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DIFFERENTIABILITY OF MEASURES ASSOCIATED WITH PARABOLIC EQUATIONS ON INFINITE DIMENSIONAL SPACES

BY

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ABSTRACT. The transition measures of the Brownian motion on manifolds modelled on abstract Wiener spaces locally correspond to fundamental solutions of certain infinite dimensional parabolic equations. We establish the existence of such fundamental solutions under a broad new set of hypotheses on the differential coefficients. The fundamental solutions can be approximated in total variation by fundamental solutions of "almost" finite dimensional parabolic equations. By the finite dimensional theory, the approximations are seen to be differentiable. We prove that the property of differentiability is closed under a particular type of sequential convergence, and conclude the differentiability of the fundamental solutions of the infinite dimensional parabolic equations. This result provides strong evidence in support of the conjecture that the transition measures of the Brownian motion are differentiable, and hence is of importance in the construction of infinite dimensional Laplace-Beltrami operators.

I. Introduction. Recent work of several authors [6], [9], [16], [19], [20] has been concerned with the problem of defining, for a class of infinite dimensional manifolds, a Laplace-Beltrami operator commensurate with the available definitions of exterior differentiation and integration [19]. When the manifold is a (flat) real separable Banach space, and the integration theory is associated with a Gaussian Borel measure, then the relevant Laplace-Beltrami operator on 0-forms turns out to be the infinitesimal generator of the L^2 Ornstein-Uhlenbeck process–i.e. the number of particles operator known to quantum field theorists.

In view of the established existence of the Brownian motion on infinite dimensional manifolds [14], one possible way to obtain a Laplace-Beltrami operator on 0-forms is to establish existence of the Ornstein-Uhlenbeck process and then to relate the Laplace-Beltrami operator to its infinitesimal generator. Kuo [16] has constructed such a process, assuming the validity of a conjecture that one of the transition measures of the Brownian motion

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process is differentiable. This paper presents strong evidence in support of the conjecture.

We begin with a discussion of differentiation of measures in §II. The paper of Averbuh, Smoljanov and Fomin [1] contains an excellent discussion of directional differentiation of measures on vector spaces, as well as a complete characterization of infinitely differentiable measures on finite dimensional spaces. In Proposition 1 we show that appropriately controlled sequential limits of differentiable measures remain differentiable. This enables us to show differentiability of infinite dimensional measures by establishing differentiability of an approximating sequence of (semi-) finite dimensional measures.

Locally, the transition measures of the Brownian motion correspond to fundamental solutions of parabolic equations. Fundamental solutions of some infinite dimensional parabolic equations were first studied in [17], [18]. The hypotheses on the differential coefficients assumed in [17], [18] do not hold in the cases of interest in this paper, so we will, in §III, establish the existence of fundamental solutions under a significantly different set of coefficient hypotheses. These fundamental solutions can be approximated in total variation by products of finite dimensional fundamental solutions with infinite dimensional Wiener measures (semi-finitely approximated). The semi-finite approximations form fundamental solutions to "adjoint" equations, and so can easily be seen to be differentiable. On passing to the limit, we establish differentiability of the infinite dimensional fundamental solutions.

II. Differentiable measures. Averbuh, Smoljanov and Fomin [1] have made a comprehensive study of differentiable measures on linear spaces, with a view towards their subsequent study [2] of generalized functions of infinitely many variables, their Fourier transforms, and applications to infinite dimensional differential operators. Proofs for otherwise unproven or unreferenced assertions which we make in this section may be found in [1].

Let X be a linear space, and $\mathcal{Q}$ a σ-algebra of subsets of X. A measure μ on $(X, \mathcal{Q})$ will be a real- or complex-valued σ-additive set function on $\mathcal{Q}$. μ is said to be differentiable in the direction h of X if and only if for each $A \in \mathcal{Q}$,

$$\lim_{t\to 0} \frac{1}{t}[\mu(A + th) - \mu(A)]$$

exists. This limit is denoted $d_h\mu(A)$ or $\mu'(A)h$. $d_h\mu$ is a measure on $(X, \mathcal{Q})$, and is absolutely continuous with respect to μ:

$$d_h\mu(A) = \int_A f(x)\,\mu(dx)$$

where f is μ-integrable. If $\mu_h(A)$ is defined to be $\mu(A + h)$, then differentiability of μ in the direction h implies that $\|\mu_h - \mu\| \leq \|d_h\mu\|$, where $\|\cdot\|$ denotes the variation of a measure.

Let H be a linear subspace of X, and assume that H is given a linear topological space structure. Let H^* be the space of continuous linear functionals on H. If $d_h\mu$ exists for each $h \in H$ and if $h \mapsto d_h\mu(A)$ is in H^* for each $A \in \mathcal{Q}$, then μ is said to be weakly differentiable with respect to H. If μ is weakly differentiable with respect to H, and if H is locally convex and semi-reflexive, then $A \mapsto \mu'(A)$ is a σ-additive set function from $\mathcal{Q}$ into H^*, where H^* is given the strong topology. Let β be the class of all bounded subsets of H. If

$$\frac{\mu(A + th) - \mu(A)}{t} \underset{t \to 0}{\to} d_h\mu(A)$$

uniformly with respect to $h \in B$ for each $B \in \beta$ and $A \in \mathcal{Q}$, then μ is said to be boundedly differentiable with respect to H.

Our interest in this paper lies in the case when H is a normed linear space. It is straightforward to show that when H can be normed, bounded weak differentiability is simply ordinary Fréchet differentiability at the origin of the function $h \to \mu(A + h)$. We will call μ H-differentiable if μ is weakly and boundedly differentiable.

Let us note what happens in the special case $x = \mathbf{R}^n$, $\mathcal{Q}$ = the Borel sets. Let $\{e_1, \ldots, e_n\}$ be a basis for $\mathbf{R}^n$. If μ is differentiable in each of the directions e_i $(i = 1, \ldots, n)$ then μ is absolutely continuous with respect to Lebesgue measure, μ is weakly and boundedly differentiable with respect to $\mathbf{R}^n$, and

$$\|(\mu_{th} - \mu)/t - d_h\mu\| \to 0$$

as $t \to 0$, uniformly for h in bounded sets. Any real-valued function φ on $\mathbf{R}^n$ which is integrable, differentiable in the direction h at each point and such that $d_h\varphi$ is also integrable on $\mathbf{R}^n$ is the density of a measure μ differentiable in the direction h. Moreover

$$d_h\mu(A) = \int_A d_h\varphi(x)\, dx \qquad \text{for each } A \in \mathcal{Q}. \tag{1}$$

μ is infinitely differentiable with respect to $\mathbf{R}^n$ if and only if μ is absolutely continuous with respect to Lebesgue measure and its density is (up to equivalence) an infinitely differentiable function for which all partial derivatives of all orders are integrable.

Hence, in the finite dimensional case, analysis of infinitely differentiable measures reduces (in the sense of generalized functions) to the analysis of differentiable point functions. In infinite dimensions the situation is radically different [**1**]. If X is an infinite dimensional locally convex Polish (i.e. complete, separable, metrizable) space, then there exists no nonzero real-valued Borel measure on X that is differentiable in all directions.

