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BASIC SEQUENCES IN NON-SCHWARTZ-FRÉCHET SPACES

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ABSTRACT. Obliquely normalized basic sequences are defined and used to characterize non-Schwartz-Fréchet spaces. It follows that each non-Schwartz-Fréchet space E has a non-Schwartz subspace with a basis and a quotient which is not Montel (which has a normalized basis if E is separable). Stronger results are given when more is known about E, for example, if E is a subspace of a Fréchet l_p-Köthe sequence space, then E has the Banach space l_p as a quotient and E has a subspace isomorphic to a non-Schwartz l_p-Köthe sequence space. Examples of Fréchet-Montel spaces which are not subspaces of any Fréchet space with an unconditional basis are given. The question of the existence of conditional basic sequences in non-Schwartz-Fréchet spaces is reduced to questions about Banach spaces with symmetric bases. Nonstandard analysis is used in some of the proofs and a new nonstandard characterization of Schwartz spaces is given.

This scrutiny of non-Schwartz-Fréchet spaces shows that such spaces (even when they are Montel) are both less like Schwartz spaces and more like Banach spaces than previously thought. For example, each Montel non-Schwartz-Fréchet space has a quotient space with a normalized basis (Theorem 5.1). Also, there are Fréchet-Montel spaces which are not subspaces of any Fréchet space (Montel or otherwise) with an unconditional basis (Theorem 4.1). In contrast, it is known that each Schwartz-Fréchet space is a subspace of a Schwartz-Fréchet space with an unconditional basis (see [2]).

Our basic result is Theorem 3.2 which characterizes non-Schwartz-Fréchet spaces by the existence of an obliquely normalized basic sequence (Definition 3.1). Thus each non-Schwartz-Fréchet space has a highly structured basic sequence spanning a non-Schwartz space. The proof of Theorem 3.2 uses the Mazur product construction with a twist. This twist uses nonstandard analysis. §2, which uses nonstandard analysis to obtain new proofs of some known theorems on the existence of basic sequences, is included for two reasons. First, it illuminates the role of nonstandard analysis in the proofs of §§3 and 4 and secondly, for completeness and later reference. Also, a new nonstandard characterization of Schwartz spaces is given (Corollary 3.6).

If more is known about the non-Schwartz-Fréchet space E stronger results are obtained. For example, if E is a subspace of a Fréchet l_p-Köthe sequence space, then E has l_p as a quotient (Corollary 5.5) and E has a non-Schwartz l_p-Köthe

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sequence subspace (Corollary 3.5). If E is an X-Köthe sequence space, then E has a complemented subspace with an obliquely normalized basis (Theorem 3.2) and if the basis for X is subsymmetric, then E has X as a quotient (Theorem 5.4). For a Fréchet X-Köthe sequence space E, the properties of being or not being Montel or Schwartz are related to the existence of the types of subspaces in E, spanned by subsequences of the basis (Proposition 3.7).

Finally, in §6, we consider Pełczyński's question [**11**] about the existence of conditional basic sequences. Proposition 6.1 shows that conditional basic sequences exist in each non-Schwartz subspace of a Fréchet l_p-Köthe sequence space. Also, Pełczyński's question, restricted to non-Schwartz-Fréchet spaces, is reduced to questions about Banach spaces with symmetric bases (Proposition 6.2).

1. Preliminaries. All spaces are assumed to be locally convex topological vector spaces over the real or complex field. A map is a continuous linear function. For a space E, the continuous dual (respectively, the algebraic dual) is denoted by E' (respectively, $E^\#$). If F is a subspace of $E^\#$, then $\sigma(E, F)$ will be the weak topology on E generated by F. If $\|\cdot\|_\alpha$ is a seminorm on the space E, let $E(\alpha)$ represent the seminormed space $(E, \|\cdot\|_\alpha)$. If $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$ are seminorms on E with $\|\cdot\|_\alpha \leq \|\cdot\|_\beta$, the induced map: $E(\beta) \to E(\alpha)$ is the map which is algebraically the identity on E. We find this definition of induced map more convenient than the usual one which makes $E(\alpha) = E/\ker \|\cdot\|_\alpha$.

A space E is *Schwartz*, if for each continuous seminorm $\|\cdot\|_\alpha$, there is another continuous seminorm $\|\cdot\|_\beta$, with $\|\cdot\|_\alpha \leq \|\cdot\|_\beta$ and so that the induced map $E(\beta) \to E(\alpha)$ is precompact. A space E is *Montel*, if it is barrelled and if each bounded set in E is relatively compact. A Fréchet space E (i.e. a complete metrizable space) is Montel if and only if each bounded set in E is precompact. We note that each Schwartz-Fréchet space is Montel but not conversely (see Horváth [**6**, pp. 277–279]).

We will write $\{x_n\}$ for $\{x_n\}_{n=1}^\infty$ and $[x_n]$ for the closed linear span of $\{x_n\}$. For $1 \leq p < \infty$, l_p will denote the usual Banach space of p-summable sequences. Also, c_0 will denote the usual Banach space of null sequences.

A sequence $\{x_n\}$ contained in the Fréchet space X is said to be a *basis* for X if for each $x \in X$ there is a unique scalar sequence $\{\alpha_n\}$ with $\sum \alpha_n x_n = x$. A sequence $\{x_n\}$ is a *basic sequence* if it is a basis for $[x_n]$. If $\|\cdot\|$ is a seminorm on $[x_n]$, then $\{x_n\}$ is said to be *K-basic with respect to* $\|\cdot\|$, if for all scalars $\{\alpha_n\}$ and integers p and q, $\|\sum_1^p \alpha_n x_n\| \leq K\|\sum_1^{p+q} \alpha_n x_n\|$. If $\{x_n\}$ is a basis for the Fréchet space X, then the topology on X can be defined by a sequence of seminorms $\{\|\cdot\|_k\}$, so that $\{x_n\}$ is 1-basic with respect to each $\|\cdot\|_k$. (This can be proved as in the similar statement for Banach spaces, i.e. see Singer [**16**, Proposition 3.2, p. 19].) A basic sequence $\{x_n\}$ is *normalized* if it is bounded and there is a neighborhood of the origin U, with $x_n \notin U$, for each n. Note that this is not the usual definition for Banach spaces. Two basic sequences $\{x_n\}$ and $\{y_n\}$ are said to be *equivalent* if, $\sum \alpha_n x_n$ converges if and only if $\sum \alpha_n y_n$ converges. A basic sequence $\{x_n\}$ is *unconditional* if $\{x_{\pi(n)}\}$ is a basic sequence for each permutation of the integers π, otherwise $\{x_n\}$ is said to be *conditional*.

A basic sequence $\{x_n\}$ is 1-*unconditional with respect to* the seminorm $\|\cdot\|$, if for each scalar sequences $\{\alpha_n\}$ and $\{\beta_n\}$ with $|\beta_n| \leq 1$ and for each integer N, $\|\Sigma_1^N \alpha_n \beta_n x_n\| \leq \|\Sigma_1^N \alpha_n x_n\|$. The basic sequence $\{x_n\}$ is 1-*symmetric* (respectively, 1-*subsymmetric*) *with respect to* $\|\cdot\|$ if it is 1-unconditional with respect to $\|\cdot\|$ and for all scalars $\{\alpha_n\}$, $\|\Sigma\, \alpha_n x_n\| = \|\Sigma\, \alpha_n x_{\pi(n)}\|$, for all permutations (respectively, all strictly increasing integer-valued functions) π on the integers.

Suppose X is a Banach space with norm $\|\cdot\|$ and basis $\{e_i\}$, so that $\{e_i\}$ is 1-unconditional with respect to $\|\cdot\|$. Let $\{a_n^k\}_{n,k=1}^{\infty}$ be a matrix of nonnegative reals. Then the set of all scalar sequences $\{\alpha_n\}$ with

$$\|\{\alpha_n\}\|_k = \left\| \sum_n a_n^k \alpha_n e_n \right\| < \infty,$$

for each k, together with the seminorms $\{\|\cdot\|_k\}$ defines a Fréchet space which we will call an X-*Köthe sequence space*. Replacing $\{a_n^k\}$ by $\{b_n^k\}$ where $b_n^k = \max\{a_n^i: i \leq k\}$ shows that each X-Köthe sequence space is a project limit of spaces isomorphic to X with connecting diagonal maps. A *Köthe sequence space* is a X-Köthe sequence for some X as above. In lots of cases, but not all, Fréchet X-Köthe sequence spaces are examples of perfect Fréchet spaces (see Dubinsky [4]).

We now give a brief introduction to the nonstandard analysis used in this paper. Our approach is much like that of Robinson and Zakon [15], where proofs and details may be found. Our models are set-theoretical and hence we assume that ordered pairs, ordered n-tuples, relations, functions, Cartesian products and the like are defined as sets in one of the usual ways (say like [15]).

