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SIMPLE PERIODIC ORBITS OF MAPPINGS OF THE INTERVAL

BY

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ABSTRACT. Let f be a continuous map of a closed, bounded interval into itself. A criterion is given to determine whether or not f has a periodic point whose period is not a power of 2, which just depends on the periodic orbits of f whose period is a power of 2. Also, a lower bound for the topological entropy of f is obtained.

1. Introduction. Let I denote a closed and bounded interval on the real line and let $C^0(I, I)$ denote the space of continuous maps from I into itself. This paper is concerned with periodic orbits of mappings $f \in C^0(I, I)$. Such mappings (sometimes called first order difference equations) arise as mathematical models for phenomena in the natural sciences (see [4] and [5] for some discussion and further references).

Let $f \in C^0(I, I)$. Consider the following ordering of the positive integers:

$$1, 2, 4, 8, \ldots \quad \ldots, 7 \cdot 8, 5 \cdot 8, 3 \cdot 8, \ldots, 7 \cdot 4, 5 \cdot 4, 3 \cdot 4, \ldots, 7 \cdot 2, 5 \cdot 2, 3 \cdot 2, \ldots, 7, 5, 3.$$

A. N. Šarkovskiĭ has proven that if m is to the left of n (in the above ordering) and f has a periodic point of period n, then f has a periodic point of period m (see [6] or [7]). This theorem suggests that the following property implies a rich orbit structure:

(1) f has a periodic point whose period is not a power of 2.

This suggestion is supported by the fact that (1) implies the following:

(2) f has a homoclinic point (see [1]);

(3) f has positive topological entropy (see [1], [2], or [7]).

Also, (2) is equivalent to (1) (see [1]) and it has been conjectured (and proved for a special case in [3]) that (3) is equivalent to (1).

In this paper we give a criterion for determining whether or not f satisfies (1) which just depends on the periodic orbits of f whose period is a power of 2. In the process we obtain a lower bound for topological entropy. The criterion we give is based on the following definition.

DEFINITION. Let P be a periodic orbit of $f \in C^0(I, I)$ of period m, where m

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is a power of 2 and $m \geqslant 2$. We say P is simple if for any subset $\{q_1, \ldots, q_n\}$ of P where n divides m and $n \geqslant 2$, and any positive integer r which divides m, such that $\{q_1, \ldots, q_n\}$ is periodic orbit of f^r with $q_1 < q_2 < \cdots < q_n$, we have

$$f^r(\{q_1, \ldots, q_{n/2}\}) = \{q_{n/2+1}, \ldots, q_n\}.$$

The reader may wish to see §4 where the definition of "simple" is discussed for a periodic orbit of period 8, and some examples are given.

Our main results are the following: (In this paper we include $1 = 2^0$ as a power of 2.)

THEOREM A. *Let $f \in C^0(I, I)$. f has a periodic point whose period is not a power of 2 if and only if f has a periodic orbit of period a power of 2 which is not simple.*

THEOREM B. *Let $f \in C^0(I, I)$. Suppose f has a periodic orbit P of period m (where $m = 2^k$ for some $k \geqslant 2$) which is not simple. Then f has a periodic point of period $3 \cdot 2^{k-2}$.*

The proof of Theorems A and B uses some results of [6] and [7] which will be stated in §2. Štefan in [7] also obtains the following result which improves a theorem of Bowen and Franks (see [2]).

THEOREM C. *Let $f \in C^0(I, I)$ and suppose f has a periodic point of period n, where $n = 2^d \cdot m$ and $m \geqslant 3$ is odd. Then the topological entropy of f is greater than $(1/2^d)\log\sqrt{2}$.*

Thus (using Theorem C) the following is an immediate corollary of Theorem B.

COROLLARY D. *Let $f \in C^0(I, I)$. Suppose f has a periodic orbit of period m (where $m = 2^k$ for some $k \geqslant 2$) which is not simple. Then the topological entropy of f is greater than $(1/2^{k-2})\log\sqrt{2}$.*

2. Preliminary definitions and results. Let $f \in C^0(I, I)$ and let N denote the set of positive integers. For any $n \in N$, we define f^n inductively by $f^1 = f$ and $f^n = f \circ f^{n-1}$. Let f^0 denote the identity map of I.

