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FREE STATES OF THE GAUGE INVARIANT CANONICAL ANTICOMMUTATION RELATIONS. II

BY

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ABSTRACT. A class of representations of the gauge invariant subalgebra of the canonical anticommutation relations (henceforth GICAR) is studied. These representations are induced by restricting the well-known pure, non-gauge invariant generalized free states of the canonical anticommutation relations (henceforth CAR). Denoting a state of the CAR by ω, and the unique generalized free state of the CAR such that $\omega(a(f)^*a(g)) = (f, Tg)$ and $\omega(a(f)a(g)) = (Sf, g)$ by $\omega_{S,T}$, it is shown that a pure, nongauge invariant state $\omega_{S,T}$ induces a factor representation of the GICAR if and only if Tr $T(I - T) = \infty$.

1. Introduction. The problem considered here is a natural extension of the work begun by the author in [2], where restrictions of gauge invariant generalized free states of the canonical anticommutation relations (CAR) to its $U(1)$ or gauge invariant subalgebra (GICAR) were studied. Viewing the CAR and GICAR as UHF and AF C^*-algebras in the sense of [5] and [4] respectively, a necessary and sufficient condition was obtained for such states to induce factor representations. The next simplest case is that of the nongauge invariant but pure generalized free state. Such states were analyzed by several authors [1], [3], the former directly generalizing a study of gauge invariant generalized free states [9]. In [6] the connection between this class of representations and the BCS theory of superconductivity is discussed.

Here, both the formalisms of [9] and [3] will be used when convenient, together with the obvious correspondence between them. The main result of this work, Theorem 4.17, is identical in conclusion to that of [2, Theorem 3.24]. In §3, two complex structures are analyzed; analogous procedures are found in [1] and [11]. Throughout, the techniques and results of [8], [9], [4], and [10] are relied upon heavily.

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2. Definitions, notation. Let $\mathcal{H}$ be a separable complex Hilbert space with orthonormal basis $\{f_n\}$, $n = 1, 2, \ldots$, and $\mathfrak{M}_n = \operatorname{span}\{f_1, \ldots, f_n\}$. The CAR algebra over $\mathfrak{M}_n$ is denoted by $\mathfrak{A}(\mathfrak{M}_n)$. It is constructed via a linear mapping $f \to a(f)$ satisfying the relations

$$a(f)a(g) + a(g)a(f) = 0,$$
$$a(f)a(g)^* + a(g)^*a(f) = (g, f)I$$

for all f, g in $\mathfrak{M}_n$. We denote by $\mathfrak{A}(\mathcal{H})$ the CAR algebra over $\mathcal{H}$, the completion of $\bigcup_n \mathfrak{A}(\mathfrak{M}_n)$ written $\mathfrak{A}(\mathcal{H}) = \overline{\bigcup_n \mathfrak{A}(\mathfrak{M}_n)}$. This is a UHF algebra in the sense of [5]. Note that commuting 2×2 matrix algebras are defined by the formulas $\mathfrak{B}_1 = \mathfrak{A}(\mathfrak{M}_1)$, $\mathfrak{B}_k = \mathfrak{A}(\mathfrak{M}_k) \cap \mathfrak{A}(\mathfrak{M}_{k-1})^c$, $k = 2, 3, \ldots$; it follows that $\mathfrak{A}(\mathfrak{M}_n) \approx \bigotimes_{k=1}^n \mathfrak{B}_k$, a $2^n \times 2^n$ matrix algebra (see e.g. [2]). We choose matrix units for the $\mathfrak{B}_k$ as in[2]:

$$e_{11}^{(k)} = a(f_k)a(f_k)^*, \quad e_{21}^{(k)} = a(f_k)V_k,$$
$$e_{12}^{(k)} = a(f_k)^*V_k, \qquad e_{22}^{(k)} = a(f_k)^*a(f_k)$$

where

$$V_1 = I,$$
$$V_k = \prod_{j=1}^{k-1} (I - 2a(f_j)^*a(f_j)), \qquad k = 2, 3, \ldots.$$

We denote by $\mathfrak{A}^\circ(\mathcal{H})$ the $U(1)$ or gauge invariant C^*-subalgebra of $\mathfrak{A}(\mathcal{H})$. This is the algebra invariant under the automorphisms χ_t, $0 \leq t < 2\pi$, defined by extending the map $a(f) \to e^{-it}a(f)$ to all of $\mathfrak{A}(\mathcal{H})$. $\mathfrak{A}^\circ(\mathcal{H})$ is an AF algebra in the sense of [4].

Alternatively, the CAR algebra may be constructed over a real Hilbert space [3]. From $\mathcal{H}$ we obtain a real Hilbert space (K, s) with scalar product $s(\ ,\ )$: for each vector $f \in \mathcal{H}$ there is a corresponding vector $[f] \in (K, s)$ and s is given by

$$s([f], [g]) = \operatorname{Re}(f, g)$$

which implies $s([f], [if]) = 0$. As in the case of $\mathfrak{A}(\mathcal{H})$, an algebra $\mathfrak{A}(K, s)$ is constructed via a linear mapping $[f] \to u([f])$ satisfying the new relations

$$u([f])u([g]) + u([g])u([f]) = 2s([f], [g])I,$$
$$u([f])^* = u([f])$$

for all $[f], [g] \in (K, s)$. Now if $[f]$ and $[g]$ are real orthogonal, the algebra generated by $u([f])$, $u([g])$ over $\mathbf{C}$ is isomorphic to a (complex) 2×2 matrix algebra. Further, algebras $\mathfrak{A}(M_k)$ and B_k may be constructed analogously to $\mathfrak{A}(\mathfrak{M}_k)$ and $\mathfrak{B}_k$ above.

In the sequel we will investigate a class of representations of $\mathfrak{A}^\circ(\mathcal{H})$ obtained by restricting the pure nongauge invariant generalized free states of

$\mathfrak{A}(\mathcal{H})$ to $\mathfrak{A}^\circ(\mathcal{H}) \subset \mathfrak{A}(\mathcal{H})$. A generalized free state ω of $\mathfrak{A}(\mathcal{H})$ ($\mathfrak{A}(K, s)$) is completely determined by its values on monomials of the form $a(f)^*a(g)$, $a(f)a(g)$ ($u([f])u([g])$) for all $f, g \in \mathcal{H}$ ($[f], [g] \in (K, s)$), see e.g. [7] and [3]. It follows that operators S, T on $\mathcal{H}$ and A on (K, s) may be defined by the formulas:

$$\omega(a(f)^*a(g)) = (f, Tg), \qquad \omega(a(f)a(g)) = (Sf, g)$$

and

$$\omega(u([f])u([g])) = s([f], [g]) + is(A[f], [g]).$$

From the properties of a state, we find T linear, $0 < T < I$, S antilinear, $S^* = -S$, and A skew-adjoint, i.e. $A^+ = -A$, with $+$ denoting the real adjoint on (K, s). We denote by $\omega_{S,T}$ (ω_A) the generalized free state of $\mathfrak{A}(\mathcal{H})$ ($\mathfrak{A}(K, s)$) with the two-point functions given by S, T (A) and the restriction of $\omega_{S,T}$ to $\mathfrak{A}^\circ(\mathcal{H})$ by $\omega^\circ_{S,T}$. Straightforward computation, using the isomorphism obtained by extending the mapping $u([f]) \to a(f) + a(f)^*$, yields, for a given state

$$s(A[f], [g]) = \mathrm{Re}((2iS + i(2T - I))f, g).$$

It is a consequence of [3, Theorem 1] that a complex structure J on (K, s) induces a pure generalized free state ω_J of $\mathfrak{A}(K, s)$. Defining ω_{J_1} to be the state of $\mathfrak{A}(K, s)$ corresponding to the (pure) Fock state $\omega_{0,0}$ it follows directly from the above formula that the complex structure J_1 satisfies

$$J_1[f] = -[if], \qquad J_1[if] = [f].$$

A state of $\mathfrak{A}(\mathcal{H})$ is said to factorize with respect to the algebras $\mathfrak{B}_k$, $k = 0, 1, 2, \ldots$, above if $\omega(xy) = \omega(x)\omega(y)$ for $x \in \mathfrak{B}_k, y \in \mathfrak{B}_l$, $k \neq l$; we then write $\omega = \bigotimes_{k=0}^{\infty}\omega|\mathfrak{B}_k$. If the $\mathfrak{M}_k$ are (simultaneous) invariant subspaces for S and T, it follows from the properties of a generalized free state that ω factorizes with respect to the associated $\mathfrak{B}_k$; similarly if M_k are invariant subspaces for $\mathfrak{A}(K, s)$, then ω_A factorizes on the associated B_k. For further elaboration of notation, definitions, or results see [7], [2]–[4].