If Y is a set, define $P(Y)$ to be the power set of Y (i.e., the set of all subsets of Y). If Y is a set, let $Y_0 = Y$, inductively define $Y_{n+1} = P(\bigcup_{i=0}^{n} Y_i)$ and let $\hat{Y} = \bigcup_{n=0}^{\infty} Y_n$. The set $\hat{Y}$ is called the *superstructure* over the *ground set* Y. It will be convenient to assume that the elements of the ground set are "atoms" or "individuals" and hence are not sets themselves.

Our standard model starts with a ground set X, so that $\hat{X}$ contains all the objects under discussion. (Note that $\hat{X}$ is very large, in particular, if $A, B \in \hat{X}$, then $\hat{X}$ contains all functions from A to B, the set of all such functions, all n-ary relations on A and the like.) If K is the scalar field, almost all mathematics is contained in $\hat{K}$. However, if E, an abstract space, is under discussion, it is sometimes convenient to let X be the disjoint union of E and K.

Next we single out a "good" collection of mathematical statements about $\hat{X}$. Our language has exactly the elements of $\hat{X}$ as constants. There are also variables. Well-formed formulas (*wff*) are defined inductively. Atomic wff are either $x = y$, $x \in y$ or $(x_1, \ldots, x_n) \in y$, where $x, y, x_1, \ldots, x_n$ are any collection of constants and variables with repetitions allowed. If A and B are wff, then so are "A and B", "A or B", "$A \Rightarrow B$", "$A \Leftrightarrow B$" and "not A". If $A(x)$ is a wff, where x is a free variable in $A(x)$ (i.e., x does not occur in the scope of a quantifier) and C is any constant, then "$\exists x \in C\ A(x)$" and "$\forall x \in C\ A(x)$" are wff. A well-formed sentence (*wfs*) is a wff with no free variables. This completes the standard model.

The nonstandard model is based on a ground set Y with $X \subset Y$ and a 1-1 map $\Phi: \hat{X} \to \hat{Y}$, with $\Phi|_X$ = identity and $\Phi(X) = Y$. For notational reasons, if $A \in \hat{X}$, then $\Phi(A)$ is written $^*A$. The reader is warned that if A is an infinite set, then $^*A \supsetneqq \{^*a: a \in A\}$. That is, the value of Φ at A is different from the image of A by Φ. We now list some properties of Φ which follow from the above and from the transfer principle below: $^*X = Y$; $^*\varnothing = \varnothing$; if $A, B \in \hat{X}$, then $^*(A \cup B) = {^*A} \cup {^*B}$, $^*(A \cap B) = {^*A} \cap {^*B}$, $^*(A \setminus B) = {^*A} \setminus {^*B}$, $^*(A \times B) = {^*A} \times {^*B}$; if $A \in \hat{X}$ then $x \in A \Leftrightarrow {^*x} \in {^*A}$ and $(x_1, \ldots, x_n) \in A \Leftrightarrow ({^*x_1}, \ldots, {^*x_n}) \in {^*A}$; if $x_1, \ldots, x_n \in \hat{X}$, then $^*\{x_1\} = \{^*x_1\}$, $^*\{x_1, \ldots, x_n\} = \{^*x_1, \ldots, {^*x_n}\}$, and $^*(x_1, \ldots, x_n) = ({^*x_1}, \ldots, {^*x_n})$; if $R \in \hat{X}$, R is a function $\Leftrightarrow {^*R}$ is a function and R is an n-ary relation $\Leftrightarrow {^*R}$ is an n-ary relation; if $R \in \hat{X}$ and R is a binary relation, then $^*(\text{Domain } R) = \text{Domain}(^*R)$ and similarly with ranges.

The next step is to use Φ to transform wffs on $\hat{X}$ to wffs on $^*\hat{X} = \bigcup_{n=0}^{\infty} {^*X_n} \subset \hat{Y}$. If α is a wff on X, let $^*\alpha$ be the wff on $^*\hat{X}$ that is obtained by replacing each constant C in α with $^*C$. For example, if

$$\alpha \text{ is "}\forall x \in A\ \exists y \in B\ x \leqslant y \text{ and } y \leqslant x + 1\text{", then}$$
$$^*\alpha \text{ is "}\forall x \in {^*A}\ \exists y \in {^*B}\ x \mathrel{^*\!\!\leqslant} y \text{ and } y \mathrel{^*\!\!\leqslant} x \mathrel{^*\!\!+} {^*1}\text{".}$$

Note that logical symbols, $\in$ and variables are not starred, but the relations $\leqslant$, $+$ and the constant 1 are starred.

Convention. It follows from the transfer principle below that if f is a function from C to D then $^*f$ is a function from $^*C$ to $^*D$ which extends f. A similar statement is true for relations. Thus we will not generally star functions or relations. Also since $1 \in X$, $\Phi(1) = 1$ and thus we will not need to star the elements in our ground set. Thus we will write the above $^*\alpha$ as

$$\text{"}\forall x \in {^*A}\ \exists y \in {^*B}\ x \leqslant y \text{ and } y \leqslant x + 1\text{".}$$

The transfer principle. (1) If α is a wfs, then α is true in $\hat{X}$, if and only if $^*\alpha$ is true in $^*\hat{X}$. (2) If $\alpha(x_1, \ldots, x_n)$ is a wff and $x_1, \ldots, x_n$ is the list of free variables occurring in $\alpha(x_1, \ldots, x_n)$ and if $C \in \hat{X}$, then $^*\{(x_1, \ldots, x_n) \in C: \alpha(x_1, \ldots, x_n)\} = \{(x_1, \ldots, x_n) \in {^*C}: {^*\alpha}(x_1, \ldots, x_n)\}$.

The transfer principle makes $^*\hat{X}$ a model for $\hat{X}$; to make $^*\hat{X}$ nonstandard, we add lots of new elements. A binary relation $R \in \hat{X}$ is said to be *concurrent* if for each $d_1, \ldots, d_n$ in the domain of R (= set of first elements), there is an e in the range of R (= set of second elements) with $(d_i, e) \in R$, for $i = 1, 2, \ldots, n$.

The enlargement principle. If $R \in \hat{X}$ is a concurrent relation with domain D and range E, then there is an $e \in {^*E}$, so that for each $d \in D$, we have $({^*d}, e) \in {^*R}$.

REMARK. If $D \in \hat{X}$ is a set and if E is the set of finite subsets of D, then the relation $\in \subset B \times E$ is concurrent. Hence there is a set $e \in {^*E}$ with $^*d \in e$ for each $d \in D$. Such a set e is called *finite (or star-finite). This just means that there is an $\omega \in {^*N}$ and an onto function f: $\{n \in {^*N}: n \leqslant \omega\} \to e$ and $f \in {^*\hat{X}}$. The existence of such sets e, and the transfer principle, together yield the enlargement principle.

It is common to put additional conditions on the nonstandard model, but we will not use them here.

An element $A \in \hat{Y}$ is said to be *standard*, if there is a $B \in \hat{X}$ with ${}^*B = A$. An element $A \in \hat{Y}$ is said to be *internal*, if there is a $B \in \hat{X}$ with $A \in {}^*B$. Otherwise $A \in \hat{Y}$ is said to be *external*. We note that standard elements are internal and elements of internal sets are internal. As examples: ${}^*R$ is standard, a nonzero infinitesimal is internal but not standard, and the set of all infinitesimals in ${}^*R$ is external. (In particular, this says that ${}^*P(R) \subsetneq P({}^*R)$.)

We will make one "use" of external sets. If $\{a_n\}$ is an internal sequence in ${}^*R$ and a_n is infinitesimal for each standard integer n, then there is an infinite integer ω, so that $\nu \leqslant \omega$ implies that a_ν is infinitesimal [**14**, p. 65]. For the definitions below see [**14**] and [**5**].

A *real number r is infinitesimal (written $r \simeq 0$) if $|r| < s$, for each positive standard real s. The *real r is finite if $|r| < s$ for some standard real number s. Let E be a space and let P be a set of continuous seminorms which generate the topology on E. In ${}^*E$, a nonstandard model for E, we identify certain subsets below. Let $F \subset E^\#$.

$$\mu = \mu_E = \{e \in {}^*E\colon \|e\| \simeq 0 \text{ for each } \|\cdot\| \in P\}.$$

$$\text{fin}_E = \{e \in {}^*E\colon \|e\| \text{ is finite for each } \|\cdot\| \in P\}.$$

$$\mu_{\sigma(E,F)} = \{e \in {}^*E\colon |f(e)| \simeq 0 \text{ for each } f \in F\}.$$

The *nonstandard hull* of E (written $\hat{E}$) is a standard space of μ_E-equivalence classes of fin_E. Each seminorm in P becomes a real-valued seminorm on fin_E/μ_E and these generate the topology on $\hat{E}$. Let $\perp_{E'}$ be the subspace of $\hat{E}$ given by $(\text{fin}_E \cap \mu_{\sigma(E,E')})/\mu_E$.