Let $x \in I$. x is said to be a periodic point of f if $f^n(x) = x$ for some $n \in N$. In this case the smallest element of $\{n \in N: f^n(x) = x\}$ is called the period of x.

We define the orbit of x to be $\{f^n(x): n = 0,1,2, \ldots\}$. If x is a periodic point we say the orbit of x is a periodic orbit, and we define the period of the orbit to be the period of x. Clearly, if x is a periodic point of period n, then the orbit of x contains n points and each of these points is a periodic point of f of period n.

Note that a periodic point of f is always a periodic point of f^n (for any $n \in N$), but the periods may be different. The following proposition (which follows immediately from the definitions) gives an example of this.

PROPOSITION 1. *Let $f \in C^0(I, I)$. Suppose P is a periodic orbit of f of period n where n is even. Then there are disjoint subsets P_1 and P_2 of P which are periodic orbits of f^2 of period $n/2$.*

Now, let P be a periodic orbit of f containing at least two points. Let $P_{\min}(f)$ denote the smallest element of P and $P_{\max}(f)$ denote the largest element of P. Let

$$U(f) = \{x \in I: f(x) > x\} \quad \text{and} \quad D(f) = \{x \in I: f(x) < x\}.$$

Let $P_U(f)$ denote the largest element of $P \cap U(f)$ and $P_D(f)$ denote the smallest element of $P \cap D(f)$.

We will use the following lemma, proved by Štefan in [7] (see (9) in §B of [7]).

LEMMA 2. *Let $f \in C^0(I, I)$ and let P be a periodic orbit of f. If f has a fixed point between $P_{\min}(f)$ and $P_U(f)$ (or between $P_D(f)$ and $P_{\max}(f)$) then f has periodic orbits of every period.*

The following corollary to Lemma 2 also appears in [7].

LEMMA 3. *Let $f \in C^0(I, I)$ and let P be a periodic orbit of f. If $P_D(f) < P_U(f)$ then f has periodic orbits of every period.*

PROOF. Suppose $P_D(f) < P_U(f)$. Since $f(P_D(f)) < P_D(f)$ and $f(P_U(f)) > P_U(f)$, f has a fixed point between $P_D(f)$ and $P_U(f)$. Thus, the hypothesis of Lemma 2 is satisfied. Q.E.D.

LEMMA 4. *Suppose $f \in C^0(I, I)$. Let $J \subset I$ and $K \subset I$ be closed intervals with $f(J) \supset K$. There is a closed interval $H \subset J$ with $f(H) = K$.*

PROOF. Let $K = [a, b]$ and let $A = f^{-1}(a) \cap J$ and $B = f^{-1}(b) \cap J$. Let d denote the usual metric on the real line. Since A and B are nonempty disjoint compact sets, there are points $a_1 \in A$ and $b_1 \in B$ such that $d(a_1, b_1) = d(A, B)$. Let H be the closed interval with endpoints a_1 and b_1. Then $H \cap A = \{a_1\}$ and $H \cap B = \{b_1\}$. Hence $f(H) = K$. Q.E.D.

LEMMA 5. *Let $f \in C^0(I, I)$. Suppose H and K are closed intervals with $H \subset K \subset I$ and $f(H) = K$. Then f has a fixed point in H.*

PROOF. Let $K = [a, b]$. For some $x \in H$ and $y \in H$, $f(x) = a$ and $f(y) = b$. Hence $f(x) \leqslant x$ and $f(y) \geqslant y$. Thus, f has a fixed point between x and y. Q.E.D.

LEMMA 6. *Let $f \in C^0(I, I)$. Let $g = f^r$ for some positive integer r which is a power of 2. Suppose there is a periodic orbit $P_0 = \{q_1, \dots, q_n\}$ of g of period n, where n is a power of 2 and $n \geq 2$. Suppose $q_1 < q_2 < \cdots < q_n$ and for some $i \leq n/2$ and $j \leq n/2$, $g(q_i) = q_j$. Then there is a periodic orbit P of f of period a power of 2 which is not simple.*

PROOF. Let P be the orbit of q_1 with respect to f. Then P is a periodic orbit and $P_0 \subset P$. Let m be the period of P. We will show that $m = n \cdot r$.

