CYCLIC EXTENSIONS OF PARAFREE GROUPS[1]

BY

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ABSTRACT. Let $1 \to F \to G \to T \to 1$ be a short exact sequence where F is parafree and T is infinite cyclic. We examine some properties of G when F/F' is a free ZT-module. Here F' is the commutator subgroup of F and ZT is the integral group ring of T. In particular, we show G is parafree and $\gamma_n F/\gamma_{n+1} F$ is a free ZT-module for every $n > 1$ (where $\gamma_n F$ is the nth term of the lower central series of F).

1. Introduction. A group G is termed parafree if
(i) G is residually nilpotent and
(ii) G has the same lower central sequence as some free group.
This paper is mainly concerned with some of the module structure involved with the action of an infinite cyclic group on a parafree group. More precisely, let $1 \to F \to G \to T \to 1$ be a short exact sequence where F is a parafree group and $T = \langle t \rangle$ is an infinite cyclic group generated by t. Here we assume T to be a subgroup of G. If F' denotes the commutator subgroup of F and ZT denotes the integral group ring of T, then F/F' can be regarded as a ZT-module once the action of t on F/F' is defined (by conjugation):

$$aF' * t = t^{-1}atF', \quad a \in F.$$

We shall prove

THEOREM 1. *Let $1 \to F \to G \to T \to 1$ be a short exact sequence where F is parafree and $T = \langle t \rangle$ is infinite cyclic. If F/F' is a free ZT-module then G is parafree*

and

THEOREM 2. *Let $1 \to F \to G \to T \to 1$ be a short exact sequence where F is parafree and $T = \langle t \rangle$ is infinite cyclic. If F/F' is a free ZT-module then $\gamma_n F/\gamma_{n+1} F$ is a free ZT-module for every $n \geqslant 1$. (Here $\gamma_n F$ is the nth term of the lower central series of F.)*

Such extensions G discussed above are not rare. Indeed it is clear that if G is free on $X \cup \{t\}$ and F is the normal closure of X, then the short exact sequence $1 \to F \to G \to T \to 1$ where $T = \langle t \rangle$, satisfies the hypothesis of Theorems 1 and 2.

Even in this case the result seems to be new and sheds additional light on the nature of free groups themselves.

Furthermore the one-relator groups $G = \langle a, b, t; a = [t^i, a][t^j, b]\rangle$, $i \cdot j \neq 0$, are also examples of such extensions (see §6).

Now according to G. Baumslag [7], the 2-generator subgroups of parafree groups are free. G. Baumslag's proof uses machinery from Lie algebras. For these parafree groups covered in Theorems 1 and 2 there is a straightforward proof which comes out of Theorem 2. Indeed we shall easily deduce the following corollary of Theorem 2.

THEOREM 3. *Let* $1 \to F \to G \to T \to 1$ *be a short exact sequence where* F *is parafree and* $T = \langle t \rangle$ *is infinite cyclic. If* F/F' *is a free ZT-module then every two-generator subgroup of* G *is free.*

Similarly we have

THEOREM 4. *Let* $1 \to F \to G \to T \to 1$ *be a short exact sequence where* F *is parafree and* $T = \langle t \rangle$ *is infinite cyclic. If* F/F' *is a finitely generated free ZT-module, freely generated by* $\{cF', dF', \ldots, eF'\}$, *then the group* $H = gp(t, c, d, \ldots, e)$ *is free.*

Many authors have discussed the existence, in a group satisfying appropriate conditions, of a finitely generated normal subgroup (see, for example, B. Baumslag [1], G. Baumslag [11], R. G. Burns [14], A. Karrass and D. Solitar [18]–[20]). In particular, O. Schreier [22] proved that if F is a free group and N is a nontrivial finitely generated normal subgroup of F, then N is of finite index in F. The corresponding possibilities for parafree groups has not been investigated so far. Here we note, as an easy consequence of Theorem 2,

THEOREM 5. *Let* $1 \to F \to G \to T \to 1$ *be a short exact sequence where* F *is parafree and* $T = \langle t \rangle$ *is infinite cyclic. If* F/F' *is a free ZT-module and* N *is a finitely generated normal subgroup of* G *contained in* F, *i.e.,* $N \lhd G$, $N \leqslant F$, *then* N *is trivial, i.e.,* $N = 1$.

The arrangement of the rest of this paper is as follows. In §2 we prove Theorem 1. In §3 we prepare the ground for the proof of Theorem 2 which takes up §4. In §5 we prove Theorems 3, 4 and 5. Finally in §6 we give an example of a nonfree parafree group satisfying the hypothesis of Theorem 1.

The notations used here are common in the literature (see W. Magnus, A. Karrass and D. Solitar [21], G. Baumslag [8]). In particular, if G is a group and X is a subset of G, then $gp(X)$ and $gp_G(X)$ denote respectively the subgroup and the normal subgroup generated by X. Furthermore we have $[x, y] = x^{-1}y^{-1}xy$, $x^y = y^{-1}xy$ for $x, y \in G$ and Z denotes the ring of integers.

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2. The proof of Theorem 1. Theorem 1 follows immediately from the proof of the following theorem.

THEOREM $1'$. *Let* $1 \to F \to G \to T \to 1$ *be a short exact sequence where F is parafree and $T = \langle t \rangle$ is infinite cyclic. If F/F' is a finitely generated free ZT-module, then G is parafree.*

PROOF. First we show that G is residually a finite p-group for all primes p.

Let p be a prime. We can assume $p \neq 2$ since the proof for $p = 2$ is similar. Let $g \in G$, $g \neq 1$. We have to construct a homomorphism ϕ of G onto a finite p-group such that $g\phi \neq 1$.

If $g \notin F$, we can easily construct the homomorphism.

Now suppose $g \in F$. We choose an integer $k > 1$ so that $g \notin \gamma_{k+1}F$. Observe that $F/\gamma_{k+1}F$ is a free nilpotent group of class k and hence a residually finite p-group. Then there exist a normal subgroup R of $F/\gamma_{k+1}F$ such that $g\gamma_{k+1}F \notin R$ and $(F/\gamma_{k+1}F)/R$ has order p^d, $d > 1$. This implies $g\gamma_{k+1}F \notin (F/\gamma_{k+1}F)^{p^d}$ or equivalently, $g \notin (\gamma_{k+1}F)F^{p^d}$.

