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SOME CONSTRUCTIONS OF INFINITE MÖBIUS PLANES

BY

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ABSTRACT. New infinite Möbius planes are constructed using transfinite induction. Any infinite affine plane A can be embedded in a Möbius plane M and the construction allows some groups of perspectivities of A to be extended to automorphism groups of M. Given $\{A_\alpha, \alpha \in J\}$, an infinite collection of s affine planes each of order s, there exist a Möbius plane M and a bijection b from J to the point set of M so that for each $\alpha \in J$, $M_{b(\alpha)}$ is isomorphic to A_α.

1. Introduction. Abstract Möbius planes were first introduced by van der Waerden and Smid in 1935 [11]. The primitive notions are "point", "circle" and "incidence", based on the incidence properties of points and nontrivial planar sections of the unit sphere in real 3-space. The theory for the infinite case has been developed and extended by Benz [1], [2], and by Ewald [4]–[6].

In this work some new infinite Möbius planes are constructed. The technique uses the axiom of choice in its well-ordering and transfinite induction version. Examples are produced of Möbius planes with a given group of automorphisms acting in a prescribed way [Theorem 3.1]. In particular [Corollary 3.2], every infinite translation plane can be embedded in a Möbius plane such that the translations extend to automorphisms of the Möbius plane. Next the question of what types of affine planes can be the associated affine planes of a Möbius plane is considered. This question is satisfactorily answered in Theorem 3.3. As corollaries there exist Möbius planes with no nontrivial automorphisms and there exist Möbius planes which are not ovoidal but all of whose associated affine planes are pappian and isomorphic to each other. In the proof of Theorem 3.1 many ovals are constructed in an infinite affine plane. The concepts needed for the construction are isolated in §2 in the notion of an "infinite space" and the definition of oval is generalized. The appplication of this small section to the main thrust of the work (apart from a gentle introduction to the complexities of the proof of Theorem 3.1) is Corollary 4, which guarantees many examples of ovoidal Möbius planes.

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The following notational conventions are observed: small Latin letters refer to elements of a set, capital Latin letters refer to sets, capital Greek letters refer to families of sets, and small Greek letters refer to ordinal numbers.

Several standard definitions are listed here as the axioms are referred to by number later in the text. For a more complete discourse see [3] or [9].

DEFINITION 1.1. An *affine plane* $\bar{A}$ is an incidence structure (P, Λ, I) of points $p \in P$, $L \in \Lambda$ satisfying the conditions: (A1) each pair of distinct points of P is on exactly one line $L \in \Lambda$, (M2) given a point q, a line L with q not on L, there is a unique line M with q on M such that no point of L is on M, (M3) there are three points not all on one line.

DEFINITION 1.2. A *projective plane* Π is an incidence structure (P, Λ, I) of points $p \in P$ and lines $L \in \Lambda$ satisfying the conditions: (P1) each pair of distinct points x, y are on exactly one line (denoted xy), (P2) each pair of distinct lines L, M is on exactly one point (denoted $L \cap M$), (P3) there are four points no three of which are on a line.

DEFINITION 1.3. A *Möbius plane* $\bar{M}$ is an incidence structure (P, Ξ, I) of points $p \in P$ and circles $C \in \Xi$ satisfying the conditions (M1) each triple of points are on exactly one circle C, (M2) given $P \; I \; C$, $q \not I \; C$, there is a unique circle D containing p and q meeting C only at p (D is called tangent to C at p), (M3) there are four points not all on one circle and each circle contains at least one point.

THEOREM 1.4. *$\bar{M}$ is a Möbius plane if and only if for each point p, the internal structure $M_p = (P_p, \Lambda_p, I_p)$ is an affine plane where $P_p = P \setminus \{p\}$, $\Lambda_p = \{C \in \Xi : p \; I \; C\}$, $I_p = (P_p \times \Lambda_p) \cap I$.*

DEFINITION 1.5. The *order* of an affine plane is the cardinal number of points on one line. The *order* of a projective plane is the order of one of the affine planes associated with it. The *order* of a Möbius plane is the order of one of the affine planes associated with it.

DEFINITION 1.6. In a projective or affine plane, an *oval* is a nonempty set of points Ω such that (i) no three distinct points of Ω are on a line, and (ii) for $p \in \Omega$, there is a unique line containing p which meets Ω only at P (called the tangent line to Ω at p).

DEFINITION 1.7. In a projective 3-space an *ovoid* is a nonempty set of points Ω such that (i) no three points of Ω are on a line, and (ii) for each $p \in \Omega$, the set of lines meeting Ω only at p is the set of all lines through p in some plane $\Pi(p)$. The plane $\Pi(p)$ is called the tangent plane to Ω at p.

It is easy to verify that from an ovoid Ω, with the points of Ω as points, the nontrivial planar intersections of Ω as circles and containment as incidence, a Möbius plane may be constructed, denoted $M(\Omega)$. Any Möbius plane isomorphic to $M(\Omega)$ for some ovoid Ω, will be called *ovoidal*.

2. Ovals.

OVALS IN INFINITE SPACES.

DEFINITION 2.1. Let P be a set of points and let Λ, Φ be two collections of subsets of P, whose members are called lines and flats, respectively. The point p is on the line L (flat F) if and only if $p \in L$ ($p \in F$), and the line L is on the flat F if and only if $L \subset F$. Let s be an infinite cardinal. Then $\bar{S} = (P, \Lambda, \Phi)$ is *an infinite space of cardinality* s if and only if the following conditions are satisfied:

(1) $|P| = |\Lambda| = |\Phi| = s$,

(2) each pair of distinct points is on exactly one line,

(3) each point is on s lines and each line contains s points,

(4) each point is on s flats and each flat contains s points,

(5) if $F_1, F_2 \in \Phi$ and $F_1 \subset F_2$, then $F_1 = F_2$,

(6) if $L \in \Lambda$, $F \in \Phi$ and if $L \not\subseteq F$, then $|L \cap F| = 0$ or 1,

(7) if $B \subseteq P$, $p \in P \setminus B$ and $|B| < s$, then there exists $F \in \Phi$ such that $p \in F$ and $F \cap B = \varnothing$.

Note that $\Lambda \cap \Phi = \varnothing$ is not required. If $\Lambda = \Phi$ it is easily seen that (1)–(3) imply (4)–(7).

EXAMPLES. (a) In any affine or projective space R of dimension $n > 2$ and of infinite cardinality s, let P be the set of all points, Λ the collection of all lines and Φ the collection of all d-dimensional subspaces, where d is fixed and $1 < d < n - 1$. Then $\bar{S} = (P, \Lambda, \Phi)$ is an infinite space of cardinality s. (b) If in (a) R is Euclidean, then the structure induced by $\bar{S}$ on any nonempty open subset of R is also an infinite space of cardinality s.

DEFINITION 2.2. A *semioval* in the infinite space $S = (P, \Lambda, \Phi)$ is a pair (Ω, T) with $\varnothing \neq \Omega \subseteq P$, $T \subseteq \Phi$ such that (i) no three distinct points of Ω are collinear, (ii) there is a bijection φ between Ω and T such that if $p \in \Omega$, then $\varphi(p) \cap \Omega = \{p\}$. A *tangent flat* to the semioval Ω at p is a flat F such that $|F \cap \Omega| = 1$. For $p \in \Omega$, $\varphi(p)$ shall be called *the* tangent flat to Ω at p. An *oval* is a semioval such that (iii) if $F \in \Phi$ and if $|F \cap \Omega| = 1$, then $F \in T$.

Condition (iii) guarantees that the tangent-flat at a given point of Ω is unique. Note that if $\bar{S}$ is a projective or affine plane with P the set of all points and $\Lambda = \Phi$ the set of all lines, then this definition of oval is equivalent to the standard definition (Definition 1.6).

The class of semiovals in a given infinite space can be partially ordered by defining $(\Omega_1, T_1) < (\Omega_2, T_2)$ if and only if $\Omega_1 \subset \Omega_2$ and $T_1 \subseteq T_2$.

LEMMA 2.1. *Let* $\Xi = \{(\Omega_i, T_i): i \in I\}$ *be an ascending chain of semiovals in the infinite space* $\bar{S}$. *Then* $(\bigcup_{i \in I}\Omega_i, \bigcup_{i \in I}T_i)$ *is a semioval.*

PROOF. Write $\Omega = \bigcup_{i \in I}\Omega_i$, $T = \bigcup_{i \in I}T_i$. (i) If $p, q, r \in \Omega, p \neq q \neq r \neq p$, then, for some $i \in I$, $p, q, r \in \Omega_i$, and hence they are not collinear. (ii) If

$(\Omega_i, T_i) < (\Omega_j, T_j)$ and if φ_i, φ_j are the associated bijections, then $p \in \Omega_i$ implies that $\varphi_j(p) = \varphi_i(p)$, and hence that $\varphi_i(p) \cap \Omega_j = \{p\}$. For, since $\varphi_i(p) \in T_j$, $\varphi_i(p)$ is the tangent-flat to Ω_j at some point, and this can only be p. If $p \in \Omega$, define $\varphi(p) = \varphi_i(p)$, where i is the first index for which $p \in \Omega_i$. It is then easily verified that φ is a bijection between Ω and T which satisfies the required conditions.

THEOREM 2.1. *In an infinite space* (P, Λ, Φ) *of cardinality* s, *every semioval* (Ω_0, T_0) *such that* $|\Omega_0| < s$ *can be embedded in an oval* (Ω, T).

