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THE INTERIOR OPERATOR LOGIC AND PRODUCT TOPOLOGIES

BY

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ABSTRACT. In this paper we present a model theory of the interior operator on product topologies with continuous functions. The main results are a completeness theorem, an axiomatization of topological groups, and a proof of an interpolation and definability theorem.

0. Introduction and basic results. This paper develops a model theory for $L(I^n)_{n\in\omega}$, the interior operator logic, which is analogous to the author's development for $L(Q^n)_{n\in\omega}$, the "open set" quantifier logic, in **[16]**. The significant difference is that $L(I^n)_{n\in\omega}$ satisfies an interpolation and definability theorem in marked contrast to $L(Q^n)_{n\in\omega}$. Another contrasting result is an omitting types theorem for $L(I^n)_{n\in\omega}$.

These results were obtained by the author shortly after the results contained in **[16]** on $L(Q^n)_{n\in\omega}$ were proved. More recently, J. A. Makowsky and M. Ziegler have obtained similar results for $L(I)$ using proof theory.

In §1 we prove a completeness theorem for $L(I^n)_{n\in\omega}$. This proof uses the ideas developed for $L(Q^n)_{n\in\omega}$. We also show that if an $L(I^n)_{n\in\omega}$ theory has a Hausdorff topological model then it has a 0-dimensional normal topological model. Another result presented in this section is an axiomatization for the $L(I)$ theory of topological groups.

We conclude this paper by presenting a proof of a Robinson joint-consistency theorem for $L(I^n)_{n\in\omega}$. This result implies both interpolation and definability. Thus $L(I^n)_{n\in\omega}$ has a more "smooth" model theory than $L(Q^n)_{n\in\omega}$. Another result is the omitting types theorem.

We will assume that the reader is familiar with the basic results of model theory.

DEFINITION 1. If we take a first order model $\mathfrak{A}$ and $q_i \subseteq \mathcal{P}(A)$ then we call $(\mathfrak{A}, q_1, q_2, q_3, \dots)$ a *weak model* for $L(I^n)_{n\in\omega}$. If each q_i is a topology on A then $(\mathfrak{A}, q_1, q_2, q_3, \dots)$ is called *topological*. A topological model $(\mathfrak{A}, q_1, q_2, \dots)$ is called *complete* if each q_k is the kth topological product of q_1 on A. We will abbreviate $(\mathfrak{A}, q_1, q_2, q_3, \dots)$ by $(\mathfrak{A}, \mathbf{q})$.

The formulas of $L(I^n)_{n\in\omega}$ are defined analogously to first order logic with the additional clause:

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If φ is a formula and $x_1, \ldots, x_n$ are distinct variables, then $I^n x_1, \ldots, x_n \varphi$ is also a formula ($x_1, \ldots, x_n$ occur free in $I^n x_1, \ldots, x_n \varphi$). Again we abbreviate $I^n x_1, \ldots, x_n \varphi$ by $I\mathbf{x}\varphi$.

Since the language $L(I^n)$ is not closed under substitution of free variables, we introduce the following notation to take the place of substitution.

DEFINITION. Given n-tuples of variables $\mathbf{x}$ and substitutable terms $\mathbf{k}$, $\varphi[\mathbf{k}]$ denotes the formula $\exists\mathbf{x}(\bigwedge_{i=1}^n x_i = k_i \wedge \varphi)$.

The notion of an n-tuple $a_1, \ldots, a_n \in A$ satisfying a formula $\varphi(v_1, \ldots, v_n)$ of $L(I^n)_{n\in\omega}$ in a weak model $(\mathfrak{A}, \mathbf{q})$ is defined in the usual manner by induction on the complexity of φ and is denoted by $(\mathfrak{A}, \mathbf{q}) \models \varphi[\mathbf{a}]$. The only difficult case is the $I\mathbf{x}\varphi$ case:

$$(\mathfrak{A}, \mathbf{q}) \models I\mathbf{x}\varphi[\mathbf{a}] \quad \text{iff} \quad \mathbf{a} \in \mathcal{O} \subseteq [\varphi(\mathbf{x})]^{(\mathfrak{A},\mathbf{q})} \quad \text{for some } \mathcal{O} \in q_n$$

($[\varphi(\mathbf{x})]^{(\mathfrak{A},\mathbf{q})}$ denotes the set $\{\mathbf{a} \in A^n: (\mathfrak{A}, \mathbf{q}) \models \varphi[\mathbf{a}]\}$).

It can be shown by induction that for $\mathbf{a} \in A^n$, a sentence $\varphi[\mathbf{a}]$ holds in $(\mathfrak{A}, \mathbf{a}, \mathbf{q})$ iff $(\mathfrak{A}, \mathbf{q}) \models \varphi[\mathbf{a}]$ as previously defined.

REMARK. Notice that by our definition of $\models$, $I\mathbf{x}\varphi$ is the interior of the set defined by φ in a topological model.

The axioms for $L(I^n)_{n\in\omega}$ are the standard first order ones where we replace $\varphi(\mathbf{c})$ by $\varphi[\mathbf{c}]$, e.g.

$$\bigwedge_{i=1}^{n} x_i = y_i \rightarrow (I\mathbf{x}\varphi[\mathbf{x}] \leftrightarrow I\mathbf{y}\varphi[\mathbf{y}])$$

and

$$Ax\colon \forall\mathbf{x}(\varphi \leftrightarrow \psi) \rightarrow (I\mathbf{x}\varphi \leftrightarrow I\mathbf{x}\psi).$$

The rules of inference are:

(i) Modus ponens, i.e. from φ, $\varphi \rightarrow \psi$ infer ψ,

(ii) Generalization, i.e. from φ infer $\forall x\varphi$.

We can now prove the following completeness theorem for topological models.

THEOREM 2. *An $L(I^n)_{n\in\omega}$ theory Σ has a topological model* $(\mathfrak{A}, \mathbf{q})$, *if and only if Σ is consistent with*

(A0) *axioms for $L(I^n)_{n\in\omega}$,*

(A1) $I\mathbf{x}(y = y) \leftrightarrow y = y$,

(A2) $I\mathbf{x}\varphi \rightarrow \varphi$,

(A3) $I\mathbf{x}\varphi \rightarrow I\mathbf{x}I\mathbf{x}\varphi$,

(A4) $(I\mathbf{x}\varphi \wedge I\mathbf{x}\psi) \leftrightarrow I\mathbf{x}(\varphi \wedge \psi)$.

PROOF (ONLY IF). Straightforward because the axioms hold in every topology.

(IF) Let Σ^* be a maximal consistent extension of Σ with witnesses using $\varphi[c]$ instead of $\varphi(c)$. As in the proof of the completeness theorem for first order logic take $\mathfrak{A}$ to be the model generated by Σ^*.

Define $q_k = \{\{\mathbf{a} | I\mathbf{x}\varphi[\mathbf{a}] \in \Sigma^*\} | \varphi \text{ is a formula of } L(I^n)_{n\in\omega}(A)\}$.

We claim $(\mathfrak{A}, \mathbf{q})$ models Σ^*. We prove this by induction on the I complexity of φ that, for every formula $\varphi(x)$ and n-tuple $\mathbf{a}$, $\varphi[\mathbf{a}] \in \Sigma^*$ iff $(\mathfrak{A}, \mathbf{q}) \models \varphi[\mathbf{a}]$. The only difficult clause is $I\mathbf{x}\varphi[\mathbf{a}]$.

If $Ix\varphi[\mathbf{b}] \in \Sigma^*$ then $a \in \{\mathbf{b} \in A^n | Ix\varphi[\mathbf{b}] \in \Sigma^*\} \in q_n$. Hence $(\mathfrak{A}, \mathbf{q}) \models Ix\varphi[\mathbf{a}]$ using (A2).