By hypothesis, F/F' is a finitely generated free ZT-module, freely generated say by

$$c_{1,0}F', c_{2,0}F', \ldots, c_{n,0}F'.$$

Put

$$c_{i,j} = c_{i,0}^{t^j}, \qquad 1 \leqslant i \leqslant n, j \in Z.$$

Then modulo $(\gamma_{k+1}F)F^{p^d}$, the set $X = \{c_{i,j} | 1 \leqslant i \leqslant n, \ j \in Z\}$ freely generates $F/(\gamma_{k+1}F)F^{p^d}$ viewed as a free group in the variety of all nilpotent groups of class at most k and exponent dividing p^d.

Now let us put
(a) $\bar{F} = F/(\gamma_{k+1}F)F^{p^d}$,
(b) $\bar{g} = g(\gamma_{k+1}F)F^{p^d}$,
(c) $\bar{c}_{i,j} = c_{i,j}(\gamma_{k+1}F)F^{p^d}$, $1 \leqslant i \leqslant n, j \in Z$,
(d) $\bar{X} = \{\bar{c}_{i,j} | 1 \leqslant i \leqslant n, j \in Z\}$.
Since $\bar{X}$ freely generates $\bar{F}$, there exists an integer m such that the group

$$\bar{N} = gp\left(\bar{c}_{i,j} | 1 \leqslant i \leqslant n, \ -\frac{(p^m - 1)}{2} \leqslant j < \frac{(p^m - 1)}{2}\right)$$

contains $\bar{g}$. Since $\bar{N}$ is a finitely generated nilpotent group of class at most k and exponent dividing p^d, $\bar{N}$ is a finite p-group.

We define an automorphism τ of $\bar{N}$ by

$$\left.\begin{array}{rcl}
\bar{c}_{i,-(p^m-1)/2}\tau & = & \bar{c}_{i,-(p^m-1)/2+1} \\[4pt]
\bar{c}_{i,-(p^m-1)/2+1}\tau & = & \bar{c}_{i,-(p^m-1)/2+2} \\
\vdots & & \\
\bar{c}_{i,0}\tau & = & \bar{c}_{i,1} \\
\vdots & & \\
\bar{c}_{i,(p^m-1)/2}\tau & = & \bar{c}_{i,-(p^m-1)/2}
\end{array}\right\}, \quad 1 \leqslant i \leqslant n.$$

It is obvious that τ is of order p^d. Now let $\overline{T} = \langle \bar{t} \rangle$ be a cyclic group of order p^d generated by $\bar{t}$. Finally let $\overline{G}$ be the split extension of $\overline{N}$ by $\langle \bar{t} \rangle$ with $\bar{t}$ inducing the automorphism τ on $\overline{N}$, i.e.,

$$\overline{G} = gp(\overline{N}, \bar{t}; \bar{t}^{-1}u\bar{t} = u\tau, u \in \overline{N}).$$

The group $\overline{G}$ is clearly a finite p-group.

It is now clear that the mapping ϕ of G onto $\overline{G}$ defined by

$$\phi: t \to \bar{t}, \quad c_{i,0} \to \bar{c}_{i,0}, \quad 1 \le i \le n,$$

extends to a homomorphism of G onto $\overline{G}$ which we again denote by ϕ. Notice that $g\phi = \bar{g} \neq 1$. Hence G is residually a finite p-group.

Now, we show that G has the same lower central sequence as a free group.

Let $X' = \{c'_{1,0}, c'_{2,0}, \ldots, c'_{n,0}\}$ and A be the free group freely generated by

$$X' \cup \{t'\} = \{t', c'_{1,0}, c'_{2,0}, \ldots, c'_{n,0}\}.$$

We define a map θ from A to G by

$$\phi: t' \to t, \quad c'_{i,0} \to c_{i,0}, \quad 1 \le i \le n.$$

Then θ extends to a homomorphism, also denoted θ, from A to G. We claim G has the same lower central sequence as A.

To see this, put $B = gp_A(X')$, i.e., B is the normal closure of X' in A. Then B is free on $Y = \{c'_{i,j} = (t')^{-j}c'_{i,0}(t')^j; 1 \le i \le n, j \in Z\}$. Observe that θ induces a homomorphism of B into F and hence noting the actual effect of θ on the free set of generators Y of B, it follows that θ induces an isomorphism of $B/\gamma_2 B$ to $F/\gamma_2 F$. Since F is parafree, it follows that θ induces an isomorphism of $B/\gamma_{k+1}B$ to $F/\gamma_{k+1}F$ for every $k \ge 1$.

We next show $A/\gamma_{k+1}B$ is isomorphic to $G/\gamma_{k+1}F$. To do this let θ_1 be homomorphism of A to $G/\gamma_{k+1}F$ induced by θ. We show $\ker \theta_1 = \gamma_{k+1}B$. It is obvious that $\gamma_{k+1}B \le \ker \theta_1$. Now let $x \in \ker \theta_1$. If $x \in B$, then $x \in \gamma_{k+1}B$ by the observation in the preceding paragraph. Suppose $x \in A$, $x \notin B$. Then x can be written uniquely $x = t'^m w(c'_{i,j})$ where $m \neq 0$ and $w(c'_{i,j})$ is a word in the $c'_{i,j}$'s. Then

$$x\theta_1 = t^m w(c_{i,j})\gamma_{k+1}F \neq \gamma_{k+1}F.$$

Hence it follows that $\ker \theta_1 \le \gamma_{k+1}B$. Therefore $A/\gamma_{k+1}B \cong G/\gamma_{k+1}F$ and hence

$$A/\gamma_{k+1}A \cong A/\gamma_{k+1}B/\gamma_{k+1}(A/\gamma_{k+1}B)$$
$$\cong G/\gamma_{k+1}F/\gamma_{k+1}(G/\gamma_{k+1}F) \cong G/\gamma_{k+1}G.$$

This completes the proof of the theorem.