3. Factor condition. We begin with a proposition indicating the correspondence between pure generalized free states of $\mathfrak{A}(K, s)$ and complex structures on (K, s); this is essentially [3, Theorem 3].

PROPOSITION 3.1. *Let ω_A be a generalized free state of $\mathfrak{A}(K, s)$. Then ω_A is pure $\Leftrightarrow$ A is a complex structure on (K, s).*

PROOF. Since ω_A is a generalized free state $A^+ = -A$; hence we need only show $A^2 = -I$. In [3, Theorem 3] it is concluded that ω_A is pure $\Rightarrow A^2 = -I$ assuming $\mathrm{Ker}(A)$ has even or infinite dimension, and that $A^2 = -I \Rightarrow \omega_A$ is pure. Thus we need only consider the case of odd, finite dimensional kernel.

In this case we show ω_A is not pure. Choose a unit vector $[f] \in \text{Ker}(A)$ and define $M = \text{span}([f])$. Straightforward computation shows ω_A factorizes with respect to $\mathfrak{A}(M, s)$ and $\mathfrak{A}(M^\perp, s)$. Assuming ω_A pure, it follows from the proof of [7, Theorem 5.5] that $\omega_A|\mathfrak{A}(M, s)$ and $\omega_A|\mathfrak{A}(M^\perp, s)$ are also pure. Now $\mathfrak{A}(M, s)$ is a two-dimensional C^*-algebra generated by I and $u([f])$ with two ideals generated by $I \pm u(f)$; it follows that the pure states satisfy $\omega(u([f])) = \pm 1$. However, for any generalized free state $\omega_A(u[f]) = 0$ contradicting the hypothesis that ω_A is pure with Ker(A) finite dimensional and odd. Done.

REMARK. The argument above shows ω_A pure $\Rightarrow \ker(A) = 0$.

We will now consider an arbitrary pure generalized free state on $\mathfrak{A}(K, s)$ and its associated complex structure, call it J_2. The aim is to factorize ω_{J_2} by choosing finite dimensional subspaces of (K, s) which are simultaneously invariant under J_2 and the complex structure J_1 of §1.

LEMMA 3.2. *Let J_2 be an arbitrary complex structure on (K, s) and J_1 be as above (see §1). Then the operator $M = -(J_1 - J_2)^2$ is a real selfadjoint operator on (K, s) with the properties*

(i) $0 < M/4 < I$,

(ii) $[J_1, J_2]_+ = M - 2I$.

PROOF. Since $(J_1 - J_2)^+ = -(J_1 - J_2)$, M is clearly selfadjoint. Further,

$$0 < (J_1 - J_2)^+ (J_1 - J_2) = -(J_1 - J_2)^2$$

and

$$\| -(J_1 - J_2)^2 \| = \|J_1 - J_2\|^2 \leqslant (\|J_1\| + \|J_2\|)^2 = 4$$

giving (i). For (ii), simply multiply out $-(J_1 - J_2)^2$. Done.

We now specialize to the case where M has pure point spectrum.

DEFINITION 3.3. Suppose $M/4$ has pure point spectrum $\{\mu_n\}$, $n = 1, 2, \ldots$; note by 3.2, $0 \leqslant \mu_n \leqslant 1$. We define θ_n by the formula

$$\theta_n = \cos^{-1}(1 - 2\mu_n).$$

The following proposition obtains simultaneous invariant subspaces for J_1, J_2 on which these operators take a simple form:

PROPOSITION 3.4. *Let $M = -(J_1 - J_2)^2$ as above, and K_{μ_n} be the spectral subspace of K associated with the spectrum point $4\mu_n$ of the operator M. Then there is a basis for K_{μ_n} satisfying*

(i) $$J_1|K_{\mu_n} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \otimes I\mu_n, \qquad \mu_n = 0, 1,$$

$$J_2|K_{\mu_n} = \begin{cases} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \otimes I_0, & \mu_n = 0, \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes I_1, & \mu_n = 1, \end{cases}$$

where I_{μ_n}, I_0, I_1 *are identities on real Hilbert spaces whose dimensions equal the multiplicities of eigenvalues* μ_n, 0, 1 *respectively and*

(ii)
$$J_1|K_{\mu_n} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \otimes I_{\mu_n}, \qquad 0 < \mu_n < 1,$$
$$J_2|K_{\mu_n} = \begin{pmatrix} 0 & R \\ -R^T & 0 \end{pmatrix} \otimes I_{\mu_n}, \qquad 0 < \mu_n < 1,$$

where 0, I, R *represent* 2×2 *matrices*, 0, I *as usual and*

$$R = \begin{pmatrix} \cos\theta_n & -\sin\theta_n \\ \sin\theta_n & \cos\theta_n \end{pmatrix}$$

with θ_n *defined as in* 3.3.

PROOF. The basis for this proof is to make the ansatz implicit in (i) and (ii) and carry out the computations.

(i) Choose a unit vector v in K_0; if there are none, let I_0 be the identity on a zero-dimensional space. Since v is in K_0, we have $-(J_1 - J_2)^2 v = 0$ and straightforward computation shows $-(J_1 - J_2)^2 J_i v = 0$, hence $J_i v$ in K_0, $i = 1, 2$. Clearly, $(v, J_i v) = 0$ and $\|J_i v\| = \|v\| = 1$, $i = 1, 2$. Finally, using $[J_1, J_2]_+ = -2I$ we have $-(J_1 J_2 v, v) = (v, J_1 J_2 v) + 2$ or $+1 = (J_2 v, J_1 v)$ giving $J_2 v = J_1 v$. Hence, v, $J_1 v$ are orthonormal and J_1 and J_2 are given by 2×2 matrices of the form appearing in (i). Now choose w in the orthogonal complement of v, $J_1 v$ and repeat the process for w, $J_1 w$. Continuing until K_0 is exhausted, we obtain (i), for $\mu_n = 0$. Similar computations to those above complete (i).

(ii) Given a unit vector v_1 in K_{μ_n}, we wish to find four orthonormal vectors v_1, v_2, v_3, v_4 such that the following are satisfied:

$$\begin{array}{llll} (1) & J_1 v_1 = -v_3, & (5) & J_2 v_1 = -v_3 \cos\theta + v_4 \sin\theta, \\ (2) & J_1 v_2 = -v_4, & (6) & J_2 v_2 = -v_3 \sin\theta - v_4 \cos\theta, \\ (3) & J_1 v_3 = v_1, & (7) & J_2 v_3 = v_1 \cos\theta + v_2 \sin\theta, \\ (4) & J_1 v_4 = v_2, & (8) & J_2 v_4 = -v_1 \sin\theta + v_2 \cos\theta, \end{array}$$

suppressing the subscript n on θ for convenience. Clearly, we can choose an arbitrary unit vector v_1 in K_{μ_n} (if not, let I_{μ_n} be the identity on a zero-dimensional space), and invert (3) to get $v_3 = -J_1 v_1$; substitute in (7) to get

$$v_2 = (I \cot\theta + J_1 J_2 \csc\theta) v_1$$

and in (5) to obtain

$$v_4 = (-J_1 \cot\theta + J_2 \csc\theta) v_1.$$

(Note $0 < \mu_n < 1$ implies $\sin\theta \neq 0$.) We now claim v_i, $i = 1, 2, 3, 4$, are orthonormal and satisfy (1) through (8) above. Given this, we may choose w_1 in the orthogonal complement of v_1, $i = 1, \ldots, 4$, and repeat the process, obtaining (ii). Straightforward computation using $J_i^+ = -J_i$, $J_i^2 = -I$, $i = 1, 2$, and $[J_1, J_2]_+|K_{\mu_n} = -2\cos\theta_n I$ shows that the v_i, $i = 1, 2, 3, 4$, are in K_{μ_n}, are orthonormal, and do satisfy (1) through (8). Done.