Assume $(\mathfrak{A}, \mathbf{q}) \models Ix\varphi[\mathbf{a}]$. That is, there is an $\mathcal{O} \in q_n$ such that $\mathbf{a} \in \mathcal{O} \subseteq [\varphi(\mathbf{x})]^{(\mathfrak{A},\mathbf{q})}$. By the definition of q_n there is an $Ix\psi$ such that

$$\mathbf{a} \in \mathcal{O} = [Ix\psi]^{(\mathfrak{A},\mathbf{q})} \subseteq [\varphi(\mathbf{x})^{(\mathfrak{A},\mathbf{q})}].$$

Using (A4), (A3) and the fact that $\forall\mathbf{x}((Ix\psi) \wedge \varphi \leftrightarrow (Ix\psi)) \in \Sigma^*$ we obtain that $\forall\mathbf{x}(Ix\psi \rightarrow Ix\varphi) \in \Sigma^*$. Thus we have $Ix\varphi[\mathbf{a}] \in \Sigma^*$.

Let q_k^* be the topology generated by q_k. By axioms (A1), (A2), (A4) we know that q_k is a basis for q_k^*. We claim

$$(\mathfrak{A}, \mathbf{q}) \equiv_{L(I^n)_{n\in\omega}(A)} (\mathfrak{A}, \mathbf{q}^*).$$

We will prove this by showing that, for each $\psi(x_1, \ldots, x_n)$,

$$[\psi(\mathbf{x})]^{(\mathfrak{A},\mathbf{q})} = [\psi(\mathbf{x})]^{(\mathfrak{A},\mathbf{q}^*)}.$$

Again this is proved by induction on the complexity of ψ. The only difficult case is $Ix\varphi$. Since q_n is a basis for q_n^* and from the definition of $\models$ we easily obtain

$$[Ix\varphi]^{(\mathfrak{A},\mathbf{q})} \subseteq [Ix\varphi]^{(\mathfrak{A},\mathbf{q}^*)}.$$

Let $\mathbf{a} \in [Ix\varphi]^{(\mathfrak{A},\mathbf{q}^*)}$. Then $a \in \mathcal{O} \subseteq [\varphi(\mathbf{x})]^{(\mathfrak{A},\mathbf{q}^*)}$ where $\mathcal{O} \in q_n^*$. Since q_n is a basis for q_n^*, there is an $\mathcal{O}^\# \in q_n$ such that $\mathbf{a} \in \mathcal{O}^\# \subseteq \mathcal{O}$.

Using the induction hypothesis we see that $\mathbf{a} \in \mathcal{O}^\# \subseteq [\varphi(\mathbf{x})]^{(\mathfrak{A},\mathbf{q})}$ whence $\mathbf{a} \in [Ix\varphi]^{(\mathfrak{A},\mathbf{q})}$.

Definition 3 (Tarski and Vaught). $(\mathfrak{B}, \mathbf{r})$ is said to be an *elementary extension* of $(\mathfrak{A}, \mathbf{q})$, in symbols $(\mathfrak{A}, \mathbf{q}) \prec (\mathfrak{B}, \mathbf{r})$, if and only if $A \subseteq B$ and for all formulas $\varphi(x_1, \ldots, x_n)$ of $L(I^n)_{n\in\omega}$ and all $a_1, \ldots, a_n \in A$ we have $(\mathfrak{A}, \mathbf{q}) \models \varphi[\mathbf{a}]$ iff $(\mathfrak{B}, \mathbf{r}) \models \varphi[\mathbf{a}]$. A sequence $(\mathfrak{A}_\alpha, \mathbf{q}^\alpha)$, $\alpha < \gamma$, of weak models is said to be an *elementary chain* if and only if we have $(\mathfrak{A}_\alpha, \mathbf{q}^\alpha) \prec (\mathfrak{A}_\beta, \mathbf{q}^\beta)$ for all $\alpha < \beta < \gamma$.

The union of an elementary chain $(\mathfrak{A}_\alpha, \mathbf{q}^\alpha)$, $\alpha < \gamma$, is the weak model $(\mathfrak{A}, \mathbf{q}) = \bigcup_{\alpha<\gamma}(\mathfrak{A}_\alpha, \mathbf{q}^\alpha)$ such that $\mathfrak{A} = \bigcup_{\alpha<\gamma}\mathfrak{A}_\alpha$ and $q_n = \{S \subset A^n |$ for some $\beta < \gamma$, $\beta < \alpha < \gamma$ implies $S \cap A_\alpha^n \in q^\alpha\}$.

These definitions enable us to state:

Theorem 4. *Let* $(\mathfrak{A}_\alpha, \mathbf{q}^\alpha)$, $\alpha < \gamma$, *be an elementary chain of weak models of* (A0)–(A4) *and let* $(\mathfrak{A}, \mathbf{q})$ *be the union. Then, for all* $\alpha < \gamma$, $(\mathfrak{A}_\alpha, \mathbf{q}^\alpha) \prec (\mathfrak{A}, \mathbf{q})$.

We conclude this introduction by stating an omitting types theorem whose proof is similar to the one found in Keisler [7].

Theorem 5. *Let* Γ *be a set of sentences of* $L(I^n)_{n\in\omega}$ *containing* (A0)–(A4) *and* $\Sigma_n(y_{n_1}, \ldots, y_{n_{p_n}})$, $n \in \omega$, *be sets of formulas of* $L(I^n)_{n\in\omega}$. *If* Γ *is consistent and omits each* Σ_n *then* Γ *has a weak model which omits each* Σ_n.

Remark. In this section we could have developed a theory of weak models using fewer axioms than (A0)–(A4) for our definition of $\models$. For brevity and directness we restricted the scope of the results to interior operators.

1. Main Theorem and applications. We will show that the $L(I^n)_{n\in\omega}$ theory of continuous functions (relations) on product spaces has the following axiomatization:

(A0), . . . , (A4),

(A5) $I\mathbf{x}\mathbf{y}\varphi \to I\mathbf{x}\varphi$,

(A6) $Ix_{i_1}, \ldots, x_{i_k}(\varphi(\sigma \circ \mathbf{x})) \leftrightarrow I\mathbf{x}(\varphi(\sigma \circ \mathbf{x}))$ where σ: $n + 1 \to n + 1$ and range $\sigma = \{i_1 < \cdots < i_k\}$,

(A7) $I\mathbf{x}\varphi \wedge I\mathbf{y}\psi \leftrightarrow I\mathbf{x}\mathbf{y}(\varphi \wedge \psi)$,

($\mathrm{A8}_\varphi$) $I\mathbf{y}\psi(\mathbf{y}) \wedge \varphi(\mathbf{x}, \mathbf{y} \upharpoonright k) \to I\mathbf{x}y_{k+1}, \ldots, y_m\exists\mathbf{z}(\varphi(\mathbf{x}, \mathbf{z}) \wedge \psi(\mathbf{z}, y_{k+1}, \ldots, y_m))$ where $\mathrm{lh}\ \mathbf{y} = m$ and $k \leqslant m$.

That is, the inverse image of a segment of an interior point concatenated with its tail is an interior point which is stronger than the usual definition of continuous but is equivalent to the usual definition of a continuous function (relation) on product spaces. Here, if $\mathbf{x} = \langle x_1, \ldots, x_n\rangle$ and σ: $n + 1 \to n + 1$, we have that $\sigma \circ x(i) = x_{\sigma(i)}$.

Suppose we have a topological model $(\mathfrak{A}, \mathbf{q})$ satisfying (A0)–($\mathrm{A8}_{\varphi_\alpha}$), $\alpha \in J$. Let $\{\mathcal{V}_\beta\}_{\beta\in D}$ be a collection of subsets of A. If we add the $\{\mathcal{V}_\beta\}_{\beta\in D}$ to q_1 and still expect to have a model satisfying (in the expanded language with a V_β for each $\mathcal{V}_\beta$) (A0)–($\mathrm{A8}_{\varphi_\alpha}$), $\alpha \in J$, what do we need to add to q_n, $n \in \omega$?

Let φ_α, $\alpha \in J$, be a collection of (n_α, m_α)-ary relations which satisfy ($\mathrm{A8}_{\varphi_\alpha}$), $\alpha \in J$. There is trouble later on if the φ_α are formulas of $L(I^n)$ since $L(I^n)$ is not closed under substitution. We will restrict the φ_α to be relations defined by formulas of L for convenience and clarity.