PROOF. Let σ be the smallest ordinal number such that the cardinal number associated with σ is s. Then there is a one-one correspondence between the ordinals α with $1 \leqslant \alpha < \sigma$ and the lines $L \in \Lambda$, which serves to well order Λ. It will be shown that for each α with $0 \leqslant \alpha < \sigma$ we can find Ω_α, T_α satisfying: (I) $(\Omega_\alpha, T_\alpha)$ is a semioval, (II) if $0 \leqslant \alpha' < \alpha < \sigma$ then $(\Omega_{\alpha'}, T_{\alpha'}) < (\Omega_\alpha, T_\alpha)$, (III) $|\Omega_\alpha| = |\Omega_0| + |\{\gamma\colon \gamma \leqslant \alpha\}| < s$. For $\alpha = 0$, the conditions are just the hypotheses of the theorem. Let $0 < \beta < \sigma$. Suppose that for each $\alpha < \beta$ we have constructed $(\Omega_\alpha, T_\alpha)$ satisfying (I)–(III). Define $\Omega_\beta^* = \bigcup_{\alpha<\beta}\Omega_\alpha$, $T_\beta^* = \bigcup_{\alpha<\beta}T_\alpha$. By Lemma 2.1, $(\Omega_\beta^*, T_\beta^*)$ is a semioval, and $|\Omega_\beta^*| \leqslant |\Omega_0 \cup \{\alpha\colon \alpha \leqslant \beta\}| < s$. Let $A_\beta = \{L \in \Lambda\colon$ there exists $p \in \Omega_\beta^*$ such that $p \in L$, $|L \cap \Omega_\beta^*| = 1$ and $L \not\subseteq \varphi_\beta(p)\}$, $B_\beta = \{L \in \Lambda\colon L = p'p''$, where $p' \neq p''$ and $p', p'' \in \Omega_\beta^*\}$. We first show that $A_\beta \neq \varnothing$. Let $p \in \Omega_\beta^*$, $p \neq q \in \varphi(p)$. Then, by Definition 2.1(7) there is a flat F containing p such that $F \cap (\Omega_\beta^* \cup \{q\}) = \{p\}$. By (5) and since $q \notin F$, $F \not\subseteq \varphi(p)$, so let $r \in F \setminus \varphi(p)$. Then the line pr is contained in F and so meets Ω_β^* only at p. Furthermore, since $r \notin \varphi(p)$, the line pq is in A_β.

Let γ be the smallest ordinal such that $L_\gamma \in A_\beta$. Suppose $p \in \Omega_\beta^*$ and $p \in L_\gamma$. Then L_γ meets each tangent flat of T_β^* in at most one point. For, otherwise, by Definition 2.1(6), $L_\gamma \subseteq \varphi(q)$ for some $q \in \Omega_\beta^*$. Since $p \in L_\gamma$, $p \in \varphi(q)$, so $q = p$. But, since $L_\gamma \in A_\beta$, L_γ meets $\varphi(p)$ only at p. Since there are less than s tangent flats, there are less than s points of L_γ which lie on some flat in T_β^*. Since $|\Omega_\beta^*| < s$, $|B_\beta| < s$. Now, $L_\gamma \notin B_\beta$ and each line of B_β meets L_γ in at most one point by Definition 2.1(2), so there are less than s points of L_γ on any line of B_β. Thus, there are s points of L_γ on no line of B_β and on no flat of T_β^*. Let x be one of these points. By Definition 2.1(7), there is a flat F_x such that $x \in F_x$ and $F_x \cap \Omega_\beta^* = \varnothing$. Set $\Omega_\beta = \Omega_\beta^* \cup \{x\}$, $T_\beta = T_\beta^* \cup \{F_x\}$. Then, by the choice of x, F_x, (Ω_β, T_β) is a semioval, with $|\Omega_\beta| < s$. Hence, we can find an ascending chain of semiovals $(\Omega_\alpha, T_\alpha)$ which satisfy (I)–(III).

Let $\Omega = \bigcup_{\alpha<\sigma}\Omega_\alpha$, $T = \bigcup_{\alpha<\sigma}T_\alpha$. By Lemma 2.1, (Ω, T) is a semioval. In order to show that (Ω, T) is an oval, it suffices to show that $A_\sigma = \varnothing$. For suppose that F is a flat such that $F \cap \Omega = \{p\}$, and that $F \neq \varphi(p)$. Then there exists q such that $q \in F$, $q \notin \varphi(p)$, and the line $pq \in A_\sigma$. Hence, suppose $A_\sigma \neq \varnothing$. Then there exists L_γ, with $\gamma < \sigma$ such that $L_\gamma \cap \Omega = \{p\}$ and $L_\gamma \nsubseteq \varphi(p)$. For some $\alpha < \sigma, p \in \Omega_\alpha$. Then, for each $\beta > \alpha$, $L_\gamma \in A_\beta$, so the line removed from A_β at stage β has index less than γ. Thus there is an injection from the set of ordinals γ with $\alpha < \lambda < \sigma$ into the set of ordinals μ with $0 \leqslant \mu < \gamma$ which implies that $|\{\mu\colon 0 \leqslant \mu < \gamma\}| \geqslant s$, a contradiction. This completes the proof of Theorem 2.1.

COROLLARY 1. *There is a bounded oval in the real plane which is not the boundary of a convex set.*

PROOF. In the real plane, let $\bar{S}$ be the infinite space induced in the interior of the unit circle as in Example (b). Let Ω_0 consist of four points no three of which are collinear and which do not form a convex quadrangle, and let T_0 consist of four lines each on one and only one point of Ω_0. Apply Theorem 2.1 to (Ω_0, T_0) in $\bar{S}$. The resulting oval is clearly also an oval in the entire plane, and is bounded but is not the boundary of a convex set.

COROLLARY 2. *There is a bounded oval in real n-space not the boundary of a convex set.*

This is a simple generalization of Corollary 1, with the convention that flats are the $n - 1$ dimensional subspaces of real n-space (cf. Example (a)).

COROLLARY 3. *Every infinite projective plane contains an oval.*

PROOF. Every infinite projective plane can be regarded as an infinite space as in Example (a). Let (Ω_0, T_0) be any incident point-line pair and apply Theorem 2.1. (It is not known whether every finite projective plane must contain an oval.)

Let $\bar{S} = (P, \Lambda, \Phi)$ be an infinite projective 3-space, represented as in Example (a), with $d = 2$. In the course of proving Theorem 2.1, it was shown that $A_\sigma = \varnothing$, i.e., if $L \in \Lambda$ and if $L \cap \Omega = \{p\}$, then $L \subseteq \varphi(p)$. Hence if Theorem 2.1 is applied to $\bar{S}$, the oval (Ω, T) is in fact an ovoid. We define a semiovoid in $\bar{S}$ to be a semioval as defined in Definition 2.2. The next corollary now follows immediately from Theorem 2.1.

COROLLARY 4. *In an infinite projective 3-space of cardinality s, every semi-ovoid (Ω_0, T_0) with $|\Omega_0| < s$ can be embedded in an ovoid (Ω, T).*

3. Infinite Möbius planes. To a given Möbius plane there are associated affine planes (Theorem 1.4). The question then arises, "Can a given affine plane $\bar{A}$ be embedded in a Möbius plane $\bar{M}$ such that for some point u, $\bar{M}_u$ is isomorphic to $\bar{A}$?" In such a case, the affine plane $\bar{A}$ is called *embeddable*.

In this section, two different methods are given for embedding an arbitrary infinite affine plane into a Möbius plane. A second question that may be asked is: "Given a group G of collineations of $\bar{A}$, can $\bar{A}$ be embedded in a Möbius plane $\bar{M}$ so that the collineations of G extend to automorphisms of $\bar{M}$?" In such a case G will be called *extendable*. This question is partly answered in Theorem 3.1.

LEMMA 3.1. *If $\bar{M}$ is a Möbius plane and $\bar{M}_u$ an associated affine plane, then a circle C in $\bar{M}$ not containing u is an oval in $\bar{M}_u$.*

PROOF: (i) By condition (M1) no three distinct points of C are collinear, and (ii) given $p\ I\ C$, there is, by (M2), a unique circle D on u, such that D meets C only at p. But in $\bar{M}_u$, D is a tangent line to C at p. Thus C is an oval.

LEMMA 3.2. *An affine plane $\bar{A}$ is embeddable in a Möbius plane $\bar{M}$ if and only if there is a family Ξ of ovals in $\bar{A}$ satisfying the following conditions*:

(1) 3 *noncollinear points of $\bar{A}$ are on exactly one oval in Ξ,*

(2) *if $p \in C$, $q \notin C$, with $C \in \Xi$ and q is not on the tangent line to C at p, then there is a unique oval D such that $p, q \in D$ and $D \cap C = \{p\}$,*

(3) *given p on the line L and q not on L there is a unique oval $C \in \Xi$ with p, q on C and L the tangent line to C at p,*

(4) *if $p \in C$, $C \in \Xi$, $q \neq p$, and q is on the tangent line to C at p, then every oval through p and q meets C in a point different from p.*

PROOF. If $\bar{M}$ is a Möbius plane and $\bar{M}_p = \bar{A}$, then the conditions (1)–(4) follow immediately from Lemma 3.1 and the definition of a Möbius plane. Conversely, if the conditions (1)–(4) are satisfied, let u be a point not in $\bar{A}$. Let $\bar{M} = (P, \Delta, I)$ where P consists of u and all points of $\bar{A}$, Δ consists of the ovals Ξ and all lines of $\bar{A}$ each augmented by the point u, and u is incident with all lines of $\bar{A}$, and a point of $\bar{A}$ is incident with all ovals in Ξ which contain it and all lines of $\bar{A}$ which are incident with it in $\bar{A}$. Given u and two points x, y of $\bar{A}$, they are incident with a unique circle in Δ, namely the line xy. If x, y, z are in $\bar{A}$ and are collinear they are on the line xy, and if x, y, z are not collinear they are on a unique oval in Ξ by (1). Thus (M1) is satisfied. If p is on the line L and q is not on L then if $p \neq u$, by (3), there is a unique oval in Ξ containing p and q and meeting L only at p, while if $p = u$, there is a unique line M of $\bar{A}$ through q parallel to L, and thus M and L are incident in $\bar{M}$ only at u. If $p \in C$, $C \in \Xi$, and q is not on C, and q is not on the tangent line to C at p, then (2) guarantees that there is a unique oval D containing p and q and meeting C only at p. If q is on the tangent line L to C at p, then L contains p and q and meets C only at p, and (4) guarantees that no oval in Ξ has this property. Thus (M2) is satisfied. Since $\bar{A}$ contains points x, y, z not on a line, the points x, y, z, u are not on a circle, and it is evident that every circle in Ξ contains at least one point. Thus $\bar{M}$ is a Möbius plane.