THEOREM 3.5. *Let* (K, s) *be the real Hilbert space corresponding to* $\mathcal{K}$ *as in* §1, *and* $\mathfrak{A}(K, s)$ *be the CAR algebra over* (K, s). *Let* ω_{J_1}, ω_{J_2} *be the Fock state and an arbitrary pure generalized free state on* $\mathfrak{A}(K, s)$ *respectively. Suppose that* $-(J_1 - J_2)^2$ *has pure point spectrum. Then we can write*

$$K = \bigoplus_{n=0}^{\infty} K_{\mu_n}$$

with the K_{μ_n} *satisfying the following*:

(i) *the* K_{μ_n} *are mutually orthogonal,*

(ii) $\dim(K_{\mu_n}) = 4$, $0 < \mu_n < 1$,

(iii) $\dim(K_{\mu_n}) = 2$, $\mu_n = 0, 1$,

(iv) $\omega_{J_2} = \bigotimes_{r=0}^{\infty}\omega_{J_2}|B_r$,

where B_r *is a finite-dimensional matrix algebra,* $r = 1, 2, \ldots$. *In particular* ω_{J_2} *factorizes with respect to the* $\{K_{\mu_n}\}$, $n = 1, 2, \ldots$.

PROOF. By hypothesis $-(J_1 - J_2)^2$ has pure point spectrum which we denote $\{4\mu_n\}$, $n = 1, 2, \ldots$, as above. Note by 3.2, $0 \leq \mu_n \leq 1$. Letting K_{μ_n} be the spectral subspace of $-(J_1 - J_2)^2$ associated with the eigenvalue $4\mu_n$, we choose a basis for these as in 3.4. This yields two and four dimensional mutually orthogonal invariant subspaces for J_i, $i = 1, 2$, corresponding to $\mu_n = 0, 1$ and $0 < \mu_n < 1$ respectively. Consequently, we may define $M_r = \bigoplus_{n=0}^{r} K_{\mu_n}$, $B_r = \mathfrak{A}(M_{r+1}, s) \cap \mathfrak{A}(M_r, s)^c$ and obtain factorization for ω_{J_2} on the finite-dimensional $\{B_r\}$, $r = 1, 2, \ldots$. Done.

COROLLARY 3.6. *Let* $\omega_{S,T}$ *be a pure generalized free state of* $\mathfrak{A}(\mathcal{K})$, *and suppose* T *has pure point spectrum. Then we can write*

$$\mathcal{K} = \bigoplus_{n=0}^{\infty} \mathcal{K}_{\mu_n}$$

with the $\mathcal{K}_{\mu_n}$ *satisfying the following*:

(i) *the* $\mathcal{K}_{\mu_n}$ *are mutually orthogonal,*

(ii) $\dim(\mathcal{K}_{\mu_n}) = 2$, $0 < \mu_n < 1$,

(iii) $\dim(\mathcal{K}_{\mu_n}) = 1$, $\mu_n = 0, 1$,

(iv) $\omega_{S,T} = \bigotimes_{r=0}^{\infty}\omega_{S,T}|\mathfrak{B}_r$ *with* $\mathfrak{B}_r$ *finite-dimensional matrix algebras*; *in particular* $\omega_{S,T}$ *factorizes with respect to the* $\{\mathcal{K}_{\mu_n}\}$, $n = 1, 2, \ldots$,

(v) *defining*

$$\Omega_{ij}^{(n)} = \begin{cases} \omega_{S,T}\left(e_{i_n j_n}^{(n)}\right), & \mu_n = 0, 1, \\ \omega_{S,T}\left(e_{i_1 j_1}^{(n1)} e_{i_2 j_2}^{(n2)}\right), & 0 < \mu_n < 1, \end{cases}$$

with the arguments chosen to be the usual matrix units for

$$\mathfrak{B}_r = \mathfrak{A}(\mathfrak{M}_{r+1}) \cap \mathfrak{A}(\mathfrak{M}_r)^c$$

(and $\mathfrak{M}_r = \oplus_{n=0}^{r} \mathfrak{K}_{\mu_n}$*) we have*

$$\Omega_{ij}^{(n)} = \begin{cases} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, & \mu_n = 0, \\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, & \mu_n = 1, \\ \begin{bmatrix} \mu_n & 0 & 0 & \sqrt{\mu_n(1-\mu_n)} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sqrt{\mu_n(1-\mu_n)} & 0 & 0 & 1-\mu_n \end{bmatrix}, & 0 < \mu_n < 1, \end{cases}$$

in matrix form.

Proof. Consider ω_{J_2} on $\mathfrak{A}(K, s)$ corresponding to $\omega_{S,T}$ above. Straightforward computation (see §1) shows the operators S, T on $\mathfrak{K}$ correspond to $-\frac{1}{4}[J_1, J_2]_-$, $I/2 - \frac{1}{4}[J_1, J_2]_+$ respectively. Hence $[J_1, J_2]_+$ and $-(J_1 - J_2)^2$ have pure point spectrum, since T does. Invoking Proposition 3.4, we can get simultaneous invariant subspaces K_{μ_n} for J_i, $i = 1, 2$. Now with the (K, s) to $\mathfrak{K}$ correspondence, we obtain subspaces $\mathfrak{K}_{\mu_n}$ for K_{μ_n}; these are two or one (complex) dimensional given K_{μ_n} four or two (real) dimensional respectively, i.e. $0 < \mu_n < 1$ or $\mu_n = 0, 1$ respectively. By construction, the $\mathfrak{K}_{\mu_n}$ are S, T invariant, and $\omega_{S,T}$ therefore factorizes on the associated $\mathfrak{B}_n$ yielding (i), (ii) and (iii). Finally, explicit computation gives (iv). Done.

Lemma 3.7. *Let* $\omega = \omega_{S,T}$ *be a pure generalized free state of* $\mathfrak{A}(\mathfrak{K})$, *such that* $\operatorname{Tr} T(I - T) < \infty$. *Let* $(\Pi, \mathfrak{K}, f)$ *be the representation induced by* ω *via the GNS construction, and define*

$$N_n = N(\mathfrak{M}_n) = N\left(\bigoplus_{r=0}^{n} \mathfrak{K}_{\mu_r}\right)$$

as in 3.6 *and*

$$V_n = N_n - \omega(N_n)I.$$

Then the sequence of operators $\{e^{i\Pi(V_n)t}\}$, $n = 1, 2, \ldots,$ *converges strongly to an isometry, call it* U_t, *on the closure of* $\Pi(\mathfrak{A}^\circ(\mathfrak{K}))f$, *call it* $\mathfrak{K}^\circ$, *for all* t *in* (0, 1/4).

Proof. We note that V_n is selfadjoint, and $e^{i\Pi(V_n)t}$ is unitary for all integers n and real numbers t. Hence if the limit exists on $\mathfrak{K}^\circ$, it will be an isometry. Since convergence on a dense set in $\mathfrak{K}^\circ$ allows us to extend to an isometry on all of $\mathfrak{K}^\circ$, we need only show the former. Since $\mathfrak{K}^\circ$ is complete, this will follow if the sequence

$$\{e^{i\Pi(V_n)t}g\}, \qquad n = 1, 2, \ldots,$$

is Cauchy for all g in a dense set. We first show that

(i) $\{e^{i\Pi(V_n)t}f\}$, $n = 1, 2, \ldots,$

is Cauchy, with f the cyclic vector and finally that

(ii) $\{e^{i\Pi(V_n)t}\Pi(x)f\}$, $n = 1, 2, \ldots,$

is Cauchy for all x in $\mathfrak{A}°(\mathcal{H})$. Since $\Pi(\mathfrak{A}°(\mathcal{H}))$ is dense in $\mathcal{H}°$, the proof will be complete. To show (i) we first note that

$$\|e^{i\Pi(V_n)t}f - e^{i\Pi(V_n)t}f\|^2 = 2 - 2\mathrm{Re}(f, e^{i\Pi(V_n)t}f).$$