Let $\varphi_\alpha^{-1} = \{(\mathbf{b}, \mathbf{a}) | (\mathbf{a}, \mathbf{b}) \in \varphi_\alpha\}$ be the inverse relation of φ_α, $\alpha \in J$. We then define a collection of (definable) relations as follows (cf. **[16]**): $WT_0 = \{\varphi_\alpha^{-1}\}_{\alpha\in J} \cup$ {identity relation on each $n \in \omega$}, $WT_{n+1} = WT_n \cup \{\varphi(\sigma \circ \mathbf{x}, \mathbf{y})$ where σ maps n_φ into n_φ where $\varphi \in WT_n\} \cup \{\varphi(\mathbf{x}, \sigma \circ \mathbf{y})$ where σ maps m_φ into m_φ where $\varphi \in WT_n\} \cup \{\varphi(x_1, \ldots, x_{n_\varphi}, y_1, \ldots, y_{m_\varphi}) \wedge \psi(z_1, \ldots, z_{n_\psi}, t_1, \ldots, t_{m_\psi})$ where $\varphi, \psi \in WT_n\} \cup \{\varphi(x_1, \ldots, c, \ldots, x_{n_\varphi}, y_1, \ldots, y_{m_\varphi})$ where $\varphi \in WT_n$ and c is an individual constant symbol$\} \cup \{\varphi(x_1, \ldots, x_{n_\varphi}, y_1, \ldots, c, \ldots, y_{m_\varphi})$ where $\varphi \in WT_n$ and c is an individual constant symbol$\} \cup \{(\exists\mathbf{y}(\varphi(x_1, \ldots, x_{n_\varphi}, y_1, \ldots, y_{m_\varphi}) \wedge \psi(y_1, \ldots, y_{n_\psi}, z_1, \ldots, z_{m_\psi}))$, $k \leqslant m_\varphi$, $k \leqslant n_\psi$, i.e. the composition of the two relations, where $\varphi, \psi \in WT_n\}$.

Let $WT = \bigcup_{n\in\omega} WT_n$.

The intuitive meaning of WT is that it is the smallest collection of definable relations containing WT_0 and closed under composition, projection, products and mappings of the variables. Hence, since each φ_α satisfies ($\mathrm{A8}_{\varphi_\alpha}$), $\alpha \in J$, and $(\mathfrak{A}, \mathbf{q})$ models (A0)–($\mathrm{A8}_{\varphi_\alpha}$), $\alpha \in J$, we have that each $\varphi \in WT$ takes definable open sets to definable open sets.

Define q_n^*, $n \in \omega$, as follows: q_n^* is the topology generated by $\{\varphi(\prod_{j=1}^k B_j) |$ each $B_j \in q_{k_j}$ or $B_j \in \{\mathcal{U}_{\beta_j}\}_{\beta\in D}$ for $1 \leqslant j \leqslant k$, $\varphi \in WT$ and φ maps into $A^n\}$. $\varphi(C)$ means $\{\mathbf{b} | (\mathfrak{A}, \mathbf{q}) \models \varphi(\mathbf{c}, \mathbf{b})$ and $\mathbf{c} \in C\}$.

We now state the following important lemma whose proof is analogous to Lemma 2.2 in **[16]**.

LEMMA 6. *Let* $(\mathfrak{A}, \mathbf{q}^*)$ *be as above. Then* $(\mathfrak{A}, \mathbf{q}^*)$ *models* (A0)–($A8_{\varphi_\alpha}$), $\alpha \in J$, *in the expanded language containing a* V_β *for each* $\mathcal{V}_\beta$.

Now we will proceed to prove the main completeness theorem by presenting the following lemma which tells us that for each $\mathbf{c} \in \mathcal{O}$ (an open set in the I^n interpretation) we can add a $\prod_{i=1}^n o_i$ (an open n-box) to the I^n interpretation such that $\mathbf{c} \in \prod_{i=1}^n o_i \subseteq \mathcal{O}$ and still keep (A0)–($A8_{\varphi_\alpha}$), $\alpha \in J$.

LEMMA 7. *Let* Σ *be an* $L(I^n)_{n\in\omega}$ *theory consistent with* (A0)–($A8_{\varphi_\alpha}$), $\alpha \in J$. *If* $\mathbf{c} = \langle c_1, \ldots, c_n\rangle$ *is an n-tuple so that* $(I\mathbf{x}\varphi)[\mathbf{c}]$ *is consistent with* Σ *then* $V_i(c_i)$, $\forall x(IxV_i(x) \leftrightarrow V_i(x))$, $1 \leqslant i \leqslant n$, *and* $\forall \mathbf{x}(\bigwedge_{i=1}^n V(x_i) \to \varphi(\mathbf{x}))$ *is consistent with* Σ *and* (A0)–($A8_{\varphi_\alpha}$), $\alpha \in J$. *Here the* $V_i(x_i)$, $1 \leqslant i \leqslant n$, *are new one-place predicate symbols.*

PROOF. We need only prove this for countable Σ since then by using Theorem 2 we have it for all Σ and (A0)–($A8_{\varphi_\alpha}$), $\alpha \in J$. Let $(\mathfrak{A}, \mathbf{q})$ be a topological model of Σ. This is possible since Σ is consistent with (A0)–(A4) and Theorem 2.

We will assume that we have some countable enumeration of the "potential" basic open sets of $(\mathfrak{A}, \mathbf{q}^*)$, i.e., $\mathcal{O}_\beta^* = \bigcap \varphi_{\beta_j}(\prod B_k)$ where without loss of generality we can assume that $B_l = \mathcal{V}_l$ for $1 \leqslant l \leqslant n$ and $B_l = \mathcal{O}_{\delta_l} \in q_{\gamma_l}$ for $k > l > n$. Also we take an enumeration of $\sigma_k(x_1, \ldots, x_{m_k})$ of the formulas of $L(I^n)_{n\in\omega}(A)$.

We want to define $o_i \subseteq A$, $1 \leqslant i \leqslant n$, such that

$$\mathbf{c} \in \prod_{i=1}^n o_i \subseteq [I\mathbf{x}\varphi]^{(\mathfrak{A},\mathbf{q})}$$

and forming $(\mathfrak{A}, \mathbf{q}^*)$ from the $\{o_i\}_{1\leqslant i\leqslant n}$ we have that $(\mathfrak{A}, \mathbf{q}) \prec (\mathfrak{A}, \mathbf{q}^*)$. To do this we will construct the o's by induction.

Suppose we have picked $b_i^1, \ldots, b_i^{f(k)}$ for each o_i so that

$$\prod_{i=1}^n \{b_i^1, \ldots, b_i^{f(k)}\} \subseteq [I\mathbf{x}\varphi]^{(\mathfrak{A},\mathbf{q})}$$

and $\mathbf{c} = \mathbf{b}^1$. Now to pick the b_i^m, $1 \leqslant i \leqslant n$, $m \leqslant f(k+1)$, we want to insure that

$$\prod_{i=1}^n \{b_i^1, \ldots, b_i^k, b_i^{f(k+1)}\} \subseteq [I\mathbf{x}\varphi]^{(\mathfrak{A},\mathbf{q})}$$

and also to somehow guarantee that if $\sigma(\mathbf{x})$ is a formula of $L(I^n)_{n\in\omega}$, $\mathbf{a} \in A^m$, and $(\mathfrak{A}, \mathbf{q}) \models \neg I\mathbf{x}\sigma(\mathbf{x})[\mathbf{a}]$ then we do not get $(\mathfrak{A}, \mathbf{q}^*) \models I\mathbf{x}\sigma(\mathbf{x})[\mathbf{a}]$. This is equivalent to $[\sigma(\mathbf{x})]^{(\mathfrak{A},\mathbf{q}^*)} = [\sigma(\mathbf{x})]^{(\mathfrak{A},\mathbf{q})}$ for every $\sigma(\mathbf{x})$.

We claim that the $b_i^{k+1}, \ldots, b_i^{f(k+1)}$ can be picked such that

$$\prod_{i=1}^n \{b_i^1, \ldots, b_i^{f(k+1)}\} \subseteq [I\mathbf{x}\varphi]^{(\mathfrak{A},\mathbf{q}^*)} \qquad (*)$$

and (**) if $\mathcal{O}_\beta^* \in q_m^*$ and $\mathcal{O}_\beta^*$ is the $(k+1)$th basic open set $\mathcal{O}_\beta^* \subseteq [\sigma_{k+1}(\mathbf{x})]^{(\mathfrak{A},\mathbf{q}^*)}$ then there is an $\mathcal{O}_\beta \in q_m$ such that $\mathcal{O}_\beta^* \subseteq \mathcal{O}_\beta \subseteq [\sigma(\mathbf{x})]^{(\mathfrak{A},\mathbf{q})}$.