If G is a group of collineations of $\bar{A}$, then G is extendable if and only if $\Xi^G = \Xi$.

THEOREM 3.1. *Let $\bar{A}$ be an infinite affine plane of cardinality s. Let G be a group of collineations of $\bar{A}$ such that either G contains only translations, or else G consists of dilatations with a fixed center z and G is transitive on the points not equal to z on each line through z. Then $\bar{A}$ is embeddable and G is extendable.*

The theorem will be proved by constructing ovals Ω and including all images of Ω under G in classes Ξ_α such that the uniqueness statements of Lemma 3.2 are satisfied. Lemma 3.3 shows how this can be done, and the lemma will then be repeatedly applied to ensure the existence statements of Lemma 3.2.

For convenience of notation, rewrite the conditions (1), (2), (3) of Lemma 3.2 as uniqueness and existence statements (ia) and (ib), $i = 1, 2, 3$, respectively.

Note that each element of G preserves parallels, and that if G consists of translations, then only the identity e of G fixes a point of $\bar{A}$, while if G consists of dilatations with center z, then only e fixes a point $p \neq z$.

LEMMA 3.3. *Let $\bar{A}$ and G satisfy the hypotheses of Theorem* 3.1. *Let Ξ be a collection of ovals in $\bar{A}$ such that* (I) *$\Xi^G = \Xi$, and Ξ contains t G-orbits with $t < s$,* (II) *each oval in Ξ contains at most 2 point pairs from any G-orbit on the ordered pairs of distinct points of $\bar{A}$,* (III) *Ξ satisfies the conditions* (1a), (2a), (3a), *and* (4) *of Lemma* 3.2, (IV) *if $C \in \Xi$ then no two distinct tangent lines of C are parallel. Suppose that p, q, r, L are given where $p \in L$, $q, r \notin L$, and if $q \neq r$, then p, q, r are neither collinear nor all on an oval in Ξ. Suppose further that neither q nor r is on any oval in Ξ whose tangent line at p is L. Then there exists an oval Ω on p, q, r with tangent line L at p such that $\Xi \cup \{\Omega^g : g \in G\}$ satisfies* (I)–(IV).

PROOF OF LEMMA 3.3. Note that any oval family Φ with $\Phi^G = \Phi$, which satisfies (II) and (IV) has the property (∗): if x and y are distinct points of $\bar{A}$, there are at most two ovals of Φ on x and y in the same G-orbit, and if there are two, they have distinct tangent lines at both x and y. Suppose Ω, Ω^g are on x, y with $\Omega \in \Phi$, $e \neq g \in G$. The point x cannot be the center of dilatations z, as then z, y, y^g would be 3 collinear points on Ω^g, contrary to the assumption Ω is an oval. Thus $x^{g^{-1}} \neq x$ and $x^{g^{-1}}, x \in \Omega$. By (IV) the tangent lines to Ω at $x^{g^{-1}}, x$ are nonparallel. Since G preserves parallels the tangent line to Ω^g at x is parallel to the tangent line to Ω at $x^{g^{-1}}$, hence distinct from the tangent line to Ω at x. The same argument works for y (even if $x^g = y$).

Write $\Xi = \Delta \cup \Gamma$, where $\Delta \cap \Gamma = \emptyset$, and Δ contains all and only those ovals in Ξ with tangent line L at p. If Ω is any oval not in Ξ with tangent line L at p, then $\Xi' = \Xi \cup \{\Omega^g : g \in G\}$ satisfies (I)—(IV) provided that Ω

satisfies (II), (IV), (V) and (VI):

V(a) if $\bar{\Omega} \in \Delta$, then $\Omega \cap \bar{\Omega} = \{p\}$;

(b) if $\bar{\Omega} \in \Gamma$, then $|\Omega \cap \bar{\Omega}| \leqslant 2$;

(c) if $\Omega \cap \bar{\Omega} = \{x, y\}$ with $x \neq y$, then the tangents to Ω, $\bar{\Omega}$ are distinct at both x and y;

(d) if $\Omega \cap \bar{\Omega} = \{x\}$, then Ω and $\bar{\Omega}$ have the same tangent line at x, and

(VI) if $x \neq y \in \Omega$ and $g \in G$ such that $x^g = y$, then there is a unique $w \neq x$ such that $w, w^g \in \Omega$.

Then Ξ' obviously satisfies (I), (II) and (IV). Since $\Xi^G = \Xi$, if $C \in \Xi$ and $g \in G$, then C and Ω^g will violate one of (1a), (2a), (3a) or (4) if and only if C^g and Ω violate one of these conditions. But (V) ensures that they do not. For, given $C \in \Xi$, by (Va) and (Vb), $|C \cap \Omega| \leqslant 2$ so that (1a) is satisfied. Suppose that $r \in C$, $t \notin C$, contains r and t and meets C only at r. If there is a circle D through r and t, by (Vc), Ω and D have distinct tangent lines at r, while by (Vd) C and Ω have the same tangent line at r. Thus the tangent line to D at r meets C in a second point, and so by (III4) C and Ω have 2 points of intersection. Thus (2a) is satisfied. Similarly, if r is on the line M and $t \notin M$ and C, D meet at r and t they cannot both have the tangent line M at r by (Vc) so (3a) is satisfied. If C and Ω have only x in common, they have the same tangent line at x by (Vd), whence Ω cannot have a second point on the tangent line to C at x, so condition (4) is satisfied. Thus it only remains to show that for $e \neq g \in G$, Ω and Ω^g do not violate any of (1a), (2a), (3a) or (4). If a, b, c are three distinct points on Ω and Ω^g and G is a group of dilatations with center $z = a$, then a, b, b^g are collinear and on Ω^g, a contradiction. If $z \neq a, b, c$ or if G is a group of translations, then a, a^g, b, b^g, c, c^g are all on Ω^g with $a \neq a^g$, a contradiction to (VI), whence (1a) is satisfied. If $x^g \in \Omega \cap \Omega^g$ with $x^g \neq x$, then Ω and Ω^g have distinct tangent lines at x^g by (IV), whence (3a) is satisfied, and by (VI) the existence of w ensures that $\Omega \cap \Omega^g = \{x^g, w^g\}$ so that (2a) and (4) are satisfied. If $x^g \in \Omega \cap \Omega^g$ and if $x^g = x$, then G is a group of dilatations with center $z = x$ so that $\Omega \cap \Omega^g = \{x\}$, and Ω, Ω^g have the same tangent line at x, so again (2a), (3a) and (4) are satisfied. Thus, it suffices to construct an oval Ω containing p, q, r with tangent line L at p satisfying (II), (IV), (V), and (VI).

First choose L_q containing q such that $p \notin L_q$ and $r \notin L_q$ unless $q = r$, L_q is not parallel to L, and L_q is not the tangent line at q to any oval in Ξ on p and q. This choice of L_q is possible since there are at most two ovals of Ξ in the same G-orbit on p and q by (*), so there are less than s lines through q which are tangent lines at q to some oval in Ξ through p and q. If $q = r$, let $(\Omega_0, T_0) = (\{p, q\}, \{L, L_q\})$. If $q \neq r$, by a similar argument we can choose L_r containing r with $p, q \notin L_r$, L_r not parallel to L or to L_q such that L_r is not the tangent at r to any oval in Ξ on p or q containing r. In this case let $(\Omega_0, T_0) = (\{p, q, r\}, \{L, L_q, L_r\})$.

Let σ be the first ordinal number such that the cardinal number associated with σ is s. Well order the set of lines of $\bar{A}$ and the set of ordered pairs of distinct points of $\bar{A}$ with order type σ. We shall construct an ascending chain of semiovals $(\Omega_\alpha, T_\alpha)$ with $|\Omega_\alpha| < s$ ($|\Omega_\alpha| < |\omega|$ if both α and t are finite and $|\Omega_\alpha| \leqslant \max(|\alpha|, t)$ if one of α, t is infinite) for $0 \leqslant \alpha < \sigma$, each satisfying (II)$'$: Ω_α contains at most two point pairs from any pair orbit; (IV)$'$: no two distinct lines of T_α are parallel; (V)$'$: (a) if $\bar{\Omega} \in \Delta$, then $\Omega_\alpha \cap \bar{\Omega} = \{p\}$; (b) if $\bar{\Omega} \in \Gamma$, then $|\Omega_\alpha \cap \bar{\Omega}| \leqslant 2$; (c) if $|\Omega_\alpha \cap \bar{\Omega}| = 2$, then the tangent lines to $\bar{\Omega}$ and Ω_α are distinct at each point of intersection; (VI)$'$: if $e \neq g \in G$ such that for some $x \neq y \in \Omega_\alpha$ with $x^g = y$, then there exists at most one $w \in \Omega_\alpha$ such that w, $w^g \in \Omega_\alpha$.