We claim that, for any positive integer $s < r$ and any $q_j \in P_0$, $f^s(q_j) \notin P_0$. To prove this, suppose that, for some positive integer $s < r$ and some $q_j \in P_0$, $f^s(q_j) \in P_0$. We may assume (by choosing s smaller if necessary) that, for any positive integer $t < s$, $f^t(q_i) \notin P_0$ for $i = 1, \dots, n$. Note that for $k = 0, \dots, n-1$,

$$f^s\big(f^{k\cdot r}(q_j)\big) = f^{k\cdot r}\big(f^s(q_j)\big) \in P_0.$$

Hence $f^s(P_0) \subset P_0$. Since f restricted to P is one-to-one, $f^s(P_0) = P_0$. Since $f^r(P_0) = P_0$ and $s < r$, it follows from the choice of s that s divides r.

Now, $f^s(P_0) = P_0$ implies that some subset P_1 of P_0 is a periodic orbit of f^s. Hence $f^s(P_1) = P_1$. Since s divides r, $f^r(P_1) = P_1$. Hence $P_1 = P_0$. Thus P_0 is a periodic orbit of f^s and a periodic orbit of f^r. Since r is a power of 2, s divides r, $s < r$, and P_0 has at least two elements, we obtain a contradiction by repeated application of Proposition 1. This contradiction establishes our claim.

Next, we will show that the points

$$q_1, \dots, q_n, f(q_1), \dots, f(q_n), \dots, f^{r-1}(q_1), \dots, f^{r-1}(q_n)$$

are all distinct. Suppose $f^a(q_i) = f^b(q_j)$ where $0 \leq a \leq r-1$, $0 \leq b \leq r-1$, $1 \leq i \leq n$, and $1 \leq j \leq n$. We may assume that $a \leq b$. By applying f^{r-b} to the points $f^a(q_i)$ and $f^b(q_j)$, we see that $f^{r-b+a}(q_i) \in P_0$. Since $a \leq b$, $r - b + a \leq r$. By our claim above, $r - b + a = r$. Hence $a = b$. Again, applying f^{r-b} to the points $f^a(q_i)$ and $f^b(q_j)$ we see that $f^r(q_i) = f^r(q_j)$. Since f^r restricted to P_0 is one-to-one, $q_i = q_j$. Hence $i = j$.

Clearly,

$$P = \big\{q_1, \dots, q_n, f(q_1), \dots, f(q_n), \dots, f^{r-1}(q_1), \dots, f^{r-1}(q_n)\big\}.$$

Hence $m = n \cdot r$. Thus m is a power of 2 and r divides m. It follows from this and our hypothesis that P is not simple. Q.E.D.

3. Proof of Theorems A and B.

LEMMA 7. *Let $f \in C^0(I, I)$. Let $n \geq 3$ be an odd integer and suppose that f^n does not have any periodic orbits of period 3. Let P be a periodic orbit of f of period k, where k is a power of 2 and $k \geq 2$. Then $P_U(f^n) = P_U(f)$ and $P_D(f^n) = P_D(f)$.*

PROOF. Note that $P_U(f^n)$ and $P_D(f^n)$ are well defined because P is a periodic orbit of f^n.

Our hypothesis implies that f does not have any periodic orbits of period $3 \cdot n$. By Lemma 3, $P_U(f) < P_D(f)$. It follows from this (and the definitions of $P_U(f)$ and $P_D(f)$) that there are no elements of P between $P_U(f)$ and $P_D(f)$. Also, by Lemma 3, $P_U(f^n) < P_D(f^n)$ and there are no elements of P between $P_U(f^n)$ and $P_D(f^n)$.

It suffices to prove that $P_U(f^n) = P_U(f)$. Suppose $P_U(f^n) \neq P_U(f)$. We have two cases.

Case 1. $P_U(f) < P_U(f^n)$. Then $P_U(f) < P_D(f) \leqslant P_U(f^n) < P_D(f^n)$. Since $f(P_{\min}(f)) > P_{\min}(f)$ and $f(P_D(f)) < P_D(f)$, f has a fixed point between $P_{\min}(f)$ and $P_D(f)$. Hence, f^n has a fixed point between $P_{\min}(f^n) = P_{\min}(f)$ and $P_U(f^n)$. By Lemma 2, f^n has periodic orbits of every period, a contradiction.

Case 2. $P_U(f^n) < P_U(f)$. Then $P_U(f^n) < P_D(f^n) \leqslant P_U(f) < P_D(f)$. It follows that f has a fixed point between $P_U(f)$ and $P_{\max}(f)$, so f^n has a fixed point between $P_D(f^n)$ and $P_{\max}(f^n)$. By Lemma 2, f^n has periodic orbits of every period, a contradiction. Q.E.D.