3. Preparations for the proof of Theorem 2. We shall develop some notions from the commutator calculus (see G. Baumslag [8], P. Hall [16]). Unless stated differently, we assume the integer $n \ge 1$.

Let F be a group, not necessarily free, and equipped with a fixed set of generators $X = \{x_1, x_2, \ldots, x_q\}, q < \infty$.

DEFINITION 1. *Commutation.* We define a binary operation $\circ$, termed commutation, in F by $g \circ h = [g, h], g, h \in F$.

DEFINITION 2. *Commutators of weight* $n, n \geqslant 1$. We define a commutator of weight $n, n \geqslant 1$, in $x_1, x_2, \ldots, x_q$ inductively as follows:

(i) $x_1, x_2, \ldots, x_q$ are the commutators of weight 1;

(ii) if u, v are commutators of weight i, j respectively and $i + j = n$, then $u \circ v$ is a commutator of weight n in $x_1, x_2, \ldots, x_q$.

REMARKS. (i) $(F, \circ)$ is a groupoid but not a semigroup since commutation is not associative in general;

(ii) The commutators of weight n belong to $\gamma_n F$;

(iii) If F is the free group on $x_1, x_2, \ldots, x_q$ then the weight of a commutator c in $x_1, x_2, \ldots, x_q$ will, in general, differ from the length of c viewed as an element in the free group F. For example the weight of $x_1 \circ x_2$ is two but the length of $x_1 \circ x_2$ is four.

Next we define free groupoid.

DEFINITION 3. *Free groupoid*. Let $Y = \{y_1, y_2, \ldots, y_q\}$, $1 \leqslant q < \infty$, be a set of symbols. We define a groupoid-word of length $n, n \geqslant 1$, in $y_1, y_2, \ldots, y_q$ inductively as follows:

(i) $y_1, y_2, \ldots, y_q$ are the groupoid-words of length 1;

(ii) if u, v are groupoid-words of length i, j respectively and $i + j = n$, then $u \circ v = (uv)$ is a groupoid-word of length n in $y_1, y_2, \ldots, y_q$ and (uv) is just the juxtaposition of u and v with brackets.

Let $(P, \circ)$ be the collection of all groupoid-words of length $n, n \geqslant 1$. Then $(P, \circ)$ is defined to be the free groupoid on $y_1, y_2, \ldots, y_q$.

Now we define the form of a commutator by using Definition 3.

DEFINITION 4. *Form of a commutator*. Let $(F, \circ)$ be defined as in Definition 1 above and let $(M, \circ)$ be the free groupoid on x. We define, inductively, a map ϕ from the set of commutators of weight $n, n \geqslant 1$, in $x_1, x_2, \ldots, x_q$ into $(M, \circ)$ as follows:

(i) if $x_i, 1 \leqslant i \leqslant q$, is a commutator of weight 1, then $x_i \phi = x, 1 \leqslant i \leqslant q$;

(ii) if $w = u \circ v$ is a commutator of weight $n, n \geqslant 2$, where u, v are commutators of weight i, j respectively and $i + j = n$, then $w\phi = (u \circ v)\phi = u\phi \circ v\phi$.

We call the image $w\phi$ of the commutator w, the form of w.

REMARKS. Since $(M, \circ)$ is a free groupoid we have the following observations:

(i) the form of a commutator of weight n is unique;

(ii) the form of a commutator w of weight n, depends upon the order in which w is formed;

(iii) if u, v are any two commutators of a fixed weight n, we write $u \sim v$ if and only if u and v have the same form, i.e., $u\phi = v\phi$. Then $\sim$ is an equivalence relation. Hence the set of commutators of a fixed weight n can be divided into disjoint subsets where each subset contains commutators of a fixed form.

Notation. We will adopt the following notations for the remainder of §§3, 4. Let F be a group.

(i) If $g_1, g_2, \ldots, g_n \in F$, $n < \infty$, we write $g_1 \circ g_2 \circ g_3 \circ \cdots \circ g_n = (\cdots((g_1 \circ g_2) \circ g_3) \circ \cdots \circ g_n)$.

(ii) If g is a commutator of weight n, we denote the weight of g by $|g|$; so here $|g| = n$.

The above two notations are not standard but turn out to be convenient here.

Theorem 2 will depend in large measure on a carefully selected sequence of elements of F (called a basic sequence). From this sequence the free generators of $\gamma_n F / \gamma_{n+1} F$ will emerge. More precisely, let F be a free group freely generated by a finite set $X = \{x_1, x_2, \ldots, x_q\}$, $q < \infty$.

DEFINITION 5. *Basic sequence.* A sequence $b_1, b_2, \ldots$ of commutators in x_1, $x_2, \ldots, x_q$ (viewed as elements in the groupoid $(F, \circ)$) is called a basic sequence if the following conditions are satisfied:

(i) every element of X occurs in the sequence;

(ii) if $|b_i| < |b_j|$, then $i < j$;

(iii) if b_r is of weight at least two, then

   (a) $b_r = b_i \circ b_m$, $m < i$ and

   (b) either $|b_i| = 1$ or $b_r = b_k \circ b_l \circ b_m$ where $k > l$, $m \geqslant l$ and b_k, b_l and b_m are of smaller weights than b_r.

DEFINITION 6. *Basic commutator.* The elements $b_1, b_2, \ldots$ in a basic sequence are called basic commutators and the index of the basic commutator b_j is defined to be j.

We now construct a basic sequence in F.

LEMMA 1. *Let F be a free group freely generated by a finite set $X = \{x_1, x_2, \ldots, x_q\}$, $q < \infty$. Then a basic sequence $b_1, b_2, \ldots$ in F exists. Furthermore, the basic sequence can be arranged in the following way*:

(3.1) *if b_i is the ith term in the basic sequence and $i \leqslant q$, then $b_i = x_i$, i.e., the indices of b_i and x_i are the same for $i = 1, 2, \ldots, q$.*

(3.2) *if $b_i = b_{l_i} \circ b_{m_i}$ and $b_j = b_{l_j} \circ b_{m_j}$ are both basic commutators of weight 2, then $i < j$ if and only if one of the following holds*:

(i) $l_i < l_j$;

(ii) $l_i = l_j$, $m_i < m_j$.

