APPROXIMATE TORUS FIBRATIONS OF
HIGH DIMENSIONAL MANIFOLDS
CAN BE APPROXIMATED BY TORUS BUNDLE PROJECTIONS

BY

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ABSTRACT. In this paper, we prove that approximate torus fibrations of high dimensional manifolds can be approximated by torus bundle projections. The principal tools are the torus trick developed by Kirby and Siebenmann, a surgery theorem concerning homotopy structures on torii due to Hsiang and Wall, a theorem on the space of homeomorphisms of the torus due to Hamstrom and a generalization of hereditary homotopy equivalence developed by the author.

An old question in topology is how to identify a product or local product structure in a space. Many results have been obtained which tell whether a continuous function is the projection of a local product (fibre bundle) or may be approximated by one. Most of these criteria involve some sort of regularity condition. That is, nearby fibres must lie in the domain in somewhat similar fashion. With only weak hypotheses on the spaces involved, a fairly strong form of regularity (topological) produces a product theorem [D-H]. Chapman and Ferry show that this criterion is satisfied under a "homotopy regularity" condition (that of being an ANR fibration with Q-manifold fibres ([C-F 1], [C-F 2])) and so obtain a product theorem for nicer spaces with a weaker regularity hypothesis. Coram and Duvall have shown that an approximate fibration possesses a still weaker form of regularity (shape) called complete movability [C-D 2].

The problem we consider here is whether, for sufficiently nice spaces, such a map may be approximated by a fibre bundle projection.

This paper is divided into two parts. In the first, we develop many of the tools which will be used in the second. These deal, generally, with various constructions and a characterization of approximate fibrations. Some of the details are omitted, but these can be found in the author's Ph. D. thesis [Go] and, hopefully, in a forthcoming paper.

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To simplify the exposition, we will deal only with separable metric spaces and, when convenient, we will restrict our attention to proper maps, i.e. those for which the inverse image of a compact set is compact.

I. Local homotopy properties. An approximate fibration is a proper map of metric spaces which has the approximate homotopy lifting property with respect to all metric spaces. That is, any commutative diagram

$$
\begin{array}{ccccc}
A \times \{0\} & \stackrel{h}{\to} & E & & \\
\downarrow & & \downarrow p & & \\
A \times I & \stackrel{H}{\to} & B & \stackrel{\varepsilon}{\to} & (0, \infty)
\end{array}
$$

extends to a diagram

$$
\begin{array}{ccccc}
A \times \{0\} & \stackrel{h}{\to} & E & & \\
\downarrow & {}^{G} \nearrow & \downarrow p & & \\
A \times I & \stackrel{H}{\to} & B & \stackrel{\varepsilon}{\to} & (0, \infty)
\end{array}
$$

so that $G(a, 0) = h(a, 0)$ and $d(pG(a, i), H(a, i)) < \varepsilon H(a, i)$ for all (a, i) in $A \times I$. Although Coram and Duvall have defined the term in a more general setting using open covers rather than a metric [C-D 1], this definition suffices in our context.

A proper map $p: E \to B$ has local homotopy product structure with fibre F (LHP-F) if for each $b \in B$, there is a neighborhood U of b and map $f: p^{-1}(U) \to F$ so that, for each neighborhood V of b in U, $p \times f: p^{-1}(V) \to V \times F$ is a proper homotopy equivalence. p has HP-F if we may take $U = B$ in the definition of LHP-F. A useful theorem is the following:

I.A. THEOREM. *If $p: E \to B$ is a proper map of ANR's and there is a b in B so that $p^{-1}(b)$ has the shape of a compact ANR F, then p is an approximate fibration if and only if p has property LHP-F.*

PROOF. In case p is an approximate fibration, there is an exact sequence of shape groups [C-D 1]. The shape isomorphism of the fibre with the ANR F provides a map s defined on a neighborhood U' of $p^{-1}(b)$. Choose a neighborhood U of b so that $p^{-1}(U) \subseteq U'$. This is to be the U required by property LHP-F. Now, the sequence is still exact if the map is restricted to $p: p^{-1}(U) \to U$ and the shape map provides a splitting:

$$
\to \mathrm{Sh}_k(F) \underset{S_*}{\to} \mathrm{Sh}_k(p^{-1}(U)) \stackrel{P_*}{\to} \mathrm{Sh}_k(U) \to \mathrm{Sh}_{k-1}(F).
$$

Further, since all spaces involved are ANR's, shape groups may be replaced by homotopy groups. This splitting provides isomorphisms

$$
P_* \oplus S_*: \pi_k(p^{-1}(U)) \to \pi_k(F) \oplus \pi_k(U)).
$$

Thus, by the Whitehead theorem, $p \times s$ is a homotopy equivalence. The argument works equally well for $V \subseteq U$ and so it is easily seen, say, via a proper Whitehead theorem [F-T-W], that $p \times s$ is a proper homotopy equivalence of $p^{-1}(V)$ to $V \times F$ for each open V contained in U.