The setting of an abstract Wiener space has proven to be quite well suited to the study of infinite dimensional parabolic and elliptic differential operators. An abstract Wiener space consists of a pair (H, B), where H is a real separable Hilbert space with norm $|\cdot|$ and inner product $(\cdot,\cdot)$ and B is the completion of H with respect to a weaker norm $\|\cdot\|$ which is measurable in the sense defined by L. Gross **[10]**. A cylinder set in H is a set Γ of the form $\Gamma = P^{-1}(E)$ where P is a finite dimensional projection on H and E is a Borel set in the range of P. The Gaussian cylinder set measure ν_t (with variance parameter $t > 0$) is a finitely additive set function on the ring of cylinder sets, and is given by

$$\nu_t(\Gamma) = (2\pi t)^{-n/2}\int_E \exp\left[-|x|^2/2t\right] dx$$

where n is the dimension of the range of P. The natural injection of H into B and the natural embedding by restriction of H^* into B^* are continuous, and B^* is dense in H^*. We will, when appropriate, use these mappings to identify elements of B^* with elements of H^*, elements of H^* (via the Riesz representation) with elements of H, and elements of H with elements of B. That is, $B^* \hookrightarrow H^* \approx H \hookrightarrow B$. ν_t induces a cylinder set measure p_t on B by

$$p_t\{x \in B|\,(y_1(x), \dots, y_n(x)) \in E\} = \nu_t\{x \in H|(y_1(x), \dots, y_n(x)) \in E\}$$

for all finite subsets $\{y_1, \dots, y_n\}$ of B^* and Borel sets $E \subseteq \mathbf{R}^n$. The measurability of the norm $\|\cdot\|$ of B is necessary and sufficient for p_t to be countably additive on the ring of cylinder sets of B and hence to have a unique countably additive extension (which we will also denote by p_t) to the Borel σ-field of B. p_1 is often known as Wiener measure on B. For each x in B^*, we have

$$\int_B \langle x, y\rangle^2 p_1(dy) = |x|^2,$$

where $\langle\cdot,\cdot\rangle$ is the natural B^*–B pairing. Hence the identity map on B^* extends to a mapping of all of H^* (or H) to $L^2(p_1)$. For $h \in H$ we will write the corresponding element of $L^2(p_1)$ as $\langle h, y\rangle$, or later, as (h, y). (For $x \in B^*$ and $y \in H$, $\langle x, y\rangle = (x, y)$.)

For $x \in B$, $p_t(x, dy)$ is defined by $p_t(x, \Gamma) = p_t(\Gamma - x)$ for all Borel sets Γ of B. Each $p_t(x, dy)$ is differentiable only in the directions of H; however it is infinitely differentiable with respect to H **[1**, Theorem 5.3.1]. Moreover each $p_t(x, dy)$ is weakly and boundedly differentiable with respect to H and

$$d_h p_t(\Gamma) = -t^{-1}\int_\Gamma \langle h, y\rangle p_t(dy). \tag{2}$$

The family of measures $\{p_t(x, dy): x \in B, t > 0\}$ constitutes the transition function for the Brownian motion on B and also constitutes a fundamental solution for an infinite dimensional version of the heat equation [**11**]. Wiener measure appears to assume the fundamental importance for integral and differential calculus on real separable nonfinite dimensional Banach spaces that Lebesgue measure assumes on finite dimensional Banach spaces.

A real-valued function f on B is said to be H-differentiable at $x \in B$ if the function $g(h) \equiv f(x + h)$ with domain H is Fréchet differentiable at the origin of H. The H-derivative $Df(x)$ is defined to be $g'(0)$, and may be regarded as either a member of H^* or of H.

L t μ be a measure on B which is H-differentiable. Since the derivative of μ in the direction of an $h \in H$ is absolutely continuous with respect to μ, we have the existence of a Borel measurable function ξ_h such that for each Borel set Γ,

$$d_h\mu(\Gamma) = \int_\Gamma \xi_h(y)\,\mu(dy).$$

Kuo defines μ to have a logarithmic derivative ξ if there exists a Borel-measurable B-valued function ξ such that, for all $h \in B^*$, $\langle h, \xi\rangle = \xi$ a.e. and $\|\xi\|_B$ is μ-integrable over each bounded Γ. Notationally, $dD\mu/d\mu \equiv \xi$. From equation (1) we see that if φ is the density of a measure μ on $\mathbf{R}^1$, and if φ is positive and continuously differentiable, then

$$dD\mu/d\mu = \varphi'/\varphi = (\log \varphi)'.$$

For Wiener measure p_t, we have by (2):

$$\langle h, dDp_t/dp_t\rangle = -t^{-1}\langle h, y\rangle.$$

These two examples motivate the terminology "logarithmic derivative" and the use of B as the range of the logarithmic derivative.

PROPOSITION 1. *Assume* $\{\mu_n\}$, μ *and* ν *are measures on a linear space such that* (i) *each* μ_n *is differentiable in the direction* h, (ii) *for each measurable set* A, $\mu_n(A) \to \mu(A)$, (iii) $d_h\mu_n \to \nu$ *in variation. Then* μ *is differentiable in the direction* h *and* $d_h\mu = \nu$.

PROOF. We adapt a standard proof pertaining to the interchange of differentiation and limits of sequences of real valued functions. For t real [1, Theorem 1.3.1] gives

$$|\{\mu_m(A + th) - \mu_n(A + th)\} - \{\mu_m(A) - \mu_n(A)\}|$$
$$\leqslant \operatorname{var}\{\mu_m'(th) - \mu_n'(th)\} = |t|\operatorname{var}\{\mu_m'(h) - \mu_n'(h)\}.$$

Thus given $\varepsilon > 0$, there exists N such that $m, n > N$ implies

$$\left|\frac{\mu_m(A + th) - \mu_m(A)}{t} - \frac{\mu_n(A + th) - \mu_n(A)}{t}\right| < \varepsilon.$$

Letting $n \to \infty$,

$$\left| \frac{\mu_m(A + th) - \mu_m(A)}{t} - \frac{\mu(A + th) - \mu(A)}{t} \right| \leqslant \varepsilon \quad \text{for all } m > N. \tag{3}$$

Fix m_0 so that $m_0 > N$ and

$$|\mu'_{m_0}(A)h - \nu(A)h| < \varepsilon. \tag{4}$$

There exists $\delta > 0$ so that $|t| < \delta$ implies

$$\left| \frac{\mu_{m_0}(A + th) - \mu_{m_0}(A)}{t} - \mu'_{m_0}(A)h \right| < \varepsilon. \tag{5}$$

Using the triangle inequality on (3), (4), (5) yields

$$\left| \frac{\mu(A + th) - \mu(A)}{t} - \nu(A)h \right| < \varepsilon \tag{6}$$

for $|t| < \delta$, completing the proof. □

COROLLARY 1.1. *Assume the hypotheses of Proposition* 1 *and also assume that each* μ_n *is* H*-differentiable and that there exists* $\delta_0 > 0$ *such that* $d_h\mu_n \to \nu$ *in variation uniformly for all* $h \in H$ *with* $|h| < \delta_0$. *Then* μ *is* H*-differentiable.*

PROOF. Under these additional hypotheses, (3), (4) and (5), and consequently (6) are all uniform estimates for $|h| < \delta_0$. Hence $|\mu(A + h) - \mu(A) - \nu(A)h| = o(|h|)$. □

REMARK. Often an infinite dimensional μ (in particular, a μ which belongs to a fundamental solution of a parabolic equation) may be approximated by products $\{\mu_n \times p_n\}$. The μ_n are members of fundamental solutions of appropriate finite dimensional parabolic equations, which usually can be easily seen to be differentiable. The p_n are infinite dimensional Wiener measures, which are known to be infinitely H-differentiable.