LEMMA 8. *Let $f \in C^0(I, I)$. Let $P = \{p_1, \ldots, p_n\}$ be a periodic orbit of f of period n, where n is a power of* 2 *and $n \geqslant 2$, and $p_1 < p_2 < \cdots < p_n$. Suppose that, for every odd positive integer $m < n$, f^m does not have any periodic orbits of period* 3. *Then*

$$f(\{p_1, \ldots, p_{n/2}\}) = \{p_{n/2+1}, \ldots, p_n\}$$

and

$$f(\{p_{n/2+1}, \ldots, p_n\}) = \{p_1, \ldots, p_{n/2}\}.$$

PROOF. We claim that $f(P \cap U(f)) \subset P \cap D(f)$. Suppose the claim is false. Then for some $p_0 \in P \cap U(f)$, $f(p_0) \in P \cap U(f)$. Let k be the smallest nonnegative integer with $f^k(f(p_0)) = p_1$. Note that $1 < k < n$.

If k is odd then $p_1 < f(p_0)$ and f^k has a fixed point between p_1 and $f(p_0)$. Now $f(p_0) \leqslant P_U(f)$ and, by Lemma 7, $P_U(f) = P_U(f^k)$. Hence, f^k has a fixed point between $P_{\min}(f^k)$ and $P_U(f^k)$. By Lemma 2, f^k has periodic orbits of every period. This contradicts our hypothesis.

If k is even then $k + 1$ is odd, $k + 1 < n$, and $f^{k+1}(p_0) = p_1$. Hence, $p_1 < p_0$ and f^{k+1} has a fixed point between p_1 and p_0. By Lemma 7, f^{k+1} has a fixed point between $P_{\min}(f^{k+1})$ and $P_U(f^{k+1})$. Again, using Lemma 2, we obtain a contradiction.

This establishes our claim that $f(P \cap U(f)) \subset P \cap D(f)$. By a similar proof, it follows that $f(P \cap D(f)) \subset P \cap U(f)$. Since the restriction of f to P is a bijection it follows that

$$f(P \cap U(f)) = P \cap D(f) \quad \text{and} \quad f(P \cap D(f)) = P \cap U(f).$$

Hence $P \cap U(f)$ and $P \cap D(f)$ have an equal number of points. Since $P_U(f) < P_D(f)$ by Lemma 3, this proves Lemma 8. Q.E.D.

PROPOSITION 9. *Let $f \in C^0(I, I)$. Suppose $\{p_1, \ldots, p_n\}$ is a periodic orbit of f of period n, where n is a power of 2 and $n \geqslant 2$. Suppose $p_1 < p_2 < \cdots < p_n$ and $f(\{p_1, \ldots, p_{n/2}\}) \neq \{p_{n/2+1}, \ldots, p_n\}$. Then f has a periodic point of period s, where s is odd and $3 \leqslant s \leqslant 3(n-1)$.*

PROOF. By hypothesis and Lemma 8, for some odd integer $m < n$, f^m has a periodic point of period 3. The conclusion follows easily from this. Q.E.D.

LEMMA 10. *Let $f \in C^0(I, I)$. Suppose f has a periodic orbit $\{p_1, p_2, p_3, p_4\}$ with $p_1 < p_2 < p_3 < p_4$ and $f(\{p_1, p_2\}) \neq \{p_3, p_4\}$. Then f has a periodic point of period 3.*

PROOF. Let $I_1 = [p_1, p_2]$, $I_2 = [p_2, p_3]$, and $I_3 = [p_3, p_4]$. Our hypothesis implies that either $f(p_1) = p_2$ or $f(p_2) = p_1$.

Case 1. $f(p_1) = p_2$. Since $f(p_2) = p_3$ or $f(p_2) = p_4$, we have $f(I_1) \supset I_2$. Also, since $f(p_2) = p_4$ or $f(p_3) = p_4$. we have $f(I_2) \supset I_3$. Finally, since $f(p_3) = p_1$ or $f(p_4) = p_1$, we have $f(I_3) \supset I_1$.