Conversely, suppose p has property LHP-F. It is easily seen how to ε-lift the homotopy over any one of the U's for which the map is homotopically trivial. To

obtain a global lifting, one simply mimics the induction argument used to show that a coordinate bundle is a (Hurewicz) fibration [St].

We are now free to work with LHP-F maps rather than approximate fibrations whenever it seems more convenient. We will build on the analogy between LHP-F maps and fibre bundles to obtain the tools we will need for approximations. Many of these tools are founded on the following theorem, which has appeared whenever one wished to obtain global homotopy information from local information ([Ko], [Mi] and [Su]).

I.B. THEOREM. *If F is a closed ANR subspace of the ANR E and $U = \{U_j\}_{j=1}^{\infty}$ is a locally finite open cover of E such that for each nonempty finite subfamily S of U there is a strong deformation retraction of $U_S = \bigcap_{U_i \in S} U_i$ onto $F \cap U_S$, then there is a strong deformation retraction of E onto F which is limited by U.*

PROOF. The proof given by Milnor in [Mi] actually yields this result. One must observe that the deformation he constructs is limited by U.

The next corollary follows routinely from I.B. by use of the mapping cylinder of $p \times f$:

I.B.1. COROLLARY. *Suppose X, Y and F are ANR's, $p: X \to Y$, $f: X \to F$ are maps and $\{U_i\}$ is a locally finite open cover of Y such that, for every U_i, $p: p^{-1}(U_i) \to U_i$ has property HP-F via f. Then $p: X \to Y$ has property HP-F via f. Further, we may find a homotopy inverse g for $p \times f$ so that $g(U_i \times F) \subseteq p^{-1}(U_i)$ and homotopies*

$$h: g \circ (p \times f) \simeq 1_X, \qquad k: (p \times f) \circ g \simeq 1_{Y \times F}$$

so that $h(p^{-1}(U_i) \times I) \subseteq p^{-1}(U_i)$ and $k(U_i \times F \times I) \subseteq U_i \times F$.

I.B.2. COROLLARY. *If p has property HP-F via $f: X \to F$ and $\alpha: B \to Y$ is an open immersion, then the topological pullback p' of p by α has property HP-F via $f \circ \alpha'$:*

$$
\begin{array}{ccccc}
E & \xrightarrow{\alpha'} & X & \xrightarrow{f} & F \\
\downarrow p' & & \downarrow p & & \\
B & \xrightarrow{\alpha} & Y & &
\end{array}
$$

PROOF. The open sets openly embedded by α clearly have HP-F via $f \circ \alpha'$. A locally finite refinement of this cover provides the necessary cover of B.

The next proposition provides a means to alter HP trivializations:

I.C. PROPOSITION. *Suppose $p: X \to Y$ has property HP-F via $f: X \to F$. If H is a homotopy of f to f', then P has HP-F via f'.*

PROOF. $1 \times H$ is a proper homotopy of $p \times f$ to $p \times f'$, even when restricted to $p^{-1}(U)$.

One can glue HP trivializations according to the following proposition:

I.D. PROPOSITION. *Suppose $p\colon X \to Y$ has property HP-F via $f\colon X \to F$, and that H is a homotopy from f to f'. If B_1 and B_2 are two disjoint closed subsets of X, then there is a third map $g\colon X \to F$ such that*

(1) *p has HP-F via g, and*

(2) *$g|B_1 = f|B_1$ and $g|B_2 = f'|B_2$.*

PROOF. Choose a Urysohn function $u\colon X \to [0, 1]$ such that $u(B_1) = 0$ and $u(B_2) = 1$. Set $g(x) = H(x, u(x))$. If x is in B_1, then $g(x) = H(x, 0) = f(x)$ and if x is in B_2, then $g(x) = H(x, 1) = f'(x)$. Thus condition (2) above is satisfied. To verify condition (1) we must show that g is homotopic to f. Set $L(x, t) = H(x, (1 - t)u(x))$. Note that $L(x, 0) = H(x, u(x)) = g(x)$ and that $L(x, 1) = H(x, 0) = f(x)$.

