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CELL-LIKE 0-DIMENSIONAL DECOMPOSITIONS OF S^3 ARE 4-MANIFOLD FACTORS

BY

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ABSTRACT. The main result is the title theorem asserting that if G is any upper semicontinuous decomposition of S^3 into cell-like sets which is 0-dimensional, in the sense that the image of the nondegenerate elements in S^3/G is 0-dimensional, then $G \times S^1$ is shrinkable, and $(S^3/G) \times S^1$ is homeomorphic to $S^3 \times S^1$.

Late in 1973 the independently functioning teams of R. D. Edwards and R. T. Miller **[E-M]** and of W. T. Eaton and C. P. Pixley **[E-P]** showed that if G is a cell-like upper semicontinuous decomposition of the 3-sphere S^3 such that the closure of the image of the nondegenerate elements, in the decomposition space S^3/G, is 0-dimensional, then $(S^3/G) \times S^1$ is topologically $S^3 \times S^1$. More recently J. W. Cannon **[C_2]** has devised an alternate proof of the same result. Their theorem is expanded here to one anticipated ever since (eliminating the hypothesis concerning closure) based upon a new argument unlike either of theirs.

Investigations concerning products of decompositions with a line (or, essentially equivalently, a circle) trace their source to R. H. Bing's groundbreaking work **[B]** proving that the product of his nonmanifold dogbone space with a line is topologically E^4. After a subsequent history rich with a variety of results, R. D. Edwards derived a momentous theorem **[E_2]**, providing nonsimply connected homology n-spheres whose double suspensions are S^{n+2}, by showing that if X is any cell-like subset of E^n ($n \geqslant 4$), then $(E^n/X) \times E^1$ is topologically E^{n+1} (which quickly translates to a higher dimensional analogue of the Edwards-Miller and Eaton-Pixley work). Very recently, J. W. Cannon **[C_1]** completed the double suspension business, proving that the double suspension of any homology n-sphere is an $(n + 2)$-sphere (Edwards had done all except certain examples where $n = 3$), and analyzed in a profound manner the cell-like decompositions of n-manifolds ($n \geqslant 5$) that yield the same manifold, in terms of a Disjoint Disk

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Property (e.g., two maps of a 2-cell into the given decomposition space Q can be approximated by maps having disjoint images). Edwards [$\mathbf{E}_3$] then characterized such decompositions (of manifolds having dimension at least 5) by means of two features inherent in the decomposition space: finite dimensionality and this Disjoint Disk Property.

The paper at hand is a direct outgrowth of Edwards' latest effort, imitating, insofar as possible, its underlying strategy and invoking its lemmas and methods. In fact, our work originated with the overwhelming suspicion that his approach should be adaptable to 4-manifold decomposition problems–at the very least, to provide a more elementary proof of the Edwards-Miller and Eaton-Pixley Theorem.

1. Essential terminology, main results, and organizational outline. Generally, all decompositions referred to are upper semicontinuous. In particular, a 0-*dimensional decomposition* is one for which the image of the nondegenerate elements, in the associated decomposition space, is 0-dimensional; a CE *decomposition* G is a cell-like one, that is to say, each element $g \in G$ is a cell-like set [**L**]. Similarly, a CE *map* is a cell-like one (each point inverse is a cell-like set).

Given a decomposition G, we use H_G to denote its set of nondegenerate elements and N_G, the union of these nondegenerate elements. Similarly, given a map $f\colon X \to Y$, we use N_f to denote the union of all the nondegenerate sets $f^{-1}(y), y \in Y$, and call N_f the nondegeneracy set of f.

Explicitly, the main result is the following:

THEOREM 1. *If G is a* 0-*dimensional CE decomposition of S^3, then $(S^3/G) \times S^1$ is homeomorphic to $S^3 \times S^1$. In particular, the decomposition $G \times S^1$ of $S^3 \times S^1$ is shrinkable in the following sense: for each open set U in S^3 containing N_G and each $\varepsilon > 0$, there exists a homeomorphism h of $S^3 \times S^1$ to itself such that*

(1) $h|(S^3 - U) \times S^1 = 1$ (*the identity*),

(2) *h changes S^1-coordinates less than ε,*

(3) diam $h(g \times s) < \varepsilon$ *for each $g \times s \in G \times S^1$.*

The first argument showing that such a shrinkability property leads to a homeomorphism between $S^3 \times S^1$ and $(S^3/G) \times S^1$ is due to Bing [**B**]. Edwards [$\mathbf{E}_2$] has set forth a notably elegant alternative, stemming from the Baire Category Theorem. Their arguments provide a fundamental measure of shrinkability: $G \times S^1$ is shrinkable (as above) if and only if the decomposition map $\pi \times 1\colon\ S^3 \times S^1 \to (S^3/G) \times S^1$ can be approximated arbitrarily closely by homeomorphisms that agree with $\pi \times 1$ off $U \times S^1$.

As a corollary to the proof, one can readily decompactify the manifolds involved.

COROLLARY 1A. *If G is a 0-dimensional CE decomposition of an open subset U of S^3, then $(U/G) \times E^1$ is homeomorphic to $U \times E^1$.*

However, in considering more general 3-manifolds, one must avoid the possible failure of the 3-dimensional Poincaré Conjecture.

COROLLARY 1B. *If G is a 0-dimensional CE decomposition of a 3-manifold M (without boundary) such that each $g \in G$ has a neighborhood embeddable in S^3, then $(M/G) \times E^1$ is homeomorphic to $M \times E^1$.*

The extra restriction in Corollary 1B arises from the need to express decomposition elements as intersections of thin cubes with handles from the overlying 3-manifold. The terminology employed for describing that thinness is Štan'ko's notion of embedding dimension [**Šk$_1$**], [**Šk$_2$**]: a compact set X in a PL m-manifold M is said to have *embedding dimension* $\leqslant k$, abbreviated as dem $X \leqslant k$, if X can be described as a nested intersection of thin regular neighborhoods of k-dimensional subpolyhedra in M. It turns out, even in the more general setting where X is σ-compact, that dem $X \leqslant k$ if and only if each $(m - k - 1)$-dimensional subpolyhedron of M can be pushed off X via arbitrarily small adjustments of M (cf. [**E$_1$**]).

Two other minor matters of terminology: following current fashion, we say that a map f is 1-1 *over a subset R* of the range if $f|f^{-1}(R)$ is 1-1 and that a map θ of a space S to itself *equals the identity* (usually written simply as 1) *near a subset D* of S if there exists a neighborhood U of D in S such that $\theta|U = 1$.

To outline the contents of this paper, the initial step, found in §2, reduces the problem to that of a decomposition K consisting of a null sequence of nondegenerate elements, each of embedding dimension 1 (hence, dem $N_K =$ 1). Next, in §3, we state the Main Lemma, adapted from the Edwards approach [**E$_3$**], to expose the framework for the attack. Then we detail, in §4, a careful rearrangement of thin cell-like sets from levels $S^3 \times \{s\}$ of $S^3 \times S^1$ to allow a meticulously controlled shrinking, after which, in §5, we perform that shrinking. The combination of these two maneuvers, given in Lemmas 5 and 6, constitutes the heart, the novel features, of this paper. Finally, in the last section, we sketch a proof for the Main Lemma, delineating the correspondence between the standard shrinking and iterated approximation processes.

2. Reduction to a simple null sequence decomposition. In this section Theorem 1 is reduced to the case in which the nondegenerate elements of the decomposition form a null sequence of cell-like continua of embedding dimension 1. The reduction is based upon methods of Edwards [**E$_3$**] and the following result recently proved by Starbird [**Sr**, Theorem 2.1].