III. Fundamental solutions of parabolic equations. The transition measures $\{\beta_t(x, dy): t > 0, x \in B\}$ of the Brownian motion on manifolds modelled on an abstract Wiener space (H, B) locally correspond to fundamental solutions of the Cauchy problem $\{L_{x,t}u = 0 \ (x \in B, t > 0);\ u(0, x) = f(x)\}$ for the operator

$$\begin{aligned} L_{x,t}u(x, t) = \operatorname{trace}\left[a(x)D^2_{xx}u(x, t)\right] \\ + (b(x), D_xu(x, t)) + c(x)u(x, t) - \partial u/\partial t \end{aligned} \tag{7}$$

where a: $B \to \mathfrak{L}^2(H)$, b: $B \to H$, c: $B \to \mathbf{R}^1$. Assuming that one of these measures is H-differentiable, with locally Lipschitzian logarithmic derivative, Kuo [16] has described the construction of an Ornstein-Uhlenbeck process.

Although Kuo's definition of H-differentiable measure looks very different from ours, it is in fact equivalent.

In this section we establish the existence and differentiability of fundamental solutions for a large class of parabolic equations on B of the form (7). A fundamental solution of the Cauchy problem is defined to be a family $\{q_t(x, dy): x \in B, t > 0\}$ of finite Borel measures on B which will produce a solution (of the Cauchy problem) for each f in the set $\mathcal{C}$ of all bounded and uniformly Lip 1 real-valued functions on B in the following manner:

Define $q_t f(x) = \int_B f(y) q_t(x, dy)$. Then for each f in $\mathcal{C}$:

(A-i) $q_t f(x)$ is differentiable with respect to t, twice H-differentiable with respect to x, and $D^2_{xx} q_t f(x)$ is of trace class,

(A-ii) $u(x, t) \equiv q_t f(x)$ satisfies $L_{x,t} u = 0$ for $x \in B$, $t > 0$,

(A-iii) $\lim_{t \searrow 0} q_t f(x) = f(x)$ uniformly for $x \in B$.

When $b(x)$ and $c(x)$ vanish identically, and $a(\cdot)$ is uniformly positive definite, of the form $I +$ a trace class operator, and satisfies some rather technical differentiability and summability hyptheses, a fundamental solution has been developed in **[17]**, **[18]**, based on the parametrix method of E. E. Levi. The parametrix used is also a family $\{s_t(x, dy): x \in B, t > 0\}$ of finite Borel measures on B, defined by

$$s_t(x, dy) = \left[\det a(y)\right]^{-1/2} \cdot \exp\left[-\left(\left[a(x)^{-1} - I\right](x - y), x - y\right)/4t\right] p_{2t}(x, dy). \quad (8)$$

The parametrix is used as an initial approximation to the fundamental solution. Specifically, a third family $\{r_t(x, dy): x \in B, t > 0\}$ of finite Borel measures is sought so that $\{q_t(x, dy)\}$ will satisfy

$$\int_B f(y) q_t(x, dy) = \int_B f(y) s_t(x, dy) + \int_0^t \int_B \int_B f(z) r_u(y, dz) m_{t-u}(x, dy)\, du \quad (9)$$

for all f in $\mathcal{C}$. We rewrite (9) as

$$q_t f(x) = s_t f(x) + \int_0^t s_{t-u}[r_u f](x)\, du. \quad (10)$$

Since $q_t f$ must be a solution of $L_{x,t} u = 0$, we attempt to find $r_u f$ by applying $L_{x,t}$ to each side of (10). This is the stage which most influences the choice of parametrix and of differentiability and summability hypotheses on the coefficients of $L_{x,t}$. The objective is to show that for $f \in \mathcal{C}$, $s_t f$ is sufficiently smooth so that the mapping M_t: $f \to L_{x,t} s_t f$ may be defined and so that application of $L_{x,t}$ to (10) results in an integral equation of the form

$$0 = M_t f(x) + \int_0^t M_{t-u}[r_u f](x)\, du - r_t f(x). \quad (11)$$

This equation is then solved for $r_t f(x)$ by a Picard iteration technique. It can be seen from the proofs of Proposition 2 of **[17]** and Lemma 2.1 of **[18]** that we may proceed from (10) to (11) and obtain convergence of the Picard approximations if the application of $L_{x,t}$ to $s_t f(x)$ produces a family of measures $M_t(x, dy)$ such that $L_{x,t} s_t f(x) = \int_B f(y) M_t(x, dy)$ and $\{M_t(x, dy)\}$ satisfies: For each $T > 0$ there exists a constant Q, and for each $0 < \delta < T$ there exists a constant Q_δ so that

(B-i) $\int_B |M_t|(x, dy) \leqslant Q t^{-1/2}$ for $0 < t \leqslant T$.

(B-ii) The map $f \to M_t f$ is bounded linear operator on $\mathscr{C}$, with $\|M_t f\|_{\mathscr{C}} \leqslant Q t^{-1/2} \|f\|_{\mathscr{C}}$ for $0 < t \leqslant T$. Here

$$\|f\|_{\mathscr{C}} = \|f\|_\infty + \inf\{c \colon |f(x) - f(y)| \leqslant c\|x - y\|_B \text{ for all } x, y \text{ in } B\}.$$

(B-iii) For $\delta \leqslant t_1, t_2 \leqslant T$ we have $\|M_{t_1} f - M_{t_2} f\|_\infty \leqslant Q_\delta |t_1 - t_2| \cdot \|f\|_{\mathscr{C}}$ for all $f \in \mathscr{C}$.

Proposition 1 of **[17]** gives explicit formulas for the first and second H-derivatives of $s_t f(x)$. Namely,

$$\begin{aligned}(Ds_t f(x), h) = (4t)^{-1} \int_B f(y)\Big[&-\big(D\big[a(x)^{-1}\big]h(x - y), x - y\big) \\ &+ 2\big(\big[a(x)^{-1}\big]h, x - y\big)\Big] s_t(x, dy),\end{aligned} \tag{12}$$

$$\begin{aligned}(D^2 s_t f(x)k, h) = -(4t)^{-1} \int_B f(y)\Big\{&\Big[\big(D^2\big[a(x)^{-1}\big]kh(x - y), x - y\big) \\ &- 2\big(D\big[a(x)^{-1}\big]hk, x - y\big) \\ &- 2\big(D\big[a(x)^{-1}\big]kh, x - y\big) + 2\big(\big[a(x)^{-1}\big]h, k\big)\Big] \\ &- (4t)^{-1}\Big[\big(D\big[a(x)^{-1}\big]h(x - y), x - y\big) - 2\big(a(x)^{-1}h, x - y\big)\Big] \\ &\cdot \Big[\big(D\big[a(x)^{-1}\big]k(x - y), x - y\big) - 2\big(a(x)^{-1}k, x - y\big)\Big]\Big\} s_t(x, dy).\end{aligned} \tag{13}$$

In order to ensure that $D^2 s_t f(x)$ is of trace class, and, even more importantly, to be able to commute the trace summation with the integral operation and thereby to obtain $M_t(x, dy)$, considerable summability hypotheses must be placed upon $a(x)$. A set of sufficient conditions is described in **[17]**, **[18]**, focusing about an assumption that $a(x) - I$ is decomposable into a product $E_1(e(x))E_2$, where E_1 and E_2 are Hilbert-Schmidt operators. $e(x)$ is required to be twice H-differentiable with uniformly bounded and B-Lip 1 derivatives. This product decomposition is then inherited by $a(x)^{-1}$ and its first two derivatives, producing desirable estimates of the magnitudes of the terms within the integrand of (13).