Note that these conditions are satisfied by each (Ω_0, T_0) defined above. Suppose that $0 < \beta < \sigma$ and that for each α with $0 < \alpha < \beta$ semiovals $(\Omega_\alpha, T_\alpha)$ have been constructed with $|\Omega_\alpha| < s$ and satisfying (II)$'$, (IV)$'$, (V)$'$, and (VI)$'$. Let $\Omega_\beta^* = \bigcup_{\alpha<\beta}\Omega_\alpha$, $T_\beta^* = \bigcup_{\alpha<\beta}T_\alpha$. Then $(\Omega_\beta^*, T_\beta)$ is a semioval. Note that if t and β are finite, then Ω_β^* is a finite set, while if β or t is infinite, then $|\Omega_\beta^*| \leqslant \Sigma_{\alpha<\beta}|\Omega_\alpha| \leqslant \max(|\beta|, t)$. In either case, $|\Omega_\beta^*| < s$, and Ω_β^* satisfies (II)$'$, (IV)$'$, (V)$'$ and (VI)$'$. Define $A_\beta^* = \{L\colon |L \cap \Omega_\beta^*| = 1 \text{ and } L \notin T_\beta^*\}$, $B_\beta^* = \{L\colon L = p'p'', \text{with } p' \neq p'' \in \Omega_\beta^*\}$, $C_\beta^* = \{C \in \Xi\colon C \cap \Omega_\beta^* = \{m\}$ and C, Ω_β^* have the same tangent line M at $m\}$, $D_\beta^* = \{(x, y)\colon x \neq y \in \Omega_\beta^*$ and there exists $C \in \Xi$ with $x \in C$ and y on the tangent line to C at x and $C \cap \Omega_\beta^* = \{x\}\}$, $E_\beta^* = \{(x, y)\colon x \neq y \in \Omega_\beta^*$ and there exists $g \in G$, $g \neq e$ such that $x^g = y$ and $\Omega_\beta^* \cap (\Omega_\beta^*)^g = \{y\}\}$.

The proof will be completed if the chain $(\Omega_\alpha, T_\alpha)$ is such that $(\Omega_\sigma, T_\sigma) = (\bigcup_{\alpha<\sigma}\Omega_\alpha, \bigcup_{\alpha<\sigma}T_\alpha)$ satisfies (II)$'$, (IV)$'$, (V)$'$, and (VI)$'$ and $A_\sigma = D_\sigma = E_\sigma = \varnothing$. For $A_\sigma = \varnothing$ implies that $(\Omega_\sigma, T_\sigma)$ is an oval and hence Ω_σ satisfies (II), (IV), (Va), (Vb), and (Vc). $D_\sigma = \varnothing$ implies that Ω_σ satisfies (Vd). If Ω_σ satisfies (VI$'$a) and $E_\sigma = \varnothing$, then Ω_σ satisfies (VI) (so Ω_σ will be the desired oval). We prove 2 preliminary results VII and VIII.

(VII) Let M be a line not in B_β^* or T_β^* and C an oval in Ξ containing $a \in \Omega_\beta^*$ with $C \cap \Omega_\beta^* = \{a\}$, and let a point $b \in \Omega_\beta^*$ lie on the tangent line to C at a. Then there is a point x on M and a point (also denoted by x) on C such that (i) x is on no line of B_β^* or T_β^*, (ii) x is on no oval in Δ, (iii) $|(\{x\} \cup \Omega_\beta^*) \cap E| \leqslant 2$ for each $E \in \Gamma$, (iv) if $c, d, y \in \Omega_\beta^*$ and $e \neq g \in G$ with $c^g = d$, then $y^g \neq x$ and $x^g \neq y$, (v) x is on no element of C_β^*.

It suffices to show that there are less than s points on M or C violating any one of the above conditions, for then there will be s points on M or C satisfying (i)–(v). (i) Since $|\Omega_\beta^*| < s$, $|B_\beta^*| < s$ and $|T_\beta^*| < s$ whence there are less than s points of M or C on any line of B_β^* or T_β^*. (ii) Note that $C \notin \Delta$, since otherwise C meets Ω_β^* at p and has tangent line L at p, but there is no second point of Ω_β^* on L. If G is a group of dilatations with center z and $z = p$, then either Δ is empty or every point $q \notin L$ is already on some oval in

Δ. In the first case there is no element of Δ meeting M or C, and in the second case the hypotheses of Lemma 3.3 are violated by the given p, q, r, L. If G is a group of translations or dilatations with center $z \neq p$, then only e fixes p. There is at most one oval in each G-orbit with tangent line L at p by (IV), and since there are less than s G-orbits, $|\Delta| < s$. Since each oval in Δ meets M or C in at most two points (hypothesis III states that (1a) is satisfied) there are less than s points of M or C on some oval in Δ. (iii) Let $c \neq d \in \Omega_\beta^*$. By (*) there are at most two ovals on a and b in the same G-orbit of Ξ. Hence there are less than s ovals of Ξ on a and b. Since $|\Omega_\beta^*| < s$ there are less than s such pairs (c, d). Thus there are less than s ovals of Ξ containing two points of Ω_β^*. Since each of these ovals meets M or C in at most two points, there are less than s points y on M or C with $|(\{y\} \cup \Omega_\beta^*) \cap E| > 2$ for $E \in \Gamma$. (iv) Let $c, d, y \in \Omega_\beta^*$, $e \neq g \in G$, and $c^g = d$. There is at most one element of G mapping c to d unless G is a group of dilatations with center $z = a = b$. In the latter case if $x^g \in \Omega_\beta^*$ then c, x, x^g would be collinear which is excluded by (i). Similarly $c, x, x^{g^{-1}}$ would also be collinear. Thus, since there are less than s pairs $\{c, d\} \subseteq \Omega_\beta^*$, there are less than s points y on M or C with y^g or $y^{g^{-1}}$ in Ω_β^*. (v) There are less than s ovals D in C_β^* with $D \cap \Omega_\beta^* = \{y\}$ unless G is a group of dilatations with center $z = y$, since if $y \neq z$, there is at most one oval in each G-orbit containing y with the same tangent line at y as the tangent line to Ω_β^* at y, and there are less than s G-orbits in Ξ. In the latter case since $|\Omega_\beta^*| > 1$, there is a point n on Ω_β^* and a point k on $zn \cap D$. Then by the transitivity of G there is $g \in G$ with $k^g = n$, and so D^g meets Ω_β^* in z and n contradicting (V′c) since D and hence D^g have the same tangent line at z as Ω_β^*. Thus $|C_\beta^*| < s$ and so there are less than s points on M or C which are on some member of C_β^*. Hence, there is a point x on M or C such that (i)–(v) hold.

(VIII) Let x be a point satisfying (i)–(v) of (VII). Then there is a line M containing x such that (vi) M is not parallel to any line in T_β^*, (vii) if $(x \cup \Omega_\beta^*) \cap E = \{x, y\}$ for some $E \in \Gamma$ with $x \neq y$, then the tangent line to E at x is not M, (viii) M contains no element of Ω_β^*.

Again it suffices to show that there are less than s lines on x violating (vi), (vii), or (viii). (vi) Since $|\Omega_\beta^*| < s$, $|T_\beta^*| < s$. (vii) As shown in the proof of (VII)(iii), there are less than s ovals of Ξ on a given pair of points $\{x, y\}$. Since there are less than s points y in Ω_β^*, there are less than s lines on x which violate (vii). (viii) This condition is immediate since $|\Omega_\beta^*| < s$.

Step 1. Let L_γ be the first element of A_β^*. (Since $|\Omega_\beta^*| < s$, $A_\beta^* \neq \varnothing$.) Then by (VII) there is a point x on L_γ satisfying (i)–(v). Choose a line M on x by (VIII) satisfying (vi)–(viii). Let $\Omega'_\beta = \{x\} \cup \Omega_\beta^*$, $T'_\beta = \{M\} \cup T_\beta^*$. Since x satisfies (i), no three points of Ω'_β are collinear and x is on no element of T_β^*. Since M satisfies (viii), $(\Omega'_\beta, T'_\beta)$ is a semioval, and $|\Omega'_\beta| < s$. If there are two pairs (a, y) and (c, w) with $e \neq h \in G$ such that $a^h = c$, $y^h = w$, a, c, y,

$w \in \Omega^*_\beta$, and $e \neq g \in G$ such that $a^g = b \in \Omega^*_\beta$, then (iv) states that $y^g \neq x$ so that (II)′ holds. Ω'_β also satisfies (IV)′ by (vi), (VI)′ by (iv), (V′a) by (ii), (V′b) by (iii) and (V′c) by (v) and (vii). Define A'_β, B'_β, C'_β, D'_β, E'_β analogously to A^*_β, B^*_β, C^*_β, D^*_β, E^*_β respectively. Then $L_\gamma \in B'_\beta$, $L_\gamma \notin A'_\beta$.

Step 2. Define $D'_{a,b} = \{C \in \Xi\colon (a, b) \in D'_\beta$ with $a \in C$, b on the tangent line to C at a, and $\Omega'_\beta \cap C = \{a\}\}$.

Step 2a. Let (a, b) be the first element of D'_β such that $|D'_{a,b}| < s$, and let $C \in D'_{a,b}$. If $a \neq z$ where G is a group of dilatations with center z, or if G is a group of translations, then there is at most one circle in each G-orbit of Ξ in $D'_{a,b}$. Hence, $|D'_{a,b}| \leqslant t$. By (VII) choose $\bar{x}$ on C satisfying (i)–(v), and by (VIII) choose $\bar{M}$ on $\bar{x}$ satisfying (vi)–(viii). Let $\Omega''_\beta = \{\bar{x}\} \cup \Omega'_\beta$, $T''_\beta = T'_\beta \cup \{\bar{M}\}$. As in Step 1, $(\Omega''_\beta, T''_\beta)$ is a semioval with $|\Omega''_\beta| < s$ satisfying (II)′, (IV)′, (V)′, and (VI)′. Note that $C \notin D''_{a,b}$. Repeat this procedure for each circle in $D'_{a,b}$. After at most t steps, the semioval $(\Omega'''_\beta, T'''_\beta)$ will have been constructed, so that $(a, b) \notin D'''$, $|\Omega'''_\beta| < s$ and Ω'''_β satisfies (II)′, (IV)′, (V)′, and (VI)′.