THEOREM S [STARBIRD]. *Let G denote a 0-dimensional CE decomposition of S^3, U an open subset of S^3 containing N_G, and A a tame finite graph in S^3. Then there exists a homeomorphism h of S^3 to itself such that $h|S^3 - U = 1$ and $h(A) \cap N_G = \varnothing$.*

COROLLARY. *Let G denote a 0-dimensional CE decomposition of S^3, with decomposition map π: $S^3 \to S^3/G$. Then for each $\varepsilon > 0$ and open set $U \supset N_G$ there exists a CE map P of S^3 to S^3/G such that $P|S^3 - U = \pi|S^3 - U$, $N_P \subset U$, $\rho(P, \pi) < \varepsilon$,* dem $N_P \leqslant 1$, *and* dim $P(N_P) \leqslant 0$.

This corollary is a consequence of Theorem S and Edwards' methodology. We give some details to familiarize the reader with this methodology in a setting that is elementary enough to allow a brief description.

PROOF. Let $V_1, V_2, \ldots$ be a monotone decreasing sequence of open sets in S^3/G, each containing $\pi(N_G)$, such that V_1 is contained in U, the diameter of each component of V_1 is less than ε, and $\bigcap V_i$ is 0-dimensional. Set $U_i^* = \pi^{-1}(V_i)$. Determine triangulations $T_1, T_2, \ldots$ of S^3 with 1-skeleton A_i of T_i ($i = 1, 2, \ldots$) and with mesh $T_i \to 0$. The idea in the proof is to produce a map P as a limit of CE maps P_i (which then implies that P itself is CE [L, Theorem 1.4]) such that $P(x) = \pi(x)$ for each $x \in \pi^{-1}((S^3/G) - V_1)$ and P is 1-1 over each $P(A_i)$, for then N_P will be of embedding dimension $\leqslant 1$, since N_P will miss $\bigcup A_i$.

To get started, use Theorem S to obtain a homeomorphism h_1 of S^3 to itself such that $h_1|S^3 - U_1^* = 1$ and $h_1(A_1) \cap N_G = \varnothing$. Set $P_1 = \pi h_1$ and note that P_1 is 1-1 over $P_1(A_1)$.

Construct an open subset U_2 of $U_2^* - h_1(A_1)$ containing N_G for which the closure of no component of U_2 meets $h_1(A_1)$. Use Theorem S again to obtain a homeomorphism h_2 of S^3 to itself such that $h_2|S^3 - U_2 = \text{id}$ and $h_2(h_1(A_2)) \cap N_G = \varnothing$. Set $P_2 = \pi h_2 h_1$ and note that P_2 is 1-1 over $P_2(A_1 \cup A_2)$.

Continue this process. That is, given a CE map P_j expressed as $P_j = \pi h_j \ldots h_1$ which is 1-1 over $P_j(A_1 \cup \ldots \cup A_j)$, construct an open subset U_{j+1} of $(U_j \cap U_{j+1}^*) - h_j \ldots h_1(A_j)$ which contains N_G and for which the closure of no component intersects $h_j \ldots h_1(A_1 \cup \ldots \cup A_j)$. Use Theorem S again to obtain a homeomorphism of S^3 to itself such that $h_{j+1}|S^3 - U_{j+1} = 1$ and $h_{j+1}h_j \ldots j_1(A_{j+1}) \cap N_G = \varnothing$. Set $P_{j+1} = \pi h_{j+1} h_j \ldots h_1$ and note that P_{j+1} is 1-1 over $P_{j+1}(A_1 \cup \ldots \cup A_{j+1})$.

Finally, let $P = \lim P_j$ and check that it has the desired properties.

THEOREM 2 (NULL SEQUENCE THEOREM). *Let K be a CE decomposition of S^3 such that H_K forms a null sequence of cell-like sets, each of embedding dimension 1. Then $K \times S^1$ is shrinkable; that is, for each open set U in S^3 containing N_K and each $\varepsilon > 0$ there exists a homeomorphism h of $S^3 \times S^1$ to itself satisfying*

(1) $h|(S^3 - U) \times S^1 = 1$,
(2) *h changes S^1 coordinates less than ε,*
(3) diam $h(k \times s) < \varepsilon$, *for each $k \in K$ and $s \in S^1$.*

The proof of the Null Sequence Theorem will be the subject of the remaining sections of this paper. At this point we describe the promised reduction, which is a straightforward adaptation of Edwards' proof that his Countable Shrinking Theorem implies his 0-dimensional Shrinking Theorem [$\mathbf{E}_3$, §2]. The crucial observation in that argument can be isolated and stated in the following manner.

THEOREM E [$\mathbf{E}_3$]. *Let G denote a 0-dimensional CE decomposition of S^n, with* dem $N_G \leqslant m$, *and let U denote an open set in S^n containing N_G. Then there exists a CE decomposition K of S^n such that* (i) *H_K is a null sequence,* (ii) *each $k \in K$ has* dem $k \leqslant m$, *and* (iii) $N_G \subset N_K \subset U$.

PROPOSITION 3. *Theorem* 2 *implies Theorem* 1.

PROOF. Let G be any 0-dimensional CE decomposition of S^3, with π: $S^3 \to S^3/G$ the associated decomposition map, and let U be an open subset of S^3 containing N_G. To prove Theorem 1 it suffices to show (see the remark following Theorem 1) that for each $\varepsilon > 0$ there is a homeomorphism h of $S^3 \times S^1$ to $(S^3/G) \times S^1$, agreeing with $\pi \times 1$ off $U \times S^1$, such that $\rho(h, \pi \times 1) < \varepsilon$.

Fix $\varepsilon > 0$. There exists a CE map P: $S^3 \to S^3/G$ satisfying the conclusions of the Corollary to Theorem S. We shall establish, for the decomposition $G' = \{P^{-1}(q)|q \in S^3/G\}$ induced by P, that $G' \times S^1$ is shrinkable, implying that $P \times 1$: $S^3 \times S^1 \to (S^3/G) \times S^1$ is approximable arbitrarily closely by homeomorphisms, and therefore leading to a homeomorphism h approximating $\pi \times 1$ within ε, as required.

To see that $G' \times S^1$ is shrinkable, consider any open subset U' of S^3 containing $N_{G'}$. By Theorem E, there exists a null sequence CE decomposition K of S^3 such that each $k \in H_K$ has embedding dimension 1 and $N_{G'} = N_P \subset N_K \subset U'$. Finally to shrink elements of $G' \times S^1$ to small size, apply Theorem 2 to shrink elements of $K \times S^1$ to small size. Since each $g \in G'$ is contained in some $k \in K$, this has the desired effect.

3. The Main Lemma. The following result should suggest how Theorem 2 will be attacked. It has variations applicable to more general 4-dimensional decomposition problems.