The following facts will be needed in the sequel. Let X be a Fréchet space whose topology is defined by the seminorms $\{\|\cdot\|_k\}$, where $\|\cdot\|_k \leqslant \|\cdot\|_{k+1}$, and let $\{x_n\} \subset X$.

Fact 1.1. $\{x_n\}$ is an unconditional basic sequence if and only if for each j, there are k, K, so that $\|\Sigma_{n\in F}\, \alpha_n x_n\|_j \leqslant K\|\Sigma_{n\in G}\, \alpha_n x_n\|_k$, is true for each scalar sequence $\{\alpha_n\}$ and finite subsets of integers, $F \subset G$.

Fact 1.2. If $\{x_n\}$ is basic with respect to $\|\cdot\|_1$, $\|x_n\|_1 \neq 0$, for each n, and if for each k, there is an N_k, so that $\{x_n\colon n \geqslant N_k\}$ is basic with respect to $\|\cdot\|_k$, then $\{x_n\}$ is basic with respect to each $\|\cdot\|_k$.

2. Non-Montel spaces. This is the first of two sections showing how nonstandard analysis can be used to select "nice" basic sequences in certain Fréchet spaces. This method can be easily divided into two parts. First, it is shown that there is an ideal element ξ, which will be some element in a nonstandard model, with certain properties. Then the existence of ξ and the transfer principle allow for the inductive definition of a basic sequence using the usual Mazur product construction (see [**9**, p. 10]). The use of nonstandard analysis could be avoided at the cost of increased complexification. (Compare Proposition 3.7 with Theorem 3.1 of [**2**].)

The fundamental observation is fairly simple, although the statements of our results can be long-winded. This observation is Lemma 2.1.

LEMMA 2.1. *If* $(E, \|\cdot\|)$ *is a seminormed space and if* $\xi \in \mu_{\sigma(E,E')}$ *with* $\|\xi\| \not\simeq 0$, *then for each finite-dimensional subspace* $F \subset E$, *the projection* P: $[{}^*F \cup \{\xi\}] \to {}^*F$, *given by* $P(f + \lambda\xi) = f$, *where* $f \in {}^*F$, *has norm* $\|P\| \simeq 1$.

PROOF. First we may assume that $\|\xi\| = 1$, otherwise, $\eta = \xi/\|\xi\|$ satisfies $\eta \in \mu_{\sigma(E,E')}$ and yields the same projection P. Suppose the conclusion is false, then there exists a standard positive real $K < 1$, an $f \in {}^*F$, and a *scalar λ, so that $\|f\| = 1$, but $\|f + \lambda\xi\| < K$. Thus $\|\lambda\xi\| = |\lambda|$ is finite and so $\lambda\xi \in \mu_{\sigma(E,E')}$. Now, let $f' \in E'$ so that $\|f'\| = 1$ and $f'(f) \simeq 1$. This can be done by choosing f' to norm $g \in F$, where f is near standard to g. Finally $\|f + \lambda\xi\| \geqslant |f'(f + \lambda\xi)| \simeq |1 + 0|$, a contradiction.

REMARK. It is instructive to show how Lemma 2.1 and the transfer principle yield a basic sequence in an infinite-dimensional norm space E. The fact that there is a $\xi \in \mu_{\sigma(E,E')}$ with $\|\xi\| = 1$, follows from $\dim E = \infty$ (or more precisely, since the relation $S = \{(f, x) \in E' \times E: f(x) = 0\}$ is concurrent). We want to inductively choose a sequence $\{x_n\}$ of nonzero elements of E, so that the obvious projection P_{n+1}: $[x_i]_1^{n+1} \to [x_i]_1^n$ has $\|P_{n+1}\| < 1 + \varepsilon_{n+1}$, where $\Pi(1 + \varepsilon_n) < \infty$. The transfer principle states that this can be done if

$$\exists x_{n+1} \in {}^*E \setminus \{0\} \quad \text{with } \|P_{n+1}\| < 1 + \varepsilon_{n+1}. \tag{$*$}$$

Lemma 2.1 says that ($*$) is true for $x_{n+1} = \xi$. The advantage of this technique is that additional properties of ξ can also be transferred to the sequence $\{x_n\}$. A simple example, since in the above $\|\xi\| = 1$, ($*$) can become $\exists x_{n+1} \in {}^*E$ with $\|x_{n+1}\| = 1$ and $\|P_{n+1}\| < 1 + \varepsilon_{n+1}$, so each x_n could have been chosen to be norm one. More complicated examples are given in Proposition 2.2 and Theorem 3.2.

PROPOSITION 2.2. *For a Fréchet space E the following are equivalent.*
(i) *E is not Montel.*
(ii) $\perp_{E'} \neq \{0\}$.
(iii) *There is a normalized basic sequence in E.*

PROOF. (i) $\Rightarrow$ (ii). Since E is not Montel, there is a bounded sequence $\{x_n\}$, a continuous seminorm $\|\cdot\|$, and $\delta > 0$, so that $\|x_n - x_m\| \geqslant \delta$ for $n \neq m$. Now for each finite sequence $e'_1, \ldots, e'_j \in E'$ and each $\varepsilon > 0$, the set $A = A(e'_1, \ldots, e'_k, \varepsilon)$ is nonvoid, where

$$A = \{(m, n): m \neq n \text{ and } |e'_i(x_n - x_m)| < \varepsilon \text{ for } i \leqslant j\}.$$

To see this, note that the boundedness of $\{x_n\}$ implies there is a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ so that, for $i \leqslant j$, $\{e'_i(x_{n(k)})\}_k$ converges.

Therefore, in a nonstandard model of E, there exists $(m, n) \in A(e'_1, \ldots, e'_j, \varepsilon)$ where $\varepsilon \simeq 0$ and the *finite sequence $e'_1, \ldots, e'_j$ includes each $e' \in E'$. Let $\xi = x_m - x_n$. Since $\|x_m - x_n\| \geqslant \delta$, for $m \neq n$, it follows that $\|\xi\| \geqslant \delta$ and $\xi \notin \mu_E$. The boundedness of $\{x_n\}$ implies that $\xi \in \text{fin}_E$ and the choice of $e'_1, \ldots, e'_j$ implies $\xi \in \mu_{\sigma(E,E')}$. Thus the image of ξ in the nonstandard hull is a nonzero element of $\perp_{E'}$.

(ii) $\Rightarrow$ (iii). Since $\perp_{E'} \neq \{0\}$ we can choose $\xi \in {}^*E$ so that its image in the nonstandard hull is a nonzero element of $\perp_{E'}$. Let $\{\|\cdot\|_k\}$ be a sequence of seminorms whch define the topology on E. Since $\xi \in \text{fin}_E \setminus \mu_E$, the sequence $\{\|\cdot\|_k\}$ may be chosen so that $\|\xi\|_k$ is finite and not infinitesimal, for each standard k. Thus we may assume (by multiplication of $\|\cdot\|_k$ by a standard positive scalar if necessary) that $\|\xi\|_k \simeq 2$, for standard k.

Inductively choose $\{x_n\} \subset E$ so that

(a) $1 < \|x_{n+1}\|_k < 3$, for $k \leq n+1$ and

(b) the projection P_{n+1}: $[x_i]_1^{n+1} \to [x_i]_1^n$ satisfies $\|P_{n+1}y\|_k \leq (1 + \varepsilon_{n+1})\|y\|_k$ for $k \leq n+1$.

(Here ε_n are chosen in advance so that $\Pi(1 + \varepsilon_n) < \infty$.) This can be done, since ξ satisfies (a) and (b) and can be transferred back to $x_{n+1} \in E$. Condition (b) implies $\{x_n: n \geq k\}$ is basic with respect to $\|\cdot\|_k$. Thus, Fact 1.2 implies that $\{x_n\}$ is a basic sequence. The fact that $\{x_n\}$ is normalized follows from (a).

(iii) $\Rightarrow$ (i). A normalized basic sequence is a bounded set which is not precompact.

REMARKS. (1) The implication (i) $\Rightarrow$ (ii) just used the existence of a nonprecompact bounded set and not the fact that E was Fréchet. Since $\perp_{E'} = \{0\}$ is a necessary condition for E to have invariant nonstandard hulls [**1**], this is a very slight improvement of [**5**, p. 417].

(2) There are standard proofs of the equivalence of (i) and (iii) (see [**10**, Theorem 3.5]).

3. Non-Schwartz spaces. This section contains the main result (Theorem 3.2) which characterizes non-Schwartz-Fréchet spaces by the basic sequences they contain. The following definition is used.