Using the factorization

$$\omega_{S,T} = \bigotimes_{r=0}^{\infty} \omega_{S,T}|\mathfrak{B}_r$$

of 3.6, we begin the computation. Recall

$$\mathcal{H} = \bigoplus_{n=0}^{\infty} \mathcal{H}_{\mu_n};$$

let f_k be a basis for $\mathcal{H}_{\mu_k}$, $\mu_k = 0, 1$, and $f_k^{(1)}, f_k^{(2)}$ be a basis for $\mathcal{H}_{\mu_k}$, $0 < \mu_k < 1$. Analogously we define

$$n_k = \begin{cases} e_{22}^{(k)}, & \mu_k = 0, 1, \\ e_{22}^{(k1)} + e_{22}^{(k2)}, & 0 < \mu_k < 1, \end{cases}$$

and observe that in matrix form

$$n_k = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, & \mu_k = 0, 1, \\ \begin{bmatrix} 2 & & & \bigcirc \\ & 1 & & \\ & & 1 & \\ \bigcirc & & & 0 \end{bmatrix}, & 0 < \mu_k < 1, \end{cases}$$

ignoring the rest of the tensor product, which are repeated identity matrices. Thus

$$V_n - V_m = \sum_{k=m+1}^{n} (n_k - \omega(n_k)I)$$

and

$$e^{i(V_n - V_m)t} = \prod_{k=m+1}^{n} e^{it(n_k - \omega(n_k)I)}$$

for $[n_k, n_j]_- = 0$. Since ω factorizes with respect to $\{\mathcal{H}_{\mu_n}\}$, $n = 1, 2, \ldots,$ we also have

$$\omega(e^{i(V_n - V_m)t}) = \prod_{k=m+1}^{n} (e^{it(n_k - \omega(n_k)I)}).$$

Now using the matrix form of $\omega|\mathfrak{B}_n$ (namely $\Omega_{ij}^{(n)}$ of 3.6) and n_k, and straightforwardly computing we obtain

$$\omega(e^{i(V_n - V_m)t}) = \prod_{k=m+1}^{n} \{(1 - \mu_k)e^{-2it\mu_k} + \mu_k e^{2it(1-\mu_k)}\}.$$

Defining

$$\omega(e^{i(V_n - V_m)t}) = \prod_{k=m+1}^{n} (1 + z_k(t))$$

we have

$$\lim_{m,n\to\infty} \omega(e^{i(V_n - V_m)t}) = 1 \Leftrightarrow \prod_{k=0}^{\infty} (1 + z_k(t)) \text{ converges}$$

$$\Leftrightarrow \sum_{k=0}^{\infty} |z_k(t)| < \infty$$

by standard infinite product results. Direct computation shows that

$$|z_k(t)|^2 = -2(1 - \mu_k)\cos(2t\mu_k) - 2\mu_k\cos(2t(1 - \mu_k)).$$

We claim there exists an integer N such that

$$|z_k(t)|^2 \leq 4t\mu_k^2(1 - \mu_k)^2$$

for all $k > N$. To see this, first observe that if

$$f'(s) \leq g'(s) \quad \text{and} \quad f(0) = g(0) = 0$$

(assuming f and g are differentiable with derivatives f', g') then

$$\int_0^t f'(s)\, ds \leq \int_0^t g'(s)\, ds \quad \text{or} \quad f(t) \leq g(t).$$

Since $|z_k(0)| = 0$, we need only show

$$\frac{d}{dt} |z_k(t)|^2 \leq 4\mu_k^2(1 - \mu_k)$$

for $k > N$, or

$$-\sin(2t) + \sin(2t\mu_k) + \sin(2t(1 - \mu_k)) \leq \mu_k(1 - \mu_k).$$

Now

$$\operatorname{Tr} T(I - T) = 2\sum_{k=0}^{\infty} \mu_k(1 - \mu_k) < \infty$$

implies there exists an N such that $\mu_k(1 - \mu_k) < \frac{1}{4}$ for all $k > N$. Hence, there are two cases, $0 \leq \mu_k < \frac{1}{2}$, and $\frac{1}{2} < \mu_k \leq 1$, when $k > N$. The former gives

$$-\sin(2t) + \sin(2t\mu_k) + \sin(2t(1-\mu_k)) \leqslant -\sin(2t) + 2t\mu_k + \sin(2t) = 2t\mu_k < \tfrac{1}{2}\mu_k < \mu_k(1-\mu_k)$$

for all t in $(0, \frac{1}{4})$, since $x \leqslant \sin x$ and $\sin x$ is increasing for all x in $(0, \pi/2)$. Likewise, for the latter case we obtain

$$-\sin(2t) + \sin(2t\mu_k) + \sin(2t(1-\mu_k)) \leqslant \tfrac{1}{2}(1-\mu_k) < \mu_k(1-\mu_k)$$

for all t in $(0, \frac{1}{4})$. Thus,

$$|z_k(t)| \leqslant 2t^{1/2}\mu_k(1-\mu_k) \leqslant \mu_k(1-\mu_k)$$

for all $k > N$ and t in $(0, \frac{1}{4})$. Therefore

$$\sum_{k=0}^{\infty} \mu_k(1-\mu_k) < \infty \Rightarrow \sum_{k=0}^{\infty} |z_k(t)| < \infty$$

$$\Rightarrow \prod_{k=0}^{\infty} (1 + z_k(t)) \to a \neq 0 \Rightarrow \lim_{m,n\to\infty} \omega(e^{i(V_n - V_m)t}) = 1$$

$$\Rightarrow \lim_{m,n\to\infty} \|e^{i\Pi(V_m)t}f - e^{i\Pi(V_n)t}f\|^2 = 0$$

$$\Rightarrow \{e^{i\Pi(V_n)t}f\}, \qquad n = 1, 2, \ldots,$$

is Cauchy for all t in $(0, \frac{1}{4})$. Now defining $\mathfrak{D} = \bigcup_k \mathfrak{A}^\circ(\mathfrak{M}_k)$ we have $\mathfrak{D}$ dense in $\mathfrak{A}^\circ(\mathcal{K})$ and $\Pi(\mathfrak{D})f$ dense in $\Pi(\mathfrak{A}^\circ(\mathcal{K}))f$ dense in $\mathcal{K}^\circ$. Consequently, for (ii) we need only show

$$\{e^{i\Pi(V_n)t}\,\Pi(x)f\}, \qquad n = 1, 2, \ldots,$$

is Cauchy for all t in $(0, \frac{1}{4})$ and x in $\mathfrak{D}$ to complete the proof. But x in $\mathfrak{D}$ implies x in $\mathfrak{A}^\circ(\mathfrak{M}_p)$ for some p, and clearly V_n is in $\mathfrak{T}(\mathfrak{A}^\circ(\mathfrak{M}_n))$. Hence,

$$[\Pi(V_n), \Pi(x)]_- = [e^{i\Pi(V_n)t}, \Pi(x)]_- = 0$$

for all $n \geqslant p$, and for all $m, n \geqslant p$ we have

$$\|e^{i\Pi(V_n)t}\Pi(x)f - e^{i\Pi(V_m)t}\Pi(x)f\| \leqslant \|\Pi(x)\|\,\|e^{i\Pi(V_n)t}f - e^{i\Pi(V_m)t}f\| \to 0$$

as $n, m \to \infty$. This completes (ii), and the lemma. Done.

Remark. As $|z_k(t)| = |z_k(-t)|$, the proof of 3.7 gives strong convergence of $e^{i\Pi(V_n)t}$ for all t in $(-\frac{1}{4}, \frac{1}{4})$; thus (1) for fixed t in this interval, the adjoint sequence also converges strongly and hence (2) the limiting U_t are actually unitary.

Lemma 3.8. *Let $\omega = \omega_{S,T}$ be a pure generalized free state of $\mathfrak{A}(\mathcal{K}) = \overline{\bigcup_n \mathfrak{A}(\mathfrak{M}_n)}$ and suppose T has pure point spectrum. Given A in $\mathfrak{A}(\mathfrak{M}_n)$ define*

$$q_A(t) = \omega(Ae^{iN_nt}) - \omega(A)\omega(e^{iN_nt}).$$

Then the Fourier components of q_A are all even or all odd.