MAIN LEMMA 4. *Suppose K is a 0-dimensional CE decomposition of S^3, equipped with decomposition map π and with product map $f_0 = \pi \times 1$ of $S^3 \times S^1$ to $(S^3/K) \times S^1$. Suppose that for each open set U in S^3 containing*

N_K and each $\varepsilon > 0$ there exists a sequence $\{T_i\}$ of triangulations of $S^3 \times S^1$ with mesh $T_i \to 0$, and there exists a sequence $\{f_i\}$ of CE maps of $S^3 \times S^1$ to $(S^3/K) \times S^1$ satisfying

(1) $f_i|(S^3 - U) \times S^1 = f_0|(S^3 - U) \times S^1$,

(2) $\lim f_i = f$ *is a CE map with* $\rho(f, f_0) < \varepsilon$,

(3) f *is* 1-1 *over* $f(T_i^{(2)})$ *for each* i,

(4) $f(N_f)$ *is* 0*-dimensional.*

Then f_0 can be approximated arbitrarily closely by a homeomorphism F such that $F|(S^3 - U) \times S^1 = f_0|(S^3 - U) \times S^1$. Moreover, $K \times S^1$ is shrinkable (in the sense of Theorems 1 *and* 2).

Proof. Fix $\varepsilon > 0$, and apply the hypotheses to obtain the sequence $\{T_i\}$ of triangulations and the sequence $\{f_i\}$ of CE maps converging to a map f for which $\rho(f, f_0) < \varepsilon$. Hypothesis (3) above implies that the nondegeneracy set N_f is of embedding dimension $\leqslant 1$ in $S^3 \times S^1$, since it misses each of the 2-skeleta $T_i^{(2)}$. According to the $(n - 3)$-dimensional Shrinking Lemma of [E_3], the induced decomposition

$$G_f = \{f^{-1}(q) | q \in (S^3/K) \times S^1\}$$

is shrinkable–in other words, the decomposition map p: $S^3 \times S^1 \to (S^3 \times S^1)/G_f$ is approximable by homeomorphisms (agreeing with p off $U \times S^1$). To construct the desired homeomorphism F, we merely choose such an approximation h so close to p that the map F defined as $F = fp^{-1}h$ is within ε of f_0. It is a routine verification that F is indeed a homeomorphism agreeing with f_0 off $U \times S^1$.

The reader interested in the shrinkability of $K \times S^1$ is encouraged to check that a required shrinking homeomorphism of $S^3 \times S^1$ to itself can be determined as $F^{-1}F^*$, where F^* approximates f_0 so closely that any $F^*(k \times s)$ has such small diameter that $F^{-1}F^*(k \times s)$ is sufficiently small. This completes the proof.

This argument displays a form of the basic, venerable inversion trick. For a simple illustration of the attack to be used later on, consider a cellular map f of a manifold M to a space Q. To produce another map F which is 1-1 over some point $q \in Q$, one constructs a pseudo-isotopy θ_t of M to itself shrinking the cellular set $f^{-1}(q)$ to a point $x \in M$. This is exploited by defining $F = f(\theta_1)^{-1}$; since $f^{-1}(q) = (\theta_1)^{-1}(x)$, f shrinks out what $(\theta_1)^{-1}$ blows up, and $F^{-1}(q) = \{x\}$.

4. The rearrangement of cubes with handles.

Definitions. A compact 3-manifold M is a *cube with handles* if $M = \bigcup_{i=1}^{s} B_i$, where (a) each B_i is a 3-cell, (b) $B_i \cap B_j$ is either empty or a 2-cell D_{ij} (for $i \neq j$), and (c) the intersection of any three distinct B_i's is empty. The

B_i's and D_{ij}'s are called *chambers* and *partitions* of M, respectively. If each chamber of M has diameter $<\varepsilon$, then M is called an *ε-thin* cube with handles; if, in addition, M is a 3-cell, then it is called an *ε-thin cube*.

We assume each partition D_{ij} of the cube with handles M comes equipped with a preferred product structure $D_{ij} \times [-1, 1]$ in M such that $(\partial D_{ij}) \times [-1, 1] \subset \partial M$ (where D_{ij} is identified with $D_{ij} \times 0$). Given cubes with handles M_1 and M_2 with $M_2 \subset \text{Int } M_1$, we say (M_1, M_2) is a *standard handle pair* if, for every partition D of M_1 (with its preferred product structure), the components of $D \cap M_2$ are partitions (with induced product structure) of M_2.

When we consider a *spine* L for a PL cube with handles M endowed with chambers B_i and partitions D_{ij}, we require L to be a finite PL graph in M having exactly one vertex in the interior of each B_i and each D_{ij} and such that $L \cap B_i$ is a subcone from the vertex in Int B_i over the vertices in ∂B_i, with respect to some PL cone structure over ∂B_i. Of course, we choose L to be compatible with the preferred product neighborhoods of the D_{ij}'s.

The next lemma sets forth controls pertinent to the rearranging of cubes with handles in $S^3 \times 0$ into a cell from the same level. Although the methods involved may appear familiar, the combination of the precise controls that result and of the interlocking shrinking described in Lemma 6 forms the crux of this paper.

LEMMA 5. *Suppose X is a compact, cell-like subset of S^3 having embedding dimension* 1, $\varepsilon > 0$, *and M is a* PL *ε-thin cube with handles, having chambers B_i and partitions D_{ij}, such that $X \subset$* Int M. *Then there exists an arbitrarily small* PL *homeomorphism μ of M to itself, fixed on ∂M, and there exists a* PL *ε-thin cube C contained in* Int M *such that (M, C) is a standard handle pair (with respect to the partitions $\mu(D_{ij})$) and for each $\delta > 0$ there is an isotopy θ_t of $S^3 \times E^1$ to itself satisfying*

(1) $\theta_t|(S^3 - M) \times E^1 = 1$,

(2) $\theta_t|M \times (E^1 - (-\delta, \delta)) = 1$,

(3) $\theta_t = 1$ *near* $(\mu(D_{ij}) \times E^1)$ *(the nearness being independent of δ)*,

(4) $\theta_1(X \times 0) \subset \text{Int } C \times 0$.

PROOF. Suppose $\gamma \in (0, \varepsilon)$ is given. Since dem $X = 1$, X is definable by thin handlebodies (cf. [**R**, Theorem 13]); furthermore, because X is cell-like, there exists a PL γ-thin cube with handles M_1 such that

$$X \subset \text{Int } M_1 \subset M_1 \subset \text{Int } M$$

and M_1 is contractible in M. We can obtain a PL γ-homeomorphism μ of M to itself fixed on ∂M such that (M, M_1) is a standard handle pair with respect to the chambers $\mu(B_i)$ for M (adding extra partitions for M_1, if necessary), by putting the partitions of M in general position with respect to a spine for M_1

and expanding a small regular neighborhood of that spine to all of M_1. During the remainder of this proof, any description of chambers and partitions for M will refer to the structure associated with $\mu(B_i)$.

Now we find C and a PL embedding $g\colon M_1 \to \operatorname{Int} C$ that is the identity near the partitions of M. Let $p\colon \tilde{M} \to M$ denote a universal covering map; call the components of the inverse image of chambers and partitions of M, chambers and partitions of $\tilde{M}$. Since M_1 is contractible in M, there exists a lift $g'\colon M_1 \to \tilde{M}$ of the inclusion, and $\tilde{M}$ contains a 3-cell C', the union of some chambers of $\tilde{M}$, that contains $g'(M_1)$ in its interior.

Let S denote a spine for C'. There exists a PL embedding $f\colon C' \to \operatorname{Int} M$ such that

(5) $(M, f(C'))$ is a standard handle pair,

(6) $f(B) \subset p(B)$ and $f(D) \subset p(D)$ for all chambers B and partitions D of C',

(7) $(p(B), p(S \cap B))$ and $(p(B), f(S \cap B))$ are homeomorphic pairs for all chambers B of C',

(8) fg' is the identity near $D^* \cap M_1$ for all partitions D^* of M.