DEFINITION 3.1. A basic sequence $\{x_n\}$ contained in the Fréchet space E is said to be *obliquely normalized* if there are

(i) an increasing sequence of seminorms $\{\|\cdot\|_k\}$ which define the topology on E,

(ii) a sequence of positive reals $\{b_{kj}: k \leq j, j = 1, 2, \ldots\}$, and

(iii) a partition of the set of integers $\{A_j\}$ with each A_j being infinite,

so that,

(a) $\{x_n\}$ is basic with respect to each $\|\cdot\|_k$, and

(b) if $n \in A_j$ and $k \leq j$, then $1 < b_{kj}\|x_n\|_k < 2$.

Before stating Theorem 3.2 we make two observations. First, a normalized basic sequence in a Fréchet space is obliquely normalized. Second, the span of an obliquely normalized basic sequence is a non-Schwartz space.

THEOREM 3.2. (I) *A Fréchet space E is non-Schwartz if and only if E has an obliquely normalized basic sequence.*

Let E be a non-Schwartz-Fréchet space.

(II) *If E is a subspace of the Fréchet space F with a basis $\{e_n\}$, then E has an obliquely normalized basic sequence $\{x_n\}$, with $\{x_n\}$ equivalent to a block basic sequence of $\{e_n\}$.*

(III) *If $E = F$ in* (II), *then $\{x_n\}$ can be chosen to be a block basic sequence of $\{e_n\}$.*

(IV) *If E is an X-Köthe sequence space with basis $\{e_n\}$, then the obliquely normalized basic sequence may be chosen to be a subsequence of $\{e_n\}$.*

The following lemma is needed for the proof.

LEMMA 3.3. *If $T: X \to Y$ is a continuous linear map between seminormed spaces and $\{x_n\} \subset X$ satisfies $\|x_n\| \leq K$ and $\|Tx_n - Tx_m\| \geq \delta > 0$ $(n \neq m)$, then for each $f \in X^{\#}$ either*

(1) *there is a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ so that $\{f(x_{n(k)})\}$ converges to a finite scalar, or*

(2) *for each ε with $\delta > \varepsilon > 0$, there is a sequence $\{y_n\} \subset$ lin span$\{x_n\}$, so that, $f(y_n) = 0$, $\|y_n\| \leq K + \varepsilon$, and $\|Ty_n - Ty_m\| \geq \delta - \varepsilon$ $(n \neq m)$.*

PROOF. Suppose (1) is false, then $\{f(x_n)\}$ has no limit point which is a finite scalar. Thus, by passing to a subsequence if necessary, we may assume

$$2(\|T\| + 1)K|f(x_n)| \leq \varepsilon|f(x_{n+1})|.$$

Define $y_n = x_n - \alpha_n|f(x_n)|\,|f(x_{n+1})|^{-1}x_{n+1}$, where α_n is the scalar so that $|\alpha_n| = 1$ and $f(y_n) = 0$. It is straightforward to check that $\|y_n\| \leq K + \varepsilon$ and $\|Ty_n - Ty_m\| \geq \delta - \varepsilon$ $(m \neq n)$.

PROOF OF THEOREM 3.2. (I) Let E be a Fréchet space which is not a Schwartz space. Let $\{\|\cdot\|_k\}$ be a sequence of seminorms which define the topology on E with $\|\cdot\|_k \leq \|\cdot\|_{k+1}$, for each k. We may and do assume that, for each k, the induced map $E(k) \to E(1)$ is not precompact.

Next, we claim that there is a sequence of ideal elements $\{\xi_k\} \subset {}^*E$, so that

$$\xi_k \in \mu_{\sigma(E,E')}, \quad \xi_k \in \text{fin}_{E(k)} \quad \text{and} \quad \|\xi_k\|_1 \not\simeq 0. \tag{$*$}$$

Before showing $(*)$, we observe that since $E \to E(i)$ is continuous and thus weakly continuous, it follows that $\xi_k \in \mu_{\sigma(E(i),E(i)')}$, for each i [**14**, Theorem 4.27]. Also since continuous linear maps send finites to finites and infinitesimals to infinitesimals, $\|\xi_k\|_j$ is finite and not infinitesimal, for $j \leq k$ [**5**, Corollary 1.5].

Let us prove $(*)$. Let k be given. Since $E(k) \to E(1)$ is not precompact, there is a sequence $\{x_n\}$, contained in the unit semiball of $E(k)$ so that $\|x_n - x_m\|_1 \geq \delta$ $(n \neq m)$, for some $\delta > 0$. Let $\varepsilon > 0$ and $f_1, \ldots, f_j$ be elements of E'. Define $A = A(f_1, \ldots, f_j, \varepsilon)$ to be the set,

$$A = \{x \in E: \|x\|_k \leq 4; |f_i(x)| < \varepsilon, \text{for } i \leq j; \text{and } \|x\|_1 \geq \delta/2\}.$$

It suffices to prove that A is nonvoid. For, by taking $\varepsilon \simeq 0$, $f_1, \ldots, f_j$ a *finite sequence containing E', the transfer principle yields a $\xi_k \in A$ with $\xi_k \in \mu_{\sigma(E,E')}$, $\|\xi_k\|_k \leq 4$ and $\|\xi_k\|_1 \geq \delta/2 \not\simeq 0$.

We show that A is nonvoid by a strange induction proof. Either we can pass to subsequences of subsequences, so that $\{f_i(x_n)\}_n$ converges for each $i \leq j$, or there is an $i(1) \leq j$, where $\{f_{i(1)}(x_n)\}_n$ has no finite scalar limit point. In the latter case, we apply Lemma 3.3, replacing $\{x_n\}$ by $\{y_n\}$, so that $f_{i(1)}(y_n) = 0$, $\|y_n\|_k \leq 3/2$ and $\|y_n - y_m\|_1 \geq 3\delta/4$ $(n \neq m)$.

We start over again. Either we can pass to subsequences of subsequences, so that $\{f_i(y_n)\}_n$ converges for each $i \leq j$, or there is an $i(2) \leq j$, where $\{f_{i(2)}(y_n)\}_n$ has no finite scalar limit point. Again we apply Lemma 3.3, replacing $\{y_n\}$ by $\{z_n\}$, so that $f_{i(2)}(z_n) = 0$, $\|z_n\|_k \leq 7/4$ and $\|z_n - z_m\|_1 \geq 5\delta/8$ $(n \neq m)$. Note that $f_{i(1)}(z_n) = 0$. In any case, after a finite number of repetitions, we have a sequence $\{x_n\}$ so that

$\|x_n\|_k \leq 2$, $\|x_n - x_m\|_1 \geq \delta/2$ $(n \neq m)$ and $\{f_i(x_n)\}_n$ converges for each $i \leq j$. Thus for large n and m, $x_n - x_m \in A$.

We are ready to pick our obliquely normalized basic sequence. Let $\{A_j\}$ be any partition of the set of integers with each A_j infinite. Let b_{kj} be any positive real number so that $1 < b_{kj}\|\xi_j\|_k < 2$, for $j \geq k$. Let $\{\varepsilon_n\}$ be a sequence of positive reals so that $\Pi(1 + \varepsilon_n) < \infty$.

Inductively define $\{x_n\} \subset E$, so that

(1) if $n + 1 \in A_j$ and $k \leq j$, then $1 < b_{kj}\|x_{n+1}\|_k < 2$ and

(2) the projection $P_{n+1}: [x_i]_1^{n+1} \to [x_i]_1^n$ satisfies

$$\|P_{n+1}y\|_k \leq (1 + \varepsilon_{n+1})\|y\|_k, \quad \text{for } k \leq n + 1.$$

The induction is possible since ξ_j would work in $^*E$ by Lemma 2.1 and thus, by the transfer principle, there must be $x_{n+1} \in E$, which will also satisfy (1) and (2). Fact 1.2 implies that $\{x_n\}$ is basic with respect to each $\|\cdot\|_k$ and condition (1) implies that $\{x_n\}$ is obliquely normalized. The proof of (I) is complete.

To complete the proof of the theorem, we show how the proof of (I) can be modified to handle each of the other cases. Addition assumptions, besides the maps: $E(k) \to E(1)$ not being precompact, can and need to be placed on the seminorms $\{\|\cdot\|_k\}$. In (II) and (III), we assume that the seminorms $\{\|\cdot\|_k\}$ generate the topology on F and that $\{e_n\}$ is 1-basic with respect to each $\|\cdot\|_k$. Let $\{e_n'\}$ be the coefficient functionals of $\{e_n\}$.

