R}; Y)$ which are (strongly) countably additive and have $\|\xi\| < +\infty$.

The following theorem establishes an isometric isomorphism for the case $p = 1$, which is analogous to that obtained for $1 < p < \infty$ in Theorem 6.6.

6.8 THEOREM. *Let $p = 1$. The correspondence $[f] \to \xi_f$ is an isometric isomorphism on Banach space $\tilde{U}_1(\mathbf{R}; Y)$ onto Banach space $\mathrm{BVCA}(\mathfrak{R}; Y)$.*

PROOF. Obviously the correspondence is linear and one-one (see Lemma 3.5).

Let $f \in U_1(\mathbf{R}; Y)$ and $\xi =_d \dot{\xi}_f$. By Corollary 4.11 $\dot{f}$ is right continuous and such that $v_f(\mathbf{R}) \leqslant \|f\|_1$. Obviously for each $(a, b] \in \mathscr{P}$,

$$\big| \xi(a, b] \big|_Y = \big| \dot{f}(b) - \dot{f}(a) \big|_Y \leqslant v_f(a, b]. \tag{1}$$

Let $E \in \mathscr{R}$ and $\pi = (E_i)_{i=1}^n \in \Pi_E$. Then there is a finer partition $\pi' = (J_{i,j}) \in \Pi_E$ such that $J_{i,j} \in \mathscr{P}$, for each i, j, and $E_i = \bigcup_j J_{i,j}$, for each i. Then by (1) and finite additivity of v_f,

$$\sum_{E_i \in \pi} \big| \xi(E_i) \big|_Y \leqslant \sum_{J_{i,j} \in \pi'} \big| \xi(J_{i,j}) \big|_Y \leqslant v_f(E).$$

Therefore $|\xi|(\cdot) \leqslant v_f(\cdot)$ on $\mathscr{R}$ and

$$\|\xi\| \leqslant v_f(\mathbf{R}) \leqslant \|f\|_1 < +\infty. \tag{2}$$

Next by the argument of [4, Lemma III.5.16],

$$0 \leqslant \lim_{\varepsilon \to 0+} |\xi|(a, a + \varepsilon] \leqslant \lim_{\varepsilon \to 0+} v_f(a, a + \varepsilon] = 0. \tag{3}$$

It is easily seen by (3) that $|\xi|(\cdot)$ is regular on $\mathscr{R}$. Also $|\xi|(\cdot)$ is finitely additive on $\mathscr{R}$. Hence by a theorem of A. D. Alexandroff, cf. [4, Theorem III.5.13], $|\xi|(\cdot)$ is countably additive. Thus $\xi \in \mathrm{BVCA}(\mathscr{R}; Y)$.

To show the correspondence is onto $\mathrm{BVCA}(\mathscr{R}; Y)$ let $\xi \in \mathrm{BVCA}(\mathscr{R}; Y)$ and let

$$f(t) =_d \begin{cases} -\xi(t, 0], & t < 0, \\ \xi(0, t], & t \geqslant 0. \end{cases}$$

Then $\forall a, b \in \mathbf{R}, a < b$,

$$f(b) - f(a) = \xi(a, b]. \tag{4}$$

Next let $a \in \mathbf{R}$ and $(b_n)_{n=1}^\infty$ be any sequence such that $b_n \to a +$. Then by (4)

$$\big| f(b_n) - f(a) \big|_Y = \big| \xi(a, b_n] \big|_Y \leqslant |\xi|(a, b_n] \to 0.$$

Therefore f is right continuous. Let $(a_k)_{k=0}^n \in \Pi'_{\mathbf{R}}$. Then by (4),

$$\sum_{k=1}^n \big| f(a_k) - f(a_{k-1}) \big|_Y = \sum_{k=1}^n \big| \xi(a_{k-1}, a_k] \big|_Y \leqslant |\xi|(a_0, a_n] \leqslant \||\xi\||.$$

Therefore the total variation of f, $v_f(\mathbf{R}) \leqslant \|\xi\| < +\infty$.

Since f is right continuous and the set of rationals is a countable dense set in $\mathbf{R}$, the range of f is separable. Thus to show that $f(\cdot)$ is Bochner measurable it is sufficient to show that $y'f(\cdot)$ is Borel measurable for each $y' \in Y'$, the dual of Y, cf. [10, Theorem 2.13]. Let $y' \in Y'$. Then $\forall a, b \in \mathbf{R}$,

$$\big| y'f(b) - y'f(a) \big|_Y \leqslant |y'|_{Y'} \cdot |f(b) - f(a)|_Y$$

Obviously then since f has bounded variation, $y'f$ is a scalarly valued function with bounded variation. Hence $y'f$ is Borel measurable for all $y' \in Y'$ and $f(\cdot)$ is Bochner measurable.