This is done as follows. Define $(\mathbf{x}/\mathbf{b})$ to be $\prod_{i=1}^n \{x_i, b_i\}$ and let

$$C = \bigcap_{\mathbf{b}\in\prod_{i=1}^n\{b_i^1, \ldots, b_i^k\}} \ \bigcap_{\mathbf{t}\in(\mathbf{x}/\mathbf{b})} \{I\mathbf{x}\varphi[\mathbf{t}]\}^{(\mathfrak{A},\mathbf{q})}$$

where $\{Ix\varphi[\mathbf{t}]\}^{(\mathfrak{A},\mathbf{q})} = \{(a_1, \ldots, a_n) | (\mathfrak{A}, \mathbf{q}) \models Ix\varphi[k_1, \ldots, k_n]$ where $k_i = a_i$ if $t_i = x_i$, $k_i = b_i$ otherwise$\}$. (Notice that in using axioms (A5)–(A7) that C is a definable open set.)

Taking $\mathcal{O}^*_{\beta_{k+1}} = \bigcap_{j=1}^{n_\beta} \varphi_{\beta_j}(\prod_{k=1}^{n_k} B_k)$ we define

$$C^{n_\beta} = \{\mathbf{c} \in A^{n \cdot n_\beta} | \langle c(1 \cdot k), \ldots, c(n \cdot k)\rangle \in C \text{ for each } k,\ 1 \leqslant k \leqslant n_\beta\},$$

C^{n_β} is open because C is and by axiom (A5).

Let $\sigma_i\colon n_\beta \to n_\beta$ be a permutation for each i, $1 \leqslant i \leqslant n$. Then define $\boldsymbol{\sigma} = \langle \sigma_1, \ldots, \sigma_n \rangle * \mathbf{c}$ where $\mathbf{c} \in C^{n_\beta}$ by $\boldsymbol{\sigma} * \mathbf{c}(i) = \mathbf{c}(\sigma_k^{-1}(i))$ if $i \equiv k \bmod n$. That is, we look at the coordinates of the vector $\mathbf{c}$ as a collection of n_β vectors of length n and permute each of its ith coordinates by σ_i. Define $\boldsymbol{\sigma} * C^{n_\beta}$ to be $\{\boldsymbol{\sigma} * \mathbf{c} | \mathbf{c} \in C^{n_\beta}\}$ and again we have that $\boldsymbol{\sigma} * C^{n_\beta}$ is open by the axioms. Hence $C^* = \bigcap_{\boldsymbol{\sigma}} \boldsymbol{\sigma} * C^{n_\beta}$ is open because the finite intersection of open sets is open.

We claim that if $\mathbf{b} \in C^*$ then

$$\prod_{i=1}^{n} \{b_i', \ldots, b_i^{k+n_\beta}\} \subseteq [Ix\varphi]^{(\mathfrak{A},\mathbf{q})}$$

where $b_i^{k+l} = \mathbf{b}(i \cdot n + l)$.

If $\mathbf{t} \in \prod_{i=1}^{n} \{b_i', \ldots, b_i^{k+n_\beta}\}$ then it is straightforward to see that there is a $\boldsymbol{\sigma}$ such that

$$\langle \boldsymbol{\sigma} * \mathbf{b}(n - i + 1), \ldots, \boldsymbol{\sigma} * \mathbf{b}(n - (i + k))\rangle = \mathbf{t}$$

for some $1 \leqslant i \leqslant n_\beta$. Then $\boldsymbol{\sigma} * \mathbf{b} \in C^{n_\beta}$ since $\mathbf{id} * C^{n_\beta} = C^{n_\beta}$. Thus $\mathbf{t} \in C$ which implies that $\mathbf{t} \in [Ix\varphi]^{(\mathfrak{A},\mathbf{q})}$.

Remembering that

$$\mathcal{O}^*_{\beta_{k+1}} = \bigcap_{j=1}^{n_\beta} \varphi_{\beta_j}\left(\prod_{i=1}^{n} \mathcal{V}_i \times \prod_{l=1}^{k} \beta_k\right),$$

define $C^\# = \{\mathbf{c} \in A^{n_\beta(n+k)} | \mathbf{c} \upharpoonright n \cdot n_\beta \in C^*$ and $\langle \mathbf{c}(n_\beta \cdot n + k \cdot i + 1), \ldots, \mathbf{c}(n_\beta \cdot n + k \cdot (i + 1))\rangle \in \prod_{l=1}^{k} B_l$ for each $1 \leqslant i \leqslant n_\beta\}$.

Take $\mathbf{c} \in C^\#$ and define $\boldsymbol{\varphi} * \mathbf{c}$ to be the vector defined by

$$\boldsymbol{\varphi} * \mathbf{c}(m \cdot i + l) = \varphi_{\beta_i}(\langle \mathbf{c}(n \cdot i + 1), \ldots, \mathbf{c}(n \cdot (i + 1)), \mathbf{c}(n \cdot n_\beta + n \cdot i + 1), \ldots, \mathbf{c}(n \cdot n_\beta + n(i + 1))\rangle)(l).$$

By axiom (A8) and the other axioms we know that $\boldsymbol{\varphi} * C^\#$ is open where $\boldsymbol{\varphi} * C^\# = \{\boldsymbol{\varphi} * \mathbf{c} | \mathbf{c} \in C^\#\}$.

Define $\bigcap \boldsymbol{\varphi} * C^\# = \{\mathbf{c} \in A^m |$there is a $\mathbf{b} \in C^\#$ such that $\mathbf{c}(i) = \boldsymbol{\varphi} * \mathbf{b}(k \cdot i)$ for each $1 \leqslant k \leqslant n_\beta\}$. Again by the axioms $\bigcap \boldsymbol{\varphi} * C^\#$ is an open subset of A^m. Now if

$$\bigcap \boldsymbol{\varphi} * C^\# - [\sigma_{k+1}(\mathbf{x})]^{(\mathfrak{A},\mathbf{q})} \neq \varnothing$$

then there is a $\mathbf{b}^{k+1}, \ldots, \mathbf{b}^{k+n_\beta} \in C^*$ such that

$$\bigcap_{j=1}^{n_\beta} \varphi_{\beta_j}(\mathbf{b}^{k+j} \times \prod \beta_k) - [\sigma_{k+1}(\mathbf{x})]^{(\mathfrak{A},\mathbf{q})} \neq \varnothing.$$

Otherwise let $\mathbf{b}^{k+i} = \mathbf{b}^k$ for $1 \leqslant i \leqslant n_\beta$. Set $f(k + 1) = k + n_\beta$ and we are done.

Note. The proof of the corresponding result in **[16]** contains an error at this point. The correct proof is obtained by a direct adaptation of this one to $L(Q^n)_{n\in\omega}$.

Now we will show that if $o_i = \{b_i^k | k \in \omega\}$ then we have the conclusion to the lemma. $\mathbf{c} \in \prod_{i=1}^n o_i \subseteq [I\mathbf{x}\varphi]^{(\mathfrak{A},\mathbf{q})}$ by (*). To show $(\mathfrak{A}, \mathbf{q}) \prec (\mathfrak{A}, \mathbf{q}^*)$ we use induction. The difficult case is the $I\mathbf{x}$ clause.

Since $q_i \subseteq q_i^*$ for all i we have that if

$$(\mathfrak{A}, \mathbf{q}) \models I\mathbf{x}\chi(\mathbf{x})[\mathbf{a}]$$

then

$$(\mathfrak{A}, \mathbf{q}^*) \models I\mathbf{x}\chi(\mathbf{x})[\mathbf{a}]$$

for arbitrary $\mathbf{a} \in A^m$.