PROOF. We decompose $\mathcal{K}$ as in Corollary 3.6, and let $\mathfrak{M}_n = \bigoplus_{r=0}^{n} \mathcal{K}_{\mu_r}$. Now we may compute $q_A(t)$ as a product of finite-dimensional traces. For the two-dimensional $\mathcal{K}_{\mu_r}$ we observe only even powers of e^{it} appear in the associated trace; for the one-dimensional $\mathcal{K}_{\mu_r}$, only odd powers appear. Hence we have two cases (1) $\mathfrak{M}_n$ contains an odd number of one-dimensional $\mathcal{K}_{\mu_r}$, (2) $\mathfrak{M}_n$ contains an even number of one-dimensional $\mathcal{K}_{\mu_r}$. In case (1) all Fourier components of $q_A(t)$ are seen to be odd, in case (2) all even. Done.

COROLLARY 3.9. *Let $q_A(t)$ be as above. Then either*
(i) $q_A(t) = \sum_{k=0}^{p} a_k(1 - e^{2ikt})$ *or*
(ii) $q_A(t) = \sum_{k=0}^{q} b_k(1 - e^{2ikt})e^{it}$
with a_k, b_k complex numbers and $p, q \leqslant n$.

PROOF. By the above lemma, all Fourier components are odd or even, simultaneously. But by definition $q_A(0) = 0$; this enables us to write

$$q_A(t) = \sum_{k=0}^{r} c_k(1 - e^{ikt})$$

with $r \leqslant 2n$, since $\mathfrak{M}_n$ has at most n two-dimensional subspaces. Combining these two facts we can define a_k or b_k to obtain (i) or (ii). Done.

We are now ready to prove

THEOREM 3.10. *Let $\omega_{S,T}$ be a pure generalized free state of $\mathfrak{A}(\mathcal{K})$, and suppose T has pure point spectrum. Suppose further that T is not a projection. Then $\omega_{S,T}^{\circ}$ is a factor state $\Leftrightarrow$ $\operatorname{Tr} T(I - T) = \infty$.*

PROOF. ($\Rightarrow$) To show this, we suppose $\operatorname{Tr} T(I - T) < \infty$ and demonstrate that $\omega_{S,T}$ is not a factor. To this end we consider the U_t of Lemma 3.7. Since U_t is a strong limit of operators in $\Pi(\mathfrak{A}^{\circ}(\mathcal{K}))$, we have U_t in $\Pi(\mathfrak{A}^{\circ}(\mathcal{K}))''$ for all t in $(0, \frac{1}{4})$, which is the strong and weak closure of $\Pi(\mathfrak{A}^{\circ}(\mathcal{K}))$. Further, for all x in $\mathfrak{D} = \bigcup_k \mathfrak{A}^{\circ}(\mathfrak{M}_k)$ we saw that

$$\left[e^{i\Pi(V_n)t}, \Pi(x)\right]_- = 0$$

for all $n > p$, with x in $\mathfrak{A}^{\circ}(\mathfrak{M}_p)$. Thus

$$\lim_{n\to\infty} \left[e^{i\Pi(V_n)t}, \Pi(x)\right]_- g = 0$$

for all g in $\mathcal{K}^{\circ}$. But then

$$0 = \lim_{n\to\infty} \left[e^{i\Pi(V_n)t}, \Pi(x)\right]_- g = \left[\lim_{n\to\infty} e^{i\Pi(V_n)t}, \Pi(x)\right]_- g$$

for all g in $\mathcal{K}^{\circ}$, since operator multiplication is continuous. Hence

$$\left[U_t, \Pi(x)\right]_- g = 0$$

for all g in $\mathcal{H}^\circ$ or

$$[U_t, \Pi(x)]_- = 0.$$

But $\Pi(\mathcal{D})$ is dense in $\Pi(\mathfrak{A}^\circ(\mathcal{H}))$ so

$$[U_t, \Pi(y)]_- = 0$$

for all y in $\mathfrak{A}^\circ(\mathcal{H})$, i.e., U_t is in $\Pi(\mathfrak{A}^\circ(\mathcal{H}))'$. Summarizing, we have U_t in $\Pi(\mathfrak{A}^\circ(\mathcal{H}))' \cap \Pi(\mathfrak{A}^\circ(\mathcal{H}))''$ or U_t in $\mathfrak{Z}(\Pi(\mathfrak{A}^\circ(\mathcal{H})))$ for all t in $(0, \frac{1}{4})$. To prove $\omega_{S,T}$ is not a factor we need only show U_t is not a multiple of the identity. Since U_t is unitary, this reduces to $U_t \neq e^{i\phi}I$; for this it is sufficient to show $(f, U_t f) \neq e^{i\phi}$. However,

$$\begin{aligned}(f, U_t f) &= \left(f, \lim_{n\to\infty} e^{i\Pi(V_n)t} f\right)\\ &= \lim_{n\to\infty} (f, e^{i\Pi(V_n)t} f) = \lim_{n\to\infty} \omega_{S,T}(e^{iV_n t})\\ &= \prod_{k=1}^{\infty} \left\{(1-\mu_k) + \mu_k e^{2it}\right\} e^{-2it\mu_k}\end{aligned}$$

and therefore

$$\lim_{n\to\infty} |\omega_{S,T}(e^{iV_n t})|^2 = \prod_{k=1}^{\infty} \left\{1 - 2\mu_k(1-\mu_k)(1-\cos(2t))\right\}.$$

For all t in $(0, \frac{1}{4})$ we have

$$\lim_{n\to\infty} |\omega_{S,T}(e^{iV_n t})|^2 = a \neq 1$$

by the standard infinite product result, given $\sum_{k=1}^{\infty} \mu_k(1-\mu_k) \neq 0$. If not, we have $\mu_k(1-\mu_k) = 0$ for all k, and this implies T is a projection, contradiction the hypothesis. Done $(\Rightarrow)$.

$(\Leftarrow)$ Following [2, Proposition 3.14], we define

$$\begin{aligned}q_A(t) &= \omega(Ae^{iN_n t}) - \omega(A)\omega(e^{iN_n t}),\\ P_{mn}(t) &= \omega(e^{i(N_m - N_n)t})\end{aligned}$$

with A in $\mathfrak{A}^\circ(\mathfrak{M}_n)$, $m \geqslant n$, and the subscripts S, T suppressed for convenience. Using the identical calculation, we conclude that

$$\lim_{m\to\infty} \|q_A(t)P_{mn}(t)\|_1 = 0$$

implies ω° is a factor. As in [2], $\| \ \|_1$ denotes the Fourier-one norm. Since n is fixed and finite, this is equivalent to

$$\lim_{m\to\infty} \|q_A(t)P_m(t)\|_1 = 0$$

defining $P_m(t)$ to be the product from 1 to m and relabeling the nth term as the first. By Corollary 3.9 we can write

$$q_A(t)P_m(t) = \begin{cases} \sum_{k=0}^{p} a_k(1 - e^{2ikt})P_m(t), \\ \sum_{k=0}^{q} b_k(1 - e^{2ikt})e^{it}P_m(t) \end{cases}$$

and since for any function $f(t)$

$$\|e^{it}f(t)\|_1 = \|f(t)\|_1$$

it is sufficient to show

$$\lim_{m\to\infty} \|(1 - e^{2ikt})P_m(t)\|_1 = 0,$$

the total number of terms in either sum being finite. But now note that

$$(1 - e^{2ikt}) = \sum_{s=0}^{k-1} e^{2ist}(1 - e^{2it});$$

hence we need only show

$$\lim_{m\to\infty} \|(1 - e^{2it})P_m(t)\|_1 = 0$$

again due to the finiteness of the sum limit. Finally recall

$$P_m(t) = \omega^\circ\left(\prod_{k=1}^{m} e^{in_k t}\right) = \prod_{k=1}^{m} \omega^\circ(e^{in_k t})$$

by the factorization properties of ω°. Now straightforward computation shows

$$\omega^\circ(e^{in_k t}) = \begin{cases} 1, & \mu_k = 0, \\ e^{it}, & \mu_k = 1, \\ (1 - \mu_k) + \mu_k e^{2it}, & 0 < \mu_k < 1, \end{cases}$$

and we observe the presence of the $\mu_k = 0$, 1 terms do not alter the Fourier-one norm, since they are modulus one. Thus

$$\lim_{m\to\infty} \|(1 - e^{2it})P_m(t)\|_1 = \lim_{m\to\infty} \left\|(1 - e^{2it}) \prod_{\{k:\, 0<\mu_k<1\}}^{m} \{(1 - \mu_k) + \mu_k e^{2it}\}\right\|_1 = 0$$

since

$$\sum_{k=1}^{\infty} \mu_k(1 - \mu_k) = \infty \Rightarrow \sum_{\{k:0<\mu_k<1\}}^{\infty} \mu_k(1 - \mu_k) = \infty$$

and the above Fourier-one norm goes to zero by the arguments of [2, Theorem 3.20] with $2t$ replacing t. Done.