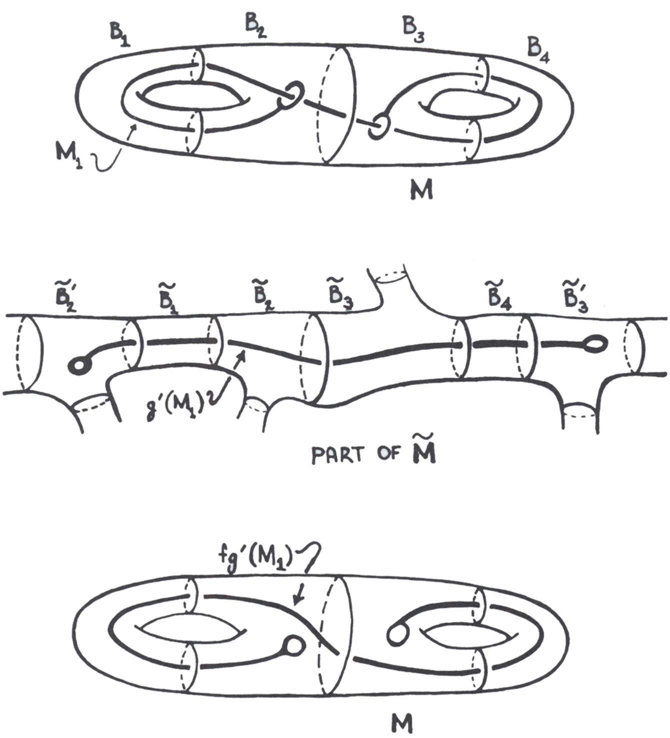

FIGURE 1

The embedding f is found by squeezing C' close to S, approximating the resulting image under p by an embedding that respects chambers and partitions of M, and matching up $D^* \cap M_1$ with its image for each partition D^* of M. The result is pictured in Figure 1. Letting $C = f(C')$ and $g = fg'$, we see from (5), (6) and (8) that they have the properties required above.

Let $\delta > 0$. We shall obtain an ambient isotopy θ_t of $S^3 \times E^1$ satisfying (1), (2), (3) and such that $\theta_1(M_1 \times 0) = g(M_1) \times 0$, which implies (4). We shall describe θ_t on $\mu(B_i) \times [-\delta, \delta]$ for one fixed chamber $\mu(B_i)$ of M.

Let L denote a spine for M_1 and Q a component of $M_1 \cap \mu(B_i)$. Note that Q is an ε-thin cube with handles, with spine $L \cap Q$. Conditions (7) and (8) for f imply that the pairs $(\mu(B_i), Q)$ and $(\mu(B_i), g(Q))$ are PL homeomorphic under an ambient isotopy fixed on a neighborhood U_Q of $\partial\mu(B_i)$. We let U_i denote a neighborhood of $\partial\mu(B_i)$ in $\mu(B_i)$ contained in each such U_Q, and we let $E_i = \mu(B_i) - U_i$.

Finally we describe θ_t. In what follows a *vertical* isotopy of $\mu(B_i) \times [-\delta, \delta]$ will mean an isotopy of $S^3 \times E^1$ that preserves S^3 coordinates and is fixed near $(S^3 \times E^1) - (\text{Int}\, \mu(B_i) \times (-\delta, \delta))$. For $0 \leqslant t \leqslant \frac{1}{3}$, θ_t is a vertical isotopy that places $(Q \cap E_i) \times 0$ in a distinct level s of $(-\delta, \delta)$, for each component Q of $M_1 \cap \mu(B_i)$. For $\frac{1}{3} \leqslant t \leqslant \frac{2}{3}$, θ_t moves each such $(Q \cap E_i) \times s$ to $(g(Q) \cap E_i) \times s$ by the ambient isotopy of pairs mentioned above; we can require this portion of the isotopy to be the identity over $U_i \times [-\delta, \delta]$ and off pairwise disjoint neighborhoods of the s-levels. For $\frac{2}{3} \leqslant t \leqslant 1$, θ_t replaces each $(g(Q) \cap E_i) \times s$ in the 0-level by a vertical isotopy that is the inverse of θ_t, $0 \leqslant t \leqslant \frac{1}{3}$, over $U_i \times [-\delta, \delta]$. Thus, $\theta_1(Q \times 0) = g(Q) \times 0$, for each component Q, as required.

REMARK. As an implicit corollary, any cell-like subset of S^3 with embedding dimension 1 admits a cellular embedding in S^3. It is natural to ask whether the restriction on dimension is necessary (R. H. Bing and M. Starbird raised this question with us at the same time as we were asking them). That there exists a cell-like subset X of S^n, $n > 3$, having no cellular re-embedding in S^n is easily seen, by considering a compact contractible n-manifold X in S^n for which $\pi_1(\partial X)$ is nontrivial. It is not known, however, whether each codimension 2 cell-like subset of S^n, $n > 3$, admits a cellular re-embedding.

5. The shrinking of decomposition elements that intersect 2-skeleta. Throughout the remainder of the paper we shall assume that K represents a fixed CE decomposition of S^3 such that H_K forms a null sequence of cell-like sets, each of embedding dimension $\leqslant 1$.

We find it convenient to work with some special "triangulations" of $S^3 \times S^1$. A *prismatic triangulation* R of $S^3 \times S^1$ is a subdivision into cells, expressible as $R = R_* \times R_{**}$, where R_* is a (possibly curvilinear) triangulation of S^3 and R_{**} is a triangulation of S^1. The transformation of the original decomposition to the new decomposition K, for which the embedding dimension of $N_K \leqslant 1$, permits an adjustment of any given triangulation to a new (prismatic) triangulation $R = R_* \times R_{**}$ such that the 1-skeleton $R_*^{(1)}$ of R_* misses N_K [Šk$_1$], [Šk$_2$], [E$_1$]. No significant attributes are lost in the change from ordinary triangulations.

For a prismatic triangulation $R = R_* \times R_{**}$ such that $R_*^{(1)} \cap N_K = \varnothing$, the only part of its 2-skeleton $R^{(2)}$ intersecting $N_K \times S^1$ is contained in $T_*^{(2)} \times T_{**}^{(0)}$. More generally, a curvilinear triangulation T of $S^3 \times S^1$ (which may be the homeomorphic image of a prismatic one) is said to be *tractable* if $T^{(2)} \cap (N_K \times S^1)$ is covered by the interiors of finitely many 2-cells $\{b_i\}$ in $T^{(2)}$ such that those interiors are pairwise disjoint and each b_i lies in some level $S^3 \times \{s_i\}$ of $S^3 \times S^1$.

LEMMA 6. *Let T denote a tractable triangulation of $S^3 \times S^1$ with $T^{(1)} \cap (N_K \times S^1) = \varnothing$, W a $(K \times S^1)$-saturated open set in $S^3 \times S^1$ containing $T^{(2)} \cap (N_K \times S^1)$, and ε a positive number.*

Then there exist an ε-homeomorphism g of $S^3 \times S^1$ to itself and an ambient isotopy ψ_t of $S^3 \times S^1$ satisfying

(1) $g|(S^3 \times S^1) - W = 1$,

(2) $\psi_t = 1$ *on some neighborhood of* $g(T^{(2)})$,

(3) $\psi_t|(S^3 \times S^1) - W = 1$,

(4) *for each* $\psi_1(k \times s) \in \psi_1(K \times S^1)$ *meeting* $g(T^{(2)})$, diam $\psi_1(k \times s) < \varepsilon$,

(5) $g(T)$ *is tractable.*

PROOF. From the null sequence decomposition K we select the finitely many elements having diameter $\geqslant \varepsilon$. Because we will perform similar (and nonoverlapping) constructions near each one, we lose no generality by supposing $k_1 \in K$ to be the only element of this size. Referring to the tractable triangulation T, we find finitely many level 2-cells in $T^{(2)}$ whose interiors are pairwise disjoint and cover $T^{(2)} \cap (N_K \times S^1)$, and once again, because we will perform similar constructions near each, we lose no generality by focusing on just one 2-cell, which we denote as $b \times 0 \subset S^3 \times 0$. (We shall use interval notation from here on to denote certain subsets of S^1, so perhaps one should mentally equate S^1 with E^1.)