Suppose that $(\mathfrak{A}, \mathbf{q}) \models \neg I\mathbf{x}\chi(\mathbf{x})[\mathbf{a}]$ and $(\mathfrak{A}, \mathbf{q}^*) \models I\mathbf{x}\chi(\mathbf{x})[\mathbf{a}]$. $\mathbf{a} \in \mathcal{O}_\beta^* \subseteq [\chi(\mathbf{x})]^{(\mathfrak{A},\mathbf{q}^*)}$ is a basic open set of q_m^*. Thus by (**)

$$\mathcal{O}_\beta^* \subseteq \mathcal{O}_\beta \subseteq [\chi(\mathbf{x})]^{(\mathfrak{A},\mathbf{q})}$$

for some $\mathcal{O}_\beta \in q_m$. Whence $(\mathfrak{A}, \mathbf{q}) \models I\mathbf{x}\chi(\mathbf{x})[\mathbf{a}]$ which is a contradiction. Thus the lemma is shown.

Now we are able to prove the main completeness theorem.

THEOREM 8. *Let Σ be an $L(I^n)_{n\in\omega}$ theory. Then Σ is consistent with* (A0)–(A8$_{\varphi_\alpha}$), *$\alpha \in J$, if and only if Σ has a complete topological model* $(\mathfrak{B}, \mathbf{r})$ *such that each φ_α, $\alpha \in J$, is continuous.*

PROOF (IF). Straightforward since (A0)–(A8$_{\varphi_\alpha}$), $\alpha \in J$, are true in every complete topological model where the φ_α, $\alpha \in J$, are continuous.

(ONLY IF) Assume Σ is consistent with (A0)–(A8$_{\varphi_\alpha}$), $\alpha \in J$. Then by Theorem 2 we have a topological model $(\mathfrak{A}, \mathbf{q})$ of Σ and (A0)–(A8$_{\varphi_\alpha}$), $\alpha \in J$, which is generated by the definable open set. By repeated applications of Lemma 7 we obtain a topological model $(\mathfrak{B}, \mathbf{r})$ of Σ, (A0)–(A8$_{\varphi_{\alpha_n}}$), $\alpha \in J$, and if $\mathbf{b} \in [\varphi(\mathbf{x})]^{(\mathfrak{B},\mathbf{r})} \in r_n$ then there is a $o_1, \ldots, o_n \in r_1$ such that $\mathbf{b} \in \prod_{i=1}^n o_i \subseteq [\varphi(\mathbf{x})]^{(\mathfrak{B},\mathbf{r})}$. Hence $(\mathfrak{B}, \mathbf{r})$ is complete.

If we omit (A8$_{\varphi_\alpha}$), $\alpha \in J$, then we obtain the following interesting corollary.

COROLLARY 9. *Let Σ be an $L(I^n)_{n\in\omega}$ theory. Then Σ is consistent with* (A0)–(A7) *if and only if Σ has a complete topological model.*

PROOF. This is a direct application of Theorem 8.

COROLLARY 10 (COMPACTNESS THEOREM). *Let Σ be an $L(I^n)_{n\in\omega}$ theory. Then Σ has a complete topological model where each φ_α, $\alpha \in J$, is continuous if and only if every finite subset of Σ has a complete topological model where φ_α, $\alpha \in J$, is continuous.*

PROOF. An easy application of the main completeness theorem.

COROLLARY 11. *The set of $L(I^n)_{n\in\omega}$ sentences valid in every complete topological model* (*with φ_α, $\alpha \in J$, continuous*) *is recursively enumerable in the signature of L and the set $\{\varphi_\alpha | \alpha \in J\}$.*

PROOF. Theorem 8 shows that a sentence is provable from (A0)–($A8_{\varphi_\alpha}$), $\alpha \in J$, if and only if it is valid, so we are done.

We can now state a Löwenheim-Skolem theorem for complete topological models with continuous functions whose proof is analogous to the author's for $L(Q^n)_{n\in\omega}$ in **[16]**.

THEOREM 12. (a) *Let* $(\mathfrak{A}, \mathbf{q})$ *be an infinite complete topological model where each* φ_α, $\alpha \in J$, *is continuous. Then for any* $\aleph \geqslant |L| + |A|$ *there is a complete topological model* $(\mathfrak{B}, \mathbf{r})$ *such that* $(\mathfrak{A}, \mathbf{q}) \prec (\mathfrak{B}, \mathbf{r})$, $|B| = \aleph$, *and each* φ_α *is continuous in* $(\mathfrak{B}, \mathbf{r})$.

(b) *Let* $(\mathfrak{A}, \mathbf{q})$ *be a complete topological model where each* φ_α, $\alpha \in J$, *is continuous. Then for any* $|L| \leqslant \aleph \leqslant |A|$ *there is a complete topological model* $(\mathfrak{B}, \mathbf{r}) \prec (\mathfrak{A}, \mathbf{q})$ *such that* $|B| = \aleph$, *and each* φ_α, $\alpha \in J$, *is continuous in* $(\mathfrak{B}, \mathbf{r})$.

We finish the applications by presenting two other theorems whose proofs are adaptations to $L(I^n)_{n\in\omega}$ (as in the main completeness theorem) of the proof for $L(Q^n)_{n\in\omega}$ in **[16]**.

THEOREM 13. *Let* Σ *be an* $L(I^n)_{n\in\omega}$ *theory and* κ *an infinite regular cardinal. Then* Σ *is consistent with* (A0)–($A8_{\varphi_\alpha}$), $\alpha \in J$, *and* $Ixy(x \neq y) \leftrightarrow x \neq y$ *if and only if* Σ *has a* 0-*dimensional normal complete topological model of cardinality* κ *where each* φ_α, $\alpha \in J$, *is continuous* (*complete in the model theoretic sense*).

The proof is a straightforward adaptation of the proof in **[16]** using the following lemma.

LEMMA. *Let* Σ *be an* $L(I^n)$ *theory consistent with* (A0)–($A8_{\varphi_\alpha}$), $\alpha \in J$. *Then if* Σ *is consistent with* $\forall xy(Ixy(x \neq y) \leftrightarrow x \neq y)$, $\forall x(\neg\psi(x) \Leftrightarrow Ix\neg\psi(x))$, $\forall x(\neg\varphi(x) \leftrightarrow Ix\neg\varphi(x))$, *and* $\neg\exists x(\psi(x) \wedge \varphi(x))$, *i.e.*, ψ *and* φ *define disjoint closed sets, then* $\forall x(\psi(x) \to U^{\psi,\varphi}(x))\forall x(\varphi(x) \to U^{\psi,\varphi}(x))$, $\forall x(IxU^{\psi,\varphi}(x) \leftrightarrow U^{\psi,\varphi}(x))$, *and* $\forall x(Ix\neg U^{\psi,\varphi}(x) \leftrightarrow U^{\psi,\varphi}(x))$ *are consistent with* Σ *and* (A0)–($A8_{\varphi_\alpha}$), $\alpha \in J$. *Here* $U^{\psi,\varphi}(x)$ *is a new one-place predicate symbol. The conclusion means that* $U^{\psi,\varphi}$ *and* $\neg U^{\psi,\varphi}$ *define complete open sets which separate* ψ *and* φ.

PROOF. We need only show the lemma for countable Σ; then using the compactness theorem we obtain it for all Σ. Let $(\mathfrak{A}, \mathbf{q})$ be a countable topological model of Σ and (A0)–($A8_{\varphi_\alpha}$), $\alpha \in J$, where the q_i are generated by the definable open sets.

As in the proof of Lemma 7 we want to obtain $(\mathfrak{A}, \mathbf{q}^*)$ from $\mathfrak{U}$, $A - \mathfrak{U}$ and $(\mathfrak{A}, \mathbf{q})$ such that $[\psi]^{(\mathfrak{A},\mathbf{q}^*)} \subseteq \mathfrak{U}$ and $[\varphi]^{(\mathfrak{A},\mathbf{q}^*)} \subseteq A - \mathfrak{U}$ and $(\mathfrak{A}, \mathbf{q}) \prec (\mathfrak{A}, \mathbf{q}^*)$.

We will define, as in Lemma 7, $\mathfrak{U}$ and $A - \mathfrak{U}$ by induction. To do this, suppose we have defined $r_1, \ldots, r_{f(k)}$ for $\mathfrak{U}$ and $s_1, \ldots, s_{f(k)}$ for $A - \mathfrak{U}$ up to stage k. Now we will define $r_{f(k)+1}, \ldots, r_{f(k+1)}$ for $\mathfrak{U}$ and $s_{f(k)+1}, \ldots, s_{f(k+1)}$ for $A - \mathfrak{U}$.