(3.3) *if $b_i = b_{k_i} \circ b_{l_i} \circ b_{m_i}$ and $b_j = b_{k_j} \circ b_{l_j} \circ b_{m_j}$ are both basic commutators of weight n, $n > 2$, then $i < j$ if and only if one of the following holds*:

(i) $|b_{m_i}| < |b_{m_j}|$;

(ii) $|b_{m_i}| = |b_{m_j}|$, $|b_{l_i}| < |b_{l_j}|$;

(iii) $|b_{m_i}| = |b_{m_j}|$, $|b_{l_i}| = |b_{l_j}|$, $k_i < k_j$;

(iv) $|b_{m_i}| = |b_{m_j}|$, $|b_{l_i}| = |b_{l_j}|$, $k_i = k_j$, $l_i < l_j$;

(v) $|b_{m_i}| = |b_{m_j}|$, $|b_{l_i}| = |b_{l_j}|$, $k_i = k_j$, $l_i = l_j$, $m_i < m_j$.

PROOF. The following method will be used in constructing the basic sequence in F (see G. Baumslag [8, p. 30]).

Let A be a subset of F and let $b \in A$. Define

$$A \text{ rep } b = \{ aib \mid i = 0, 1, 2, \ldots, a \in A \setminus b \}$$

where

$$a \circ b = a \quad \text{and} \quad a(i + 1)b = (aib) \circ b.$$

Thus A rep b consists of all elements of A excluding b and those obtained from them by repeated commutation on the right by b.

We now define a basic sequence $b_1, b_2, \ldots$ in F. Put $b_1 = x_1, b_2 = x_2, \ldots, b_q = x_q$ and $X_1 = X$, $X_2 = X_1$ rep $b_1, \ldots, X_q = X_{q-1}$ rep b_{q-1}. Suppose inductively, $b_{q+1}, \ldots, b_{q+r}$ have been defined and that

$$X_{q+i} = X_{q+i-1} \text{ rep } b_{q+i-1} \qquad (i = 1, 2, \ldots, r + 1)$$

so that conditions (3.2) and (3.3) are satisfied.

We have to define b_{q+r+1}. Observe that since $q < \infty$, each X_i, $i > 1$, contains only finitely many commutators of any fixed weight. Consider the finitely many commutators of the smallest weight in X_{q+r+1}. We now carefully select one of them to be b_{q+r+1}.

We note first that, using only the information above, it is now easy to verify that $b_1, b_2, \ldots$ is a basic sequence in F, i.e., the conditions of Definition 4 are satisfied.

Continuing the proof, let $\mathcal{C}$ be any set of commutators. Define

$$\mu(\mathcal{C}) = \text{minimum}(|c| \ |c \in \mathcal{C})$$

where $|c|$ is the weight of c.

We note, now that there are two possibilities in selecting b_{q+r+1}.

(I) $\mu(X_{q+r+1}) = \mu(X_{q+r})$. In this case, the commutators of smallest weight in X_{q+r+1} are also the commutators of smallest weight in X_{q+r}. Hence, by induction, they have been ordered to satisfy one of the conditions (3.2), (3.3). Define b_{q+r+1} to be the first of these commutators in X_{q+r+1}.

(II) $\mu(X_{q+r+1}) > \mu(X_{q+r})$. There are two subcases.

(a) $\mu(X_{q+r+1}) = 2$. Then the commutators of smallest weight in X_{q+r+1} are of the form $b_l \circ b_m$, $l > m$ and $|b_l| = |b_m| = 1$. We order these commutators $b_l \circ b_m$ according to (3.2) and denote the ordering by α. More precisely, if $b_{l_i} \circ b_{m_i}$ and $b_{l_j} \circ b_{m_j}$ are any two commutators of smallest weight in X_{q+r+1}, then $b_{l_i} \circ b_{m_i}$ α $b_{l_j} \circ b_{m_j}$ if and only if one of the following holds:

(i) $l_i < l_j$;

(ii) $l_i = l_j$, $m_i < m_j$.

Now choose b_{q+r+1} to be the first of these commutators of smallest weight in X_{q+r+1}.

(b) $\mu(X_{q+r+1}) > 2$. Then the commutators of smallest weight in X_{q+r+1} are of the form $b_k \circ b_l \circ b_m$ where $k > l$, $m \geqslant l$ and b_k, b_l and b_m are of smaller weights than $\mu(X_{q+r+1})$. By induction, the b_k's, b_l's and b_m's are well ordered and hence their indices are well defined.

We define an ordering, denoted α, on these commutators $b_k \circ b_l \circ b_m$ of smallest weight according to (3.3). More precisely, if $b_{k_i} \circ b_{l_i} \circ b_{m_i}$ and $b_{k_j} \circ b_{l_j} \circ b_{m_j}$ are any two commutators of smallest weight in X_{q+r+1}, then $b_{k_i} \circ b_{l_i} \circ b_{m_i}$ α $b_{k_j} \circ b_{l_j} \circ b_{m_j}$ if and only if one of the following holds:

(i) $|b_{m_i}| < |b_{m_j}|$;

(ii) $|b_{m_i}| = |b_{m_j}|$, $|b_{l_i}| < |b_{l_j}|$;

(iii) $|b_{m_i}| = |b_{m_j}|$, $|b_{l_i}| = |b_{l_j}|$, $k_i < k_j$;

(iv) $|b_{m_i}| = |b_{m_j}|$, $|b_{l_i}| = |b_{l_j}|$, $k_i = k_j$, $l_i < l_j$;

(v) $|b_{m_i}| = |b_{m_j}|$, $|b_{l_i}| = |b_{l_j}|$, $k_i = k_j$, $l_i = l_j$, $m_i < m_j$.

Now choose b_{q+r+1} to be the first of these commutators in X_{q+r+1}.