As in §4, there exists an $(\varepsilon/3)$-thin cube with handles M in S^3 such that

$$k_1 \times 0 \subset \operatorname{Int} M \times 0 \subset M \times 0 \subset W - \left(T^{(1)} \cup (\partial b \times 0)\right),$$

with M expressed as the union of chambers B_i, determined by partitions D_{ij}. Allowing minor changes in these chambers and partitions, we identify the 3-cell C in M satisfying the conclusions of Lemma 5.

Since M is $(\varepsilon/3)$-thin, there is a homeomorphism g_1 of $S^3 \times S^1$ to itself, moving points less than $\varepsilon/3$, fixed outside $W - T^{(1)}$, and preserving S^1-coordinates, which essentially puts $b \times 0$ in general position with respect to $M \times 0$, so that $g_1(T^{(2)}) \cap (M \times 0)$ consists of finitely many pairwise disjoint disks, each of which misses all partitions $D_{ij} \times 0$ but lies parallel to and close enough to some partition that any isotopy θ_t as described by Lemma 5 is the identity near $g_1(T^{(2)}) \cap (M \times 0)$ and that $g_1(T^{(2)}) \cap (C \times 0)$ is close and parallel to the partitions induced on $C \times 0$ (see conclusion (3) of Lemma 5). It follows from the 1-demensionality of N_K that, in addition, $g_1(b \times 0) \cap (N_K \times S^1)$ can be made 0-dimensional and that, for each 2-cell Z of $g_1(T^{(2)}) \cap (C \times 0)$, ∂Z misses $(N_K \times S^1)$ (this may involve further minor adjustments to the structures on M and C). For definiteness, let $\{Z_j \times 0\}$ denote the final set of components of $g_1(T^{(2)}) \cap (C \times 0)$.

Next we spread these components to distinct levels of $S^3 \times S^1$, via a homeomorphism g_2 moving points less than $\varepsilon/3$, fixed on $(S^3 \times S^1) - W$ and on $g_1(T^{(2)}) - (\text{Int}\, C \times S^1)$, and almost preserving S^3-coordinates, so that if $g_2 g_1(T^{(2)}) \cap (\text{Int}\, C \times s) \neq \varnothing$ then $s = s_j \in (0, \varepsilon/3)$ and this intersection equals $(\text{Int}\, Z_j \times s_j)$, and that the associated levels s_j are all distinct. To prevent unnecessary intersections with elements of $H_K \times S^1$, we insist that the component of $g_2 g_1(T^{(2)}) \cap (C \times S^1)$ containing $(Z_j \times s_j)$ be

$$\left(\partial Z_j \times [0, s_j]\right) \cup (Z_j \times s_j).$$

(See Figure 2a.) Note that S^1 contains only a finite subset F of points s_j, all in $(0, \varepsilon/3)$ as above, for which

$$g_2 g_1(T^{(2)}) \cap (\text{Int}\, C \times s_j) \neq \varnothing.$$

We choose $\delta > 0$ so small that when we set $(\alpha_j, \beta_j) = (s_j - \delta, s_j + \delta)$ for each $s_j \in F$, then $(\alpha_j, \beta_j) \subset (0, \varepsilon/3)$ and that the sets $\{[\alpha_j, \beta_j]\}$ are pairwise disjoint. Now, as a preliminary rearrangement, we apply the isotopy of Lemma 5, once for each $s_j \in F$, to obtain an ambient isotopy θ_t of $S^3 \times S^1$ such that

$$\theta_t | (S^3 - M) \times S^1 = 1,$$

$$\theta_t | M \times \left(S^1 - \bigcup (\alpha_j, \beta_j)\right) = 1,$$

$$\theta_t = 1 \quad \text{near } g_2 g_1(T^{(2)}) \cup \left(\bigcup \left(D_{ij} \times S^1\right)\right),$$

$$\theta_1 (k_1 \times s_j) \subset \text{Int}\, C \times s_j \quad \text{for each } s_j \in F.$$

The guides for the next, large-scale isotopy are some 3-cells C_j, one for each $s_j \in F$, determined by deforming $C \times s_j$ slightly towards its interior along a collar pinched at $\partial Z_j \times s_j$, so that

$$\theta_1 (k_1 \times s_j) \subset C_j \subset (\text{Int}\, C \times s_j) \cup (Z_j \times s_j).$$

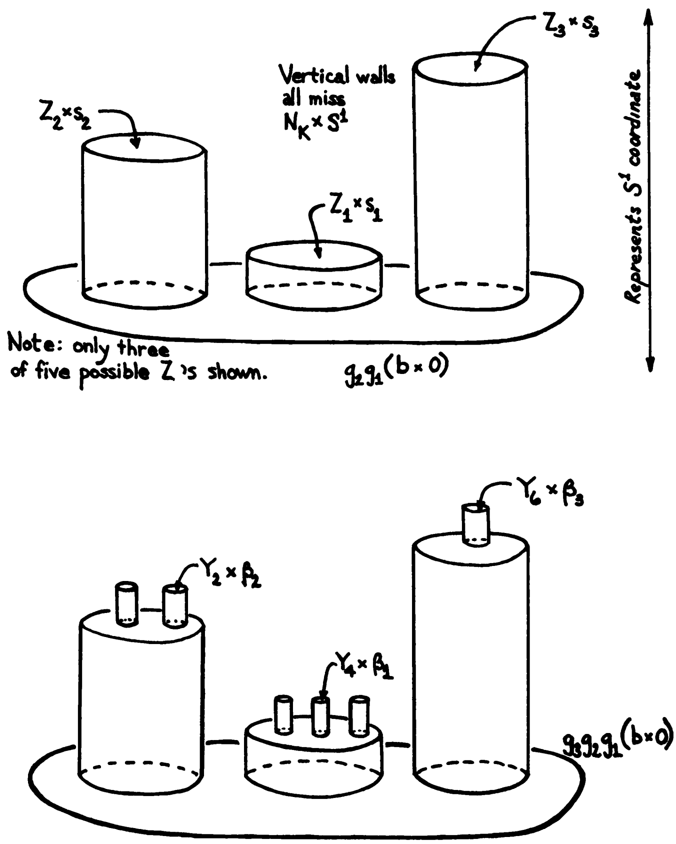

FIGURE 2

By carefully shrinking each 3-cell C_j very near its small spanning 2-cell $(Z_j \times s_j)$, we can construct, in elementary fashion, a level-preserving ambient isotopy ζ_t of $S^3 \times S^1$ such that

$$\zeta_t|(S^3 - C) \times S^1 = 1,$$

$$\zeta_t|C \times \left(S^1 - \cup\ (\alpha_j, \beta_j)\right) = 1,$$

$$\zeta_t = 1 \quad \text{near } g_2 g_1(T^{(2)}) \cup \left(\cup\ \left(Z_j \times \left[\alpha_j, \beta_j\right]\right)\right),$$

and

$$\operatorname{diam} \zeta_1\left(C_j\right) < \varepsilon \quad \text{for each } C_j.$$

Note that this final condition guarantees that diam $\zeta_1\theta_1(k_1 \times s_j) < \varepsilon$ for each $s_j \in F$. (See Figures 3 and 4.) Note also that the limitations on ζ_t near $g_2 g_1(T^{(2)})$ depend on the precise threading of $g_2 g_1(Z_i \times 0)$ through the various levels. (See Figure 5.)