By Lemma 4, there are closed intervals $J_1 \subset I_1$, $J_2 \subset I_2$, and $J_3 \subset I_3$ such that $f(J_3) = I_1$, $f(J_2) = J_3$, and $f(J_1) = J_2$. It follows that $f^3(J_1) = I_1$. By Lemma 5, f^3 has a fixed point $x \in J_1$. Since $f(x) \in I_2$, x is a periodic point of f of period 3.

Case 2. $f(p_2) = p_1$. Then $f(p_1) = p_3$ or $f(p_1) = p_4$. Thus, $f(I_1) \supset I_1$, and $f(I_1) \supset I_2$. Also, $f(p_2) = p_1$ implies that $f(I_2) \supset I_1$. By Lemma 4, there are closed intervals $J_1 \subset I_1$, $J_2 \subset I_1$, and $J_3 \subset I_2$ such that $f(J_3) = I_1$, $f(J_2) = J_3$, and $f(J_1) = J_2$. It follows that $f^3(J_1) = I_1$. By Lemma 5, f^3 has a fixed point $x \in J_1$. Since $f^2(x) \in I_2$, x is a periodic point of f of period 3. Q.E.D.

THEOREM B. *Let $f \in C^0(I, I)$. Suppose f has a periodic orbit P of period m (where $m = 2^k$ for some $k \geqslant 2$) which is not simple. Then f has a periodic point of period $3 \cdot 2^{k-2}$.*

PROOF. By hypothesis there is a subset $\{q_1, \ldots, q_n\}$ of P and a positive integer r which divides m such that $\{q_1, \ldots, q_n\}$ is a periodic orbit of f^r with $q_1 < q_2 < \cdots < q_n$ and

$$f^r(\{q_1, \ldots, q_{n/2}\}) \neq \{q_{n/2+1}, \ldots, q_n\}.$$

This implies $n > 2$. It follows from the proof of Lemma 6 that $m = n \cdot r$. Hence $r \leqslant 2^{k-2}$.

First suppose $r = 2^{k-2}$. Then $n = 4$, so $\{q_1, q_2, q_3, q_4\}$ is a periodic orbit of f^r of period 4 with $f^r(\{q_1, q_2\}) \neq \{q_3, q_4\}$. By Lemma 10, f^r has a periodic point of period 3. By the theorem of Šarkovskiĭ (stated in §1), f has a periodic point of period $3 \cdot r = 3 \cdot 2^{k-2}$.

Now suppose $r < 2^{k-2}$. Then $r \leqslant 2^{k-3}$. By Proposition 9, f^r has a periodic point of period s, where s is odd and $s \geqslant 3$. By the theorem of Šarkovskiĭ, f has a periodic point of period $3 \cdot 2^{k-2}$. Q.E.D.

THEOREM A. *Let $f \in C^0(I, I)$. f has a periodic point whose period is not a power of 2 if and only if f has periodic orbit of period a power of 2 which is not simple.*

PROOF. The "if" part of the theorem follows from Theorem B.

Suppose f has a periodic point whose period is not a power of 2. By the theorem of Šarkovskiĭ, stated in §1, for some positive integer r which is a power of 2, f^r has a periodic orbit P of period 3. Let $P = \{p_1, p_2, p_3\}$ with $p_1 < p_2 < p_3$.

Let $g = f^r$. Then $g(p_1) = p_2$ or $g(p_3) = p_2$. We may assume without loss of generality that $g(p_1) = p_2$. This implies that $g(p_2) = p_3$ and $g(p_3) = p_1$.

Since $g(p_2) > p_2$ and $g(p_3) < p_3$, g has a fixed point $e \in (p_2, p_3)$. Let $I_1 = [p_1, p_2]$, $I_2 = [p_2, e]$, and $I_3 = [e, p_3]$. Then $g(I_1) \supset I_2$, $g(I_1) \supset I_3$, $g(I_2) \supset I_3$, $g(I_3) \supset I_1$, and $g(I_3) \supset I_2$. By Lemma 4, there are closed intervals $J_8 \subset I_3$ with $g(J_8) = I_1$, $J_7 \subset I_2$ with $g(J_7) = J_8$, $J_6 \subset I_3$ with $g(J_6) = J_7$, $J_5 \subset I_2$ with $g(J_5) = J_6$, $J_4 \subset I_1$ with $g(J_4) = J_5$, $J_3 \subset I_3$ with $g(J_3) = J_4$, $J_2 \subset I_2$ with $g(J_2) = J_3$, and $J_1 \subset I_1$ with $g(J_1) = J_2$.