To complete the proof of (II), we use the same $\{\xi_j\}$ as in (I), but change the induction used to pick $\{x_n\}$. Let $\{\varepsilon_n\}$ be a sequence of positive reals $\leq 4^{-1}$, so that $4\Sigma\, \varepsilon_n(1 - 2\varepsilon_n)^{-1} < 1$. Let $\{A_j\}$ and $\{b_{kj}\}$ be as in the proof of (I). Let $N(0) = 0$ and inductively define a sequence of positive reals $\{\delta_n\}$, $\{x_n\} \subset E$, and an increasing sequence of integers $\{N(n)\}$, so that, if $n + 1 \in A_j$, then

(1) $N(n)\delta_{n+1} < \varepsilon_{n+1}b_{1j}^{-1}\max\{\|e_i\|_{n+1}: i \leq N(n)\}$,

(2) if $k \leq j$, then $1 < b_{kj}\|x_{n+1}\|_k < 2$,

(3) if $i \leq N(n)$, then $|e_i'(x_{n+1})| < \delta_{n+1}$, and

(4) if $x_{n+1} = \Sigma\, \alpha_i e_i$ and $k \leq n + 1$, then $\|\Sigma_{N(n+1)+1}^{\infty}\, \alpha_i e_i\|_k < \varepsilon_{n+1}\|x_{n+1}\|_k$.

Suppose $N(i)$, x_i, δ_i have been chosen for $i \leq n$. Condition (1) can be used to choose δ_{n+1}. Since ξ_j will satisfy conditions (2) and (3), the transfer principle implies that there is $x_{n+1} \in E$, which will also satisfy (2) and (3). Finally, condition (4) can be used to choose $N(n + 1)$, since $\{e_n\}$ is basic with respect to each $\|\cdot\|_k$. Let $y_n = \Sigma_{N(n-1)+1}^{N(n)}\, \alpha_i^n e_i$, where $x_n = \Sigma\, \alpha_i^n e_i$. We show that $\{x_n\}$ is a basic sequence equivalent to the block basic sequence $\{y_n\}$ by the stability property of basic sequences (see [**9**, p. 14]).

Let $k \leq n$ and $x_n = \Sigma\, \alpha_i^n e_i$. By condition (4), $\|\Sigma_{N(n)+1}^{\infty}\, \alpha_i^n e_i\|_k < \varepsilon_n\|x_n\|$. By conditions (1) and (3),

$$\left\| \sum_{1}^{N(n-1)} \alpha_i^n e_i \right\|_k \leq \sum_{1}^{N(n-1)} |\alpha_i^n|\, \|e_i\|_k \leq N(n-1)\delta_n \max\{\|e_i\|_n: i \leq N(n-1)\}$$

$$< \varepsilon_n\|x_n\|_1 \leq \varepsilon_n\|x_n\|_k.$$

Thus $\|x_n - y_n\|_k < 2\varepsilon_n\|x_n\|_k < 2\varepsilon_n(1 - 2\varepsilon_n)^{-1}\|y_n\|_k$. Therefore, by the above stability property, the sequence $\{x_n: n \geq k\}$ is a basic sequence with respect to $\|\cdot\|_k$ which is equivalent to the block basic sequence $\{y_n: n \geq k\}$ with respect to $\|\cdot\|_k$. An appeal to Fact 1.2 completes the proof of (II).

The proofs of (III) and (IV) require different choices of $\{\xi_j\}$. Note that in the case of (III), the induced map: $E(k) \to E(1)$ is a diagonal map between seminormed spaces with "bases". The nonprecompactness of this map requires that it is not the limit of the finite rank maps: $E(k) \to E(1) \to [e_i]_1^n$. Thus there is a $\xi_k \in {}^*E$ and infinite integers ω and η so that $\xi_k \in [e_i]_\eta^\omega$ and both $\|\xi_k\|_k$ and $\|\xi_k\|_1$ are finite and not infinitesimal.

Let $\{A_j\}$ and $\{b_{kj}\}$ be as in the proof of (I). Let $N(0) = 0$ and inductively defined $\{x_n\} \subset E$ and an increasing sequence of integers $\{N(n)\}$, so that

(1) if $n + 1 \in A_j$ and $k \leq j$, then $1 < b_{kj}\|x_{n+1}\|_k < 2$, and

(2) $x_{n+1} \in [e_i]_{N(n)+1}^{N(n+1)}$.

It is straightforward to complete the proof of (III).

The proof of (IV) is similar to the proof of (III). We may assume that each $E(k)$ is X and thus the map: $E(k) \to E(1)$ is a diagonal map between equivalent unconditional bases. Therefore, ξ_k can be chosen to be e_ω, for some infinite integer ω. The details are similar to (III) and are omitted. This completes the proof of Theorem 3.2.

COROLLARY 3.4. *If the non-Schwartz space E is a subspace of a Fréchet space with an unconditional basis, then E has an unconditional obliquely normalized basic sequence.*

PROOF. In the proof of (II) we could then assume that $\{e_n\}$ is 1-unconditional with respect to each $\|\cdot\|_k$. Thus both $\{y_n\}$ and $\{x_n\}$ are unconditional with respect to each $\|\cdot\|_k$.

REMARKS. (1) There are Fréchet-Montel spaces which are not isomorphic to any subspace of a Fréchet space with an unconditional basis. Indeed, examples of such spaces are constructed in the next section.

(2) If in addition, E is not a Montel space in Corollary 3.4, then E has a normalized unconditional basic sequence. The proof is much like that of (II) only using ξ, whose image in the nonstandard hull is in $\perp_{E'} \setminus \{0\}$, instead of $\{\xi_k\}$.

COROLLARY 3.5. *Let $X = l_p$, $1 \leq p < \infty$ or $X = c_0$. If E is a non-Schwartz subspace of a Fréchet X-Köthe sequence space, then E has a subspace isomorphic to a non-Schwartz X-Köthe sequence space.*

PROOF. In the proof of (II) we may assume that $\{e_i\}$ is some multiple of the usual basis for l_p in each $\|\cdot\|_k$. Thus the same is true for the block basic sequence $\{y_n\}$. It follows that $[x_n]$ is isomorphic to $[y_n]$ and the latter is an l_p-Köthe sequence space which is non-Schwartz by (I).

COROLLARY 3.6. *The space E is a Schwartz space if and only if for each continuous seminorm $\|\cdot\|_\alpha$ on E, there is a continuous seminorm $\|\cdot\|_\beta$ on E, with $\|\cdot\|_\alpha \leq \|\cdot\|_\beta$, so that $\xi \in \mu_{\sigma(E,E')}$ and $\|\xi\|_\beta$ finite imply that $\|\xi\|_\alpha \simeq 0$.*

PROOF. This follows from the proof of (I) where it was shown that $\{\xi_k\}$ exists. The metric hypothesis was not used.

PROPOSITION 3.7. *Let E be a Fréchet X-Köthe sequence space, and let $\{e_i\}$ be the usual basis for E, then*

(1) *either there is a subsequence of $\{e_i\}$ spanning a Banach space or E is Montel,*

(2) *either there is a subsequence of $\{e_i\}$ which is obliquely normalized or E is Schwartz and*

(3) *if E is Montel, either there is a subsequence spanning a Schwartz nonnuclear subspace or E is nuclear.*

PROOF. First we note that E is, in addition, a perfect Fréchet space, then (1) is a special case of Theorems 4 and 5 of [**4**].

(1) We will show that if E is not Montel, then there is an infinite integer ω, so that $\|e_\omega\|_1 \neq 0$ and $\|e_\omega\|_k/\|e_\omega\|_1$ is finite for each standard integer k. A modification of the proof of (IV) would then yield a subsequence of $\{e_n\}$ on which each pair $\|\cdot\|_k, \|\cdot\|_m$ would be equivalent norms.

If E is not Montel, then there is a $\xi \in \mu_{\sigma(E,E')}$, $\xi \notin \mu$ and $\xi \in \operatorname{fin} E$. Let $\xi = \sum \alpha_i e_i$, and since $\alpha_i \simeq 0$ for each standard integer i, then $N\|\sum_1^N \alpha_i e_i\|_N < 1$, for each standard integer N. Thus, for some infinite integer ω, $\omega\|\sum_1^\omega \alpha_i e_i\|_\omega < 1$ [**14**, Theorem 3.3.20] and $\|\sum_1^\omega \alpha_i e_i\|_\omega \simeq 0$. Also, for some infinite integer $\nu < \omega$, $\|\sum_{\nu+1}^\infty \alpha_i e_i\|_\omega \simeq 0$. Thus $\|\sum_1^\omega \alpha_i e_i\|_k$ and $\|\sum_{\nu+1}^\infty \alpha_i e_i\|_k$ are infinitesimal for each standard k. Therefore $\eta = \sum_{\omega+1}^\nu \alpha_i e_i$ satisfies $\|\eta\|_k \simeq \|\xi\|_k$ for each standard k and thus $\eta \in \mu_{\sigma(E,E')} \cap \operatorname{fin}_E$ and $\eta \notin \mu$.