It is easy to see now that the basic sequence b_1, b_2, ... satisfies (3.1), (3.2) and (3.3). This completes the proof of the lemma.

COROLLARY 1. *The basic commutators of the same weight and same form (as constructed in Lemma 1) are arranged lexicographically.*

LEMMA 2. *Let F be a free group of finite rank. Let b_1, b_2, ... be a basic sequence of F and B_n be the set of basic commutators of weight n, i.e., $B_n = \{ b_j | \, |b_j| = n \}$. Then $\gamma_n F / \gamma_{n+1} F$ is a free abelian group freely generated by the elements of B_n.*

PROOF. For a proof refer to G. Baumslag [8], P. Hall [16].

We have so far constructed a particular basic sequence for a free group of finite rank. We now extend the results to a free group F of infinite rank equipped with a free set X of generators where $X = \{c_i, d_i, e_i, \ldots ; (i \in Z)\}$. For ease of exposition we shall restrict our attention to the case where $X = \{c_i, d_i; (i \in Z)\}$. The case where there are further "batches" of generators $e_i, \ldots$ ($i \in Z$) can be handled in much the same way.

LEMMA 3. *Let $F = \langle c_i, d_i; i \in Z \rangle$ be a free group freely generated by c_i, d_i, $i \in Z$. Let $F_j, j \geqslant 0$, be the subgroup of F generated by*

$$c_{-j}, \ldots, c_0, \ldots, c_j, d_{-j}, \ldots, d_0, \ldots, d_j$$

and $B_{j,n}$ be the set of basic commutators of weight n in F_j. Put $B_n = \bigcup_{j=0}^{\infty} B_{j,n}$. Then modulo $\gamma_{n+1}F$, $\gamma_n F$ is a free abelian group freely generated by the elements of B_n.

PROOF. Notice first that, modulo $\gamma_{n+1}F_j$, $\gamma_n F_j, j \geqslant 0$, is a free abelian group freely generated by the elements of $B_{j,n}$ (Lemma 2).

We show B_n spans $\gamma_n F$ modulo $\gamma_{n+1}F$. To this end, let $x \in \gamma_n F$. Then $x \in F_j$ for some $j \geqslant 0$. Now F_j is a free factor of F which implies that

$$\gamma_r F \cap F_j = \gamma_r F_j \qquad (r = 1, 2, \ldots). \tag{3.4}$$

So $x \in \gamma_n F_j$ which means that $x \in gp(B_{j,n})$ (modulo $\gamma_{n+1}F_j$). Therefore $x \in gp(B_n)$ (modulo $\gamma_{n+1}F$).

In order to prove the elements of B_n are independent modulo $\gamma_{n+1}F$, it suffices to prove that the elements of $B_{j,n}$ ($j = 0, 1, 2, \ldots$) are independent modulo $\gamma_{n+1}F$. But this follows immediately from equation (3.4) and the fact that the elements of $B_{j,n}$ are independent modulo $\gamma_{n+1}F_j$.

LEMMA 4. *Let $F = \langle c_i, d_i; i \in Z \rangle$ be a free group freely generated by c_i, d_i, $i \in Z$. Let F_j and $B_{j,n}$ be as defined in Lemma 3. Put $B = \bigcup_{j,n} B_{j,n}$. Then the elements of B can be totally ordered by using Lemma 1.*

PROOF. For each $j = 0, 1, 2, \ldots$ we order (denoted $<_j$) the generators of F_j as follows:

$$c_{-j} <_j c_{-j+1} <_j \cdots <_j c_0 <_j \cdots <_j c_j <_j d_{-j} <_j \cdots <_j d_0 <_j \cdots <_j d_j.$$

Now put $B_{j,\infty} = \bigcup_{n=1}^{\infty} B_{j,n}$. The orderings extend by Lemma 1 to a total order (denoted $<_j$) of $B_{j,\infty}$. So each $B_{j,\infty}$ becomes a totally ordered set. Notice that $<_{j+1} | B_{j,\infty} = <_j$ and so we can define an ordering $<$ on B by $< = \bigcup_{j=0}^{\infty} <_j$, i.e., if

$x, y \in B$, then $x, y \in B_{j,\infty}$ for some $j \geqslant 0$ and by definition

$$x \leqslant y \text{ if and only if } x \leqslant_j y.$$

It follows that the order relation $\leqslant$ on B is a total order. This completes the proof of the lemma.

We call the elements of B_n the basic commutators of weight n of F.

We next extend our definition of form (Definition 3) of a basic commutator to the case when F is a free group freely generated by $c_i, d_i, i \in Z$.

Let F_j be the subgroup of F generated by $c_{-j}, \ldots, c_0, \ldots, c_j$, $d_{-j}, \ldots, d_0, \ldots, d_j$ and $(F_j, \circ)$ as before (Definition 2 and its remarks). Let $(N, \circ)$ be the free groupoid on c, d. We define, inductively, a map θ_j from the set of commutators of weight $n, n \geqslant 1$, in $c_i, d_i, -j \leqslant i \leqslant j$, of F_j into $(N, \circ)$ as follows:

(i) if $c_i, d_i, -j \leqslant i \leqslant j$, is a commutator of weight 1, then

$$c_i \theta_j = c, \quad -j \leqslant i \leqslant j; \qquad d_i \theta_j = d, \quad -j \leqslant i \leqslant j;$$

(ii) if $w = u \circ v$ is a commutator of weight $n, n \geqslant 2$, where u, v are commutators of weight p, q respectively and $p + q = n$, then $w\theta_j = u\theta_j \circ v\theta_j$. Notice that $\theta_{j+1}|F_j = \theta_j$.

DEFINITION 7. *Macro-form of a commutator.* Let F be the free group freely generated by $c_i, d_i, i \in Z$, and F_j the subgroup of F generated by

$$c_{-j}, \ldots, c_0, \ldots, c_j, d_{-j}, \ldots, d_0, \ldots, d_j.$$

Let $(N, \circ)$ be the free groupoid on c, d. We define a map θ from $(F, \circ)$ to $(N, \circ)$ by $\theta = \bigcup_{j=0}^{\infty} \theta_j$ where θ_j is as above, i.e., if $x \in F$, then $x \in F_j$ for some $j \geqslant 0$ and by definition $x\theta = x\theta_j$. Then $x\theta$ is unique and we call $x\theta$ the macro-form of x.