Next let

$$v(a) \underset{\mathrm{d}}{=} \begin{cases} -|\xi|(a, 0], & a < 0, \\ |\xi|(0, a], & a \geqslant 0. \end{cases}$$

Then $v(\cdot)$ is monotone increasing and for all $(a, b] \in \mathscr{P}$,

$$v(b) - v(a) = |\xi|(a, b].$$

Thus

$$\|\xi\| = v(+\infty) - v(-\infty). \tag{5}$$

Moreover by (4),

$$|f(b) - f(a)|_Y \leqslant |\xi|(a, b] = v(b) - v(a), \quad \forall a < b. \tag{6}$$

Then by (6), Triviality 6.5, and (5),

$$\frac{1}{h} \int_{-\infty}^{\infty} |f(s + h) - f(s)|_Y \, ds \leqslant \frac{1}{h} \int_{-\infty}^{\infty} v(s + h) - v(s) \, ds \leqslant \|\xi\|.$$

Hence $\|f\|_1 \leqslant \|\xi\| < +\infty$ and the correspondence $[f] \to \xi_f$ is onto. Since f is right continuous and $f \in \mathrm{BV}(\mathbf{R}; Y)$, $f(\cdot) = \dot{f}(\cdot)$. Hence by (4) $\xi_f = \xi$, and by (2)

$$\|f\|_1 \leqslant \|\xi\| \leqslant \|\xi_f\| \leqslant \|f\|_1. \quad \square$$

The following example provides an $f \in U_p(\mathbf{R}; Y) \setminus L_\infty(\mathbf{R}; Y)$, $1 < p < \infty$, and hence a measure $\xi_f \in \mathfrak{U}_p(\mathscr{R}; Y)$, $1 < p < \infty$, which is unbounded.

6.9 EXAMPLE. Let $1 < p < \infty$, $y \in Y$ such that $|y|_Y = 1$, and $f(t) =_{\mathrm{d}}$ $(\log t)y\chi_{[1,\infty)}(t)$, $t \in \mathbf{R}$. Obviously $f(\cdot)$ is the indefinite integral of its derivative $f'(t) = (1/t)y\chi_{(1,\infty)}(t)$, $t \neq 1$. Then by (3) of the proof of Theorem 4.7,

$$\|f\|_p \leqslant |f'|_p = 1/(p - 1) < +\infty.$$

Thus $f \in U_p(\mathbf{R}; Y)$, but $|f|_\infty = +\infty$.

7. The isometric isomorphism of $\tilde{U}_p(\mathbf{R}; Y)$ to $L_p(\mathbf{R}; Y)$ when Y has the Radon-Nikodym Property. Although our goal in this section is to deal with Banach spaces Y with the Radon-Nikodym Property, we shall not assume this in the next result.

7.1 LEMMA. *Let $1 \leqslant p < \infty$, $g \in L_1^{\mathrm{loc}}(\mathbf{R}; Y)$, and f be an indefinite integral of g, i.e. $f(t) = \int_0^t g(s) \, ds + y$, $t \in \mathbf{R}$, for some $y \in Y$. Then $\|f\|_p = |g|_p < +\infty$.*

PROOF. It follows from equation (3) in the proof of Theorem 4.7 that $\|f\|_p \leqslant |g|_p$. To show the reverse inequality we note that by the Lebesgue

Differentiation Theorem, for a.a. $s \in \mathbf{R}$

$$g(s) = \lim_{h \to 0+} \frac{1}{h} (f(s + h) - f(s)),$$

in the Y topology. Therefore

$$|g(s)|^p_Y = \lim_{h \to 0+} \left(\frac{1}{h} \right)^p |f(s + h) - f(s)|^p_Y, \quad \text{a.e.}$$

and by Fatou's lemma,

$$\int_{-\infty}^{\infty} |g(s)|^p_Y \, ds \leqslant \varliminf_{h \to 0+} \left(\frac{1}{h} \right)^p \int_{-\infty}^{\infty} |f(s + h) - f(s)|^p_Y \, ds \leqslant \|f\|^p_p. \quad \square$$

7.2 DEFINITION. For $1 \leqslant p < \infty$, we let $IL_p(\mathbf{R}; Y)$ be the linear manifold in $\tilde{U}_p(\mathbf{R}; Y)$ of equivalence classes $[\dot{f}]$ where $\dot{f}$ is an indefinite integral of some function $g \in L_p(\mathbf{R}; Y)$, i.e. $\dot{f}(t) = \int_0^t g(s) \, ds + y$, $t \in \mathbf{R}$ for some $y \in Y$ and $g \in L_p(\mathbf{R}; Y)$.