Step 2b. $|D'''_{a,b}| = s$ only if G is a group of dilatations with center $z = a$, and in this case each $D'''_{a,b}$ will be a full orbit of ovals in Ξ. Since there are less than s orbits, there are less than s possible pairs (a, b) for which $|D'''_{a,b}| = s$. Well order the set Ψ of ovals in Ξ containing z with order type σ. Let C be the first element of Ψ which is in $D'''_{a,b}$ for some $(z, b) \in D'''_\beta$. Apply (VII) and (VIII) to obtain a point x' on C and a line M' on x' such that (i)–(viii) are satisfied. Let $\Omega^{**}_\beta = \Omega'''_\beta \cup \{x'\}$, $T^{**}_\beta = T'''_\beta \cup \{M'\}$. Then $(\Omega^{**}_\beta, T^{**}_\beta)$ is a semioval with $|\Omega^{**}_\beta| < s$ and satisfying (II)′, (IV)′, (V)′, and (VI)′.

Step 3. Let (x, y) be the first element of E^{**}_β and suppose that $e \neq g \in G$ with $x^g = y$.

Case 1. G is a group of translations. Choose M parallel to xy such that $M \cap \Omega^{**}_\beta = \varnothing$. Then by (VII), there are s points b on M satisfying (i)–(v) and since there are less than s points on M violating any one of (i)–(v) there are s points b^g on M such that both b and b^g satisfy (i)–(v). Condition (V′b) can be violated for $\Omega^{**}_\beta \cup \{b, b^g\}$ only if there is an oval $E \in \Gamma$ such that $E \cap M = \{b, b^g\}$ and $|\Omega^{**}_\beta \cap E| = 1$. Since x and x^g satisfy (VII)(ii) and (VII)(iii), suppose, if possible, that there are s ovals $E \in \Gamma$ containing one point of Ω^{**}_β and meeting M in two points p, p^g with $p \in M$. Since $|\Omega^{**}_\beta| < s$ and there is at most one oval of Γ on a given point of Ω^{**}_β and a given pair q, q^g with $q \in M$, there are s distinct points p on M such that p, p^g are on $E \in \Gamma$ and $E \cap \Omega^{**}_\beta \neq \varnothing$. Further, since there are less than s G-orbits of ovals in Ξ, s of these pairs are G-images of pairs on one oval $F \in \Xi$, and s of these must contain the same point of Ω^{**}_β. Let $F \cap M = \{c, c^g\}$, $F^h \cap M = \{d, d^g\}$ and $F^k \cap M = \{z, z^g\}$ with h, $k \in G$, $a \in \Omega^{**}_\beta \cap F \cap F^h \cap F^g$ with c, d, z three distinct points of M. Suppose (a) that c, d, z are all in one orbit of points. Then h and k are translations fixing the line M and hence the line K through a parallel to M. Then there exist points u, v on K such that

$u^h = a$, $v^k = a$, and so a, u, v are three distinct collinear points of F, a contradiction. Suppose (b) that there are three distinct pairs c, c^g, u, u^g, v, v^g, with $u^h = d, v^k = z$. Then $(u^g)^h = u^{hg} = d^g$ and $v^{gk} = v^{kg} = z^g$ since translations in different directions commute. But then $F \cap F^g = \{c^g, u^g, v^g\}$, also a contradiction. But for five pairs x_i, x_i^g on $M \cap F^{j(i)}$, $j(i) \in G$, $i = 1, \ldots, 5$, either (a) or (b) must hold for some triple of pairs. Hence there are less than s points v on M with $v, v^g \in E$, $E \in \Gamma$ and $|E \cap \Omega_\beta^{**}| = 1$. Thus there are s points w on M such that w, w^g satisfy (i)–(v) and $\Omega_\beta^{**} = \{w, w^g\}$ satisfies (V′b). Choose one such pair $\{w, w^g\}$. By applying (VIII) to Ω_β^{**} and then to $\Omega_\beta^{**} \cup \{w\}$ the existence of lines N, N' with $w \in N$, $w^g \in N'$ is assured so that $(\Omega_\beta^{**} \cup \{w, w^g\}, T_\beta^{**} \cup \{N, N'\})$ is a semioval satisfying (IV)′ and (V)′.

The condition (II)′ will also be satisfied. For, if there are pairs (a, b), (c, d) with $a, b, c, d \in \Omega_\beta^{**}$ such that $a^k = c$, $b^k = d$, $c^j = w$, $d^j = w^g$, $e \neq k$, $j \in G$, then $c^g = d$ and $a^g = b$, contradicting that $(x, y) \in E_\beta'''$. Similarly, if $u, v, c, d \in \Omega_\beta^{**}$ such that $u^h = c$, $v^k = w$, $c^g = d$, $e \neq m \in G$, then w violates condition (VII)(iv). Since w, w^g each satisfies (i)–(v), these are the only cases to be considered, as (II)′ will not be violated by pairs containing at most one of x, x^g. Thus it remains to prove that condition (VI)′ is also satisfied. Note that g is not an involution, for in that case (x, y) and $(x^g, y^g) = (y, x)$ are in Ω_β^{**} and so $(x, y) \notin E_\beta^{**}$. Condition (VI)′ will be violated only if there is an involution $i \in G$ exchanging x and w. For, if there are two distinct pairs (a, b) and (c, d) with $a, b, c, d \in \Omega_\beta^{**} \cup \{w, w^g\}$ with $a^h = b$, $c^h = d$ and $w^h \in \Omega_\beta^{**} \cup \{w, w^g\}$, then by (VII)(iv), $w^h = w^g$ so that $h = g$ and $a = c = x$; if $a^h = b$, $c^h = d$, $b^k = w$ and $d^k = w^g$, then $b^g = w^{k^{-1}g} = w^{gk^{-1}} = d$. Since $(x, y) \in E_\beta^{**}$, $b = x$, $d = y$. Similarly, $a^g = c$, so $a = w$. Thus the case that $x^h = w$, $y^h = w^g$ is left and this causes a contradiction only when h is an involution, for then both pairs $\{y, w^g\}$ and $\{w^g, y\}$ satisfy the conditions of (VI)′ contradicting the uniqueness conditions of (VI)′.

Case 1.1. G is transitive on the points of $\bar{A}$. Then $\bar{A}$ is called a translation plane and, by Pickert [**9**, p. 205], every translation of G has the same order, so there are no involutions in G.

Case 1.2. G is not transitive on the points of $\bar{A}$. (a) If G is not transitive on the points of M, then there are s points on M to which x cannot be mapped by G. Hence we can choose w and w^g from among these points. (b) If G is transitive on the points of M, then there are s lines of $\bar{A}$ parallel to xy to which M cannot be mapped by G. Choose M' parallel to M not meeting Ω_β^{**} and not in the same orbit as the line xy, on which (w, w^g) may now be chosen.

Case 2. G is a group of dilatations with center z. Then $z \notin \Omega_\beta^{**}$, since otherwise z, x, y would be three distinct collinear points of Ω_β^{**}. Choose a line M through z such that M contains no point of Ω_β^{**}. Then there are s point pairs w, w^g on M such that w, w^g each satisfies conditions (i)–(v). There are less than s ovals $E \in \Gamma$ such that $E \cap M = \{w, w^g\}$ and $|E \cap \Omega_\beta^{**}| = 1$. For, $|\Omega_\beta^{**}| < s$, there are less than s orbits in Ξ and at most two ovals in one G-orbit on a given point of Ω_β^{**} since $z \notin \Omega_\beta^{**}$. Hence, there is a pair w, w^g on M and by (VIII) applied twice, two lines N, N' with $w \in N$, $w^g \in N'$ such that $(\Omega_\beta^{**} \cup \{x, x^g\}, T_\beta^{**} \cup \{N, N'\})$ is a semioval with $|\Omega_\beta^{**}| < s$ and satisfying (IV)′ and (V)′. Ω_β^{**} will always contain at most two pairs from a pair orbit for if $a, b, c, d \in \Omega_\beta^{**}$ and $a^k = c$, $b^k = d$ then either $b = c$, $d = a$, and z, a, b are collinear or else z, a, c and z, b, c are collinear, but there are at most two points of Ω_β^{**} on any line through z. Hence, if z is not on an oval or semioval, (II)′ is always satisfied. Hence, especially (II)′ is satisied for $\Omega_\beta^{**} \cup \{w, w^g\}$. There is no element of G mapping a point of Ω_β^{**} to x or to x^g so (VI)′ is satisfied.

Hence in either case, let $\Omega_\beta = \Omega_\beta^{**} \cup \{x, x^g\}$, $T_\beta = T_\beta^{**} \cup \{N, N'\}$. Then (Ω_β, T_β) is a semioval with $|\Omega_\beta| < s$, and from steps (1), (2), and (3) at most $\max(|\beta|, t + 4)$ points have been added to Ω_β^{*} to obtain Ω_β. Thus Ω_β satisfies the cardinality conditions in the induction hypothesis. Also, Ω_β satisfies (II)′, (IV)′, (V)′, and (VI)′; if $0 \leqslant \alpha < \beta$, then $(\Omega_\alpha, T_\alpha) < (\Omega_\beta, T_\beta)$, and $L_\gamma \notin A_\beta$, $(x, y) \notin E_\beta$, and if $|D_{a,b}| < s$, then $(a, b) \notin D_\beta$, and the first circle through z in some $D'_{z,b}$ is not in $D_{z,b}$.

Now, let $(\Omega_\sigma, T_\sigma) = (\bigcup_{\alpha<\sigma}\Omega_\alpha, \bigcup_{\alpha<\sigma}T_\alpha)$. Then $(\Omega_\sigma, T_\sigma)$ is a semioval and by the construction steps 1, 2, 3, $A_\sigma = D_\sigma = E_\sigma = \varnothing$. To show that $A_\sigma = \varnothing$, suppose that $L_\gamma \in A_\sigma$. Let β be the first stage of the construction such that $L_\gamma \in A_\beta$. Since $\gamma < \sigma$ there are less than s lines indexed by an ordinal less than γ, but there were s stages beyond β and at each one a line L_α with $\alpha < \gamma$ was deleted from the set A_δ, $\delta > \beta$. Since each of these can be deleted only once, we have constructed a 1-1 function from the ordinals greater than β into the set of ordinals less than γ, which implies that the set of ordinals less than γ has cardinality at least s, a contradiction. Similarly, $E_\sigma = \varnothing$. If $C \in D_\sigma$, then there exist $a, b \in \Omega_\sigma$ such that $\{a\} = C \cap \Omega_\sigma$ and b is on the tangent line to C at a. The proof that $A_\sigma = \varnothing$ is valid for each of the two cases $|D_{a,b}| < s$ and $|D_{a,b}| = s$ by Step 2, cases a and b. Hence $D_\sigma = \varnothing$. As shown above, this suffices to prove that Ω_σ is the desired oval, which completes the proof of Lemma 3.3.