Again without loss of generality we have $\mathcal{O}^*_{\beta_{k+1}} = \bigcap_{j=1}^{n_\beta} \varphi_{\beta_j}(\prod B_k)$ where $B_1 = \mathfrak{U}$ and $B_2 = A - \mathfrak{U}$. We will define the r's and s's such that $r_i \neq s_j$ for $1 \leqslant i \neq j \leqslant f(k + 1)$ and such that if

$$\mathcal{O}^*_{\beta_{k+1}} \subseteq [\sigma_{k+1}(\mathbf{x})]^{(\mathfrak{A},\mathbf{q}^*)}$$

then there is a $\mathcal{O}_\beta \in q_m$ such that

$$\mathcal{O}^*_{\beta_{k+1}} \subseteq \mathcal{O}_\beta \subseteq [\sigma(\mathbf{x})]^{(\mathfrak{A},\mathfrak{q}^*)}.$$

Define $\Delta = \{\mathbf{c} \in A^{2n_\beta} | c(i) = c(j)$ for some $1 \leqslant i \leqslant n_\beta$ and $n_\beta < j \leqslant 2n_\beta\}$. Δ is closed in A^{2n_β} by the axiom $\forall xy(Ixy(x \neq y) \leftrightarrow (x \neq y))$. Hence $A^{2n_\beta} - \Delta \cup [\varphi \vee \psi]^{(\mathfrak{A},\mathfrak{q})}$ is an open subset of A^{2n_β}. As in Lemma 7 take

$$C^* = A^{2n_\beta} - \Delta \cup [\varphi \cup \psi]^{(\mathfrak{A},\mathfrak{q})}$$

and form $C^\#$, $\varphi * C^\#$ and $\bigcap \varphi * C^\#$ which are open subsets of A^{2n_β} and A, respectively. (If C^* is finite the proof is trivial, so we will assume it is infinite.)

If

$$\bigcap \varphi * C^\# - [\sigma_{k+1}(x)]^{(\mathfrak{A},\mathfrak{q})} \neq \varnothing$$

then pick $r_{f(k)+1}, \ldots, r_{f(k)+n_\beta}$ and $s_{f(k)+1}, \ldots, s_{f(k)+n_\beta}$ so that

$$\bigcap_{i=1}^{n_\beta} \varphi_{\beta_i}(\langle r_{f(k)+i}, s_{f(k)+i}\rangle x \Pi B_k) - [\sigma_{k+1}(\mathbf{x})]^{(\mathfrak{A},\mathfrak{q})} \neq \varnothing.$$

Otherwise let $r_{f(k)+i} = r_{f(k)}$ and $s_{f(k)+i} = s_{f(k)}$ for each $1 \leqslant i \leqslant n_\beta$. Set $f(k+1) = f(k) + n_\beta$ and we are done.

Let $\mathcal{U} = [\varphi]^{(\mathfrak{A},\mathfrak{q})} \cup \{r_i\}_{i\in\omega}$, and using the fact that $\mathcal{U} \cap \{s_i\}_{i\in\omega} = \varnothing$ we can easily prove as in Lemma 3.1.5 in **[16]** that $(\mathfrak{A}, \mathfrak{q}) \prec (\mathfrak{A}, \mathfrak{q}^*)$.

Note. Again the proof of Lemma 3.1.5 in **[16]** should be corrected along the lines of this proof since it is incorrect as published. The verification of $(\mathfrak{A}, \mathfrak{q}) \prec (\mathfrak{A}, \mathfrak{q}^*)$, however, is correct.

Corollary 14. *Let Σ be a countable $L(I^n)_{n\in\omega}$ theory. Then Σ is consistent with* (A0)–(A8$_{\varphi_\alpha}$), *$\alpha \in I$, and $Ixy(x \neq y) \leftrightarrow x \neq y$ if and only if Σ has a second countable 0-dimensional metrizable complete topological model where each φ_α is continuous.*

Proof. Use the fact that a second countable, regular and Hausdorff space is metrizable.

Let $L(I)$ be the sublanguage of $L(I^n)_{n\in\omega}$ in which the interior operator Ix is only applied to the single variable x.

We now study the interrelation of $L(I)$ theories and $L(I^n)_{n\in\omega}$ theories. The reason for this is that in $L(I^n)_{n\in\omega}$ we have a method of expressing the fact that a function is continuous in a product topology. It is thus natural to ask what conditions on functions (or relations) in an $L(I)$ theory Σ are necessary to insure that they can be interpreted as continuous functions in some $L(I^n)_{n\in\omega}$ theory extending Σ.

The following definition and theorem formalize this.

Definition 15. $\varphi_\alpha(x_1, \ldots, x_n, y_1, \ldots, y_m)$, $\alpha \in J$, a collection of (n, m)-ary relations is called *$L(I)$-continuous* (in Σ) if and only if

$$\bigwedge_{i=1}^{m} Iy_i\psi_i(y_i) \wedge \theta(\mathbf{t}, \mathbf{y}) \rightarrow Iz\exists\mathbf{y}\left(\bigwedge_{i=1}^{m} \psi_i(y_i) \wedge \theta(\mathbf{t}, \mathbf{y})\right)$$

is consistent with Σ, where the $\varphi_i(y_i)$ are arbitrary formulas of $L(I)$, $\mathbf{t} \in ((\sigma \circ x)/z)$ and $\sigma\colon n+1 \to n+1$ (where θ is an arbitrary composition of the φ_α, i.e., $\theta \in WT$, and $((\sigma \circ \mathbf{x})/z)$ is the collection of k-tuples which are permuted by σ and then any number of them are replaced by z).

THEOREM 16. *Let T be an $L(I)$ theory and let $\varphi_\alpha(x_1, \ldots, x_{n_\alpha}, y_1, \ldots, y_{m_\alpha})$, $\alpha \in J$, be (n_α, m_α)-ary $L(I)$-continuous relations. Then there is an $L(I^n)_{n\in\omega}$ theory T^*, such that $T \subseteq T^*$ and* (A0)–(A8_{φ_α}), *$\alpha \in J$, are in T^*. (This is to say that we can find a complete topological model $(\mathfrak{A}, \mathfrak{q})$ of T where each φ_α is continuous in the product topology.)*

Consider a group $(G,\cdot)$. Now take a topology τ on G. We call $(G,\cdot,\tau)$ a *topological group* if $^{-1}$ and $\cdot$ are continuous maps into G. Other definitions of topological-algebraic structures, e.g. a topological vector space, often appear in mathematics. Using Theorem 16 we are now able to give an $L(I)$ axiomatization of their $L(I)$ theories. For more details on topological groups, etc., see **[5]**.

We formalize these comments in the following corollary.

COROLLARY 17. *Let T be an $L(I)$ theory. Then T has a topological group model if and only if T is consistent with the basic $L(I)$ axioms, group axioms, and $Iy\psi(y)[t] \to Ix\psi(t)$ where*

$$\mathbf{t} \in \left(\frac{y_{\sigma(1)}^{\varepsilon(1)} \cdot y_{\sigma(2)}^{\varepsilon(2)} \cdot \;\cdots\; \cdot y_{\sigma(k)}^{\varepsilon(k)}}{x}\right),$$

$\sigma\colon k+1 \to k+1$ and $\varepsilon\colon k+1 \to \{1, -1\}$.

PROOF. These axioms for topological groups are just the definition of $L(I)$-continuity for x^{-1} and

COROLLARY 18. *Let T be an $L(I)$ theory. Then T has a topological abelian group model if and only if T is consistent with the basic $L(I)$ axioms, abelian group axioms, $Iy\psi(y)[x^{-1}] \to Ix\psi(x^{-1})$, and $Iy\psi(y)[x^n \cdot y] \to Ix\psi(x^n \cdot y)$.*

2. Interpolation, definability and omitting types. In this section we will prove a Robinson-type joint-consistency theorem and an omitting types theorem for $L(I^n)_{n\in\omega}$.

THEOREM 19. *Let T_1, T_2 be $L_1(I^n)_{n\in\omega}$, $L_2(I^n)_{n\in\omega}$, theories, respectively, consistent with* (A0)–(A8_φ). *Then if $T_1 \cap T_2$ is a complete $L_1 \cap L_2(I^n)_{n\in\omega}$ theory consistent with* (A0)–(A8_φ) *then $T_1 \cup T_2$ is $L_1 \cup L_2(I^n)_{n\in\omega}$ consistent with* (A0)–(A8_φ).