COROLLARY 2. *For $F = \langle c_i, d_i; i \in Z \rangle$ be a free group freely generated by $c_i, d_i, i \in Z$. Then the basic commutators of the same weight and same macro-form (as constructed in Lemma 3) are ordered lexicographically.*

REMARKS. (i) It is convenient to relabel the element $r \in B$ (see Lemma 4) by b_r and call r the index of b_r. So, of course, we have

$$b_r \leqslant b_s \text{ if and only if } r \leqslant s, r, s \in B.$$

(ii) From Lemma 1 we have

(a) if $b_k \circ b_l$ is a basic commutator of weight 2, then $b_k > b_l$,

(b) if $b_k \circ b_l \circ b_m$ is a basic commutator of weight $n, n > 2$, then $b_k > b_l$ and $b_m > b_l$.

4. The proof of Theorem 2. Theorem 2 follows immediately from the slightly stronger

THEOREM 2′. *Let $1 \to F \to G \to T \to 1$ be a short exact sequence where F is free nilpotent and $T = \langle t \rangle$ is infinite cyclic. If F/F' is a finitely generated free ZT-module then $\gamma_n F / \gamma_{n+1} F$ is also a free ZT-module for every $n \geqslant 1$.*

PROOF. The proof is by induction on the class r of F. The hypothesis is the case $r = 1$. So we can assume $r > 1$ and the result holds for $\gamma_k F / \gamma_{k+1} F$, $1 \le k \le r - 1$.

By hypothesis F/F' is a finitely generated free ZT-module of rank s, freely generated by say cF', dF', ..., eF'. For ease of exposition we restrict our attention to the case where $s = 2$, i.e., F/F' is freely generated as a ZT-module by cF', dF'.

Now let $c_i = t^{-i} c t^i$ and $d_i = t^{-i} d t^i$, $i \in Z$. Then modulo F', c_i, d_i, $i \in Z$, generate F and c_i, d_i, $i \in Z$, are linearly independent. Since F is free nilpotent of class r, this implies c_i, d_i, $i \in Z$, freely generate F. Then by Lemma 3, $\gamma_r F$ is a free abelian group freely generated by the basic commutators of weight r.

We shall show $\gamma_r F$ is a free ZT-module by exhibiting a free set of generators for it.

We introduce the notation $*$ for the scalar multiplication in $\gamma_k F / \gamma_{k+1} F$ (qua ZT-module), $1 \le k \le r$, where

$$x \gamma_{k+1} F * t^i = t^{-i} x t^i \gamma_{k+1} F, \qquad x \in \gamma_k F, i \in Z.$$

Furthermore we shall adopt the notations of §3.

The case when the class $r = 2$. In this case, the basic commutators of weight 2 in $\gamma_2 F$ are of the form $b_k \circ b_l$, $k > l$, $k, l \in B$ and $|b_k| = |b_l| = 1$. We denote the set of basic commutators of weight 2 by B_2.

Furthermore, notice that the basic commutators of weight 1 are simply the elements c_i and d_i, $i \in Z$. It follows from the ordering of the basic commutators of weight 1, i.e.,

$$\cdots < c_{-1} < c_0 < c_1 < \cdots < \cdots < d_{-1} < d_0 < d_1 < \cdots$$

that the elements of B_2 can be divided, according to their macro-forms into the following disjoint subsets:

$$R_1 = \{ c_i \circ c_j | i > j, i, j \in Z \}, \qquad R_2 = \{ d_u \circ c_v | u, v \in Z \},$$

$$R_3 = \{ d_i \circ d_j | i > j, i, j \in Z \},$$

i.e., $B_2 = R_1 \cup R_2 \cup R_3$.

Now observe that modulo $\gamma_2 F$, the elements c_0 and d_0 from a free set of generators for the ZT-module F/F'. Furthermore notice that

$$c_0 F' * t^p = c_p F', \qquad p \in Z; \qquad d_0 F' * t^p = d_p F', \qquad p \in Z.$$

We shall make use of the above trivial observation to prove that $\gamma_2 F$ is a free ZT-module. So let

$$T_1 = \{ c_m \circ c_0 | m > 0 \}, \qquad T_2 = \{ d_m \circ c_0 | m \in Z \}, \qquad T_3 = \{ d_m \circ d_0 | m > 0 \},$$

and put $S_2 = T_1 \cup T_2 \cup T_3$. We claim that S_2 is a free set of generators for the ZT-module $\gamma_2 F$.

LEMMA 5. *$S_2 = T_1 \cup T_2 \cup T_3$ generates the ZT-module $\gamma_2 F$.*

PROOF. Observe that

$$c_m \circ c_0 * t^p = c_{m+p} \circ c_p, \qquad m > 0, p \in Z,$$

$$d_m \circ c_0 * t^p = d_{m+p} \circ c_p, \qquad m, p \in Z,$$

$$d_m \circ d_0 * t^p = d_{m+p} \circ d_p, \qquad m > 0, p \in Z. \tag{4.1}$$

The right-hand sides of the equations (4.1) comprise the whole of B_2. Hence the lemma.

LEMMA 6. T_i is linearly independent over ZT for $i = 1, 2, 3$.

PROOF. We shall discuss only the case $i = 1$. The other cases are similar and will be left to the reader. Suppose then

$$c_{m_1} \circ c_0 * f_{m_1} + c_{m_2} \circ c_0 * f_{m_2} + \cdots + c_{m_n} \circ c_0 * f_{m_n} = 0 \qquad (4.2)$$

where $0 < m_1 < m_2 < \cdots < m_n$ and $f_{m_i} \in ZT$, $i = 1, 2, \ldots, n$. We may suppose $f_{m_1} \neq 0$. On multiplying equation (4.2) by a suitable power of t, we may assume that f_{m_1} has the form

$$f_{m_1} = a_0 + a_1 t + \cdots + a_j t^j, \qquad j \geqslant 0, a_0 \neq 0.$$

Then, by virtue of equation (4.1), the basic commutator $c_{m_1} \circ c_0$ has coefficient a_0 ($\neq 0$) and arises once and only once in equation (4.2). Hence from the Z-linear independence of the elements of B_2, we must have $f_{m_1} = 0$. This contradicts the choice of f_{m_1} at the outset and so the lemma follows.