Kuo's proof of the existence of the Brownian motion on Riemann-Wiener manifolds modelled on (H, B) assumes that the norm on B is C^2 off the origin. This is a strong assumption on B, for it implies the existence of a C^2 partition of unity for B. Such a partition of unity does not exist for many separable Banach spaces, including, in particular, classical Wiener space **[3]**. Furthermore the second order coefficients of the parabolic equations related to the local Brownian flows have extension and differentiability properties stronger than those assumed in **[17]**, **[18]**. The strength of these properties can be seen to compensate for the decomposition assumptions of **[17]**, **[18]**. That is, although the parabolic operators corresponding to the Brownian motion do not satisfy the hypotheses of **[17]**, **[18]**, they do possess fundamentally different alternative properties enabling the construction of a fundamental solution by the process described in the foregoing.

THEOREM 2. *Assume that the coefficients a, b, c of the differential operator* (17) *satisfy*

(C-i) $\{a(x): x \in B\}$ *is a family of operators on B such that* $\|a(x) - I\|_{\mathfrak{L}(B,B)} \leqslant 1 - \varepsilon$ *for some* $\varepsilon > 0$;

(C-ii) $a(x)$ *is of the form* $I +$ *a member of* $\mathfrak{L}^2(B)$, *and* $a(x)|_H$ *is positive and symmetric*;

(C-iii) $a(\cdot) - I$, *considered as a mapping of B to* $\mathfrak{L}^2(B)$, *is twice B-differentiable*;

(C-iv) $\|a(\cdot) - I\|_{\mathfrak{L}^2(B)}$, $\|(a(\cdot) - I)'\|_{\mathfrak{L}^3(B)}$, *and* $\|(a(\cdot) - I)''\|_{\mathfrak{L}^4(B)}$ *are uniformly bounded and* $\mathfrak{L}$ip 1;

(C-v) $|b(\cdot)|_H$ *and* $|c(\cdot)|$ *are uniformly bounded and* $\mathfrak{L}$ip 1.

Then there exists a fundamental solution $\{q_t(x, dy)\}$ *of* $L_{x,t}u = 0$ *which satisfies properties* (A-i)–(A-iii).

REMARK 1. In the hypotheses we have used the prime (′) notation to indicate B-differentiation. As $\mathfrak{L}^2(H)$ valued functions,

$$D[a(x) - I] = D[a(x)] = [a(x) - I]'|_{H \times H \times H}.$$

Hence the H-derivative $D[a(x)]$ extends by continuity to a member of $\mathfrak{L}^3(B)$. Similarly the second H-derivative $D^2[a(x)]$ extends by continuity to a member of $\mathfrak{L}^4(B)$. In the proof of Theorem 2, so as to make our formulas resemble the formulas of **[17]**, **[18]** we will use $D[a(x)]$ and $D^2[a(x)]$ to simultaneously represent elements of $\mathfrak{L}^3(H)$ and $\mathfrak{L}^4(H)$ and their respective extensions to elements of $\mathfrak{L}^3(B)$ and $\mathfrak{L}^4(B)$.

REMARK 2. If $a(x)$ is multiplied by a positive constant α, the conclusion of Theorem 5 still holds. For we notice that $\{q_t(x, dy)\}$ is a fundamental solution of $\partial u/\partial t = (1/\alpha)L_x u$ if and only if $\{q_t'(x, dy) \equiv q_{t/\alpha}(x, dy)\}$ is a fundamental solution of $\partial u/\partial t = L_x u$.

REMARK 3. We could replace (C-iii) and (C-iv) by the seemingly weaker extendibility hypotheses

(C-iii-a) $a(\cdot)$, considered as a mapping of B to $\mathfrak{L}^2(H)$, is twice H-differentiable. The first H-derivative $D[a(x)]$ extends to a member of $\mathfrak{L}^3(B)$; the second H-derivative $D^2[a(x)]$ extends to a member of $\mathfrak{L}^4(B)$.

(C-iv-a) $\|a(\cdot) - I\|_{\mathfrak{L}^2(B)}$, $\|D[a(\cdot)]\|_{\mathfrak{L}^3(B)}$ and $\|D^2[a(\cdot)]\|_{\mathfrak{L}^4(B)}$ are uniformly bounded and Lip 1.

However, (C-iii-a) and (C-iv-a) turn out to be equivalent to (C-iii) and (C-iv). Let us assume (C-iii-a) plus the continuity of $x \to \|D[a(\cdot)]\|_{\mathfrak{L}^3(B)}$, which is assured by (C-iv-a). We have, for each x and y in B

$$a(x + y) = a(x) + \int_0^1 \big(D[a(x + sy)], y\big)\, ds$$

and so

$$\big[a(x + y) - I\big] = \big[a(x) - I\big] + \int_0^1 \big(D[a(x + sy)], y\big)\, ds.$$

The identity holds in $\mathfrak{L}^2(H)$. But since the integrand is strongly s-continuous as in $\mathfrak{L}^2(B)$-valued function, we see that the identity holds in $\mathfrak{L}^2(B)$. It now follows that $a(x) - I$ is B-differentiable as an $\mathfrak{L}^2(B)$-valued function. We similarly deduce twice B-differentiability of $a(x) - I$ as an $\mathfrak{L}^2(B)$-valued function.

We will show that the assumptions of Theorem 2 enable us to interpret each of the terms in the expressions (8), (12), (13), of $s_t f(x)$ and its H-derivatives as measurable functions on B, to demonstrate that $D^2 s_t f(x)$ is of trace class, and to obtain $\{M_t(x, dy)\}$ satisfying (B-i), (B-ii), (B-iii). The methods of [17] then apply to enable construction of a fundamental solution as earlier described in this section.

In order to analyze the terms of $D^2 s_t f(x)$, we depart from the estimates of [17] and make repeated use of the following two facts:

1. There exists an $\alpha > 0$ such that $\int_B e^{\alpha\|y\|^2} p_1(dy) < \infty$. It follows that $\|y\|^p$ is integrable with respect to $p_t(dy)$ for all $p, t > 0$.