4. This section is aimed at removing the point spectrum condition from Theorem 3.10. We begin by obtaining an estimate for $\|\omega_J - \omega_{J'}\|$ in terms of $\|J - J'\|_{\text{H.S.}}$.

LEMMA 4.1. *Let ω be a state of a C^*-algebra $\mathfrak{A}$, and $E_1, E_2 \in \mathfrak{A}$ be commuting projections. Then if $\omega(E_1) \geqslant 1 - \lambda$ and $\omega(E_2) \geqslant 1 - \mu$ we have $\omega(E_1E_2) \geqslant 1 - \lambda - \mu$.*

PROOF. Note that

$$E_1E_2 + E_1(I - E_2) + E_2(I - E_1) + (I - E_1)(I - E_2) = I.$$

Applying ω to each of the summands on the left we get four numbers, call them a, b, c, d respectively; clearly $a, b, c, d \geqslant 0$. Also

(i) $\omega(I) = 1 = a + b + c + d$,

(ii) $\omega(E_1) = a + b \geqslant 1 - \lambda$,

(iii) $\omega(E_2) = a + c \geqslant 1 - \mu$,

(iv) $\omega(E_1E_2) = a$.

Adding (ii) and (iii) gives $2a + b + c \geqslant 2 - \lambda - \mu$; thus $2 + a + b + c + d = 1 + a \geqslant 2 - \lambda - \mu$ or $\omega(E_1E_2) \geqslant 1 - \lambda - \mu$. Done.

LEMMA 4.2 [9, LEMMA 2.5]. *Suppose ω_1 and ω_2 are states of a C^*-algebra. Suppose $\{E_\gamma; \gamma \in I_0\}$ is a decreasing net of projections in $\mathfrak{A}$ (i.e. $E_\alpha \leqslant E_\beta$ for $\alpha > \beta$) with the property that $\omega_1(E_\gamma) = 1$ for all $\gamma \in I_0$, and if ω is any state of $\mathfrak{A}$ such that $\omega(E_\gamma) = 1$ for all $\gamma \in I_0$, then $\omega = \omega_1$. Let $\alpha = \inf(\omega_2(E_\gamma); \gamma \in I_0)$. Then the following inequalities are valid*:

$$2(1 - \alpha) \leqslant \|\omega_1 - \omega_2\| \leqslant 2(1 - \alpha)^{1/2}.$$

Furthermore if ω_2 is pure, then

$$\|\omega_1 - \omega_2\| = 2(1 - \alpha)^{1/2}.$$

LEMMA 4.3. *Let $\omega_{0,0}$ be the Fock state, and $\omega_{S,T}$ be a pure generalized free state on $\mathfrak{A}(\mathcal{H})$, the CAR algebra over a complex separable Hilbert space $\mathcal{H}$. Then*

$$\|\omega_{0,0} - \omega_{S,T}\| \leqslant 2\|T\|_{\text{Tr}}^{1/2}.$$

PROOF. Let $\{f_k\}$ be an orthonormal basis for $\mathcal{H}$ and

$$E_N = \prod_{k=1}^{N} a(f_k)a(f_k)^* = \prod_{k=1}^{N} e_{11}^{(k)}.$$

It follows from Lemma 4.2 and the properties of the Fock state that

$$\|\omega_{0,0} - \omega_{S,T}\| = 2(1 - \alpha)^{1/2}$$

with $\alpha = \lim_{N\to\infty}\omega_{S,T}(E_N)$. But $\omega_{S,T}(a(f_k)a(f_k)^*) = 1 - (f_k, Tf_k)$, and we will show

$$\omega_{S,T}(E_N) \geqslant 1 - \sum_{k=1}^{N} (f_k, Tf_k)$$

by induction. Equality holds for $N = 1$, so assume true for N; for $N + 1$ we obtain

$$\omega_{S,T}(E_{N+1}) = \omega_{S,T}\left(E_N e_{11}^{(N+1)}\right) \geqslant 1 - \sum_{k=1}^{N} (f_k, Tf_k) - (f_{N+1}, Tf_{N+1})$$

by Lemma 4.1. Hence $\alpha \leqslant 1 - \|T\|_{\mathrm{Tr}}$ and by Lemma 4.2

$$\|\omega_{0,0} - \omega_{S,T}\| = 2(1-\alpha)^{1/2} \leqslant 2\|T\|_{\mathrm{Tr}}^{1/2}.$$

Done.

LEMMA 4.4. *Let $\omega_{S,T}$, $\omega_{S',T'}$, be pure generalized free states of $\mathfrak{A}(\mathcal{K})$ and ω_J, $\omega_{J'}$ the corresponding states of $\mathfrak{A}(K, s)$ respectively. Then*

$$\tfrac{1}{8}\|J - J'\|_{\mathrm{H.S.}}^2 = \|S - S'\|_{\mathrm{H.S.}}^2 + \|T - T'\|_{\mathrm{H.S.}}^2$$

with H.S. *denoting the real Hilbert-Schmidt norm on (K, s) on the left and* H.S. *denoting the standard Hilbert-Schmidt norm on $\mathcal{K}$ on the right.*

PROOF. Using the relation

$$s(A[f], [g]) = \mathrm{Re}((2iS + i(2T - I))f, g)$$

(see §1) we directly compute $\|J - J'\|_{\mathrm{H.S.}}^2$ with the basis $\{[f_n], [if_n]\}$ of $\mathfrak{A}(K, s)$ to obtain the equality. Done.

PROPOSITION 4.5. *Let J, J' be complex structures on (K, s) and ω_J, $\omega_{J'}$ the corresponding generalized free states of $\mathfrak{A}(K, s)$. Then*

$$\|J - J'\|_{\mathrm{H.S.}} < \sqrt{2}\,\varepsilon \Rightarrow \|\omega_J - \omega_{J'}\| < \varepsilon.$$

PROOF. Let ω_L be the pure generalized free state of $\mathfrak{A}(K, s)$ corresponding to the state $\omega_{0,0}$ of $\mathfrak{A}(\mathcal{K})$. Straightforward computation shows that L is a complex structure on (K, s). Since any two complex structures may be related by an orthogonal transformation B, and the mapping $\tau_B(u(h)) = u(Bh)$ preserves the anticommutation relations and thus extends to an automorphism of $\mathfrak{A}(K, s)$, there is an automorphism τ_B of $\mathfrak{A}(K, s)$ such that $\omega_J \circ \tau_B = \omega_L$. Hence $\omega_{J'} \circ \tau_B = \omega_M$ with $M = B^+J'B$, a complex structure. Let $\omega_{S,T}$ on $\mathfrak{A}(\mathcal{K})$ be the state corresponding to ω_M on $\mathfrak{A}(K, s)$. It follows from Lemma 4.3 that

$$\|\omega_J - \omega_{J'}\| = \|\omega_L - \omega_M\| = \|\omega_{0,0} - \omega_{S,T}\| \leqslant 2\|T\|_{\mathrm{Tr}}^{1/2}.$$

Since $M^2 = -I$ on (K, s) we have the corresponding equation on $\mathcal{K}$:

$$(2iS + i(2T - I))^2 = -I$$

or

$$- S^2 - ST + TS + T^2 - T = 0.$$

Let $\|J - J'\|^2_{\text{H.S.}} < \sqrt{2}\,\varepsilon$; by Lemma 4.4

$$2\varepsilon^2 > \|J - J'\|^2_{\text{H.S.}} = \|L - M\|^2_{\text{H.S.}} = 8\big(\|S\|^2_{\text{H.S.}} + \|T\|^2_{\text{H.S.}}\big).$$

We conclude S and T are Hilbert-Schmidt class, hence ST and TS are trace class. From the above identity, we observe that T is trace class. Taking the trace of both sides and noting $T > 0$, we obtain

$$\|S\|^2_{\text{H.S.}} + \|T\|^2_{\text{H.S.}} = \|T\|_{\text{Tr}} < \varepsilon^2/4$$

and $\|\omega_J - \omega_{J'}\| < \varepsilon$. Done.