Proof of Theorem 3.1. Let $X = \{(p, L, q)\colon\ p \in L,\ q \in L\}$, $Y = \{(x, y, p, q)\colon x, y, p$ are not collinear and x, y, p, q are pairwise distinct$\}$. Let $\overline{X}$, $\overline{Y}$ be sets of representatives of the G-orbits of members of X, Y respectively. The ascending chain Ξ_α $(0 \leqslant \alpha < \sigma)$ of G-orbits of ovals will be constructed so that each Ξ_α satisfies the hypotheses of Lemma 3.3 and $\Xi = \bigcup_{\alpha<\sigma}\Xi_\alpha$ satisfies conditions (1b), (2b), (3b) of Lemma 3.2. Then Ξ will satisfy all the conditions of Lemma 3.2 and $\Xi^G = \Xi$, which will prove Theorem 3.2. Let $\Xi_0 = \varnothing$. Suppose that for each α with $0 \leqslant \alpha < \beta$ and $\beta < \sigma$

we have constructed oval classes Ξ_α satisfying the hypotheses of Lemma 3.3, but that Ξ_β has not yet been constructed. Let $\Xi_\beta^* = \bigcup_{\alpha<\beta}\Xi_\alpha$. Then Ξ_β^* also satisfies the hypothesis of Lemma 3.3 since from parts (a) and (b) below there are at most $|3\beta|$ G-orbits of ovals in Ξ_β, so that t, the number of G-orbits of Ξ_β is less than s. (a) Let (p, L, q) be the first element of $\bar{X}$ such that there is no oval in Ξ_β^* through p, q with tangent line L at p. With $q = r$ in Lemma 3.3 construct an oval Ω through p, q with tangent line L at p so that $\Xi_\beta^{**} = \Xi_\beta^{**} \cup \{\Omega^g: g \in G\}$ satisfies the hypotheses of Lemma 3.3. (b) Let (x, y, p, q) be the first element in $\bar{Y}$ such that there is no oval in Ξ_β^{**} through p, q tangent at p to the oval in Ξ_β^{**} through x, y, p. (The latter oval may not yet be constructed, but if it exists it is unique.) If there is no oval in Ξ_β^{**} on x, y, p, choose a line L on p such that there is no oval of Ξ_β^{**} with tangent line L at p containing either x or y. This is possible since by $(*)$ in Lemma 3.3, there are less than s ovals of Ξ_β^{**} on (x, y) or (y, p). Apply Lemma 3.3 to p, x, y, L to obtain an oval Ω' on p, x, y with tangent line L at p such that $\Xi_\beta^{***} = \Xi_\beta^{**} \cup \{\Omega'^g: g \in G\}$ satisfies the conditions (I)–(IV) of Lemma 3.3. If there is already an oval Ω' on x, y, p let $\Xi_\beta^{***} = \Xi_\beta^{**}$. By Lemma 3.2 we may assume that q is not on L, the tangent line to Ω' at p. With $q = r$ apply Lemma 3.3 to p, L, q to obtain an oval Ω'' through p, q with tangent line L at p. By condition (V)(a) in the proof of Lemma 3.3, $\Omega'' \cap \Omega' = \{p\}$. Let $\Xi_\beta = \Xi_\beta^{***} \cup \{\Omega''^g: g \in G\}$. Then Ξ_β is a family of ovals satisfying the hypotheses of Lemma 3.3. Hence, there exist Ξ_α satisfying these hypotheses for each α with $0 \leqslant \alpha < \sigma$.

Let $\Xi = \bigcup_{\alpha<\sigma}\Xi_\alpha$. Then Ξ satisfies conditions (1a), (2a), (3a), and (4) of Lemma 3.2 since each Ξ_α satisfies these conditions. Let (a, b, c) be three noncollinear points of $\bar{A}$. Then there exists (a', b', c') in the same G-orbit as (a, b, c) such that, for some q', $(a', b', c', q') \in \bar{Y}$. By (b) there is an oval in Ξ on (a', b', c') and hence there is an oval in Ξ on (a, b, c). Thus Ξ satisfies (1b). Similarly, Ξ satisfies (2b) and (3b). Hence by Lemma 3.2 $\bar{A}$ is embeddable. Moreover, since $\Xi^G = \Xi$, G is extendable. This completes the proof of Theorem 3.1.

COROLLARY 1. *If $\bar{A}$ is an infinite affine plane then $\bar{A}$ is embeddable.*

PROOF. Let $G = \{e\}$ with e the collineation fixing all points and lines of $\bar{A}$. Then G acts as a group of translations of $\bar{A}$, and by Theorem 3.1, $\bar{A}$ is embeddable.

COROLLARY 2. *Let G be an infinite group. Then there exist infinite Möbius planes $\bar{M}_1$ and $\bar{M}_2$ such that $\bar{M}_1$ is (u, C) transitive for a fixed pencil (u, C) and $\bar{M}_2$ is (z, u) transitive for a fixed pair (z, u) and the group of all (u, C) translations of $\bar{M}_1$ and (z, u) dilatations of $\bar{M}_2$ is isomorphic to G.*

PROOF. Hughes [8] and Wilker [12] have shown that given an infinite group G, there is a projective plane Π such that Π is (z, L) transitive for a fixed pair (z, L) with z on L and the group of all (p, L) perspectivities is isomorphic to G. Yaqub and Krier [13] have obtained the same result for the case z not on L. Thus there exist infinite affine planes $\bar{A}_1$ and $\bar{A}_2$ admitting G as a group of translations and dilatations respectively such that $\bar{A}_1$ is transitive on the points of each line in one fixed parallel class of $\bar{A}_1$, and $\bar{A}_2$ is transitive on the points not equal to z of each line containing z. The conclusion now follows from Theorem 3.1.

COROLLARY 3. *Every infinite translation plane* $\bar{A}$ *can be embedded in a Möbius plane of class* III.1.

PROOF. Let $\bar{A}$ be an infinite translation plane and G the group of all translations of $\bar{A}$. By Theorem 3.1, $\bar{A}$ can be embedded in a Möbius plane $\bar{M}$ such that for some point u in $\bar{M}$, $\bar{M}_u = \bar{A}$ and $\bar{A}$ is of class at least III.1. Thus, it remains to show that the construction of Theorem 3.1 can be modified so that $\bar{M}$ is not (x, u) transitive for any point x in $\bar{A}$ and there is no automorphism of $\bar{M}$ which moves u. The translation plane $\bar{A}$ when extended to a projective plane Π is (p, L_∞) transitive for each point p on L_∞ and hence satisfies the (p, L_∞) Desargues Theorem. If the automorphism g of $\bar{M}$ maps u to $c \neq u$, then $\bar{M}_u \cong \bar{M}_c$ and so $\bar{M}_c$ must also be a translation plane, so that it suffices to show that $\bar{M}_c$ does not satisfy the (p, L_∞) Desargues Theorem.

Choose a point c in $\bar{A}$ and three distinct lines L, M, L_1 containing c. With $\Xi = \varnothing$ apply Lemma 3.3 to construct an oval L_2 with tangent line L_1 at c. We make the further condition that no tangent line to L_2 is parallel to L or to M. This is possible, since tangent lines are chosen according to (VIII) in the proof of Lemma 3.3 and this restriction only removes two lines through each point from the set of those lines which may be chosen as tangent lines. Let $\Xi_1 = L_2^G$. If $\bar{A}$ admits a nonidentity dilatation d with center c, let $c \neq s$, t be two distinct points of L_2 with neither s nor t on L or M. Let $q = s^d$ and choose r on the line ct, $r \neq c$, t, t^d. By Lemma 3.3 with $\Xi = \Xi_1$, there is an oval L_3 containing c, q, r, with tangent line L_1 at c and such that no tangent line of L_3 is parallel to L or M. Let $\Xi_2 = \Xi_1 \cup L_3^G$. Choose two lines A_{23}, B_{23} containing c with $A_{23}, B_{23} \neq L_1, L, M$. Let a_2 be the point on L_2 and A_{23}, a_3 the point on L_3 and A_{23}, b_2 the point on L_2 and B_{23}, b_3 the point on L_3 and B_{23}. Since there are no ovals in Ξ_2 with tangent line L, apply Lemma 3.3 with $\Xi = \Xi_2$ to construct an oval A_{12} containing a_2, with tangent line L at c, with no tangent line parallel to M and so that A_{12} does not contain a_3, b_2, or b_3. Further, since there are at most 4 ovals in Ξ_2 containing c and a_3, we may require that the point a_1 on $L_1 \cap A_{12}$ is on no oval containing c and a_3. Let $\Xi_3 = \Xi_2 \cup A_{12}^G$. Similarly, construct the oval B_{12} containing b_2 with tangent line L at c, such that no tangent line of B_{12} is parallel to M, B_{12} does not contain a_1, a_2, a_3, or b_3 and the point b_1 on L_1 and B_{12} is on no oval in Ξ_3 containing c and b_3 and no image of B_{13} under G contains c, a_1, and a_3. Since, at each stage of the construction of B_{13} there are at most 2 elements of G mapping B_{13} onto a given pair of c, a_1, and a_3 and if g maps m, n onto a_1, a_3 then the point $c^{g^{-1}}$ cannot be chosen to add to B_{13}, the condition puts only finitely many restrictions on the set of points from which one is to be chosen in (VII). Let $\Xi_4 = \Xi_3 \cup B_{12}^G$. Construct A_{13} containing c, a_1, a_3 with tangent line M at c and such that A_{13} does not contain a_2, b, b_2, or b_3. Let $\Xi_5 = \Xi_4 \cup A_{13}^G$. With $\Xi = \Xi_5$ apply Lemma 3.3 again to construct an oval B_{13} containing c, b_1, b_3 with tangent line M at c. Let $\Xi_5 = \Xi_5 \cup B_{13}^G$. Then Ξ_5 may be extended to a class Ξ of ovals such that $\bar{A}$ is embeddable and G is extendable. The dilatation d cannot be an automorphism of $\overline{M}$, for $L_2^d \neq L_3$ since $t \in L_2$ but $t^d \notin L_3$, but $q \in L_2^d \cap L_3$ and L_2 and L_3 have the same tangent line L_1 at c contradicting condition 3 of Lemma 3.2. Thus, $\overline{M}$ cannot be (c, u) transitive and since G is transitive on the points of $\bar{A}$, $\overline{M}$ is not (z, u) transitive for any point z in $\bar{A}$. In $\overline{M}_c$ the hypotheses of Desargues Theorem are satisfied with A_{12} parallel to B_{12}, A_{13} parallel to B_{13} but A_{23} meets B_{23} in u so the conclusion is not satisfied. Hence, $\overline{M}_c$ is not a translation plane so no automorphism of $\overline{M}$ can map u to c, and since G is transitive on the points of $\bar{A}$, no automorphism of $\overline{M}$ can map u to any point of $\bar{A}$. This completes the proof of Corollary 3.