2. Recall that the restriction mapping is a continuous injection of B^* into H^*; that H^* may be identified with H via the Riesz representation; and that the injection of H into B is continuous. The restriction to H of a continuous linear mapping of B into B^* is a trace class operator on H. Moreover,

$$\|\cdot\|_{\mathrm{tr}} \leq \int_B \|y\|^2 p_1(dy) \cdot \|\cdot\|_{\mathfrak{L}^2(B)}. \tag{14}$$

1 is due to X. Fernique [8]. 2 is due to V. Goodman and makes use of 1. Proofs of 1 and 2 may be found in [15].

We will now observe that $\{a(x)^{-1}: x \in B\}$ also satisfies the conditions (C-iii) and (C-iv) of Theorem 5. By (C-i), $\{a(x)^{-1}\}$ is a uniformly bounded family in $\mathfrak{L}(B, B)$. Hence we see that $x \to a(x)^{-1}$ is twice B-differentiable as an $\mathfrak{L}(B, B)$ valued function, and the second B-derivative is uniformly bounded and Lip 1. Writing $a(x)^{-1} - I = -(a(x) - I)a(x)^{-1}$, we conclude that $a(x)^{-1}$ satisfies (C-iii) and (C-iv).

Since $a(x)$ is of the form I + a member of $\mathfrak{L}^2(B)$, $a(x)\colon H \to H$ and $[a(x) - I]|_H$ is of trace class. In order to guarantee that the $[\det a(y)]^{-1/2}$ factor in the expression (8) for $s_t(x, dy)$ is well defined we need only show that $a(x)|_H$ is uniformly positive definite. Since $[a(x) - I]|_H$ is compact and symmetric, it is diagonalizable. Hence $a(x)|_H$ is diagonalizable, and we see from (C-i) that the lowest eigenvalue must exceed ε. Hence $a(x) \geqslant \varepsilon I$ for all x. It follows from the considerations of [17, p. 98] that $y \to [\det a(y)]^{-1/2}$ is well defined, uniformly bounded and Lip 1. The exponential factor of $s_t(x, dy)$ is well defined since $a(x)^{-1} - I$ belongs to $\dot{\mathfrak{L}}^2(B)$. Since the B^*–B pairing coincides on $B^* \times H$ with the H inner product pairing $(\cdot,\cdot)$, we will use the inner product notation for both pairings. The validity of the extendibility properties (C-iii) for $a(x)^{-1}$ yields an obvious interpretation of each term in the integrands of (12) and (13) except perhaps those of the form $([a(x)^{-1}]h, y)$. But these are terms (k, y) where $k \in H$ and $y \in B$, which we earlier defined as elements of $L^2(p_1)$.

We turn now to the application of $L_{x,t}$ to $s_t f(x)$. Following [17, p. 105] we split $(D^2 s_t f(x)k, h)$ into two parts, $(V_t(x)k, h) + (W_t(x)k, h)$:

$$(V_t(x)k, h) = -(4t)^{-1}\int_B f(y)\Big\{2\big(\big[a(x)^{-1}\big]h, k\big) - t^{-1}\big(\big[a(x)^{-1}\big]h, x-y\big)\big(\big[a(x)^{-1}\big]k, x-y\big)\Big\}s_t(x, dy), \tag{15}$$

$$\begin{aligned}(W_t(x)k, h) = -(4t)^{-1}\int_B f(y)\Big\{&\big(D^2\big[a(x)^{-1}\big]kh(x-y), x-y\big) \\ &-2\big(D\big[a(x)^{-1}\big]hk + D\big[a(x)^{-1}\big]kh, x-y\big) \\ &-(4t)^{-1}\Big[\big(D\big[a(x)^{-1}\big]h(x-y), x-y\big) \\ &\cdot\big(D\big[a(x)^{-1}\big]k(x-y), x-y\big) \\ &-2\big(\big[a(x)^{-1}\big]h, x-y\big)\big(D\big[a(x)^{-1}\big]k(x-y), x-y\big) \\ &-2\big(\big[a(x)^{-1}\big]k, x-y\big) \\ &\cdot\big(D\big[a(x)^{-1}\big]h(x-y), x-y\big)\Big]\Big\}s_t(x, dy).\end{aligned} \tag{16}$$

By temporarily assigning the new inner product $\{h, k\} \equiv ([a(x)^{-1}]h, k)$ to H, $(V_t(x)k, h)$ is seen to be similar to formulas derived by Gross **[11]** for the second Fréchet derivative of the solution to the heat equation on the new inner product space. It then follows from the work of Gross that $(V_t(x)k, h)$ is of trace class, and that

$$(\partial/\partial t)s_t f(x) = \operatorname{trace}[a(x)V_t(x)]. \tag{17}$$

Only the invertibility of $a(x)$ and the fact that $a(x)$ is of the form I + a Hilbert-Schmidt operator are used for this part of the calculation.

Remark. In finite dimensions, the parametrix usually differs from ours (see equation (8)) in that the exponential term has $a(y)^{-1}$ rather than $a(x)^{-1}$. Then x-differentiation of the parametrix does not involve differentiation of the $a(\cdot)$ coefficients, and the fundamental solution of $L_{x,t}u = 0$ can be shown to exist assuming only boundedness plus a Hölder condition on $a(\cdot)$. However this modified parametrix is very difficult to work with in infinite dimensions. Equation (17) tells us that the singular behavior of the time derivative of the parametrix is exactly cancelled by part of the second derivative of the parametrix. This is why we have chosen (8) as the parametrix. Attempts to work with the modified parametrix have not as yet produced an infinite dimensionally interpretable term to cancel out the singular behavior of the time derivative of the parametrix.

We observe that $(D^2[a(x)^{-1}](\cdot)(\cdot)(x-y), x-y)$ is in $\mathfrak{L}(B, B^*)$. Hence

$$\left\|\left(D^2\left[a(x)^{-1}\right](\cdot)(\cdot)(x-y), x-y\right)\right\|_{\mathrm{tr}} \leqslant \text{const} \cdot \|x-y\|_B^2.$$

From the definition (8) of $s_t(x, dy)$, we see that it is absolutely continuous with respect to $p_{2t}(x, dy)$. Calculations on p. 99 of **[17]** show that the Radon-Nikodým derivative of $s_t(x, dy)$ with respect to $p_{2t}(x, dy)$ is in $L^{1+\lambda}(p_{2t}(x, dy))$ for all positive λ sufficiently close to zero. The $L^{1+\lambda}$ norm is uniformly bounded for all x in B. We easily conclude that

$$(-4t)^{-1}\int_B f(y)\left(D^2\left[a(x)^{-1}\right](\cdot)(\cdot)(x-y), x-y\right)s_t(x, dy)$$

is of trace class. Making the change of variable $y \to \sqrt{2t}\, y + x$, we obtain $\|\cdot\|_{\mathrm{tr}} \leqslant \text{const} \cdot \|f\|_\infty$. The constant is independent of f, t and x. Dominated convergence now enables us to interchange the trace and integral operations.

Similar considerations lead to the conclusion that

$$2(4t)^{-1}\int_B f(y)\left(D\left[a(x)^{-1}\right]hk + D\left[a(x)^{-1}\right]kh, x-y\right)s_t(x, dy)$$

extends to a member of $\mathfrak{L}(B, B^*)$, and hence is of trace class with $\|\cdot\|_{\mathrm{tr}} \leqslant \text{const} \cdot t^{-1/2} \cdot \|f\|_\infty$. As above, we can interchange the trace and integral operations.