7.3 THEOREM. *Let D be differentiation on $IL_p(\mathbf{R}; Y)$, where $1 \leqslant p < \infty$. Then*

(a) *D is an isometric isomorphism on $IL_p(\mathbf{R}; Y)$, a closed linear subspace of $\tilde{U}_p(\mathbf{R}; Y)$, onto $L_p(\mathbf{R}; Y)$.*

(b) *If Y has the Radon-Nikodym Property and $1 < p < \infty$, then $IL_p(\mathbf{R}; Y) = \tilde{U}_p(\mathbf{R}; Y)$, and D is an isometric isomorphism on the Banach space $\tilde{U}_p(\mathbf{R}; Y)$ onto the Banach space $L_p(\mathbf{R}; Y)$.*

PROOF. (a) is obvious in view of Lemma 7.1.

(b) By Definition 7.2, $IL_p(\mathbf{R}; Y) \subseteq \tilde{U}_p(\mathbf{R}; Y)$. Let $[f] \in \tilde{U}_p(\mathbf{R}; Y)$. Then by Corollary 4.10, $f(\cdot) = \dot{f}(\cdot)$, a.e., for some locally absolutely continuous function $\dot{f}$ on $\mathbf{R}$ to Y. Then since Y has the Radon-Nikodym Property, $\dot{f}$ is differentiable a.e., $\dot{f}' \in L_1^{\text{loc}}(\mathbf{R}; Y)$, and $\dot{f}$ is an indefinite integral of $\dot{f}'$, cf. [3, p. 30]. By Lemma 7.1, $|\dot{f}'|_p = \|\dot{f}\|_p = \|f\|_p < +\infty$ and $[f] \in IL_p(\mathbf{R}; Y)$. $\quad \square$

8. Remarks on a theorem of Butzer and Berens. In [2, p. 160] Butzer and Berens prove a theorem for arbitrary Banach spaces X and certain indices α, r, q, viz. 3.1.3, which for $X = L_p(\mathbf{R}; Y)$, $1 \leqslant p < \infty$, $\alpha = 1 = r$, and $q = \infty$, asserts the completeness of the space $L_p^{1,1,\infty}(\mathbf{R}; Y)$ of functions f in $L_p(\mathbf{R}; Y)$ for which (in our notation)

$$|f|_{1,1,\infty} \underset{\text{d}}{=} |f|_p + \|f\|_p < \infty.$$

We should point out that the completeness of $\tilde{U}_p(\mathbf{R}; Y)$ (Theorem 5.1) is not deducible from their completeness result, for we know that $U_p(\mathbf{R}; Y)$ has plenty of functions which are not in $L_p(\mathbf{R}; Y)$, cf. Ex. 4.6.

On the other hand, the special case of the Butzer-Berens result for $X = L_p(\mathbf{R}; Y)$, $1 \leqslant p < \infty$, $\alpha = 1 = r$, and $q = \infty$, follows easily from our

theory. To show this let $(g_n)_{n=1}^{\infty}$ be a Cauchy sequence in $L_p^{1,1,\infty}(\mathbf{R};\ Y)$. Then by Theorem 4.7(b) there exists a sequence $(f_n)_{n=1}^{\infty}$ in $U_p(\mathbf{R};\ Y)$ such that $A_p(f_n) = g_n\ \forall n \in \mathbf{N}$. By Lemma 4.5(d),

$$\|f_n - f_m\|_p \leqslant |A_p(g_n - g_m)|_p + \|A_p(g_n - g_m)\|_p = |g_n - g_m|_{1,1,\infty}\quad \forall n,m \in \mathbf{N}.$$

Thus $([f_n])_{n=1}^{\infty}$ is a Cauchy sequence in $\tilde{U}_p(\mathbf{R};\ Y)$. By completeness of $\tilde{U}_p(\mathbf{R};\ Y)$ (Theorem 5.1), there exists $[f] \in \tilde{U}_p(\mathbf{R};\ Y)$ such that $[f_n] \to [f]$ in $\tilde{U}_p(\mathbf{R};\ Y)$. Let $g =_d A_p(f)$. Then by Lemma 4.5(a), (c),

$$|g_n - g|_{1,1,\infty} = |A_p(f_n - f)|_p + \|A_p(f_n - f)\|_p \leqslant 3\|f_n - f\|_p\quad \forall n \in \mathbf{N}.$$

Thus $g_n \to g$ in $L_p^{1,1,\infty}(\mathbf{R};\ Y)$ and $L_p^{1,1,\infty}(\mathbf{R};\ Y)$ is complete.

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