DEFINITION 4.6. Let $\{[f_n], [if_n]\}$ be a basis for (K, s) and J_1 the complex structure associated with this basis as in §1. Let J_2 be an arbitrary complex structure on (K, s) and $M = -(J_1 - J_2)^2$ as in Lemma 3.2. We define subspaces K_0, K_1, K_2 of (K, s) and an operator Λ on (K, s) by the formulas

$$K_0 = \{h \in K \colon Mh/4 = 0\}, \quad \Lambda|K_0 = 0,$$
$$K_1 = \{h \in K \colon Mh/4 = h\}, \quad \Lambda|K_1 = \pi I,$$
$$K_2 = K \ominus (K_0 \oplus K_1), \quad \Lambda|K_2 = \cos^{-1}\big(I + \tfrac{1}{2}(J_1 - J_2)^2\big).$$

REMARK. It follows directly from the definitions that

$$M = 2I - 2\cos\Lambda.$$

The following definitions aim toward constructing a continuous realization of J_1, J_2 and Λ_2 as multiplication by matrices of functions on a measure space. An application of the Weyl-von Neumann theorem will then yield an appropriate sequence of complex structures approaching J_2 in Hilbert-Schmidt norm.

DEFINITION 4.7. Let $\Lambda_2 = \int_0^\pi \lambda\, dE_\lambda$, the spectral resolution of Λ_2. Let f_0 be a unit vector in K_2. We define a measure μ_0 on $[0, \pi]$ by the formula

$$\mu([0, \lambda]) = (f_0, E_\lambda f_0).$$

DEFINITION 4.8. Let h be in $\mathcal{H}_0 = \mathcal{L}^2_{\mathbf{R}}([0, \pi], \mu_0) \otimes \mathbf{R}^4$, i.e. h is a 4-tuple with components h_i real, μ_0 square-integrable functions on $[0, \pi]$, $i = 1, \ldots, 4$. We define a mapping

$$U_0 \colon \mathcal{H}_0 \to K_2$$

by the formula

$$U_0 h = \int_0^\pi h_1(\lambda)\, dE_\lambda f_0 + \int_0^\pi h_2(\lambda)\big[\csc\lambda(I\cos\lambda + J_1 J_2)\big]\, dE_\lambda f_0$$
$$+ \int_0^\pi h_3(\lambda)(-J_1)\, dE_\lambda f_0 + \int_0^\pi h_4(\lambda)\big[\csc\lambda(-J_1\cos\lambda + J_2)\big]\, dE_\lambda f_0.$$

REMARK. This mapping is clearly linear. It is also well defined when the component functions h_i vanish in a neighborhood of 0 and π. Further, with the inner product $\langle h, g\rangle_0 = \Sigma_{i=1}^{4}\int_0^{\pi} h_i(\lambda)g_i(\lambda)d\mu_0(\lambda)$ it is straightforward but tedious to show that U_0 is isometric, using the properties of the spectral integral and calculations analogous to those showing v_1, v_2, v_3, v_4 orthonormal outlined in the proof of Proposition 3.4. Since U_0 is an isometry it may be extended to arbitrary square integrable h_i by continuity; hence U_0 is well defined.

DEFINITION 4.9. Let f_0 be a unit vector in K_2, and $P[J_1, J_2]$ be the polynomial algebra generated by J_1, J_2. We define $H_0 = [P[J_1, J_2]f_0]^-$.

LEMMA 4.10. $H_0 \subset K_2$.

PROOF. By straightforward computation (see the proof of Proposition 3.4) we observe $J_iK_0 \subset K_0$ and $J_iK_1 \subset K_1$, $i = 1, 2$. Hence for all $g \in K_0(K_1)$ we have $s(g, J_if_0) = s(-J_ig, f_0) = 0$. Thus $J_if_0 \in K_2$ and it follows that $[P[J_1, J_2]f_0]^- \subset K_2$. Done.

REMARK. By the formula defining U_0 and Lemma 4.10 we conclude $U_0h \in H_0$. We may now construct a sequence of measures on $[0, \pi]$, $\{\mu_k\}$, $k = 0, 1, 2, \dots,$ and an isometry U: $\bigoplus_k \mathcal{H}_k \to K_2$ with $\mathcal{H}_k = \mathcal{L}^2_{\mathbf{R}}([0, \pi], \mu_k) \otimes \mathbf{R}^4$. Let f_0 be a unit vector in K_2 as above; we define U_0: $\mathcal{H}_0 \to H_0 \subset K_2$ as above. Now choose a unit vector $f_1 \in H_0^{\perp} \cap K_2$. Define $\mu_1([0, \lambda]) = (f_1, E_\lambda f_1)$, $H_1 = [P[J_1, J_2]f_1]^-$, and U_1 analogously to U_0; by the same argument U_1 is also an isometry. Repeating this process we must exhaust K_2 in at most countably many steps since we assume K_2 separable. That is, $K_2 = \bigoplus_k H_k$ with k running over a finite or countable set. We now give

DEFINITION 4.11. Let U_k and $\mathcal{H}_k$ be as above, and $\mathcal{H} = \bigoplus_k \mathcal{H}_k$. We define an isometry U: $\mathcal{H} \to K_2$ by the formulas $U|\mathcal{H}_k = U_k$.

We are now ready to realize J_1 and J_2 on K_2 via operators J_1' and J_2' on $\mathcal{H}$.

DEFINITION 4.12. We define the operators J_1' and J_2' on $\mathcal{H}$ by the formulas

$$J_1'|\mathcal{H}_k = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \qquad J_2'|\mathcal{H}_k = \begin{pmatrix} 0 & R \\ -R^T & 0 \end{pmatrix}$$

where

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

0 and 1 denoting multiplication by the zero and unit functions on $[0, \pi]$ respectively and

$$R = \begin{pmatrix} \cos\lambda & -\sin\lambda \\ \sin\lambda & \cos\lambda \end{pmatrix}$$

each entry denoting multiplication by the given function.

LEMMA 4.13. *Let* U, J_i, J_i', $i = 1, 2$, *be as above. Then* $UJ_i' = J_iU$, $i = 1, 2$.

PROOF. Choose $h \in \mathcal{H}$, then $h = \sum_k h_k$, $h_k \in \mathcal{H}_k$. It is clearly sufficient to show the formula holds for $h \in \mathcal{H}_k = \mathcal{L}^2_{\mathbf{R}}([0, \pi], \mu_k) \otimes \mathbf{R}^4$, so let

$$h = \begin{bmatrix} h_1(\lambda) \\ 0 \\ 0 \\ 0 \end{bmatrix} \in \mathcal{H}_k.$$

Then

$$UJ_2' \begin{bmatrix} h_1(\lambda) \\ 0 \\ 0 \\ 0 \end{bmatrix} = U \begin{bmatrix} 0 \\ 0 \\ -h_1(\lambda)\cos\lambda \\ h_1(\lambda)\sin\lambda \end{bmatrix}$$

$$= \int_0^\pi -h_1(\lambda)\cos\lambda(-J_1)\,dE_\lambda f_k$$

$$+ \int_0^\pi h_1(\lambda)\sin\lambda[\csc\lambda(-J_1\cos\lambda + J_2)]\,dE_\lambda f_k$$

$$= \int_0^\pi h_1(\lambda) J_2\, dE_\lambda f_k$$

and

$$J_2 U \begin{bmatrix} h_1(\lambda) \\ 0 \\ 0 \\ 0 \end{bmatrix} = J_2 \int_0^\pi h_1(\lambda)\,dE_\lambda f_k = \int_0^\pi h_1(\lambda)\, J_2\, dE_\lambda f_k.$$

Computing in a similar fashion for the other three components (and J_1) we obtain the desired equality. Done.