We identify the term $(D[a(x)^{-1}](\cdot)(x-y), x-y)$ with a bounded linear functional z on H, such that $|z|_H \leqslant \text{const} \cdot \|x-y\|_B^2$. Then the operator defined by

$$\left(D\left[a(x)^{-1}\right]h(x-y), x-y\right) \cdot \left(D\left[a(x)^{-1}\right]k(x-y), x-y\right)$$

is identifiable with $z \otimes z$. Since $\|z \otimes z\|_{\text{tr}} = |z|_H^2$, we can easily show that

$$(4t)^{-2}\int_B f(y)\left\{\left(D\left[a(x)^{-1}\right](\cdot)(x-y), x-y\right)\right.$$
$$\left.\otimes\left(D\left[a(x)^{-1}\right](\cdot)(x-y), x-y\right)\right\} \cdot s_t(x, dy)$$

is of trace class, with $\|\cdot\|_{\text{tr}} \leqslant \text{const} \cdot \|f\|_\infty$, and that we can interchange the trace and integral operations.

The last term of $(W_t(x)k, h)$ to consider is

$$(Sh, k) \equiv -2(4t)^{-2}\int_B f(y)\left\{\left(\left[a(x)^{-1}\right]h, x-y\right)\right.$$
$$\cdot\left(D\left[a(x)^{-1}\right]k(x-y), x-y\right)$$
$$\left.+\left(\left[a(x)^{-1}\right]k, x-y\right)\left(D\left[a(x)^{-1}\right]h(x-y), x-y\right)\right\}s_t(x, dy). \quad (18)$$

We will show that the operator S is of trace class by applying the test operator method developed by L. Gross in [11]. A test operator is a bounded operator T of finite rank from B to B, whose range is contained in B^*. The set of restrictions to H of test operators is dense in the space of completely continuous operators on H. The space of completely continuous operators on H with $\mathfrak{L}^2(H)$ norm is well known to be dual to the space of trace class operators with trace norm under the pairing $\langle A, C\rangle = \text{trace}[C^*A]$. Thus we need only show that

$$|\text{trace}[T^*S]| \leqslant \text{const} \cdot \|T\|_{\mathfrak{L}^2(H)} \quad (19)$$

for all test operators T in order to conclude that S is of trace class. Since S is symmetric, it is easy to see that without loss of generality we need only prove (19) for those T whose restriction to H is symmetric. Such a T has the form $Tx = \sum_{i=1}^n \lambda_i\langle e_i, x\rangle e_i$ where $e_1, \ldots, e_n$ is an H-orthonormal subset of elements in B^*, and each $\lambda_i \in \mathbf{R}^1$. Since T is of finite rank, T^*S is of trace class. Noting that $(T^*Se_i, e_i) = (Se_i, Te_i) = \lambda_i(Se_i, e_i)$, we write

$$\text{trace}[T^*S] = -4(4t)^{-2}\int_B f(y)\left\{\sum_{i=1}^n \lambda_i\left(\left[a(x)^{-1}\right]e_i, x-y\right)\right.$$
$$\left.\cdot\left(D\left[a(x)^{-1}\right]e_i(x-y), x-y\right)\right\}s_t(x, dy).$$

We easily observe that $([a(x)^{-1}]e_i, x - y) = (e_i, [a(x)^{-1}](x - y))$, and so $\lambda_i([a(x)^{-1}]e_i, x - y)$ may be rewritten as $(Te_i, [a(x)^{-1}](x - y)) = (e_i, T \circ [a(x)^{-1}](x - y))$. Then

$$\operatorname{trace}[T^*S] = -4(4t)^{-2}\int_B f(y)$$
$$\cdot \left(D\left[a(x)^{-1}\right]\left\{\sum_{i=1}^{n}\left(e_i, T \circ \left[a(x)^{-1}\right](x - y)\right)e_i\right\}(x - y), x - y\right)s_t(x, dy)$$
$$= -4(4t)^{-2}\int_B f(y)$$
$$\cdot \left(D\left[a(x)^{-1}\right]\left\{T \circ \left[a(x)^{-1}\right](x - y)\right\}(x - y), x - y\right)s_t(x, dy).$$

We can now easily compute that

$$|\operatorname{trace}[T^*S]|$$
$$\leqslant \text{const} \cdot \|f\|_\infty \cdot t^{-2}\int_B \left\|T \circ \left[a(x)^{-1}\right](x - y)\right\|_B \|x - y\|_B^2 s_t(x, dy). \quad (20)$$

Gross [11, p. 172] has proved that for any test operator T

$$\int_B \|T_y\|_B^p\, p_t(dy) \leqslant \|T\|_{\mathcal{L}^2(H)} \int_B \|y\|_B^p\, p_t(dy)$$

for all $p \geqslant 1$ and $t > 0$. (Note that the $\mathcal{L}^2(H)$ norm occurs on the right side of the inequality; this is the norm we require in order to establish (19).) Several applications of Hölder's inequality to (20) enable us to conclude that

$$|\operatorname{trace}[T^*S]| \leqslant \text{const} \cdot \|f\|_\infty \cdot t^{-1/2} \cdot \|T\|_{\mathcal{L}^2(H)}$$

where the constant is independent of t, f, x and T. Thus S is of trace class, with $\|S\|_{\text{tr}} \leqslant \text{const} \cdot \|f\|_\infty \cdot t^{-1/2}$, and once again we can interchange the trace and integral operations in the calculation of trace S.

We conclude that $W_t(x)$, and hence $D^2 s_t f(x)$, are of trace class. This enables application of $L_{x,t}$ to $s_t f(x)$. Making use of (12)–(17) we obtain

$$L_{x,t} s_t f(x) = \operatorname{trace}\left[a(x) W_t(x)\right] + (b(x), D s_t f(x)) + c(x) s_t f(x). \quad (21)$$

All that remains for the proof of Theorem 2 is for us to verify the existence of a family of measure $\{M_t(x, dy)\}$ satisfying the bounds (B-i)–(B-iii) and such that

$$L_{x,t} s_t f(x) = \int_B f(y) M_t(x, dy).$$

In fact, the preceding calculations show that there exists a function $g(t, x, \cdot)$ in $L^1(s_t(x, \cdot))$ such that

$$L_{x,t} s_t f(x) = \int_B f(y)\, g(t, x, y) s_t(x, dy).$$

The existence of the bounds (B-i) has been noted. Making use of the Lipschitz hypotheses (C-iv) and (C-v) on the coefficients, it is straightforward to make the estimates

$$|g(x_1, y, t) - g(x_2, y, t)| \leqslant Ct^{-1/2}\gamma_1(y) \cdot \|x_1 - x_2\|_B$$

and

$$|g(x, y, t_1) - g(x, y, t_2)| \leqslant C|t_1^{-1/2} - t_2^{-1/2}| \cdot \gamma_2(y)$$

for a.e. y, where C is a constant, independent of $0 < t_1, t_2, t \leqslant T$ and of $x, x_1, x_2, y \in B$, and γ_1 and γ_2 are in $L^p(p_1)$ for all $1 \leqslant p < \infty$. Properties (B-ii) and (B-iii) now easily follow as in the proof of [17, Proposition 2].