PROOF. Denote $L_1 \cap L_2$ by L and $T_1 \cap T_2$ by T. Let $(\mathfrak{A}, \mathfrak{q})$ be a topological model of T_1. Assume $\sigma(\mathbf{x})$, $\psi(\mathbf{x})$ are formulas of $L_1(I^n)_{n\in\omega}(A)$ such that σ defines an open set and ψ does not. Let $\mathbf{b} \in [\psi(\mathbf{x})]^{(\mathfrak{A},\mathfrak{q})} - [I\mathbf{x}\psi]^{(\mathfrak{A},\mathfrak{q})}$ and $\varphi(\mathbf{b}) \in \mathrm{Th}_{L_1}((\mathfrak{A}, \mathfrak{q}))$. Take $\chi(\mathbf{x})$ to be a formula of $L_2(I^n)_{n\in\omega}$ such that $\forall\mathbf{x}(I\mathbf{x}\chi(\mathbf{x}) \leftrightarrow \chi(\mathbf{x})) \in T_2$, i.e., χ is open, $\chi[\mathbf{b}] \in T_2 \cup T_L((\mathfrak{A}, \mathfrak{q}))$, and take $a_1, \ldots, a_n$ to be new constant symbols. Notice that $T_2' = T_2 \cup T_L((\mathfrak{A}, \mathfrak{q}))$ is consistent since T is complete and consistent.

Form

$$T_1^{\#} = \mathrm{Th}_{L_1}((\mathfrak{A}, \mathbf{q})) \cup \{\neg\psi[\mathbf{a}] \wedge \sigma[\mathbf{a}]\},$$
$$T_2^{\#} = T_2' \cup \{\chi[\mathbf{a}]\}.$$

Let $\mathrm{Cn}(T_i^{\#})$ be the set of consequences of $T_i^{\#}$ in $L(I^n)_{n\in\omega}(A \cup \{a_1, \ldots, a_n\})$.

We claim that $\mathrm{Cn}(T_1^{\#}) \cup \mathrm{Cn}(T_2^{\#})$ is consistent. To prove this, suppose not; then for some θ we have that $\theta \in \mathrm{Cn}(T_1^{\#})$ and $\neg\theta \in \mathrm{Cn}(T_2^{\#})$.

That is to say,

$$\mathrm{Th}((\mathfrak{A}, \mathbf{q})) \vdash (\neg\psi[\mathbf{a}] \wedge \sigma[\mathbf{a}]) \to \theta$$

and

$$T_2' \vdash \chi[\mathbf{a}] \to \neg\theta.$$

Replacing $a_1, \ldots, a_n$ by $z_1, \ldots, z_n$ and generalizing we obtain

$$\mathrm{Th}_L((\mathfrak{A}, \mathbf{q})) \vdash \forall\mathbf{z}((\neg\psi[\mathbf{z}] \wedge \sigma[\mathbf{z}]) \to \theta[\mathbf{z}]),$$

or equivalently

$$\mathrm{Th}_L((\mathfrak{A}, \mathbf{q})) \vdash \forall\mathbf{z}(\neg\theta[\mathbf{z}] \to (\psi[\mathbf{z}] \vee \neg\sigma[\mathbf{z}])),$$

and

$$T_2' \vdash \forall\mathbf{z}(\chi[\mathbf{z}] \to \neg\theta[\mathbf{z}]).$$

Using the monotonicity of the interior operator we show

$$\mathrm{Th}_L((\mathfrak{A}, \mathbf{q})) \vdash \forall\mathbf{z}(I\mathbf{z}\neg\theta[\mathbf{z}] \to I\mathbf{z}(\psi \vee \neg\sigma)[\mathbf{z}])$$

and

$$T_2' \vdash \forall\mathbf{z}(I\mathbf{z}\chi[\mathbf{z}] \to I\mathbf{z}\neg\theta[\mathbf{z}]).$$

But $I\mathbf{z}\chi[\mathbf{z}] \to \chi[\mathbf{z}]$, so $T_2 \vdash \chi[\mathbf{z}] \to I\mathbf{z}\neg\theta[\mathbf{z}]$, and so

$$T_2' \vdash \chi[\mathbf{b}] \to (I\mathbf{z}\neg\theta[\mathbf{z}])[\mathbf{b}].$$

Using the consistency and completeness of $T_2' \cap L(I^n)_{n\in\omega}(A) = \mathrm{Th}_L((\mathfrak{A}, \mathbf{q}))$ we have that

$$\mathrm{Th}_L((\mathfrak{A}, \mathbf{q})) \vdash (I\mathbf{z}\theta[\mathbf{z}])[\mathbf{b}] \to (I\mathbf{z})(\psi \vee \neg\sigma)[\mathbf{z}][\mathbf{b}]$$

so

$$\mathrm{Th}_L((\mathfrak{A}, \mathbf{q})) \vdash (I\mathbf{z}(\psi \vee \neg\sigma)[\mathbf{z}])[\mathbf{b}].$$

Thus, using (A4), we get $\mathrm{Th}_L((\mathfrak{A}, \mathbf{q})) \vdash (I\mathbf{z}(\sigma \wedge (\psi \vee \neg\theta))[\mathbf{z}])[\mathbf{b}]$. However, $\vdash \sigma \wedge (\psi \vee \neg\varphi) \leftrightarrow \psi$, hence $\mathrm{Th}_L((\mathfrak{A}, \mathbf{q})) \vdash (I\mathbf{z}\psi[\mathbf{z}])[\mathbf{b}]$, a contradiction since $[I\mathbf{z}\psi[\mathbf{z}]]^{(\mathfrak{A},\mathbf{q})} = [I\mathbf{x}\psi]^{(\mathfrak{A},\mathbf{q})}$.

Thus $\mathrm{Cn}(T_1^{\#}) \cup \mathrm{Cn}(T_2^{\#})$ is consistent. From this we easily conclude that $\Delta = \mathrm{Cn}(T_1^{\#}) \cup \mathrm{Cn}(T_2^{\#}) \cup T_2$ is consistent.

Let $(\mathfrak{B}, \mathbf{r})$ be a topological model of Δ generated by the definable open sets. Then $(\mathfrak{A}, \mathbf{q}) \prec_{L(I^n)_{n\in\omega}} (\mathfrak{B}, \mathbf{r})$ and $(\mathfrak{B}, \mathbf{r}) \vDash T_2$.

Notice for the following induction that $T_1^{\#} \cup \mathrm{Th}_L((\mathfrak{B}, \mathbf{r}))$, $\mathrm{Th}_{L_2}((\mathfrak{B}, \mathbf{r}))$ are consistent and $\mathrm{Th}_L((\mathfrak{B}, \mathbf{r}))$ is complete.

Interchanging T_1 and T_2 in the construction and iterating ω times through all the triples of L_1 definable open and nonopen sets and the definable L_2 open sets and vice versa we produce an elementary chain $(\mathfrak{A}_i, \mathbf{q}^i)$, $i \in \omega$, of models of T_1, $(\mathfrak{B}_i, \mathbf{r}^i)$, $i \in \omega$, of models of T_2 whose topologies are generated by definable open sets such that

$$\ldots (\mathfrak{A}_i \restriction L, \mathbf{q}^i) \prec (\mathfrak{B}_i \restriction L, \mathbf{r}^i) \prec (\mathfrak{A}_{i+1} \restriction L, \mathbf{q}^{i+1}) \prec (\mathfrak{B}_{i+1} \restriction L, \mathbf{r}^{i+1}), \ldots$$

forms an elementary chain. We also have that if $\psi(\mathbf{x})$ is a formula of $L_1(I^n)_{n\in\omega}(A_k)$ such that $[\psi(\mathbf{x})]^{(\mathfrak{A}_k, \mathbf{q}^k)}$ is not open in q_n^k, $\mathcal{O} \in q_n^k$ and $\mathcal{O}^\# \in r_n^k$ then either

$$\mathcal{O} \cap \mathcal{O}^\# \subseteq [I\mathbf{x}\psi]^{(\mathfrak{A}_k, \mathbf{q}^k)} \quad \text{or} \quad \mathcal{O} \cap \mathcal{O}^\# - [\psi(\mathbf{x})]^{(\mathfrak{A}_k, \mathbf{q}^k)} \neq \varnothing.$$

Similarly for $L_2(I^n)_{n\in\omega}$ in place of $L_1(I^n)_{n\in\omega}$.