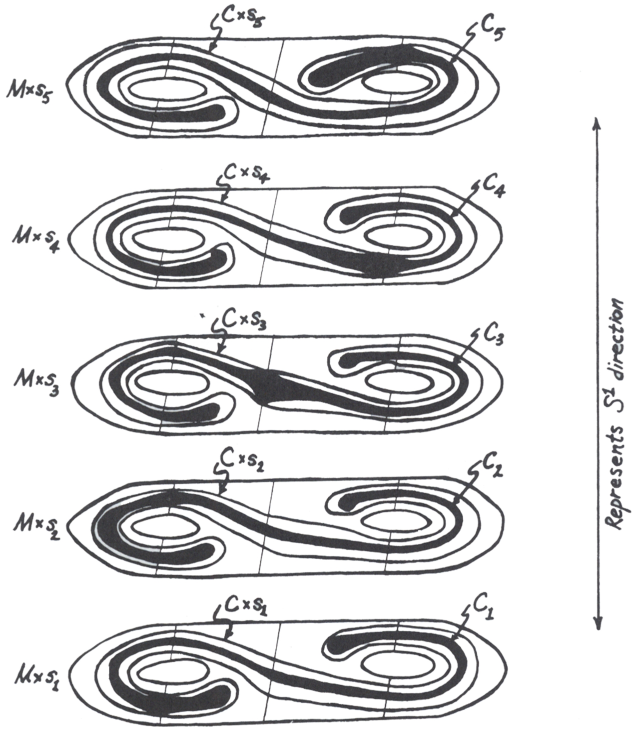

FIGURE 3

The isotopy $\zeta_t\theta_t$ is the one we want. However, we must modify $g_2g_1(T)$ once more to produce the promised ε-homeomorphism g of $S^3 \times S^1$.

At this point the adjusted triangulation T has the desirable feature that if $\zeta_1\theta_1(k_1 \times s)$ intersects $g_2g_1(T^{(2)})$, then diam $\zeta_1\theta_1(k_1 \times s) < \varepsilon$, since

$$g_2g_1(T^{(2)}) \cap \zeta_1\theta_1(k_1 \times s) = g_2g_1(T^{(2)}) \cap (k_1 \times s)$$

and the intersection is nonvoid if and only if $s = s_j \in F$. Nevertheless, the isotopy $\zeta_t\theta_t$ can stretch far enough to cause some $\zeta_1\theta_1(k \times s)$ meeting $g_2g_1(T^{(2)})$ to have large diameter. To remedy this, one should verify that, because $\zeta_t\theta_t$ is the identity near $g_2g_1(T^{(2)})$, the intersection of $g_2g_1((b \cap \operatorname{Int} M) \times 0)$ with $\zeta_1\theta_1(N_K \times S^1)$ is 0-dimensional and is contained in

$$\big(\cup\ (\operatorname{Int} Z_j \times s_j)\big) \cup \big((\operatorname{Int} b - \cup\ Z_j) \times 0\big).$$

In particular, because $\zeta_t\theta_t$ is the identity outside $M \times S^1$ and on the $S^3 \times 0$ level, the subset of $g_2g_1(T^{(2)}) \cap \zeta_1\theta_1(N_K \times S^1)$ arising from those $\zeta_1\theta_1(k \times s) \in \zeta_1\theta_1(K \times S^1)$ having diameter at least ε is a compact 0-dimensional subset

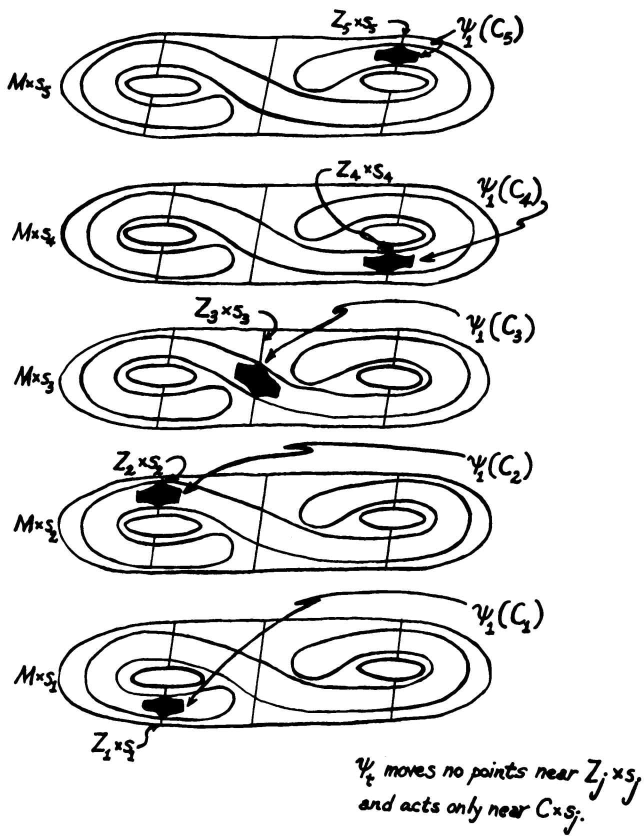

FIGURE 4

of $\cup$ (Int $Z_j \times s_j$) and, as such, can be covered by the interiors of finitely many pairwise disjoint 2-cells $\{Y_i\}$ in

$$\cup\left((\operatorname{Int} Z_j \times s_j) - \zeta_1\theta_1(k_1 \times S^1)\right)$$

whose boundaries miss $\zeta_1\theta_1(N_K \times S^1)$. Consequently, we can produce an $(\varepsilon/3)$-homeomorphism g_3 of $S^3 \times S^1$ fixed on $g_2 g_1(T^{(2)}) - (\cup\, Y_i)$ and lifting these disks off big elements, exactly as in the construction of g_2. To be specific, those Y_i in $Z_j \times s_j$ can be identified with $Y_i \times s_j$ and g_3 can be defined so that

$$g_3(Y_i) = \left(\partial Y_i \times [s_j, \beta_j]\right) \cup (Y_i \times \beta_j).$$

(See Figure 2b.) Now it is crucial to recall that $\zeta_t\theta_t = 1$ near $Z_j \times [\alpha_j, \beta_j]$. As a result, $g_3(Y_i)$ can intersect $\zeta_1\theta_1(k \times s) \in \zeta_1\theta_1(H_K \times S^1)$ only at points of $Y_i \times \beta_j$, implying that $\zeta_1\theta_1(k \times s) = k \times \beta_j$ (because $\zeta_t\theta_t | S^3 \times \beta_j = 1$), where $k \neq k_1$, and $k \times \beta_j$ necessarily has small size.