We may assume $\|\eta\|_1 = 1$ and let $\|\eta\|_k = a(k)$ for each integer k. Define an outer measure μ on the internal subsets of $B = \{\omega + 1, \ldots, \nu\}$ by $\mu(A) = \|\sum_{i\in A} \alpha_i e_i\|_1$. Let $A(1) = \{i \in B\colon\ \|e_i\|_1 = 0\}$ and let $A(k) = \{i \in B\setminus A(1)\colon \|e_i\|_k/\|e_i\|_1 > 4^k a(k)\}$ for $k \geqslant 2$. Note that each $A(k)$ is internal, $\mu(A(1)) = 0$ and $\mu(A(k)) < 4^{-k}$ for integers $k \geqslant 2$. Thus, if $A = \bigcup_{k=1}^\omega A(k)$, A is internal and $\mu(A) < 3^{-1}$. Hence, there is an $i \in B\setminus A$ so that $\|e_i\|_1 \neq 0$ and $\|e_i\|_k/\|e_i\|_1 < 4^k a(k)$, which is finite for standard k. This completes the proof of (1).

(2) This is just Theorem 3.2(IV).

(3) We may assume that E is a Montel non-Schwartz space or there is nothing to prove. We may also assume that $\{e_i\}$ is an obliquely normalized basis by (IV). We claim, by passing to subsequences $\{j(i)\}$ and $\{k(m)\}$ if necessary, that $\lim_j b_{kj} b_{k+1j}^{-1} = 0$. Otherwise, by taking subsequences of subsequences of $\{j\}$ and diagonalizing, we would have $\{b_{kj}^{-1} b_{mj}\}$, uniformly bounded for each k and m. But this would imply that $[e_{i(j)}]$ where $i(j) \in A_j$ is a normed space, which by (1) is impossible.

Now by choosing $F_j \subset A_j$ to be a finite subset with more than b_{jj}/b_{1j} elements and letting $F = \bigcup F_j$, then $G = [e_i]_{i\in F}$ is a Schwartz nonnuclear space. Indeed, G is Schwartz since $G_{k+1} \to G_k$ is a c_0-diagonal map (which is precompact) on a normed space with an unconditional basis. Also G is nonnuclear, since the map $([e_i]_{i\in F_j}, \|\ \|_k) \to ([e_i]_{i\in F_j}, \|\cdot\|_1)$ has nuclear norm (see [**13**]) $\geqslant 1$, for $j \geqslant k$. (Compare with Theorem 3.1 of [**2**].) This completes the proof.

4. An example. Fréchet-Montel spaces which are not subspaces of any Fréchet space with an unconditional basis are constructed (Theorem 4.1). For each Banach space X with a basis $\{e_n\}$, the construction below yields "the X-plank space" XP, a Montel-non-Schwartz-Fréchet space that has properties in common with X. Indeed, X is a quotient space of XP (see remark after Theorem 5.4). The construction is based on an obliquely normalized basis and is similar to a counterexample considered by Köthe and Grothendieck (see [**8**, p. 433]). For convenience, assume $\|e_n\| = 1$ for each n, and that $\{e_n\}$ is 1-basic with respect to $\|\cdot\|$, the norm on X.

Let I: $X \to X$ be the identity map and let S: $X \to X$ be the diagonal map, so that $S(\sum \alpha_n e_n) = \sum n^{-1}\alpha_n e_n$. Let Y be the l_2-sum of countably many copies of X. For each k, let T_k: $Y \to Y$, be the map, whose restriction to jth copy of X is either S if $j \leq k$, or jI if $j > k$. Define XP to be the projective limit of

$$\cdots \xrightarrow{T_3} Y \xrightarrow{T_2} Y \xrightarrow{T_1} Y.$$

An equivalent way of defining XP is as the set of all doubly indexed scalar sequences $\{\alpha_{ij}\}$, so that, each of the norms $\|\{\alpha_{ij}\}\|_k < \infty$. Here, $\|\cdot\|_k$, for $k = 0, 1, \ldots,$ is defined by

$$\|\{\alpha_{ij}\}\|_k = \left(\sum_j \left\|\sum_i \alpha_{ij} b_{ij}^k e_i\right\|^2\right)^{1/2},$$

where $b_{ij}^k = i^k$ if $j \leq k$, and $b_{ij}^k = j^k$ if $j > k$. It needs noting that although each $\|\cdot\|_k$ is equivalent to the norm on the projective limit obtained from the $(k+1)$st copy of Y, they are not the same. Indeed, in the second definition, $\|\cdot\|_{k+1}$ is not greater than $\|\cdot\|_k$, but $\|\cdot\|_{k-1} \leq k^k\|\cdot\|_k$ is true. In either case, it should be clear that XP is a non-Schwartz-Fréchet space with an obliquely normalized basis.

To see that XP is Montel, we use Proposition 2.2. Suppose $\xi = \{\xi_{ij}\} \in \mu_{\sigma(XP,XP')}$ but $\xi \notin \mu$. Thus there is a standard integer k, so that $\|\xi\|_k \not\simeq 0$. The proof will be completed by showing that $\|\xi\|_n$ is infinite for some standard n and thus $\xi \notin \text{fin}_{XP}$. Suppose $\|\xi\|_{k+1}$ is finite. Define $\xi(n)$, $\eta(\nu)$, $\zeta(\omega, m)$ and $\pi(\omega, m, n)$ to be the sequences $\{\alpha_{ij}\}$, $\{\beta_{ij}\}$, $\{\gamma_{ij}\}$ and $\{\delta_{ij}\}$, respectively, where

$$\alpha_{ij} = \begin{cases} \xi_{ij}, & \text{both } i, j \leq n, \\ 0, & \text{otherwise,} \end{cases} \qquad \beta_{ij} = \begin{cases} \xi_{ij}, & j > \nu, \\ 0, & \text{otherwise,} \end{cases}$$

$$\gamma_{ij} = \begin{cases} \xi_{ij}, & j \leq m \text{ and } i > \omega, \\ 0, & \text{otherwise,} \end{cases} \qquad \delta_{ij} = \begin{cases} \xi_{ij}, & i > \omega \text{ and } m < j \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

Let n, i, j be standard integers. Since $\alpha_{ij} \simeq 0$, $\|\xi(n)\|_k \simeq 0$. Thus for some infinite integer ω', $\|\xi(\omega')\|_k \simeq 0$ [**14**, Theorem 3.3.20]. If ν is an infinite integer then $\|\eta(\nu)\|_{k+1} \geq \nu\|\eta(\nu)\|_k$ and thus $\|\eta(\nu)\|_k \simeq 0$. Similarly, $\|\zeta(\omega', k+1)\|_{k+1} \geq \omega'\|\zeta(\omega', k+1)\|_k$ and thus $\|\zeta(\omega', k+1)\|_k \simeq 0$. If $\|\pi(\omega', K+1, n)\|_k \simeq 0$ for each standard $n > k+1$, then there is an infinite integer ν' so that $\|\pi(\omega', k+1, \nu')\|_k \simeq 0$. But this is impossible since

$$\|\xi\|_k \leq \|\xi(\omega')\|_k + \|\eta(\nu')\|_k + \|\zeta(\omega', k+1)\|_k + \|\pi(\omega', k+1, \nu')\|_k.$$

Therefore, for some standard integer n, $\|\pi(\omega', K+1, n)\|_k \not\simeq 0$, and

$$\|\pi(\omega', k+1, n)\|_n \geq \omega'\|\pi(\omega', k+1, n)\|_k.$$

Thus both $\|\pi(\omega', k + 1, n)\|_n$ and $\|\xi\|_n$ are infinite, and XP is Montel.

XP has the following interesting property.

(*) If XP is isomorphic to a subspace of a product of countably many Banach spaces $\{Z_n\}$, then X is isomorphic to a subspace of a finite product of Banach spaces $\prod_{n=1}^{N} Z_n$, for some N.

For if $XP \subset \prod Z_n$, then there are N and n integers, and constants K and L, so that, if $|\cdot|$ is the seminorm on $\prod Z_n$ obtained from the norm on $\prod_1^N Z_n$, then $\|x\|_1 \leq K|x| \leq L\|x\|_n$ for $x \in XP$. But $\|\cdot\|_1$ and $\|\cdot\|_n$ are equivalent to X on the "nth copy of X" in XP. Therefore X is a subspace of $\prod_1^N Z_n$.

THEOREM 4.1. *There are Fréchet-Montel spaces which cannot be subspaces of any Fréchet space with an unconditional basis.*

PROOF. Let X be the James space or any other Banach space which is not a subspace of any Banach space with an unconditional basis (see [**9**, p. 18]). Since a Fréchet space with an unconditional basis is a subspace of a product of Banach spaces, each having an unconditional basis, XP satisfies the theorem by (*) above.

REMARKS. (1) This is a striking difference between Fréchet-Montel spaces and Fréchet-Schwartz spaces. Each Schwartz-Fréchet space is a subspace of a Schwartz-Fréchet space with an unconditional basis [**2**].

(2) Theorem 4.1 shows that the hypothesis of Corollary 3.4 can fail to be satisfied even if the space is Montel. It is an open question if each Fréchet-Montel-non-Schwartz space has an unconditional obliquely normalized basic sequence. It is possible that this question could have an answer independent of the related question "does each Banach space have an unconditional basic sequence?".