The next proposition allows one to combine HP trivializations which do not necessarily agree on their common domain. This facilitates certain pasting constructions which are necessary to the approximation scheme.

To establish notation, suppose $p\colon X \to Y$ is a map and that Y is the union of two open sets A_1 and A_2. Set

$$B_1' = Y \setminus A_2, \quad B_2' = Y \setminus A_1,$$
$$C_1' = \{ y \in A_1 \cap A_2 \colon 2d(y, B_1') \leqslant d(y, B_2') \},$$
$$C_2' = \{ y \in A_1 \cap A_2 \colon 2d(y, B_2') \leqslant d(y, B_1') \},$$
$$C_i = p^{-1}(C_i'), \quad B_i = p^{-1}(B_i'); \quad i = 1, 2,$$
$$D_1' = \{ y \in Y \colon 2d(y, B_1') < d(y, B_2') \},$$
$$D_2' = \{ y \in Y \colon 2d(y, B_2') < d(y, B_1') \},$$
$$D_i = p^{-1}(D_i').$$

I.E. PROPOSITION. *For $i = 1, 2$ suppose $p|p^{-1}(A_i)\colon p^{-1}(A_i) \to A_i$ has property HP-F via some f_i. If f_1 is homotopic to f_2 on $p^{-1}(A_1 \cap A_2)$, then p has property HP-F via a map $g\colon X \to F$ such that, for $i \in \{1, 2\}$, $g|B_i = f_i|B_i$.*

PROOF. We first show that $C_1' \cap C_2' = \varnothing$:

Let y be an element of $C_1' \cap C_2'$, then $2d(y, B_1') < d(y, B_2') < d(y, B_1')$, so $d(y, B_1') = 0$. by the same argument, $d(y, B_2') = 0$ and so $y \in B_1' \cap B_2'$. Thus $C_1' \cap C_2' \subseteq B_1' \cap B_2'$. However, since $Y = A_1 \cup A_2$,

$$B_1' \cap B_2' = (Y \setminus A_2) \cap (Y \setminus A_1) = Y \setminus (A_1 \cup A_2) = \varnothing.$$

Thus, $B_1 \cap B_2 = \varnothing$ and $C_1 \cap C_2 = \varnothing$. By I.D., there is a function $g'\colon p^{-1}(A_1 \cap A_2) \to F$ which agrees with f_i on C_i and which is an HP trivialization for $p|p^{-1}(A_1 \cap A_2)$. Now, observe that each D_i is an open neighborhood of B_i and that $D_i \cap p^{-1}(A_1 \cap A_2) \subseteq C_i$. Again, note that D_1 and D_2 are disjoint because $D_i \subseteq B_i \cup C_i$. Thus $\{D_1, D_2, p^{-1}(A_1 \cap A_2)\}$ is an open cover of X with the properties that $D_1 \cap D_2 = \varnothing$, and $D_i \cap p^{-1}(A_1 \cap A_2) \subseteq C_i$.

Thus, setting

$$g(x) = \begin{cases} f_1(x) & \text{if } x \in D_1, \\ g'(x) & \text{if } x \in p^{-1}(A_1 \cap A_2), \\ f_2(x) & \text{if } x \in D_2, \end{cases}$$

we obtain a continuous function, since each pair of the three component functions agree on their common domains. Thus, by I.B.1, p is HP-F via g. Note that since $B_i \subseteq D_i$, we have that $g|B_i = f_i|B_i$ for $i \in \{1, 2\}$.

II. The approximation scheme. The principal result is the following:

II.A. THEOREM. *Suppose* $p\colon M^{n+2} \to N^n$ *is an LHP-T^2 map of topological manifolds and* $n \geqslant 6$. *Suppose that* $\varepsilon\colon N^n \to (0, \infty)$ *is continuous. Then there is a torus bundle projection* $p'\colon M \to N$ *with the property that, for every* m *in* M, $d(p(M), p'(M)) < \varepsilon(p(m))$.