LEMMA 7. $S_2 = T_1 \cup T_2 \cup T_3$ is linearly independent over ZT.

PROOF. Suppose that

$$\sum_m c_m \circ c_0 * f_m + \sum_u d_u \circ c_0 * f_u + \sum_v d_v \circ d_0 * f_v = 0 \qquad (4.3)$$

where the ith summation sign represents a finite ZT-linear sum of elements of T_i, $i = 1, 2, 3$.

Then, by virtue of equation (4.1) and the Z-linear independence of the elements of B_2, it follows that

$$\sum_m c_m \circ c_0 * f_m = \sum_u d_u \circ c_0 * f_u = \sum_v d_v \circ d_0 * f_v = 0.$$

Now by Lemma 6, we have $f_m = f_u = f_v = 0$. Hence the lemma.

Thus we have proved that S_2 is a free set of generators for the ZT-module $\gamma_2 F$.

The case when the class $r > 2$. In this case the basic commutators of weight r in $\gamma_r F$ are of the form $b_k \circ b_l \circ b_m$, $k > l$, $m \geqslant l$, $k, l, m \in B$ and b_k, b_l, b_m are of weights less than r. We denote the set of basic commutators of weight r by B_r.

Now, the elements of B_r can be divided into disjoint subsets according to their macro-forms as follows (see the remarks following Definition 3):

$$B_r = \bigcup_{\text{disjoint}} T_{\phi\psi}$$

where

$$T_{\phi\psi} = \{ b_k \circ b_l \circ b_m | b_k \circ b_l \circ b_m \in B_r, b_k \circ b_l \text{ has macro-form } \phi,$$
$$b_m \text{ has macro-form } \psi \}.$$

Notice that the macro-form of the element $b_k \circ b_l \circ b_m$ in $T_{\phi\psi}$ is $\phi\psi$. Notice further that the macro-forms $\phi\psi$, $\phi'\psi'$ are distinct unless ϕ, ϕ' are identical macro-forms and ψ, ψ' are identical macro-forms.

We now construct, inductively, a free set of generators for the ZT-module $\gamma_r F$. To do so put

$$S_{\phi\psi} = \left\{ \begin{array}{l} \bar{b}_k \circ \bar{b}_l \circ b_m | \bar{b}_k \circ \bar{b}_l \circ b_m \in T_{\phi\psi} \text{ and } \bar{b}_k \circ \bar{b}_l \gamma_{i+1} F \text{ ranges} \\ \text{over the already selected free set of generators} \\ \text{of macro-form } \phi \text{ of the } ZT\text{-module } \gamma_i F / \gamma_{i+1} F, 1 \leqslant i \leqslant r - 1 \end{array} \right\}.$$

Notice the definition of $S_{\phi\psi}$ makes tacit use of the obvious inductive assumptions. Now let

$$S_r = \bigcup_{\text{disjoint}} S_{\phi\psi}.$$

Then we claim that S_r is a free set of generators for the ZT-module $\gamma_r F$.

LEMMA 8. $T_{\phi\psi} = \bigcup_{n \in Z} S_{\phi\psi} * t^n$.

PROOF. Let $b_k \circ b_l \circ b_m \in T_{\phi\psi}$ where $b_k \circ b_l$ and b_m are basic commutators of weight i and j respectively with $1 \leqslant i, j < r$ and $i + j = r$. Hence, inductively in the ZT-module $\gamma_i F / \gamma_{i+1} F$, there exists an element $\bar{b}_{k_1} \circ \bar{b}_{l_1} \gamma_{i+1} F$ of the free set of generators such that

$$b_k \circ b_l \gamma_{i+1} F = \bar{b}_{k_1} \circ \bar{b}_{l_1} \gamma_{i+1} F * t^p$$

for some $p \in Z$. Now in the ZT-module $\gamma_j F / \gamma_{j+1} F$ observe that

$$b_m \gamma_{j+1} F * t^{-p} = b_{m_1} \gamma_{j+1} F$$

where b_{m_1} is the appropriate basic commutator obtained by adding $-p$ to each index in b_m (see also the comments following (i.b) below). Now form the commutator $\bar{b}_{k_1} \circ \bar{b}_{l_1} \circ b_{m_1}$ and note that

$$b_k \circ b_l \circ b_m = \bar{b}_{k_1} \circ \bar{b}_{l_1} \circ b_{m_1} * t^p.$$

It remains only to check $\bar{b}_{k_1} \circ \bar{b}_{l_1} \circ b_{m_1} \in S_{\phi\psi}$.

To do this we have to show

(i) $\bar{b}_{k_1} \circ \bar{b}_{l_1} \circ b_{m_1} \in T_{\phi\psi}$ and

(ii) $\bar{b}_{k_1} \circ \bar{b}_{l_1} \gamma_{i+1} F$ belongs to the free set of generators of the ZT-module $\gamma_i F / \gamma_{i+1} F$ where $1 \leqslant i \leqslant r - 1$.

We need only to verify (i) since (ii) is already satisfied. To do so we prove

(i.a) $\bar{b}_{k_1} \circ \bar{b}_{l_1}$ has macro-form ϕ and b_{m_1} has macro-form ψ,

(i.b) $\bar{b}_{k_1} \circ \bar{b}_{l_1} \circ b_{m_1}$ is a basic commutator of weight r, i.e., $k_1 > l_1$ and $m_1 \geqslant l_1$.