5. Quotients of non-Schwartz-Fréchet spaces. An often quoted counterexample of Köthe (see [**8**, p. 433]) is a Fréchet-Montel-non-Schwartz reflexive l_1-Köthe sequence space which has the Banach space l_1 as a quotient. It has been used to show that quotients do not preserve properties like reflexivity or being Montel. We will show that Fréchet-non-Schwartz X-Köthe sequence spaces have X as a quotient if the basis for X is subsymmetric. Also, we will show that the "pathological behavior" of Köthe's example is shared–in some sense–by all Fréchet-Montel-non-Schwartz spaces.

If $\|\cdot\|$ is a continuous seminorm on E, we will also denote by $\|\cdot\|$ the (possibly infinite-valued) norm on E' which is dual to $\|\cdot\|$ on E.

THEOREM 5.1. *Each Fréchet-Montel-non-Schwartz space has an infinite-dimensional quotient with a normalized basis.*

PROOF. Let E be a Fréchet-Montel-non-Schwartz space. By Theorem 3.2, E has an obliquely normalized basic sequence $\{x_n\}$. Let $\{A_j\}$, $\{\|\cdot\|_k\}$ and $\{b_{kj}\}$ be as required by Definition 3.1. We may assume (by normalization if necessary) that $\|x_n\|_1 = 1$ for each n. We doubly index $\{x_n\}$ by $\{x_{ij}\}$, where $x_n = x_{ij}$ exactly when n is the ith element of A_j. Let $F = [x_{ij}]$ and let K be the basis constant of $\{x_n\}$ with respect to $\|\cdot\|_1$.

Define $\{f_m\} \subset F'$ by $f_m(\sum \alpha_{ij}x_{ij}) = \sum_j \alpha_{mj}2^{-j}$. It is straightforward to check that, for each m annd k, $(*)$ $b_{kk}2^{-k-1} \leqslant \|f_m\|_k \leqslant \|f_m\|_1 \leqslant 2K$. While preserving $\|f_m\|_1$, extend f_m to all of E, these will also be denoted by $\{f_m\}$. Note that $(*)$ remains true. A Montel space is separable [8, p. 370], so let $\{e_n\}$ be a countable dense subset of E. For each n, the sequence $\{f_m(e_n)\}_m$ is bounded. By passing to subsequences and diagonalizing, there is a subsequence $\{f_{m(i)}\}$, so that $\lim_i f_{m(i)}(e_n)$ exists for each n. Thus $g_i = f_{m(2i)} - f_{m(2i-1)}$ is a $\sigma(E', E)$-null sequence so that for each k and i,

$$b_{kk}2^{-k-1} \leqslant \|g_i\|_k \leqslant \|g_i\|_1 \leqslant 4K. \tag{$**$}$$

Now, let $y_i = 2x_{m(2i),1}$, so that $g_i(y_i) = 1$ and $g_i(y_j) = 0$ if $i \neq j$. For any subsequence of the integers $\{i(n)\}$, let $L\{i(n)\} = \bigcap_n \ker g_{i(n)}$, $z_n = y_{i(n)} + L\{i(n)\}$, and let $\|\cdot\|_k$ denote the quotient norm on $E/L\{i(n)\}$ generated by the seminorm $\|\cdot\|_k$ on E. From $(**)$ it follows that for each n and k,

$$2^{-1}K^{-1} \leqslant \|z_n\|_1 \leqslant \|z_n\|_k \leqslant 2^{k+1}b_{kk}^{-1}.$$

Thus $\{z_n\}$ is normalized in $E/L\{i(n)\}$. The results of Johnson and Rosenthal [7] imply that for each k, there is a subsequence $\{i(n)\}$ so that $\{z_n\}$ is a basis for the normed space $(E/L\{i(n)\}, \|\cdot\|_k)$. Note (also by [7]) that if $\{i(n')\}$ is a subsequence of $\{i(n)\}$, then $\{z_{n'}\}$ is a basis for the normed space $(E/L\{i(n)\}, \|\cdot\|_k)$. Thus by passing to subsequences of subsequences and diagonalizing we obtain a subsequence $\{i(n)\}$ so that $\{z_n\}$ is a basis for $E/L\{i(n)\}$. The proof is complete.

Corollary 5.2. *Each non-Schwartz-Fréchet space has an infinite-dimensional quotient with a normalized basic sequence.*

Proof. Either the space is Montel or the space itself is the quotient by Proposition 2.2.

Corollary 5.3. *A Fréchet space is Schwartz if all its quotients are Montel.*

Remark. The converse is also true [6, p. 279].

Theorem 5.4. *If X is a Banach space with a subsymmetric basis $\{e_n\}$, then each non-Schwartz X-Köthe sequence space has a quotient isomorphic to X.*

Proof. Let E be a non-Schwartz-Fréchet X-Köthe sequence space with basis $\{x_n\}$. By Theorem 3.2, there is a subsequence $\{x_{n(i)}\}$ of $\{x_n\}$, which is obliquely normalized. Since X is subsymmetric, $[x_{n(i)}]$ is also a X-Köthe sequence space. Since $[x_{n(i)}]$ is complemented in E, we may assume that the basis $\{x_n\}$ is obliquely normalized.

As in the proof of Theorem 5.1, let $\{A_j\}$, $\{\|\cdot\|_k\}$ and $\{b_{kj}\}$ be as required by Definition 3.1. We may assume that for each n, $\|x_n\|_1 = 1$. We doubly index $\{x_n\}$ by $\{x_{ij}\}$, where $x_n = x_{ij}$ exactly when n is the ith element of A_j. Consider the linear function $T: E \to X$ given by $T(\sum \alpha_{ij}x_{ij}) = \sum_m(\sum_j \alpha_{mj}2^{-j})e_m$.

Let $\|\cdot\|$ be the 1-subsymmetric norm on X. Note that for each j and $k < j$, $([x_{ij}]_{i=1}^{\infty}, \|\cdot\|_k)$ is equivalent to $([e_n], \|\cdot\|)$. Since $\|x_{ij}\|_1 = 1$, the above equivalent is an isometry when $k = 1$. To see that T is continuous, let $\|\sum \alpha_{ij}x_{ij}\|_1 < 1$. Now

$$\left\|\sum_m\left(\sum_j \alpha_{mj}2^{-j}\right)e_m\right\| \leq \sum_j 2^{-j}\left\|\sum_m \alpha_{mj}e_m\right\| \leq \sum_j 2^{-j}\left\|\sum_m \alpha_{mj}x_{mj}\right\|_1 < 1.$$

Let $K = \ker T$, and identify E/K with $T(E) \subset X$. Let $\{|\cdot|_k\}$ on $T(E)$ be the quotient seminorms obtained from $\{\|\cdot\|_k\}$ on E. From the continuity of T above, we have on $T(E)$, $\|\cdot\| \leq |\cdot|_1$. Furthermore, the equivalence of $([x_{ij}]_{i=1}^{\infty}, \|\cdot\|_j)$ with $([e_n], \|\cdot\|)$ implies that on $T(E)$, $|\cdot|_j \leq 2b_{jj}^{-1}\|\cdot\|$. Thus, since $|\cdot|_k \leq |\cdot|_{k+1}$, all the seminorms $\{|\cdot|_j\}$ are each equivalent to $\|\cdot\|$ on $T(E)$. Therefore $T(E) = X$ and the proof is complete.

REMARK. A similar proof shows that XP has X as a quotient.

COROLLARY 5.5. *Let $X = l_p$, $1 \leq p < \infty$, or c_0. If E is a non-Schwartz subspace of a Fréchet X-Köthe sequence space, then X is a quotient of E.*

PROOF. Combine Corollary 3.5, Theorem 5.4 and the proof of Theorem 5.1 (the Montel property was used only to show that the space was separable).

6. Conditional basic sequences? What is the collection C of Fréchet spaces with conditional basic sequences? It is known that C contains no nuclear space **[13**, p. 173] and the intersection of C with each of the following classes of Fréchet spaces is exactly the nonnuclear spaces in that class: l_p-Köthe sequence spaces **[17]**, Banach spaces **[12]**, Hilbertian spaces **[2]**, X-Köthe sequence spaces **[2]** and "most" non-Montel spaces **[3]**. Pełczyński asked (in **[11]**) if C is the collection of non-nuclear Fréchet spaces. In this section, we show that non-Schwartz subspaces of l_p-Köthe sequence spaces are in C (Proposition 6.1) and the general question of whether each non-Schwartz space belongs in C can be reduced to the interrelationships of a finite number of Banach spaces each with the same symmetric basis (Proposition 6.2).