Here the term bundle or fibre bundle denotes a (locally trivial) fibre bundle in the sense of Steenrod [St] and the transformation group of the fibre is the group of homeomorphisms of the fibre. Also, we will use the term "over" to mean "when restricted to the inverse image of". For example, to say "p has HP-T^2 over U" means $p|p^{-1}(U)\colon p^{-1}(U) \to U$ has property HP-T^2.

Theorem II.A depends on:

II.B. THEOREM. *Let* $p\colon V^{m+2} \to B^k \times R^n$, $k + n = m \geqslant 6$, *be an HP-T^2 map of topological manifolds of the indicated dimensions. If p is a trivial T^2-bundle projection over the neighborhood* $(B^k \setminus \frac{1}{2}\mathring{B}^k) \times R^n$ *of* $\partial B^k \times R^n$, *then there is an HP-T^2 map* $p'\colon V^{m+2} \to B^k \times R^n$ *such that*
  (1) p' *is a (trivial) T^2-bundle projection over* $B^k \times B^n$ *and*
  (2) $p' = p$ *over* $(\partial B^k \times R^n) \cup [B^k \times (R^n \setminus r\mathring{B}^n)]$ *for some radius r.*
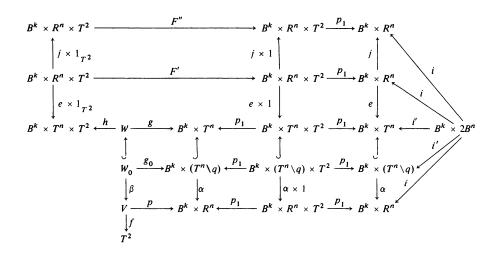
II.B depends on:

II.C. LEMMA. *Given p, V, m, k, n as above, there are maps* $\phi\colon V \to B^k \times R^n \times T^2$ *and* $F\colon B^k \times R^n \times T^2 \to B^k \times R^n$ *such that ϕ is a homeomorphism, F has property HP-T^2 and*
  (1) $F = $ *projection to $B^k \times R^n$ over* $((B^k \setminus rB^k) \times R^n) \cup (B^k \times (R^n \setminus 4B^n))$ *for some $r < 1$ and*
  (2) $F\phi = p$ *over* $(\partial B^k \times R^n) \cup (B^k \times B^n)$.

Notice that II.C is essentially II.B except that the roles of 0 and ∞ in the R^n factor are reversed. II.A follows from II.B using induction and an easy computation. II.C is proved using a modification of the main diagram argument used by Siebenmann in [Si 1].

Thus, we will prove II.A, II.B, and II.C in reverse order.

The outline of this proof is the same as that found in [Si 1] and we will assume that the reader is somewhat familiar with the "main diagram" ([Ki], [K-S] and [Si 1]).

PROOF OF II.C. The proof of II.C is based on constructing the following version of the main diagram:

$$
\begin{array}{c}
B^k \times R^n \times T^2 \xrightarrow{\quad F'' \quad} B^k \times R^n \times T^2 \xrightarrow{p_1} B^k \times R^n \\
\uparrow{\scriptstyle j \times 1_{T^2}} \qquad\qquad j \times 1 \uparrow \qquad\qquad j \uparrow \qquad\qquad i \\
B^k \times R^n \times T^2 \xrightarrow{\quad F' \quad} B^k \times R^n \times T^2 \xrightarrow{p_1} B^k \times R^n \\
\downarrow{\scriptstyle e \times 1_{T^2}} \qquad\qquad e \times 1 \downarrow \qquad\qquad e \downarrow \qquad i \\
B^k \times T^n \times T^2 \xleftarrow{h} W \xrightarrow{g} B^k \times T^n \xleftarrow{p_1} B^k \times T^n \times T^2 \xrightarrow{p_1} B^k \times T^n \xleftarrow{i'} B^k \times 2B^n \\
W_0 \xrightarrow{g_0} B^k \times (T^n\backslash q) \xleftarrow{p_1} B^k \times (T^n\backslash q) \times T^2 \xrightarrow{p_1} B^k \times (T^n\backslash q) \\
\downarrow{\scriptstyle \beta} \qquad \downarrow{\scriptstyle \alpha} \qquad \downarrow{\scriptstyle \alpha \times 1} \qquad \downarrow{\scriptstyle \alpha} \\
V \xrightarrow{p} B^k \times R^n \xleftarrow{p_1} B^k \times R^n \times T^2 \xrightarrow{p_1} B^k \times R^n \\
\downarrow{\scriptstyle f} \\
T^2
\end{array}
$$

(a) As in [Si 1], i is inclusion, $\bar{e}$ is an exponential map which is 1 to 1 on the interior of each 8×8 square, e is $1_{B^k} \times \bar{e}$, q is a point not in $\bar{e}(2B^n)$ and $ei = i'$. j is the embedding constructed in [Si 1], and $\alpha = 1 \times \bar{\alpha}$ where $\bar{\alpha}$ is an immersion of $T^n\backslash q$ in R^n.