The conclusion of Theorem 2 follows. □

We turn now to a description of the semifinite approximation of a fundamental solution as developed in [18]. We require an additional hypothesis on B, namely:

Assumption. There exists an orthonormal basis for H, consisting of vectors $\{e_n\}$ contained in B^*, such that the projections $\{P_n\}$ on B defined by $P_n x = \sum_{i=1}^{n} \langle e_i, x \rangle e_i$ $(x \in B)$ converge strongly to the identity operator on B.

This assumption is satisfied, in particular, when B is a classical Wiener space. However, we note that the assumption implies the existence of a Schauder basis for B; that any real separable Banach space can be the "B" of an abstract Wiener pair [10]; and that Enflo [7] has proved the existence of a separable Banach space which does not possess a Schauder basis. It is probable, but not presently clear to the author, that we might dispense with the assumption by making use of Kuo's construction [15, Corollary 4.2, p. 66] of an intermediate space B_0 such that $H \subset B_0 \subset B$, (H, B_0) is an abstract Wiener pair, B_0 is of full p_t measure, and such that the assumption holds with B replaced by B_0 throughout.

For a fixed n, let K denote the finite dimensional subspace $P_n B$. Set

$$a_n(x) = P_n a(P_n x) P_n, \qquad b_n(x) = P_n b(P_n x), \qquad c_n(x) = c(P_n x).$$

Then the finite dimensional parabolic equation

$$L_n u(x, t) = \operatorname{trace}_K\left[a_n(x) u_{xx}(x, t)\right] + (b_n(x), u_x(x, t)) + c_n(x) u(x, t) - u_t(x, t) = 0$$

has a fundamental solution consisting of a family of real-valued functions $\{q'(t, x, y)\colon t > 0, x, y \in K\}$. In particular, as a function of x and t, q' satisfies $L_n q' = 0$. ($\{q'(t, x, y) dy\}$ is a fundamental solution in the sense that we have been using on infinite dimensional spaces.) The property of $q'(t, x, y)$ that is of primary importance in establishing differentiability of $\{q_t(x, dy)\}$ is the smoothness in the y-variable. S. Ito [13] has shown that $\{q^*(t, x, y)\} \equiv \{q'(t, y, x)\}$ forms a fundamental solution of the adjoint differential equation

$L_n^* u = 0$, where

$$L_n^* u(x, t) = \sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j} a_n^{ij}(x) u(x, t) - \operatorname{div}\left[b_n(x) u(x, t) \right] + c_n(x) u(x, t) - u_t(x, t), \tag{22}$$

provided that the derivatives $[b_n(x)]_x$ are uniformly bounded and satisfy a Hölder condition. In particular, $q^*(t, x, y)$ is C^2 in the variable x. Hence $q'(t, x, y)$ is C^2 in the variable y.

Considering K as a subspace of B^*, let $K^\perp$ be the annihilator of K in B. Let p_t'' denote Wiener measure on $K^\perp$. For x and y in B, we make the decomposition $x = x' + x''$, $y = y' + y''$, x' and $y' \in K$, x'' and $y'' \in K^\perp$. Now we may define a family of Borel measures $\{q_t^n(x, dy)\}$ on B by

$$q_t^n(x, dy) = q'(t, x', y')dy' \times p_{2t}''(x'', dy''). \tag{23}$$

The exceedingly tedious calculations of **[18]** can be somewhat modified, by using techniques used in our proof of Theorem 2, to accommodate our current assumptions (C-i)–(C-v) on the coefficients $a(x)$, $b(x)$ and $c(x)$. In fact the calculations become considerably simpler, and we are readily able to prove the following semifinite approximation (see Proposition 6 of **[18]**):

PROPOSITION 3. *As $P_n \to I$, $q_t^n(x, dy) \to q_t(x, dy)$, in total variation norm, for each $x \in B$ and $t > 0$.*

Our next step is to establish differentiability of the approximating measures. We impose an additional hypothesis on the first order coefficients of $L_{x,t}$.

(C-vi) $b(x)$ is H-differentiable, and $|Db(\cdot)|_H$ is bounded and uniformly Lip 1 on B.

(C-vi) ensures that the adjoint equations (22) have fundamental solutions.

PROPOSITION 4. *Assume that the coefficients of $L_{x,t}$ satisfy* (C-i)–(C-v) *and also* (C-vi). *Then $\{q_t^n(x, dy)$: $t > 0, x \in B\}$ is a family of H-differentiable measures on B. Each $q_t^n(x, dy)$ has a locally Lipschitzian logarithmic derivative.*

PROOF. An excellent reference for bounds on the derivatives of fundamental solutions is the survey paper **[12]** of Il'in, Kalashnikov and Oleinik (see especially Theorem 4.1). One such bound asserts the existence of constants α and β such that

$$|(\partial/\partial x_i) q^*(t, x, y)| \leqslant \alpha |t|^{-(n+1)/2} \exp\left(-\beta\left(|x - y|^2/|t|\right)\right) \tag{24}$$

for all x, y in K, where α may depend on the maximum of $|t|$. From (24) we see that for each t and x, all directional derivatives $(\partial/\partial y_i) q'(t, x, y)$ are integrable over K with respect to y. It follows that $\{q'(t, x, y)dy\}$ is a family of K-differentiable measures on K.

From (23), we see that $q_t^n(x, dy)$ is the product of a K-differentiable measure with a $(K^\perp \cap H)$-differentiable measure. We conclude from [1, Proposition 2.3.1] that the product measure is weakly and boundedly differentiable with respect to $K \times (K^\perp \cap H)$, i.e. is H-differentiable. The expected product rule for the derivative holds: for $h = h' + h''$, $h' \in K$, $h'' \in K^\perp \cap H$, we have

$$d_h q_t^n(x, dy) = d_{h'}\left[q'(t, x', y')dy'\right] \times p''_{2t}(x'', dy'') + q'(t, x', y')dy' \times d_{h''} p''_{2t}(x'', dy'').$$

From this we see that the logarithmic derivative of $q_t^n(x, dy)$ is given by

$$\left\langle h, \frac{dDq_t^n(x, dy)}{dq_t^n(x, dy)} \right\rangle = \left\langle h', \frac{D_{y'}q'(t, x', y')}{q'(t, x', y')} \right\rangle - (2t)^{-1}\langle h'', y'' - x''\rangle. \quad (25)$$

Since $q'(t, x', y') > 0$ [**12**, p. 67] and $q'(t, x', y')$ is C^2 in y', the logarithmic derivative of $q_t^n(x, dy)$ is immediately seen from (25) to be locally Lipschitzian.