It follows that $g^8(J_1) = I_1$. By Lemma 5, g^8 has a fixed point $c \in J_1$. Hence c is a periodic point of g of period 1,2,4, or 8. Since $g(c) \in I_2$, $g^2(c) \in I_3$, and $g^4(c) \in I_2$, c is a periodic point of g of period 8.

Let $\{q_1, \ldots, q_8\}$ denote the orbit of c where $q_1 < q_2 < \cdots < q_8$. We claim that, for some $i \leqslant 4$ and $j \leqslant 4$, $g(q_i) = q_j$. Note that $c \in I_1$, $g^3(c) \in I_1$, $g(c) \in I_2$, $g^4(c) \in I_2$, $g^6(c) \in I_2$, $g^2(c) \in I_3$, $g^5(c) \in I_3$, and $g^7(c) \in I_3$. Hence, $\{q_1, q_2, q_3, q_4\}$ contains c, $g^3(c)$, and two of the points $g(c)$, $g^4(c)$, and $g^6(c)$.

First, suppose that $g(c) \in \{q_1, q_2, q_3, q_4\}$. Then the claim is true with $q_i = c$ and $q_j = g(c)$. Now, suppose that $g(c) \notin \{q_1, q_2, q_3, q_4\}$. Then $g^4(c) \in \{q_1, q_2, q_3, q_4\}$. So the claim holds with $q_i = g^3(c)$ and $q_j = g^4(c)$.

Thus, our claim holds in either case. By Lemma 6, f has a periodic orbit of period a power of 2 which is not simple. Q.E.D.

4. Some examples. Let $f \in C^0(I, I)$ and let $P = \{p_1, \ldots, p_8\}$ be a periodic orbit of f of period 8 with $p_1 < p_2 < \cdots < p_8$. Then P is simple if and only if the following two conditions hold:

(1) $f(\{p_1, p_2, p_3, p_4\}) = \{p_5, p_6, p_7, p_8\}$,

(2) $f^2(\{p_1, p_2\}) = \{p_3, p_4\}$ and $f^2(\{p_5, p_6\}) = \{p_7, p_8\}$.

Clearly, (1) and (2) and the fact that P is a periodic orbit of period 8 imply that

$$f(\{p_5, p_6, p_7, p_8\}) = \{p_1, p_2, p_3, p_4\},$$

$$f^2(\{p_3, p_4\}) = \{p_1, p_2\}, \qquad f^2(\{p_7, p_8\}) = \{p_5, p_6\},$$

$$f^4(p_1) = p_2, \quad f^4(p_2) = p_1, \quad f^4(p_3) = p_4, \quad f^4(p_4) = p_3,$$

$$f^4(p_5) = p_6, \quad f^4(p_6) = p_5, \quad f^4(p_7) = p_8, \quad f^4(p_8) = p_7.$$

EXAMPLE 1. $f(p_1) = p_5$, $f(p_2) = p_6$, $f(p_3) = p_8$, $f(p_4) = p_7$, $f(p_5) = p_3$, $f(p_6) = p_4$, $f(p_7) = p_1$, and $f(p_8) = p_2$.

In this example P is simple.

EXAMPLE 2. $f(p_1) = p_2$, $f(p_2) = p_5$, $f(p_3) = p_7$, $f(p_4) = p_8$, $f(p_5) = p_3$, $f(p_6) = p_1$, $f(p_7) = p_4$, and $f(p_8) = p_6$.

In this example P is not simple because condition (1) above does not hold. By Proposition 9, f has a periodic point of period s, where s is odd and $3 < s < 21$. By Theorem C of §1, the topological entropy of f is greater than $\log\sqrt{2}$.

EXAMPLE 3. $f(p_1) = p_5$, $f(p_2) = p_7$, $f(p_3) = p_6$, $f(p_4) = p_8$, $f(p_5) = p_2$, $f(p_6) = p_4$, $f(p_7) = p_3$, and $f(p_8) = p_1$.

In this example P is not simple because condition (2) does not hold (since $f^2(p_1) = p_2$). By Theorem B, f has a periodic point of period 6, and by Corollary D, the topological entropy of f is greater than $(\frac{1}{2})\log\sqrt{2}$.

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