(b) We wish to find a map $f: V \to T^2$ that is an HP trivialization for p and that is also a bundle trivialization over $(B^k\backslash\frac{1}{2}\mathring{B}^k) \times R^n = N$. First take any bundle trivialization $T: p^{-1}(N) \to N \times T^2$ and any LHP trivialization $f_1: V \to T^2$.

We have the following diagram:

$$
\begin{array}{ccccccc}
N \times T^2 & \xleftarrow{T} & p^{-1}(N) & \hookrightarrow & V & \xrightarrow{p \times f_1} & B^k \times R^n \times T^2 \\
\downarrow & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} & & \downarrow \\
N & \leftarrow & N & \hookrightarrow & B^k \times R^n & \rightarrow & B^k \times R^n
\end{array}
$$

We wish to extend T to all of V; however, there are certainly trivializations which will not extend. We therefore modify T as follows: Consider $(p \times f_1) \circ T^{-1}$. This is a fibre homotopy trivialization of the bundle $N \times T^2$ over N and so it is fibre homotopic to a genuine bundle trivialization T'. Let $g: B^k \times R^n \times T^2 \to E$ be a homotopy inverse to $p \times f$ which also restricts to a homotopy inverse to $p \times f|p^{-1}(N)$. We have the following homotopy commutative diagram:

$$
\begin{array}{ccc}
p^{-1}(N) & \overset{p_2 T'T}{\longrightarrow} & T^2 \\
\parallel & & \parallel \\
p^{-1}(N) & \overset{p_2 T'Tg(p \times f_1)}{\longrightarrow} & T^2 \\
\parallel & & \parallel \\
p^{-1}(N) \overset{p \times f_1}{\longrightarrow} N \times T^2 & \overset{p_2 T'(Tg)}{\longrightarrow} & T^2 \\
\parallel & & \parallel \\
p^{-1}(N) \overset{p \times f_1}{\longrightarrow} N \times T^2 & \overset{p_2}{\longrightarrow} & T^2 \\
\downarrow & & \parallel \\
V \overset{p \times f_1}{\longrightarrow} B^k \times R^n \times T^2 & \overset{p_2}{\longrightarrow} & T^2
\end{array}
$$

The bottom row is an extension of the previous row over all of V. Thus, $T' \circ T$ admits an extension $f: V \to T^2$. Since $f|p^{-1}(N) = T' \circ T$, it is a bundle trivialization.

It remains to show that f is an HP trivialization for $p: V \to B^k \times R^n$. However, since $p: V \to B^k \times R^n$ has property HP-T^2, there is a strong deformation retraction of V to the fibre over any point in $\partial B^k \times R^n$. Since f is certainly a homotopy equivalence on this fibre, it must be a homotopy equivalence on all of V.

(c) Let W_0 be the topological pullback of

$$
\begin{array}{ccc}
& & B^k \times T^n \diagdown q \\
& & \downarrow \alpha \\
V & \overset{p}{\longrightarrow} & B^k \times R^n
\end{array}
$$

and let g_0, β be the projections. These have the properties that β is an immersion [Si 1, Lemma 2.3] and so W_0 is a manifold. Since α is a codimension 0, hence open, immersion, we have that g has property HP-T^2 via $f \circ \beta$. Note also that, on $g_0^{-1} \alpha^{-1}(N)$, g_0 is a fibre bundle projection and $f \circ \beta$ is a bundle trivialization.