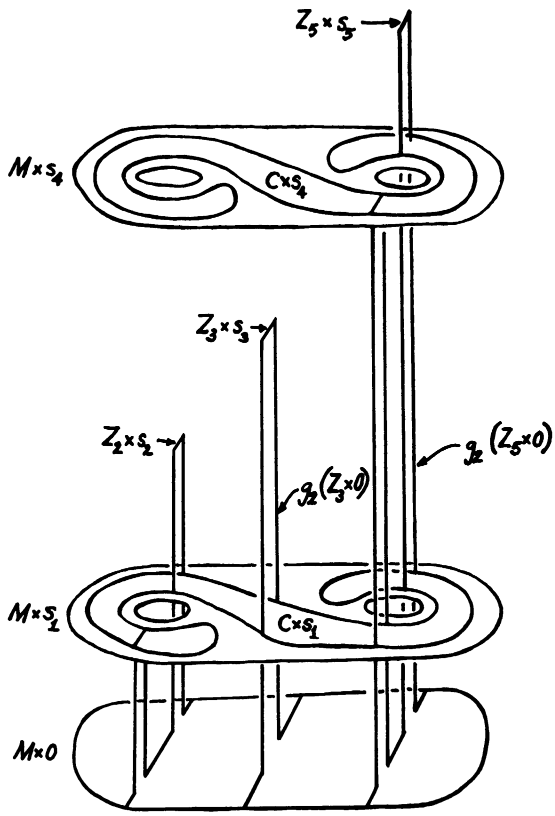

FIGURE 5

In conclusion, we define g as $g_3 g_2 g_1$. It should be clear that $g(T)$ is tractable (the disks with holes $g((b \times 0) - \bigcup (\text{Int } Z_i \times 0))$ and $g_3((Z_i \times s_i) - \bigcup (\text{Int } Y_j \times s_i))$ can be chopped into disks whose interiors cover the intersection with $N_K \times S^1$, since the intersection is 0-dimensional) and that the other requirements of Lemma 6 are satisfied.

The following variation is an easy corollary of Lemma 6.

LEMMA 6′. *In terms of the data of Lemma* 6, *let* g *denote an* ε-*homeomorphism and* ψ_t *an ambient isotopy satisfying its conclusions. Furthermore, let* W^* *denote a* $\psi_1(K \times S^1)$-*saturated open subset of* $S^3 \times S^1$ *containing* $g(T^{(2)}) \cap \psi_1(N_K \times S^1)$ *and* ε^* *a positive number.*

Then there exist an ε^*-*homeomorphism* g^* *of* $S^3 \times S^1$ *to itself and an ambient isotopy* ψ_t^* *of* $S^3 \times S^1$ *satisfying*

(1) $g^* | (S^3 \times S^1) - W = 1$,

(2) $\psi_t \psi_t^* = 1$ *on some neighborhood of* $g^* g(T^{(2)})$,

(3) $\psi_t^* | (S^3 \times S^1) - \psi_1^{-1}(W^*) = 1$,

(4) $\rho(\psi_1\psi_t^*, \psi_1) < \varepsilon$,

(5) *for each* $\psi_1\psi_1^*(k \times s) \in \psi_1\psi_1^*(K \times S^1)$ *meeting* $g^*g(T^{(2)})$,

$$\operatorname{diam} \psi_1\psi_1^*(k \times s) < \varepsilon^*,$$

(6) $g^*g(T)$ *is tractable.*

PROOF. Since $g(T)$ is tractable, and $W \cap \psi_1^{-1}W^*$ is $K \times S^1$ saturated it follows that, for $\varepsilon' > 0$, Lemma 6 applies directly to yield a homeomorphism g^* satisfying conditions (1) and (6) and an ambient isotopy ψ_t^* satisfying condition (3). After cutting back W^* so that each of its components has diameter less than ε, we obtain condition (4). Condition (5) depends merely on the uniform continuity of ψ_1: the ε' mentioned above may be chosen such that if diam $\psi_1^*(k \times s) < \varepsilon'$, then $\psi_1\psi_1^*(k \times x) < \varepsilon^*$.

After shrinking out part of $K \times S^1$, we will attempt further shrinking and then will discern the need for controls to minimize potential technical difficulties, namely, the need of having triangulations which mesh properly with the altered decomposition so that the process may be iterated. To that end, we say that a pseudo-isotopy θ_t of $S^3 \times S^1$ *abides by the special constraints* if (i) S^1 contains a compact 0-dimensional set C such that every nondegenerate inverse set of θ_1 is an element of $H_K \times C$ and (ii) S^3 has triangulations T with arbitrarily small mesh and with $T^{(1)}$ disjoint from N_K such that for each open subset V of S^1 containing C there exists $\alpha \in [0, 1)$ for which

$$\theta_t|\big(S^3 \times (S^1 - V)\big) \cup (T^{(1)} \times S^1) = \theta_\alpha$$

whenever $t \in [\alpha, 1]$.

LEMMA 7. *Let* $R = R_* \times R_{**}$ *denote a prismatic triangulation of* $S^3 \times S^1$ *such that* $R_*^{(1)} \cap N_K = \varnothing$, $z \in S^1 - R_{**}^{(0)}$, W *a* $(K \times S^1)$*-saturated open subset of* $S^3 \times S^1$ *containing both* $(T^{(2)} \cap (N_K \times S^1))$ *and* $N_K \times \{z\}$, *and* $\varepsilon > 0$.

Then there exist an ε*-homeomorphism* g *of* $S^3 \times S^1$ *to itself and a pseudo-isotopy* θ_t *of* $S^3 \times S^1$ *to itself satisfying*

(a) $g|(S^3 \times S^1) - W = 1$,

(b) $\theta_t|(S^3 \times S^1) - W = 1$,

(c) θ_t *abides by the special constraints,*

(d) *for each* $k \in K$, $\theta_1(k \times z)$ *is a point,*

(e) *for each* $\theta_1(k \times s) \in \theta_1(K \times S^1)$ *meeting* $g(R^{(2)})$, $\theta_1(k \times s)$ *is a point.*

PROOF. Two distinct processes are put to work here. One occurs in the product of S^3 with a neighborhood of z in S^1 having closure disjoint from $R_{**}^{(0)}$, and there θ_1 shrinks each set $k \times z$ to a point, keeping $R^{(2)}$ fixed and doing no unnecessary shrinking [Ev, Theorem 1], to fulfill condition (d). The other occurs in the product of S^3 with a neighborhood of $R_{**}^{(0)}$ in S^1, and there θ_1 shrinks each set $k \times s$ that meets $g(R^{(2)})$. The latter process evolves, in standard fashion, from repeated applications of Lemma 6′, and we leave out the details. Regarding the special constraints, however, we do wish to point out that, at the nth stage of the shrinking process, we can identify a finite subset F_n of S^1 such that the only nondegenerate elements of $K \times S^1$ which meet the adjusted 2-skeleton are elements of $K \times F_n$, and we can certainly restrict later adjustments to a small neighborhood of F_n so as to determine a compact 0-dimensional set C in S^1 such that θ_t is ultimately constant off $S^3 \times C$. In similar fashion, for any countable union X of compact sets in $S^3 - N_K$, Lemma 6′ (and, for the other process, Everett's proof [Ev]) provides sufficient controls to guarantee that θ_t is ultimately constant on $X \times S^1$.

6. The applicability of the Main Lemma. With the approach isolated in a remark concluding §3, the next result connects Lemma 7 and the Main Lemma (Lemma 4). Afterwards, we outline how to verify that the hypotheses of the Main Lemma are satisfied, which is merely technical dirty work that should look familiar to anyone who has read [$\mathbf{E_3}$].

As before, f_0 denotes the natural map of $S^3 \times S^1$ to $(S^3/K) \times S^1 = Q$.