The construction of Corollary 3 will not yield all infinite Möbius planes of class III.1. Ewald [6] has embedded the real plane in a Möbius plane of class III.1 such that the class Ξ of ovals (cf. Lemma 3.2) consists of the boundaries of sharply convex sets, and hence each such oval contains two parallel tangent lines, whereas Lemma 3.3 restricts the choice to only one. There may be translation planes that are embeddable with ovals in Ξ that have more than 2 parallel tangent slopes.

DEFINITION 3.1. A *partial Möbius plane* is an incidence structure $(P, \Xi, I) = M$ of points and circles satisfying (i) 3 distinct points are on at most one circle, (ii) each circle contains at least 3 points, and (iii) there are four points not all on one circle. Note that there may be points on no circle.

Partially order the class of all partial Möbius planes by defining $(P_1, \Xi_1, I_1) < (P_2, \Xi_2, I_2)$ if $P_1 \subseteq P_2$, $\Xi_1 \subseteq \Xi_2$, $I_1 \subseteq I_2$, and $p \in P_1$, $C \in \Xi_1$, and $p\ I_2\ C$ implies that $p\ I_1\ C$. The following lemma is immediate.

LEMMA 3.4. *If* $\{M^j\}$, $j \in J$, *is an ascending chain of partial Möbius planes then* $\bigcup \{M^j : j \in J\}$ *is also a partial Möbius plane.*

THEOREM 3.2. *If* M^0 *is a partial Möbius plane, then there exists a Möbius plane* $\overline{M}$ *such that* $M^0 < \overline{M}$.

PROOF. We shall define an ascending sequence of partial Möbius planes M^i and a set of equivalence relations, $\tan_p$, defined recursively for each point p in one of the M^i such that

(1) $i < j$ implies that $M^i < M^j$, and each M^i is a partial Möbius plane.

(2) $C \neq D \in \Xi_i$, then in $M^{i+1} C \tan_p D \Leftrightarrow C$ meets D only at p. (If $p \in C$, then $C \tan_p C$.)

(3) Given $p, q \in P_i$, $C \in \Xi_i$, $P\ I_i\ C$, $q \not{I}_i\ C$, there is a circle $D \in \Xi_{i+1}$ such that $q\ I_{i+1}\ D$ and $C \tan_p D$.

(4) Given $p, q, r \in P_i$, there is a circle $C \in \Xi_{i+1}$ such that $p, q, r\ I_{i+1}\ C$.

Then $\overline{M} = \bigcup_{i=0}^{\infty} M^i$ is a partial Möbius plane. (M1) is satisfied by (4); (M2) is satisfied by (2) and (3); and (M3) is satisfied by M^0 (the definition of "$<$" assures that the four points in M^0 not all on a circle are not all on one circle in $\overline{M}$).

Let $\tan_p$ be empty for each point of M^0. Then M^0 satisfies (1)–(4).

Suppose we have constructed M^i and defined $\tan_p$, $p \in P_i$, satisfying (1)–(4) for all $i < j - 1$, but that M^j has not yet been constructed. Let $P_j^* = P_{j-1}$, $\Xi_j^* = \Xi_{j-1}$, and $I_j^* = I_{j-1}$.

Step 1. Consider all triples (p, C, q) with $p, q \in P_{j-1}$, $C \in \Xi_{j-1}$, $p\ I_{j-1}\ C$, $q \not{I}_{j-1}\ C$.

Case 1. There is a circle D containing q and such that $C \tan_p D$. Then for each circle E containing p with E not $\tan_p C$, $E \cap C = \{p\}$ in M^{j-1} let x be a new point. Put $x \in P_j^*$, $x\ I_j^*\ C, E$. (This will show sufficiency in property (2).)

Case 2. There is no circle D containing p and q such that D meets only at p each circle E such that $E \tan_p C$. Let y be a new point, let D be a new circle. Put $y \in P_j^*$, $D \in \Xi_j^*$, $p, y, q\ I_j^*\ D$. Let $C \tan_p D$, $D \tan_p D$, $E \tan_p D$ for each E such that $E \tan_p C$. (Return to Case 1.)

Case 3. There are several circles containing p and q and meeting each circle tangent to C at p only in p. Let D be one of them. Set $D \tan_p E$ for each circle E such that $E \tan_p C$. Return to Case 1.

Following this procedure for each triple (p, C, q) yields the partial Möbius plane M^{j^*} which satisfies (1), (2), and (3).

Step 2. Consider all triples of pairs $(a, b, c) \in P_j^*$, not all on one circle of Ξ_j^*. We form a new circle K and let $K \in \Xi_j^*$, $a, b, c\ I_j^*\ K$. Set $M^{j^*} = M^j$. M^j satisfies the conditions (1)–(4) and so the proof is completed.

The next lemma will be useful in the proof of Theorem 3.3.

LEMMA 3.5. *If $M = (P, \Xi, I)$ is a partial Möbius plane such that all circles through the point p also contain the point q, if $|P| < s$, and if $\overline{A}$ is an affine plane of infinite order s, then there is an injection g of M_p into $\overline{A}$ such that the circles through p and q are mapped into lines through $g(q)$ and the points of any circle C not containing p are mapped into semiovals of $\overline{A}$. Furthermore, the* *image of any point of P on no circle in Ξ is not collinear with any pair of image points of P.*

Proof. Let a be a point of $\bar{A}$. Set $g(q) = a$. Well order the circles through p and all points on no circle through p. If we have constructed image lines for each circle C_α, $\alpha < \beta$, but not C_β, let $g(C_\beta)$ be a line through a distinct from $g(C_\alpha)$, $\alpha < \beta$. Then on $g(C_\beta)$ there are less than s points r such that r is on the line $g(x)g(y)$ with $x\ I\ C_\alpha$, $y\ I\ C_{\alpha'}$, $\alpha, \alpha' < \beta$, so we can find an image for each point of C_β satisfying the conclusion of the theorem. Since $|P| < s$, then $|\Xi| < s$ so this procedure may be continued until we have exhausted all circles through p. Then for each point p_β on no circle through p, let $g(p_\beta)$ be on a line through a distinct from all $g(C_\delta)$ and $ag(p_\alpha)$, $\alpha < \beta$, and not collinear with any pair of image points of P for which g has already been defined. This process may be continued until all points of P have g images, which proves the lemma.

Definition 3.2. If $M = (P, \Xi, I)$ is a partial Möbius plane with $|P| < s$ then M is *s-acceptable at p* if M_p admits an injection into any affine plane of cardinality s, such that circles not through p are mapped into semiovals. M is *s-acceptable* if M is s-acceptable at p for each point p of M. If P is a finite set and M is s-acceptable, call M *acceptable*.

Example. Let P contain $n + 1$ points with $n \geqslant 3$, let Ξ contain one circle with n points on it and one point not on it. Then by Lemma 3.5 (P, Ξ, I) is an acceptable partial Möbius plane.

Theorem 3.3. *Let $\{\bar{A}_\alpha : \alpha \in J\}$ be an infinite collection of s affine planes, each of order s. Then there exists a Möbius plane $\bar{M}$ and a bijection b from J to the point-set of $\bar{M}$ such that for each $\alpha \in J$, there is an isomorphism g_α between $M_{b(\alpha)}$ and $\bar{A}_\alpha$.*

Proof. Well order the points of each A_α with order type σ. For each γ with $0 \leqslant \gamma < \sigma$, we shall construct a partial Möbius plane $M^\gamma = (P_\gamma, \Xi_\gamma, I_\gamma)$, and bijections b^γ, g_α^γ with the following properties:

(1) $|P^\gamma| < s$, ($|P^\gamma| \leqslant |\gamma|$ if γ is infinite, and P^γ is a finite set if γ is a finite ordinal),

(2) if $0 < \beta < \gamma$, then $M^\beta < M^\gamma$,

(3) b^γ is an injection from an initial segment $J^{(\gamma)}$ of J onto P_γ; $\gamma \in J^{(\gamma)}$; if $0 \leqslant \beta < \gamma$, then $b^{(\gamma)}$ is an extension of $b^{(\beta)}$,

(4) for each $\alpha \in J^{(\gamma)}$, g_α^γ is a bijection from $M^\gamma_{b^\gamma(\alpha)}$ onto a subset A_α^γ of the point-set of $\bar{A}_\alpha$ such that $b^\gamma(\alpha), x, y, z$ are concyclic in M^γ if and only if $g_\alpha^\gamma(x)$, $g_\alpha^\gamma(y)$, $g_\alpha^\gamma(z)$ are collinear in $\bar{A}_\alpha$; if $\beta < \gamma$ and if $\alpha \in J^{(\beta)}$, then g_α^γ is an extension of g_α^β,

(5) if $\alpha < \gamma$, then the first γ points of $\bar{A}_\alpha$ are in A_α^γ.