(d) To construct $g: W \to B^k \times T^n$, we first construct $g_1: W_1 \to (B^k \times T^n) \setminus (\frac{1}{2} B^k \times q)$. To do this, let

$$
N_0 = \left(B^k \diagdown \tfrac{1}{2} B^k \right) \times (T^n \diagdown q) = \alpha^{-1}(N).
$$

Now, $g_0 \times (f \circ \beta): g_0^{-1}(N_0) \to N_0 \times T^2$ is a homeomorphism. Note that

$$
N_0 \times T^2 \subseteq \left(B^k \diagdown \tfrac{1}{2} B^k \right) \times T^n \times T^2.
$$

Thus, if we use $g_0 \times (f \circ \beta)$ as an attaching map, we obtain a manifold

$$
W_1 = W_0 \cup \left(B^k \diagdown \tfrac{1}{2} B^k \right) \times T^n \times T^2
$$

which maps via g_1 to $(B^k \times T^n) \setminus (\frac{1}{2} B^k \times \{q\})$ as follows:

$$
g_1(x) = \begin{cases} g_0(x), & x \in W_0, \\ a, & x = (a, z) \in \left(\left(B^k \diagdown \tfrac{1}{2} B^k \right) \times T^n \right) \times T^2. \end{cases}
$$

Note also that W_1 maps to T^2 as follows:

$$f_1(x) = \begin{cases} f \circ \beta(x), & x \in W, \\ z, & x = (a, z) \in \left(\left(B^k \setminus \tfrac{1}{2} B^k \right) \times T^n \right) \times T^2. \end{cases}$$

Note that each of g_1, f_1 is well defined and hence continuous. We have that, over $(B^k \setminus \tfrac{1}{2} B^k) \times T^n$, f_1 is a bundle trivialization of g_1, by construction. Over $(B^k \setminus \tfrac{1}{2} B^k) \times (T^n \setminus q)$, g_1 and f_1 agree with g_0 and $f \circ \beta$ respectively. Thus, g_1 has property HP-T^2 via f_1.

   Now, to obtain $g \colon W \to B^k \times T^n$, choose $\delta \colon B^k \times T^n \to B^k \times T^n$ to be a continuous onto map such that

   (1) $\delta(\tfrac{1}{2} B^k \times \{q\}) = \{(0, q)\}$,

   (2) δ pointwise fixes a neighborhood of $e(B^k \times 2B^n)$, and

   (3) δ is a homeomorphism on the complement of $(\tfrac{1}{2} B^k \times \{q\})$.

Then set $g_2 = \delta g_1 \colon W_1 \to (B^k \times T^n) \setminus \{(0, q)\}$ and note that g_2 has property HP-T^2 via f_1 and $g_2 = g_0$ near $e(B^k \times 2b^n)$. We wish to extend W_1 to a manifold W and extend g_2 and f_1 to maps $g \colon W \to B^k \times T^n$ and $f_2 \colon W \to T^2$ so that g has property HP-T^2 via f_2. In [Si 1], the corresponding problem is solved by taking the one-point compactification of W_1 and sending ∞ to $(0, q)$. Here, however, the problem is more delicate. It requires the addition of an entire fibre to W_1 in such a way that f_1 can be extended. Once again, it is the fact that each self homotopy equivalence of $S^k \times T^2$ is homotopic to a homeomorphism [H-W], which provides the necessary map. For the construction, let $S^{m-1} \times R$ be an annular neighborhood of the deleted point $(0, q)$ in $B^k \times T^n$ with $(0, q)$ at $+\infty$ and consider $U = g_2^{-1}(S^{m-1} \times R)$. According to property HP-T^2, $g_2 \times f_1 \colon U \to (S^{m-1} \times R) \times T^2$ is a proper homotopy equivalence. Thus, the ends of U are tame, and $\tilde{K}_0(\pi_1(E_i)) = 0$ for each end E_i of U. Thus, according to Siebenmann's thesis [Si 2], there is a manifold M_i and a homeomorphism $\zeta_i \colon M_i \times [0, 1) \to E_i$. Now, $\overline{U \setminus E_1 \cup E_2}$ is an h-cobordism of M_1 to M_2 and each M_i is homotopy equivalent to $S^{m-1} \times T^2$ by $g_2 \times f_1$. Since $\mathrm{Wh}(Z \oplus Z) = 0$, the s-cobordism theorem yields a homeomorphism of $\overline{U \setminus E_1 \cup E_2}$ to $M_i \times I$. Thus, there is a homeomorphism $\zeta \colon U \to M \times R$. Now the composition

$$M = M \times \{0\} \to M \times R \overset{\zeta^{-1}}{\to} U \overset{g_2 \times f_1}{\to} S^{m-1} \times R \times T^2 \to S^{m-1} \times T^2$$