THEOREM 5. *Assume that the coefficients of the parabolic equation $L_{x,t}u = 0$ satisfy the hypotheses of Theorem* 2 *and also satisfy* (C-vi). *Then the fundamental solution $\{q_t(x, dy)\}$ is a family of H-differentiable measures each with locally Lipschitzian logarithmic derivative.*

PROOF. For H-differentiability of $q_t(x, dy)$ it suffices to show that $\{d_h q_t^n(x, \cdot)\}_{n=1,2,\ldots}$ converges in variation uniformly for all h with $|h| < \delta_0$. In order for the logarithmic derivative of $q_t(x, dy)$ to exist and be locally Lipschitzian it suffices that the sequence of logarithmic derivatives given by (25) converges uniformly for y in B-bounded subsets and h in B^*-bounded subsets. The required convergence may be established by a lengthy but straightforward application of the estimates of the semifinite approximation [**18**] provided we alter the estimates (24) on the derivatives of the finite dimensional fundamental solutions in the following manner so that we may estimate their behavior as $n \to \infty$.

The parametrix used by Il'in, Kalashnikov and Oleinik [**12**] for the derivation of (24) is

$$W^*(t, x', y') = (4\pi t)^{-n/2}\left[\det a(y')\right]^{-1/2} \cdot \exp\left(-\left(a(y')^{-1}(x' - y'), x' - y'\right)/4t\right)$$

which we may write in dimension-free notation as

$$W^*(t, x', y')\, dy = \left[\det a(y')\right]^{-1/2} \cdot \exp\left(-\left(\left[a(y') - I\right](x' - y'), x' - y'\right)/4t\right)p'_{2t}(x', dy').$$

A brief examination of the derivation in [12] of (24) shows that the estimates on x-derivatives of $q^*(t, x', y')$ are inherited from corresponding estimates on derivatives of the parametrix. We easily see that

$$|D_{x'}W^*(t, x', y')|_K \leq \alpha\|x' - y'\|_B|t|^{-1}\exp\left(-\left(a(y')^{-1}(x' - y'), x' - y'\right)/4t\right)$$

and hence obtain

$$|D_{x'}q^*(t, x', y')|(dy) \leq \alpha\|x' - y'\|_B|t|^{-1} \cdot \exp\left(-\left(\left[a(y')^{-1} - I\right](x' - y'), x' - y'\right)/4t\right)p'_{2t}(x', dy').$$

Using the fact that

$$q_t^n(x, dy) = (\text{positive function}) \cdot \left(\exp\left(-\left(\left[a(y')^{-1} - I\right](x' - y'), x' - y'\right)/4t\right)p_{2t}(x, dy)\right),$$

some additional computation yields (for $n > m$)

$$\left|\frac{dDq_t^n(x', dy')}{dq_t^n(x', dy')} - \frac{dDq_t^m(x', dy')}{dq_t^m(x', dy')}\right|_B \leq \alpha|t|^{-1}\|(P_n - P_m)(x' - y')\|_B. \quad (26)$$

The above estimates are sufficient for us to conclude the validity of Theorem 5. □

REMARK. It is probable that the fundamental solutions of $L_{x,t}u = 0$ and an associated "adjoint" equation analogous to (22) are obtainable from one another as is the case in finite dimensions. That is, if we write the fundamental solution $\{q_t(x, dy)\}$ of $L_{x,t}u = 0$ as $\{f(t, x, y)p_t(x, dy)\}$, then $L^*_{x,t}u = 0$ has as fundamental solution $\{w_t(x, dy)\}$ where $w_t(x, dy) = f(t, y, x)p_t(x, dy)$. If this were the case, then the differentiability of the measure $q_t(x, dy)$ could be deduced from the x-differentiability of $w_t(x, dy)$. However the proof of the relationship between the fundamental solutions of L_n^* and L_n in finite dimensions depends on their being adjoint operators in $L^2(\mathbf{R}^n, dx)$. There is no reasonable extension of Lebesgue measure to infinite dimensions, and furthermore the infinite dimensional operators $L_{x,t}$ and $L^*_{x,t}$ are not adjoint operators in $L^2(\mathbf{R}^n, dp_t)$. In fact Gross' Laplacian $\Delta f(x) \equiv \text{trace } D^2f(x)$ has no adjoint in $L^2(\mathbf{R}^n, dp_t)$. It appears that the relationship, if any, between $q_t(x, dy)$ and $w_t(x, dy)$ could not be proved directly in infinite dimensions; but rather would have to be proved by a semifinite approximation, based on the validity of the finite dimensional relationship.

REFERENCES

1. V. I. Averbuh, O. G. Smoljanov and S. V. Fomin, *Generalized functions and differential equations in linear spaces*. I. *Differential measures*, Trans. Moscow Math. Soc. **24** (1971), 140–184.

2. ______, *Generalized functions and differential equations in linear spaces*. II. *Differential operators and their Fourier transforms*, Trans. Moscow Math. Soc. **27** (1972), 255–270.

3. R. Bonic and J. Frampton, *Smooth functions on Banach manifolds*, J. Math. and Mech. **15** (1966), 866–898.

4. N. Dunford and J. T. Schwartz, *Linear operators*. Part I, Interscience, New York, 1957.

5. C. M. Elson, *An extension of Weyl's lemma to infinite dimensions*, Trans. Amer. Math. Soc. **194** (1974), 301–324.

6. K. D. Elworthy, *Gaussian measures on Banach spaces and manifolds*, Proc. Summer Institute on Global Analysis, Trieste, 1972.

7. P. Enflo, *A counterexample to the approximation problem in Banach spaces*, Acta Math. **130** (1973), 309–317.

8. X. Fernique, *Intégrabilité des vecteurs Gaussians*, C. R. Acad. Sci. Paris Ser. A **270** (1970), A1698–A1699.

9. T. Goldring, *Decomposition theorems for differential forms on Hilbert spaces*, Thesis, Yeshiva University (1977).

10. L. Gross, *Abstract Wiener spaces*, Proc. Fifth Berkeley Sympos. on Mathematical Statistics and Probability, Vol. 2, Part I, Univ. California Press, Berkeley, Calif., 1967, pp. 31–42.

11. ______, *Potential theory on Hilbert space*, J. Functional Analysis **1** (1967), 123–181.

12. A. M. Il'in, A. S. Kalashnikov and O. A. Oleinik, *Linear equations of the second order of parabolic type*, Russian Math. Surveys **17** (1962), 1–143.

13. S. Ito, *The fundamental solution of the parabolic equation in a differentiable manifold*, Osaka Math. J. **5** (1953), 75–92.

14. H.-H. Kuo, *Diffusion and Brownian motion on infinite-dimensional manifolds*, Trans. Amer. Math. Soc. **169** (1972), 439–457.

15. ______, *Gaussian measures in Banach spaces*, Lecture Notes in Math., vol. 463, Springer-Verlag, Berlin and New York, 1975.

16. ______, *Uhlenbeck-Ornstein process on a Riemann-Wiener manifold*, Proceedings of a special session on vector measures, San Antonio, Texas, 1976 (to appear).

17. M. A. Piech, *A fundamental solution of the parabolic equation on Hilbert space*, J. Functional Analysis **3** (1969), 85–114.

18. ______, *A fundamental solution of the parabolic equation on Hilbert space*. II. *The semigroup property*, Trans. Amer. Math. Soc. **150** (1970), 257–286.

19. ______, *The exterior algebra for wiemann manifolds*, J. Functional Analysis **28** (1978), 279–308.

20. R. Ramer, *Integration on infinite-dimensional manifolds*, Thesis, University of Amsterdam (1974).

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