The ascending chain M^γ then yields the conclusion of the theorem. For let

$\overline{M} = \bigcup_{\gamma<\sigma} M^{\gamma}$, $P = \bigcup_{\gamma<\sigma} P_{\gamma}$. If $\gamma \in J$, then by (3) $\gamma \in J^{(\gamma)}$, $0 < \gamma < \sigma$, and we may define $b(\gamma) = b^{\gamma}(\gamma)$. Then b is a bijection from J to P. If $\alpha \in J$ and $x \in M$, suppose that M^{γ} is the first partial Möbius plane to which x belongs. If $\alpha \leqslant \gamma$, then $\alpha \in J^{(\gamma)}$ by (3), and we define $g_{\alpha}(X) = g_{\alpha}^{\gamma}(x)$. If $\alpha > \gamma$, $M^{\gamma} < M^{\alpha}$ by (2) and we define $g_{\alpha}(x) = g_{\alpha}^{\alpha}(x)$. Then g_{α} maps $M_{b(\alpha)}$ to $\overline{A}_{\alpha}$, injectively by (4) and surjectively by (5), whence by (4), g_{α} is an isomorphism from $M_{b(\alpha)}$ onto $\overline{A}_{\alpha}$, for each α with $0 \leqslant \alpha < \sigma$. It follows from Theorem 1.4 that $\overline{M}$ is a Möbius plane. (Thus it is unnecessary to define the equivalence relations $\tan_p$ in this construction.) Hence it suffices to show that a chain M^{γ} can be constructed satisfying (1)–(5).

Let M^0 be a finite acceptable partial Möbius plane containing n points. Let b^0 be a bijection from $J^{(0)}$, the first n elements of J, onto the point-set of M^0. Since M^0 is acceptable, for each $\alpha \in J^0$, there exist maps g_{α}^0 from $M_{b^0(\alpha)}^0$ into $\overline{A}_{\alpha}$ which satisfy (4). M^0 satisfies (2) and (5) vacuously, and (1), (3), (4) by construction.

Let $0 < \beta < \sigma$. Suppose that for each α with $0 \leqslant \alpha < \beta$ we have constructed $M^{\alpha}, b^{\alpha}, g_{\gamma}^{\alpha}$ satisfying (1)–(5), but that M^{β} has not yet been constructed. Write $^*M^{\beta} = \bigcup_{\alpha<\beta} M^{\alpha}$, $^*P_{\beta} = \bigcup_{\alpha<\beta} P_{\alpha}$. Then by (2) and Lemma 3.4, $^*M^{\beta}$ is a partial Möbius plane, and, by (1), $^*P_{\beta}$ satisfies condition (1). Let $^*J^{\beta} = \bigcup_{\alpha<\beta} J^{(\alpha)}$, let $b^{\beta}(\delta) = b^{\alpha}(\delta)$ if $\delta \in J^{(\beta)}$, and let $g_{\alpha}^{\beta}(x) = g_{\alpha}^{\gamma}(x)$ if $x \in M^{\gamma}$ and $\alpha \in J^{(\beta)}$ for any $\gamma < \beta$. (By (4) this definition is unambiguous.) Write $^*A_{\gamma}^{\beta} = g_{\gamma}^{\beta}(^*M_{b^{\beta}(\gamma)}^{\beta})$. Suppose that m is a point of $\overline{A}_{\alpha}$ indexed by an ordinal $\leqslant \beta$, where $\alpha < \beta$, such that $m \notin {}^*A_{\gamma}^{\beta}$. By (5), if β is a limit ordinal then such a point m must be indexed by β; if β is not a limit ordinal, then either m is indexed by β or $\alpha = \beta - 1$. In the first case there are at most β such points, and in the second case there are at most 2β such points. In any case there are less than s such points m. Suppose that m is on the lines $K, L, M, \ldots$ in $^*A_{\gamma}^{\beta}$. There are less than s such lines, since $|^*P_{\beta}| < s$. Let δ be the first element of J which is not in $^*J^{(\beta)}$. Adjoin the new point x to $^*P_{\beta}$, and define $b^{\beta}(\delta) = x$, $g_{\alpha}^{\beta}(x) = m$. Set $x\, I_{\beta}\, (g_{\alpha}^{\beta})^{-1}(K)$, $x\, I_{\beta}\, (g_{\alpha}^{\beta})^{-1}(L), \ldots$. Then $g_{\alpha}^{\beta}(x)$ is on $K, L, M \ldots$ and condition (4) is satisfied for α. We must define $g_{\eta}^{\beta}(x)$ for each $\eta \in {}^*J^{\beta} \cup \{\delta\}$. If $\eta \neq \delta$ and if $g_{\alpha}^{\beta}[b^{\beta}(n)]$ is on K, define $g_{\eta}^{\beta}(x)$ to be a point of $\overline{A}_{\eta} \setminus {}^*A_{\eta}^{\beta}$ which is on $g_{\eta}^{\beta}[(g_{\alpha}^{\beta})^{-1}(K)] = T$ but is not collinear with any pair of points of $^*A_{\eta}^{\beta}$ which are not both on T. (This is possible, since $^*A_{\eta}^{\beta}$ contains less than s points.) Then (4) is satisfied by g_{η}^{β}. Repeat this procedure for each point $b^{\beta}(\lambda)$ such that $g_{\alpha}^{\beta}[b^{\beta}(\lambda)]$ is on one of $K, L, M \ldots$. If $g_{\lambda}^{\beta}[b^{\beta}(\mu)]$ is not on any of $K, L, M \ldots$, let $g_{\mu}^{\beta}(x)$ be a point in $\overline{A}_{\mu} \setminus {}^*A_{\mu}^{\beta}$ which is not collinear with any pair of points in $^*A_{\mu}^{\beta}$. If $\eta = \delta$, then since all circles on $x = b^{\beta}(\delta)$ are also on $b^{\beta}(\alpha)$, we may apply Lemma 3.5 to define g_{δ}^{β} on $^*M_{b(\delta)}^{\beta}$ so that (4) is satisfied. Repeat this procedure for each such point m, at each step adjoining the required new elements of J to $^*J^{\beta}$ and the newly constructed points to

$^*P_\beta$. In this way we finally obtain an initial segment $J^{(\beta)}$ of J and a partial Möbius plane M^β with point-set P_β. Since there are at most 2β points m, P_β satisfies condition (1). By construction, $M^\alpha < M^\beta$ for each $\alpha < \beta$. Hence, if there exists a point m, then $\beta \in J^\beta$. If there is no point m, we apply the above construction to the first point of A_0 which is not in $^*A^0_\beta$. Thus we can always ensure that $\beta \in J^{(\beta)}$. By construction the bijections b^γ and g^γ_α, $\alpha \in J^{(\gamma)}$, satisfy (3)–(5). Hence the induction step is completed, and the theorem proved.

COROLLARY 1. *There exist infinite Möbius planes with no nontrivial automorphisms.*

PROOF. Let $\overline{A}_i$ be a collection of pairwise nonisomorphic affine planes. Apply Theorem 3.3 to obtain a Möbius plane $\overline{M}$. If ψ is an automorphism of $\overline{M}$, then $\psi(a) = b$ implies $\overline{M}_a \cong \overline{M}_b$ so $a = b$. Therefore ψ is the identity.

As an example of a family of pairwise nonisomorphic affine planes, let $\overline{A}_i$ be the pappian affine plane coordinatized by $Q(\sqrt{p_i}\,)$ where Q is the rational field and p_i is the ith prime integer. These are pairwise nonisomorphic fields, and by Pickert [9, p. 109] the $\overline{A}_i$ are pairwise nonisomorphic.

COROLLARY 2. *There is an infinite Möbius plane $\overline{M}$ such that each $\overline{M}_p$ is pappian and $\overline{M}_p \cong \overline{M}_q$ for all points p and q, but $\overline{M}$ is not ovoidal.*

PROOF. Let $\{\overline{A}_i\}$ be continuum many copies of the real affine plane. Let M^0 consist of 8 points, $p_1, \ldots, p_8$, and 6 circles, $C_1, \ldots, C_6$, with $p_1, p_2, p_3, p_4\ I_0\ C_1$; $p_3, p_4, p_5, p_6\ I_0\ C_2$; $p_5, p_6, p_7, p_8\ I_0\ C_3$; $p_7, p_8, p_1, p_2\ I_0\ C_4$; $p_1, p_2, p_5, p_6\ I_0\ C_5$; $p_3, p_4, p_7\ I_0\ C_6$; $p_8 \not I_0\ C_6$.

$M^0_{p_i}$ is acceptable for $i = 1, 2, \ldots, 6$ and $i = 8$ by Lemma 3.5. If $\overline{A}$ is an infinite affine plane, choose a point b_8 and two distinct lines L_5 and L_3 on b_8. Choose two points $b_1, b_2 \neq b_8$ on L_5 and $b_5, b_6 \neq b_8$ on L_3. Choose a line L_6 not meeting b_1, b_2, b_5, b_6, or b_8 and choose b_3, b_4 on L_6 but not on L_5 or L_3 such that neither b_3 nor b_4 is on the line b_ib_j ($i = 1, 2$; $j = 5, 6$). Then the correspondence $p_i \to b_i$ and $C_i \to L_i$ shows that M^0 is acceptable at p_7. Thus M^0 is acceptable. Apply Theorem 3.3 to $\{\overline{A}_i\}$ and M^0 to obtain $\overline{M}$. If $\overline{M}$ were ovoidal, it would satisfy the Bundle Theorem [2, p. 256], but M^0 is a counterexample to the Bundle Theorem (cf. [2, p. 255]).

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