is a homotopy equivalence, and by [H-W], it is homotopic to a homeomorphism $\lambda \colon M \to S^{m-1} \times T^2$. Thus

$$M \times R \overset{\zeta^{-1}}{\to} U \overset{g_2 \times f_1}{\to} (S^{m-1} \times R) \times T^2$$

is homotopic to $\lambda \times 1_R = \lambda'$ and, in particular, $f_1 \zeta^{-1}$ is homotopic to $p_2 \lambda'$ and so $p_2 \lambda' \zeta$ is homotopic to f_1. Now, define $a \colon U \to \mathring{B}^m \times T^2$ by the composition $U \overset{\lambda' \zeta}{\to} S^{m-1} \times R \times T^2 \hookrightarrow \mathring{B}^m \times T^2$ and set $W = W_1 \cup_a (\mathring{B}^m \times T^2)$. Define $g \colon W \to B^k \times T^n$ by the formula

$$g(w) = \begin{cases} g_2(w), & w \in W_1, \\ (0, q), & w = (0, z) \in \{0\} \times T^2. \end{cases}$$

Observe that g is continuous, because g_2 is proper. We now show that g has property HP-T^2: $g|\mathring{B}^m \times T^2$ clearly has HP-T^2 via p_2 and $g|W_1$ has HP-T^2 via f_1. On $(\mathring{B}^m \times T^2) \cap W_1$, $p_2 a = p_2 \lambda' \zeta$ which is homotopic to f_1 and so there is an HP-T^2 trivialization f_2 for g on W, which agrees with f_1 on $W_1 \setminus U$.

(e) $h: W \to B^k \times T^n \times T^2$ is a homeomorphism such that $h = g \times f_2$ on a neighborhood of ∂W and h is homotopic to $g \times f_2$. Note that gh^{-1} has property HP-T^2 via $f_2 h^{-1}$ which is homotopic to p_2: $B^k \times T^n \times T^2 \to T^2$. Thus, gh^{-1} has property HP-T^2 via p_2.

(f) F' is a covering of $gh^{-1} \times p_2$ which fixes $\partial B^k \times R^n \times T^2$. Note that $p_2 F' = p_2$ and that, since $gh^{-1} \times p_2$ is a homotopy equivalence, the diagram

$$
\begin{array}{ccc}
B^k \times R^n \times T^2 & \overset{F'}{\to} & B^k \times R^n \times T^2 \\
\downarrow e \times 1 & & \downarrow e \times 1 \\
B^k \times T^n \times T^2 & \overset{gh^{-1}p_2}{\to} & B^k \times T^n \times T^2
\end{array}
$$

is a bundle map and hence a pullback diagram. Thus, $p_1 F'$ has property HP-T^2 via p_2.

(g) F'' is obtained from F' in the usual fashion ([Si 1], [K-S], etc.) so that the top rectangle in the diagram commutes, F'' is the identity off $B^k \times 4B^n \times T^2$ and F'' has property HP-T^2 via p_2.

This completes the construction of the diagram. To complete the lemma, we must construct ϕ. Note first, that by moving vertically up the left-hand side of the diagram, one obtains a homeomorphism ψ, provided he restricts his attention to $p^{-1}(B^k \times 2B^n)$. ϕ is obtained from ψ by an Alexander isotopy used in conjunction with the collaring of ∂V [Si 1]. This completes the lemma.

For the proof of II.B, we adapt the inversion device of [Si 1] as follows:

Let $\alpha: R^n \to R^n$ be the homeomorphism given by $\alpha(x) = 4X$. Then, with the data of II.B, let $\bar{p} = (1_{B^k} \times \alpha^{-1})p$, and let $f_1 = F\phi: V \to B^k \times R^n$ be as given by Lemma II.C applied to $\bar{p}$. F and ϕ together provide a manifold $\bar{V} = B^k \times S^n \times T^2$ containing V and an HP-T^2 map $\bar{f}_1: V \to B^k \times (R^n \cup \infty)$ extending f_1 as follows:

$$\phi: V \to B^k \times R^n \times T^2 \to B^k \times (R^n \cup \infty) \times T^2 = B^k \times S^n \times T^2$$

is an embedding.