LEMMA 8. *Let U be an open subset of S^3 containing N_K, $z_1 \in S^1$, and $\varepsilon_1 > 0$. Then there exists a triangulation T_1 of $S^3 \times S^1$ having mesh less than ε_1 and there exists a CE map f_1 of $S^3 \times S^1$ to Q satisfying*

(i) $\rho(f_1, f_0) < \varepsilon_1$,

(ii) $f_1|(S^3 - U) \times S^1 = f_0|(S^3 - U) \times S^1$,

(iii) f_1 *is* 1-1 *over* $f_1(T_1^{(2)})$,

(iv) f_1 *is* 1-1 *over* $((S^3/K) \times \{z_1\})$.

PROOF. Name a prismatic triangulation $R = R_* \times R_{**}$ of $S^3 \times S^1$ having mesh less than $\varepsilon_1/3$ such that $R_*^{(1)} \cap N_K = \varnothing$ and $z_1 \notin R_{**}^{(0)}$. Restrict U, if necessary, and choose a neighborhood V of $\{z_1\} \cup R_{**}^{(0)}$ in S^1 small enough that the image under f_0 of each component from $W = U \times V$ has diameter less than ε_1. Apply Lemma 7 to obtain a pseudo-isotopy θ_t and an $(\varepsilon_1/3)$-homeomorphism g.

Define T_1 as $g(R)$ and f_1 as $f_0\theta^{-1}$, where $\theta = \theta_1$.

Properties of the special constraints (condition (c) of Lemma 7) imply that f_1 is well-defined and, hence, continuous. Furthermore, since $f_1^{-1} = \theta f_0^{-1}$, they imply that each set $f_1^{-1}(q)$ either is a singleton or is homeomorphic to $f_0^{-1}(q)$. Thus, f_1 is a CE map.

The restrictions on U and V above are designed to limit θ_t so that conditions (i) and (ii) are satisfied.

For any point $a \in S^3/K$ one sees that

$$\begin{aligned} f_1^{-1}(a \times z_1) &= \theta f_0^{-1}(a \times z_1) \\ &= \theta_1(k \times z_1) \quad \text{for some } k \in K. \end{aligned}$$

By condition (d) of Lemma 7, $\theta_1(k \times z_1)$ is a point, indicating that f_1 is 1-1 over $(S^3/K) \times \{z_1\}$.

Finally, for any point $x \in T_1^{(2)} = g(R^{(2)})$, consider $f_1^{-1}f_1(x) = \theta f_0^{-1}f_1(x)$. If nondegenerate, $f_0^{-1}f_1(x)$ itself is nondegenerate and, therefore, must be a set $k \times s$. But $k \times s$ meets $g(R^{(2)})$ at x, and, according to condition (e) of Lemma 7, $\theta_1(k \times s)$ is a singleton. Consequently, f_1 is 1-1 over $f_1(T^{(2)})$, as required.

Satisfying the hypothesis of the Main Lemma (*Outline*). Lemma 8 produces the first triangulation T_1 and CE map f_1 called for in the Main Lemma. Because successive steps in the iteration unexpectedly depend upon an accounting of all previous pseudo-isotopies, we describe the production of the second pair.

For notational ease we use $\theta_{1,t}$ to denote the pseudo-isotopy involved in the first step (Lemma 8) and C_1 the compact 0-dimensional subset of S^1 identified because $\theta_{1,t}$ abides by the special constraints.

Name a small positive number ε_2 and a point $z_2 \in S^1 - C_1$ (selected with the ultimate aim of making $\{z_i | i = 1, 2, \dots\}$ be dense in S^1). There exists $\delta > 0$ such that for any $A \subset S^3 \times S^1$ with diam $A < \delta$, diam $\theta_{1,t}(A) < \varepsilon_2$ for all t.

Next we need a prismatic triangulation $R = R_* \times R_{**}$ (different from that of Lemma 8) of mesh less than $\delta/3$ such that $R_*^{(1)} \cap N_K = \varnothing$ and $R_{**}^{(0)} \cap (\{z_2\} \cup C_1) = \varnothing$. Since $\theta_{1,t}$ abides by the special constraints, we can choose R as above so that there exists $\alpha \in [0, 1)$ for which $\theta_{1,t}|R^{(2)} = \theta_{1,\alpha}|R^{(2)}$ whenever $t \in [\alpha, 1]$.

Naturally, we apply Lemma 7 again to this triangulation R, with z_2 as the point of S^1 and with

$$W \subset \big(U \times (S^1 - C_1)\big) - (\theta_{1,1})^{-1}(T_1^{(2)}),$$

to obtain a $(\delta/3)$-homeomorphism g_2 and a pseudo-isotopy $\theta_{2,t}$. Now we define T_2 as $\theta_{1,\alpha}g_2(R)$ and f_2 as $f_0(\theta_{2,1})^{-1}(\theta_{1,1})^{-1}$. With this definition in mind we further restrict W so that $\rho(f_2, f_1) < \varepsilon_2$.

One readily sees that mesh $g_2(R) < \delta$ and, recalling the choice of δ, that mesh $T_2 < \varepsilon_2$.

It should be clear that epsilonic controls can be installed to get at hypotheses (1) and (2) of the Main Lemma. The fundamental hypothesis there is (3). To see why f_2 is 1-1 over $f_2(T_2^{(2)})$, note that

$$\begin{aligned} f_2|T_2^{(2)} &= f_2|\theta_{1,\alpha}g_2(R^{(2)}) \\ &= f_0(\theta_{2,1})^{-1}(\theta_{1,1})^{-1}\theta_{1,\alpha}|g_2(R^{(2)}) \\ &= f_0(\theta_{2,1})^{-1}|g_2(R^{(2)}); \end{aligned}$$

then f_2 is 1-1 over $f_2(T_2^{(2)})$ for the same reasons given to show, in Lemma 8, that f_1 is 1-1 over $f_1(T_1^{(2)})$. Moreover, f_2 agrees with f_1 on $T_1^{(2)} = f_1^{-1}f_1(T_1^{(2)})$, for

$$\begin{aligned} f_2|T_1^{(2)} &= f_0(\theta_{2,1})^{-1}|(\theta_{1,1})^{-1}(T_1^{(2)}) \\ &= f_0|(\theta_{1,1})^{-1}(T_1^{(2)}) \\ &= f_1|T_1^{(2)}. \end{aligned}$$

Concerning hypothesis (4), there are two significant features:

$$f_2^{-1}(a \times z_2) = \theta_{1,1}\theta_{2,1}(k \times z_2) = \theta_{1,1}(\text{point}) = \text{point},$$

and

$$f_2^{-1}(a \times z_1) = \theta_{1,1}\theta_{2,1}(k \times z_1) = \theta_{1,1}(k \times z_1) = f_1^{-1}(a \times z_1),$$

since $k \times z_1 \notin W$.

Finally, one must insert additional restrictions to control the rate of convergence, in order to guarantee that the limit map f agrees with f_1 over $((S^3/K) \times \{z_1\}) \cup f_1(T_1^{(2)})$. This is not difficult–at this stage one simply refines the set W to make f_2 so close to f_1 that for $x \in S^3 \times S^1$ such that $f(x) \notin f_1(T_1^{(2)} \cup (S^3 \times C_1))$,

$$\rho(f_2(x), f_1(x)) < \tfrac{1}{3}\rho\big(f_1(x), f_1\big(T_1^{(2)} \cup \big(S^3 \times C_1\big)\big)\big).$$

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