Let $(\mathfrak{A}^\omega, \mathbf{q}^\omega)$ be the topological model generated by $\bigcup_{i\in\omega}(\mathfrak{A}_i, q^i)$ and $(\mathfrak{B}^\omega, \mathbf{r}^\omega)$ be the topological model generated by $\bigcup_{i\in\omega}(\mathfrak{B}_i, \mathbf{r}^i)$.

Because the q_k^i and r_k^i are generated by the definable open sets we see that $(\mathfrak{A}^\omega, \mathbf{q}^\omega) \models T_1$, $(\mathfrak{B}^\omega, \mathbf{r}^\omega) \models T_2$. We also know that

$$\mathfrak{A}^\omega \restriction L = \mathfrak{B}^\omega \restriction L. \tag{$*$}$$

Define $(\mathfrak{C}, \mathbf{p}) = (\mathfrak{A}^\omega \cup \mathfrak{B}^\omega, (r_1^\omega \cup q_1^\omega)^*, (r_2^\omega \cup q_2^\omega)^*, \ldots)$ where $(r_i^\omega \cup q_i^\omega)^*$ is the topology generated by the definable sets of $r_i^\omega \cup q_i^\omega$ and $\mathfrak{A}^\omega \cup \mathfrak{B}^\omega$ is the $L_1 \cup L_2$ model formed from $\mathfrak{A}^\omega$ and $\mathfrak{B}^\omega$ using $(*)$.

We claim that

$$(\mathfrak{A}^\omega, \mathbf{q}^\omega) \prec (\mathfrak{C}, \mathbf{p})$$

and

$$(\mathfrak{B}^\omega, \mathbf{r}^\omega) \prec (\mathfrak{C}, \mathbf{p}).$$

This implies that $(\mathfrak{C}, \mathbf{p}) \models T_1 \cup T_2$.

We will show $(\mathfrak{A}^\omega, \mathbf{q}^\omega) \prec (\mathfrak{C}, \mathbf{p})$ and the other equivalence follows by analogy. This assertion follows easily from the claim that

$$[\psi(\mathbf{x})]^{(\mathfrak{A}^\omega, \mathbf{q}^\omega)} = [\psi(\mathbf{x})]^{(\mathfrak{C}, \mathbf{p})}$$

for each $\psi(\mathbf{x})$ a formula of $L_1(I^n)_{n\in\omega}(A^\omega)$ which we prove by induction on the complexity of $\psi(\mathbf{x})$.

The only difficult case is the $I\mathbf{x}$ case. Suppose $\psi(\mathbf{x}) = I\mathbf{x}\varphi$. Since $q_i^\omega \subseteq p_i$ we obtain that $[\psi(\mathbf{x})]^{(\mathfrak{A}^\omega, \mathbf{q}^\omega)} \subseteq [\psi(\mathbf{x})]^{(\mathfrak{C}, \mathbf{p})}$. Take $\mathbf{a} \in [\psi(\mathbf{x})]^{(\mathfrak{C}, \mathbf{p})}$. By definition there are $\mathcal{O} \in q_n^*$, $\mathcal{O}^\# \in r_n^*$ such that $\mathbf{a} \in \mathcal{O} \cap \mathcal{O}^\# \subseteq [\varphi(\mathbf{x})]^{(\mathfrak{C}, \mathbf{p})}$.

By the construction of q_n^ω, r_n^ω we know that there are a χ of $L_1(I^n)_{n\in\omega}(A^\omega)$ and a δ of $L_2(I^n)_{n\in\omega}(A^\omega)$ such that

$$\mathbf{a} \in [\chi(\mathbf{x})]^{(\mathfrak{A}^\omega, \mathbf{q}^\omega)} \cap [\delta(\mathbf{x})]^{(\mathfrak{B}, \mathbf{r})} \subseteq \mathcal{O} \cap \mathcal{O}^\#.$$

Hence, we know that by our construction

$$\mathbf{a} \in [\chi(\mathbf{x})]^{(\mathfrak{A}^\omega, \mathbf{q}^\omega)} \cap [\varphi(\mathbf{x})]^{(\mathfrak{B}^\omega, \mathbf{r}^\omega)} \subseteq [I\mathbf{x}\varphi(\mathbf{x})]^{(\mathfrak{A}^\omega, \mathbf{q}^\omega)}$$

since otherwise by the construction of the elementary chain, $[\chi(\mathbf{x})]^{(\mathfrak{A}^\omega, \mathbf{q}^\omega)} \cap [\delta(\mathbf{x})]^{(\mathfrak{B}^\omega, \mathbf{r}^\omega)}$ would not be a subset of $[\delta(\mathbf{x})]^{(\mathfrak{A}^\omega, \mathbf{q}^\omega)} = [\varphi(\mathbf{x})]^{(\mathfrak{C}, \mathbf{p})}$. Thus we are done because $(\mathfrak{C}, \mathbf{p}) \models T_1 \cup T_2$ and (A0)–(A8_φ).

COROLLARY 20. *$L(I^n)_{n\in\omega}$ with* (A0)–(A8$_\varphi$) *has an interpolation theorem.*

PROOF. Straightforward since the Robinson joint-consistency theorem implies interpolation for compact logics.

COROLLARY 21. *$L(I^n)_{n\in\omega}$ with* (A0)–(A8$_\varphi$) *has a Beth definability theorem.*

PROOF. Again the Robinson joint-consistency theorem implies the definability theorem.

THEOREM 22. *$L(I)$ with* (A0)–(A4) *has a Robinson joint-consistency theorem.*

PROOF. The proof is the same as the $L(I^n)_{n\in\omega}$ case. In place of I^n just use I^1.

COROLLARY 23. *$L(I)$ with* (A0)–(A4) *has an interpolation and a Beth definability theorem.*

PROOF. Use Theorem 22.

REMARK. In **[15]** we show an interpolation theorem for a system weaker than $L(I)$. Also our methods apply to give an interpolation theorem for "ideal models" (see **[0]**) which was done independently by S. Shelah.

We now will state and prove an omitting types theorem for $L(I^n)_{n\in\omega}$ with (A0)–(A8$_\varphi$).

DEFINITION 24. A set of sentences Γ *topologically omits* $\Sigma(x_1, \ldots, x_n)$ if and only if $\Gamma \cup \{(\text{A0})\text{–}(\text{A8}_\varphi)\}$ has a model which omits Σ.

THEOREM 25. *Let Γ be a countable set of sentences of $L(I^n)_{n\in\omega}$ and $\Sigma_n(y_{n_1}, \ldots, y_{n_{p_n}})$, $n \in \omega$, be sets of formulas of $L(I^n)_{n\in\omega}$. If Γ is consistent with* (A0)–(A8$_\varphi$) *and Γ topologically omits each Σ_n, the Γ has a topological model where each φ is continuous and omits each Σ_n.*

PROOF. By Theorem 5 we obtain a countable weak model $(\mathfrak{A}, \mathbf{q})$ of Γ and (A0)–(A8$_\varphi$) which omits each Σ_n. If we take $q_k^\#$ to be the topology generated by the definable subsets of q_k, it is straightforward to see as in Theorem 2 that $(\mathfrak{A}, \mathbf{q}) \prec (\mathfrak{A}, \mathbf{q}^\#)$.

We actually proved in the proof of the main completeness theorem that for a countable topological model $(\mathfrak{A}, \mathbf{q}^\#)$ there is a complete model $(\mathfrak{A}, \mathbf{r})$ such that $(\mathfrak{A}, \mathbf{r}) \prec (\mathfrak{A}, \mathbf{q}^\#)$. Hence $(\mathfrak{A}, \mathbf{r})$ models Γ, each φ is continuous, and it omits each Σ_n, $n \in \omega$.

REMARK. We would like to point out that we showed in **[14]** that $L(I^n)_{n\in\omega}$ cannot have an isomorphic ultrapowers theorem so our Theorem 19 is sharp.

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