Define $\bar{V} = V \cup_\phi B^k \times S^n \times T^2$ and define $\bar{f}_1: \bar{V} \to B^k \times (R^n \cup \infty)$ according to the rule

$$\bar{f}_1(x) = \begin{cases} F(x) & \text{if } x \in B^k \times R^n \times T^2, \\ p_1(x) & \text{if } x \in N_\delta(B^k \times \infty \times T^2), \end{cases}$$

where δ is chosen small enough that $F = p_1$ on $N_\delta(B^k \times \infty \times T^2) \cap V$.

Define $\theta: B^k \times \{(R^n \cup \infty) \setminus 0\} \to B^k \times R^n$ according to the rule $\theta(x, y) = (x, y/|y|^2)$ if $y \neq \infty$ and $\theta(x, \infty) = (x, 0)$. Note that $\theta^{-1}(B^k \times \partial rB^n) = B^k \times \partial(r^{-1}B^n)$.

Let g_1: $\overline{V} \setminus p^{-1}(B^k \times 0) \to B^k \times \{(R^n \cup \infty) \setminus 0\}$ be the restriction of $\bar{f}_1$ and let $\bar{g}_1$ be the following composition:

$$\overline{V} \setminus p^{-1}(B^k \times 0) \xrightarrow{g_1} B^k \times \{(R^n \cup \infty) \setminus 0\} \xrightarrow{\theta} B^k \times R^n.$$

Applying Lemma II.C to $\bar{g}_1$, we obtain $\bar{g}_2$: $V \setminus p^{-1}(B^k \times 0) \to B^k \times R^n$. Setting $g_2 = \theta^{-1}\bar{g}_2$, we have that g_2 is a bundle projection over $B^k \times (\frac{1}{4}B^n \setminus 0)$ and $g_2 = g_1$ over $B^k \times \{(R^n \cup \infty) \setminus B^n\}$ and over $\partial B^k \times \{(R^n \cup \infty) \setminus 0\}$.

We can now define h: $V \setminus p^{-1}(B^k \times 0) \to B^k \times (R^n \setminus 0)$ by setting h equal to $\tilde{p}$ over $B^k \times (R^n \setminus \mathring{B}^n)$ and setting h equal to g_2 over $B^k \times (B^n \setminus 0)$. Then h is well defined and continuous because over $B^k \times \partial B^n$, one has $g_2 = g_1 = \bar{f}_1 = \tilde{p}$. Clearly h is HP-T^2 and equals $\tilde{p}$ over $\partial B^k \times (R^n \setminus 0)$. Thus we can extend h to

$$h': V \setminus \tilde{p}^{-1}\left(\tfrac{1}{2}B^k \times 0\right) \to B^k \times R^n \setminus \tfrac{1}{2}(B^k \times 0)$$

by setting $h' = \tilde{p}$ on $\tilde{p}^{-1}((B^k \setminus \frac{1}{2}B^k) \times 0)$.

Now, consider $h'^{-1}(\partial(B^k \times \frac{1}{10}B^n))$. This is a copy of $S^{m-1} \times T^2$ and so, by [H-W] and the h-cobordism theorem, it bounds a topological $B^m \times T^2$.

Thus, the restriction of h' to $V \setminus (\mathring{B}^m \times T^2)$ extends as follows to an HP-T^2 map p': $V \to B^k \times R^n$:

On $V \setminus (\mathring{B}^m \times T^2)$, set $p'(x) = h'(x)$, and on $B^m \times T^2$, first project to B^m and then cone. $\tilde{p}' = \alpha p'$ is the approximation to p asserted by II.B.

We obtain II.A from II.B by induction over the index of handles in a small handle decomposition of the range. The details are left to the reader. The only new problem is that one must show that the bundle which has been constructed inductively over the boundary of a k-handle is, indeed, trivial. Notice that property HP-T^2 assures us that it will be fibre homotopically trivial. In the case at hand, that of T^2 fibres, this is sufficient because the group of homotopy equivalences of T^2 is homotopy equivalent to the group of homeomorphisms of T^2 [Ha].

Note that the proof given for T^2 fibres is sufficient for T^k fibres with the exception of the last paragraph. The last paragraph is valid for $T^1 = S^1$ fibres and so the corresponding approximation results hold [Go]. The problem remains substantially open for $k > 2$.

ADDED IN PROOF. Recent work of F. S. Quinn provides a sequence of obstructions for approximating with topological block bundle projections. These obstructions all vanish for T^n. See F. S. Quinn, *Ends of maps and applications*, Bull. Amer. Math. Soc. 1 (1979), 270–272.

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