LEMMA 4.14. *Let* $\{A_n\}$, $n = 1, 2, \ldots,$ *be a sequence of bounded selfadjoint operators on a separable Hilbert space. Suppose there is an operator* A *such that* $A_n \to A$ *in Hilbert-Schmidt norm. Then* $\cos A_n$, $\sin A_n$ *converge to* $\cos A$, $\sin A$ *in Hilbert-Schmidt norm respectively.*

PROOF. First observe

$$\frac{d}{dt}\left(e^{itA}e^{-itA_n}\right) = ie^{itA}(A - A_n)e^{-itA_n};$$

integrating from 0 to 1 we obtain

$$e^{iA}e^{-iA_n} - I = \int_0^1 ie^{itA}(A - A_n)e^{-itA_n}\,dt.$$

Now

$$\|e^{iA} - e^{iA_n}\|_{\text{H.S.}} = \|(e^{iA} - e^{iA_n})e^{-iA_n}\|_{\text{H.S.}} = \|e^{iA}e^{-iA_n} - I\|_{\text{H.S.}}$$
$$= \left\| \int_0^1 ie^{itA}(A - A_n)e^{-itA_n}\,dt \right\|_{\text{H.S}} \leqslant \int_0^1 \|e^{itA}(A - A_n)e^{-itA_n}\|_{\text{H.S.}}\,dt$$
$$= \int_0^1 \|A - A_n\|_{\text{H.S.}}\,dt = \|A - A_n\|_{\text{H.S.}}$$

since Hilbert-Schmidt norm is preserved under multiplication by a unitary. Summarizing, for every $\varepsilon > 0$, there is an N such that

$$\|e^{iA} - e^{-iA_n}\|_{\text{H.S.}} \leqslant \|A - A_n\|_{\text{H.S.}} < \varepsilon \quad \text{for } n > N.$$

It follows that $\|\cos A - \cos A_n\|_{\text{H.S.}} < \varepsilon$, $\|\sin A - \sin A_n\|_{\text{H.S.}} < \varepsilon$ for $n > N$. Done.

Lemma 4.15. *Let J_2 be an arbitrary complex structure on a real Hilbert space (K, s) and J_1 be the complex structure associated with a paired basis of (K, s) as in* §1. *There is a sequence of complex structures $\{J_{2,n}\}$, $n = 1, 2, \ldots,$ such that $-(J_1 - J_{2,n})^2$ has pure point spectrum and $\|J_2 - J_{2,n}\|_{\text{H.S.}} \to 0$ as $n \to \infty$.*

Proof. Consider $J_2'|\mathcal{K}_k$ as above; as in the Weyl-von Neumann theorem we may approximate multiplication by λ on $\mathcal{L}^2_{\mathbf{R}}([0, \pi], \mu_k)$ by an operator Λ_k with pure point spectrum such that $\|\Lambda_k - \lambda \cdot\|_{\text{H.S.}} < 1/8n2^k$. By the proof of Lemma 4.14, $\|\cos \Lambda_k - \cos \lambda \cdot\|_{\text{H.S.}} < 1/8n2^k$. Now define

$$J_{2,n}'|\mathcal{K}_k = \begin{pmatrix} 0 & R \\ -R^+ & 0 \end{pmatrix}$$

with

$$R = \begin{pmatrix} \cos \Lambda_k & -\sin \Lambda_k \\ \sin \Lambda_k & \cos \Lambda_k \end{pmatrix}.$$

Now $\|(J_{2,n}' - J_2')|\mathcal{K}_k\|_{\text{H.S.}} < 1/n2^k$, by straightforward computation. This yields

$$\|J_{2,n}' - J_2'\|_{\text{H.S.}} \leqslant \sum_{k=0}^{\infty} \frac{1}{n2^{k+1}} = \frac{1}{n}.$$

Finally, define

$$J_{2,n}|K_2 = UJ_2'U^*, \quad J_{2,n}|K_0 = J_2, \quad J_{2,n}|K_1 = J_2.$$

Then

$$\|J_2 - J_{2,n}\|_{\text{H.S.}} = \|(J_2 - J_{2,n})|K_2\|_{\text{H.S.}} = \|J_{2,n}' - J_2'\|_{\text{H.S.}} < 1/n.$$

Further, $-(J_1' - J_{2,n}')^2$ has pure point spectrum since $-(J_1' - J_{2,n}')^2|\mathcal{K}_K = 2I - 2\cos \Lambda_k$ and Λ_k has pure point spectrum. From the unitarity of U and the definition of $J_{2,n}$ on K_0 and K_1 we conclude $-(J_1 - J_{2,n})^2$ has pure point spectrum. Done.

LEMMA 4.16. *Let $\omega_{S,T}$ and $\omega_{S',T'}$ be pure generalized free states of $\mathfrak{A}(\mathcal{K})$ such that $S - S'$ is a Hilbert-Schmidt class operator. Then* $\operatorname{Tr} T(I - T) = \infty \Leftrightarrow \operatorname{Tr} T'(I - T') = \infty$.

PROOF. Equivalently, we show $\operatorname{Tr} T(I - T) < \infty \Leftrightarrow \operatorname{Tr} T'(I - T') < \infty$. Observing that it is sufficient to prove ($\Rightarrow$) we assume $\operatorname{Tr} T(I - T) < \infty$. From the proof of Proposition 4.5 we have

$$\|S\|^2_{\text{H.S.}} = \|T\|_{\text{Tr}} - \|T\|^2_{\text{H.S.}} = \operatorname{Tr} T(I - T);$$

by hypothesis $\operatorname{Tr} T(I - T) < \infty$ and therefore

$$\|S'\|_{\text{H.S.}} \leqslant \|S\|_{\text{H.S.}} + \|S - S'\|_{\text{H.S.}} < \infty$$
$$\Rightarrow \operatorname{Tr} T'(I - T') < \infty,$$

since the above equality holds for S', T' as well. Done.

We are now ready to remove the point spectrum condition.

THEOREM 4.17. *Let $\omega_{S,T}$ be a pure nongauge invariant generalized free state of $\mathfrak{A}(\mathcal{K})$. Then $\omega^\circ_{S,T}$ is a factor state* $\Leftrightarrow \operatorname{Tr} T(I - T) = \infty$.

PROOF. ($\Rightarrow$) Suppose $\operatorname{Tr} T(I - T) < \infty$; then T has pure point spectrum. In [7, Lemma 4.8] it is shown that if T is a projection, $S = 0$; from the definitions it follows that $\omega_{S,T}$ is gauge invariant. Hence, T is not a projection and the proof reduces to that of Theorem 3.10.

($\Leftarrow$) Let ω_{J_2}, ω_{J_1} be the states of $\mathfrak{A}(K, s)$ corresponding to $\omega_{S,T}$, $\omega_{0,0}$ respectively. As we noted in the proof of Corollary 3.6 if $M = -(J_1 - J_2)^2$ has pure point spectrum, T has pure point spectrum and we are done by Theorem 3.10. If not, by Lemma 4.15 there is a sequence of complex structures $J_{2,n}$ such that $-(J_1 - J_{2,n})^2$ has pure point spectrum and $\|J_2 - J_{2,n}\|_{\text{H.S.}} \to 0$ as $n \to \infty$. By Proposition 4.5, we have $\|\omega_{J_{2,n}} - \omega_{J_2}\| \to 0$ as $n \to \infty$. Let ω_{S_n,T_n} be the state of $\mathfrak{A}(\mathcal{K})$ corresponding to $\omega_{J_{2,n}}$, then

$$\|\omega^\circ_{S_n,T_n} - \omega^\circ_{S,T}\| \leqslant \|\omega_{S_n,T_n} - \omega_{S,T}\| = \|\omega_{J_{2,n}} - \omega_{J_2}\| \to 0$$

as $n \to \infty$. By Lemmas 4.16, 4.4 and Theorem 3.10, the $\omega^\circ_{S_n,T_n}$ are also factor states. Thus $\omega^\circ_{S,T}$ is the uniform limit of factor states, hence a factor state. Done.

We conclude with a

REMARK. If $\omega_{S,T}$ is a pure gauge invariant state of $\mathfrak{A}(\mathcal{K})$ it follows from [**10**, III.1, Proposition] that $\omega^\circ_{S,T}$ is pure.

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