PROPOSITION 6.1. *Each non-Schwartz subspace of a Fréchet l_p- (or c_0-) Köthe sequence space has a conditional basic sequence.*

PROOF. By Corollary 3.5, the subspace has a further subspace which is isomorphic to a non-Schwartz-Fréchet l_p-Köthe sequence space which has a conditional basis by **[17]** or **[2]**.

PROPOSITION 6.2. *Each non-Schwartz-Fréchet space has a conditional basic sequence if and only if each space E has a conditional basic sequence where E is any space which satisfies: there are*

(1) *an increasing sequence of norms on $E\{\|\cdot\|_k\}$ which define the topology on E;*

(2) *a basis $\{x_n\}$ for E;*

(3) *a partition of the set of integers $\{A_j\}$ with each A_j infinite; and*

(4) *a sequence of positive numbers $\{a_{kj}: k \leq j\}$ so that*

(a) *$\{x_n\}$ is 1-unconditional with respect to each $\|\cdot\|_k$;*

(b) *if $n \in A_j$ and $k \leq j$, then $\|x_n\|_k = a_{kj}$; and*

(c) *$\{x_n: n \in A_j\}$ is 1-symmetric with respect to each $\|\cdot\|_k$ for $k \leq j$.*

PROOF. Since such a space E is non-Schwartz the only if is trivial. Conversely, since each non-Schwartz-Fréchet space has a subspace with an obliquely normalized basis, we need only consider non-Schwartz-Fréchet spaces with an unconditional basis.

From the proof of Theorem 3.2(III), we can pass to a further subspace with an obliquely normalized basis $\{x_n\}$ so that the norms $\{\|\cdot\|_k\}$ are such that $\{x_n\}$ is 1-unconditional with respect to each $\|\cdot\|_k$. Let $\{A_j\}$ and $\{b_{kj}\}$ be as required by Definition 3.1. Replace each $\|\cdot\|_k$ by an equivalent norm $|\cdot|_k$ which satisfies all of the above conditions with the possible exception of (c) by defining $|\Sigma\, \alpha_n x_n|_k = \|\Sigma\, \alpha_n \beta_n^k x_n\|_k$. This can be done since the conditions on the $\{b_{kj}\}$ allow $\{\beta_n^k\}$ to be a sequence of scalars which is bounded and bounded away from zero.

Assume E, $\{x_n\}$, $\|\cdot\|_k$, $\{A_j\}$ and $\{a_{kj}\}$ satisfy all the conditions of Proposition 6.2 except condition (c). Divide each A_j into two disjoint infinite sets B_j and C_j. Define $E_n =$ linear span $\{x_m\colon m \in \bigcap_{j=n}^{\infty} B_j\}$ and let Q be the collection of all functions $\pi\colon E_1 \to E$, so that there is an injection $\phi\colon \bigcup B_j \to \bigcup C_j$ with $\phi(n) \in C_j$ for each $n \in B_j$ and for each j, where $\pi(\Sigma\, \alpha_i x_i) = \Sigma\, \alpha_i x_{\phi(i)}$ for $\Sigma\, \alpha_i x_i \in E_1$. For integers k, m and i with $k \leqslant m \leqslant i$, define $K(k, m, i) = K_i$ to be the smallest number $\leqslant \infty$, so that for each $x \in E_i$ and $\pi \in Q$, both

$$\|x\|_k \leqslant K_i \|\pi(x)\|_m \tag{$*$}$$

and

$$\|\pi(x)\|_k \leqslant K_i \|x\|_m \tag{$**$}$$

are true. Let $K(k, m) = \lim_i K_i$.

Suppose that for each k there is an m so that $K(k, m) < \infty$. Thus for some i, K_i is also finite. Define norms $|||\cdot|||_k^n$ on E_n, for each n and k by $|||x|||_k^n = \sup\{\|\pi(x)\|_k\colon \pi \in Q\}$. By $(**)$, $K_i < \infty$ implies that $|||\cdot|||_k^i$ is continuous on E_i. On the other hand, $(*)$ implies $\|\cdot\|_k \leqslant (\text{constant})|||\cdot|||_m^n$ on E_n for each $n \geqslant i$. Thus for some subsequence $\{n(k)\}$, the norms $\{|||\cdot|||_k^{n(k)}\}$ define the topology on $[x_m\colon m \in \bigcup_{k=1}^{\infty} B_{n(k)}]$. It is straightforward to check that this subspace satisfies the conditions of the proposition.

If there is a k, so that for each m, $K(k, m) = \infty$, and thus $K(k, m, i) = \infty$, then we will inductively construct a conditional basic sequence in E. Inductively choose $\{e_n\} \subset \{x_n\}$, $\{\pi_i\} \subset Q$ and two increasing sequences of integers $\{N(i)\}$ and $\{j(i)\}$ with $N(0) = j(0) = 0$, so that

(1) $\{e_n\colon 2N(i) < n \leqslant 2N(i+1)\} \subset \{x_m\colon m \in \bigcup_{j=j(i)+1}^{j(i+1)} A_j\}$,

(2) either

(i) for each n with $N(i-1) < n \leqslant N(i)$, $\pi_i(e_{2n-1}) = e_{2n}$, or

(ii) for each n with $N(i-1) < n \leqslant N(i)$, $\pi_i(e_{2n}) = e_{2n-1}$, is true, and

(3) there is a scalar sequence $\{\alpha_n\}$ so that

$$\left\| \sum_{n=N(i-1)+1}^{N(i)} \alpha_n e_{2n-1} \right\|_k = 1 \quad \text{and} \quad \left\| \sum_{n=N(i-1)+1}^{N(i)} \alpha_n e_{2n} \right\|_{k+1} > i.$$

The option in (2) reflects the possibility that (*) or (**) could make $K_i = \infty$. Let $\{y_n\}$ be defined by $y_{2n-1} = e_{2n-1}$ and $y_{2n} = e_{2n-1} + e_{2n}$. Lemma 6.3 (below) and Fact 1.2 imply that $\{y_n\}$ is basic, while Lemma 6.3, condition (3) and Fact 1.1 imply that $\{y_n\}$ is a conditional basic sequence. The proof is complete except for the following lemma.

LEMMA 6.3. *Let $\{e_n\}$ be 1-unconditional with respect to $\|\cdot\|_i$, $i = 1, 2$; and suppose that for each n and $i = 1, 2$: $\|e_{2n-1}\|_i = \|e_{2n}\|_i$. If $y_{2n-1} = e_{2n-1}$ and $y_{2n} = e_{2n-1} + e_{2n}$, then $\{y_n\}$ is basic with respect to $\|\cdot\|_i$, $i = 1, 2$.*

Furthermore, if for all finite subsets $F \subset G$ and for each scalar sequence $\{\alpha_n\}$, $\|\sum_{n\in F} \alpha_n y_n\|_1 \leq L\|\sum_{n\in G} \alpha_n y_n\|_2$, then for any scalar sequence $\{\alpha_n\}$, $\|\sum \alpha_n e_{2n-1}\|_1 \leq L\|\sum \alpha_n e_{2n}\|_2$.

PROOF. Note that $\|\alpha y_{2n-1}\|_i \leq 2\|\alpha y_{2n-1} + \beta y_{2n}\|_i$ and hence $\{y_n\}$ is 2-basic with respect to $\|\cdot\|_i$. A proof of the furthermore statement is essentially given in the proof of Proposition 3 of **[12**, p. 12].

REMARKS. (1) The reduction to Banach S-algebras as done in **[3]** for non-Montel spaces does not seem to work in this case as Lemma 6.3 does not apply. However, if for some reason the spaces $(\{x_n: n \in A_j\}, \|\cdot\|_k)$ have uniformly bounded "index" for $k \leq j$, $j = 1, 2, \ldots$ then E can be shown to have a conditional basic sequence in a manner similar to Theorem 4.1 of **[3]**.

(2) Let $\{e_n\}$ be 1-symmetric with respect to the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ and $\|\cdot\|_1 \leq \|\cdot\|_2$. For each finite scalar sequence $\alpha_1, \ldots, \alpha_m$, define $u_n^\alpha = \sum_{i=1}^m \alpha_i e_{(n-1)m+i}$ for each n. We will say that $\|\cdot\|_1$ and $\|\cdot\|_2$ are counterfeitly equivalent if there is an $\varepsilon > 0$ so that for each finite sequence $\alpha_1, \ldots, \alpha_m$, the norm of $([u_n^\alpha]_n, \|\cdot\|_2) \to ([u_n^\alpha]_n, \|\cdot\|_1)$ is $\geq \varepsilon$. If the symmetric norms in Proposition 6.2 are not counterfeitly equivalent then it can be shown that E has a Schwartz nonnuclear subspace with an unconditional basis. The author knows no example of nonequivalent counterfeitly equivalent symmetric norms. If counterfeit equivalence implies equivalence then Pełczyński's question **[11]** has the same answer among Schwartz-Fréchet spaces as it does among Fréchet spaces in general.

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