Now if b is a commutator of weight n and macro-form α, then $b * t^p$ is also a commutator of weight n and macro-form α for all $p \in Z$. So it follows that $\bar{b}_{k_1} \circ \bar{b}_{l_1}$ and $b_k \circ b_l$ have the same macro-form ϕ and b_{m_1} and b_m have the same macro-form ψ. Hence $\bar{b}_{k_1} \circ \bar{b}_{l_1} \circ b_{m_1}$ has macro-form $\phi\psi$. Furthermore, the weight of $\bar{b}_{k_1} \circ \bar{b}_{l_1} \circ b_{m_1}$ is obviously r.

Now if b_r and b_s are two basic commutators with $b_r < b_s$, then $b_r * t^p < b_s * t^p$ for all $p \in Z$ (see Corollary 2). Since $b_k \circ b_l \circ b_m$ is a basic commutator we have $b_k > b_l$ and $b_m \geqslant b_l$. Since $\bar{b}_{k_1} = b_k * t^{-p}$, $\bar{b}_{l_1} = b_l * t^{-p}$ and $b_{m_1} = b_m * t^{-p}$, it follows that $\bar{b}_{k_1} > \bar{b}_{l_1}$ and $b_{m_1} \geqslant \bar{b}_{l_1}$. In other words $k_1 > l_1$ and $m_1 \geqslant l_1$.

Hence $\bar{b}_{k_1} \circ \bar{b}_{l_1} \circ b_{m_1} \in T_{\phi\psi}$ and the proof of the lemma is complete.

LEMMA 9. $S_r = \bigcup_{\text{disjoint}} S_{\phi\psi}$ *generates the ZT-module* $\gamma_r F$.

PROOF. Since

$$B_r = \bigcup_{\text{disjoint}} T_{\phi\psi}$$

and $T_{\phi\psi} = \bigcup_{n \in z} S_{\phi\psi} * t^n$ by Lemma 8, the result follows.

We observe that $S_{\phi\psi}$ can be further divided into disjoint subsets $S_{k \circ l}$ where

$$S_{k \circ l} = \left\{ \bar{b}_k \circ \bar{b}_l \circ b_m \middle| \bar{b}_k \circ \bar{b}_l \circ b_m \in S_{\phi\psi} \text{ and all the } \bar{b}_k \circ \bar{b}_l \text{ are identical} \right\}.$$

Hence

$$S_{\phi\psi} = \bigcup_{\text{disjoint}} S_{k \circ l}.$$

Thus a finite ZT-linear sum of elements of $S_{\phi\psi}$ will be a ZT-linear sum of elements from a finite number of subsets $S_{k \circ l}$.

LEMMA 10. $S_{\phi\psi} = \bigcup_{\text{disjoint}} S_{k \circ l}$ *is linearly independent over ZT.*

PROOF. Suppose that

$$\sum_u \bar{b}_{k_1} \circ \bar{b}_{l_1} \circ b_u * f_u + \sum_v \bar{b}_{k_2} \circ \bar{b}_{l_2} \circ b_v * f_v$$

$$+ \cdots + \sum_w \bar{b}_{k_s} \circ \bar{b}_{l_s} \circ b_w * f_w = 0 \qquad (4.4)$$

is a finite ZT-linear sum of elements of $S_{\phi\psi}$ where the ith summation sign represents a finite linear ZT-linear sum of elements of $S_{k_i \circ l_i}$, $i = 1, 2, \ldots, s$. Writing in full the first summation sign of equation (4.4) we get

$$\bar{b}_{k_1} \circ \bar{b}_{l_1} \circ b_{u_1} * f_{u_1} + \bar{b}_{k_1} \circ \bar{b}_{l_1} \circ b_{u_2} * f_{u_2} + \cdots + \bar{b}_{k_1} \circ \bar{b}_{l_1} \circ b_{u_z} * f_{u_z}$$

$$+ \sum_v \bar{b}_{k_2} \circ \bar{b}_{l_2} \circ b_v * f_v + \cdots + \sum \bar{b}_{k_s} \circ \bar{b}_{l_s} \circ b_w * f_w = 0 \qquad (4.5)$$

where
  (i) $u_i \in B$, $1 \leqslant i \leqslant z$,
  (ii) $u_1 < u_2 < \cdots < u_z$,
  (iii) $f_{u_i} \in ZT$, $1 \leqslant i \leqslant z$.
We may suppose $f_{u_1} \neq 0$. On multiplying equation (4.5) by a suitable power of t we may assume that f_{u_1} has the form

$$f_{u_1} = a_0 + a_1 t + \cdots + a_j t^j, \qquad j \geqslant 0, a_0 \neq 0.$$

Now $\bar{b}_{k_i} \circ \bar{b}_{l_i} * t^p \not\equiv \bar{b}_{k_i} \circ \bar{b}_{l_i}$ for all $p \in Z$, $2 \leqslant i \leqslant s$, since $\{\bar{b}_{k_1} \circ \bar{b}_{l_1}\gamma_{i+1}F$, $\bar{b}_{k_2} \circ \bar{b}_{l_2}\gamma_{i+1}F, \ldots, \bar{b}_{k_s} \circ \bar{b}_{l_s}\gamma_{i+1}F\}$ is inductively part of a free set of generators of the appropriate ZT-module. It follows that the basic commutator $\bar{b}_{k_1} \circ \bar{b}_{l_1} \circ b_{u_1}$ has coefficient a_0 ($\neq 0$) and arises once and only once in equation (4.5). Hence from the Z-linear independence of the elements of B_r, we have $f_{u_1} = 0$. This contradicts the choice of f_{u_1} at the outset and so the lemma follows.

LEMMA 11. $S_r = \bigcup_{\text{disjoint}} S_{\phi\psi}$ is linearly independent over ZT.

PROOF. Suppose that

$$\sum \bar{b}_k \circ \bar{b}_l \circ b_m * f_\alpha + \sum \bar{b}_u \circ \bar{b}_v \circ b_w * f_\beta$$
$$+ \cdots + \sum \bar{b}_x \circ \bar{b}_y \circ b_z * f_\gamma = 0 \qquad (4.6)$$

is a finite ZT-linear sum of elements of S_r, where the first summation sign represents a finite ZT-linear sum of elements of $S_{\phi\psi}$, the second summation sign represents a finite ZT-linear sum of elements of $S_{\phi'\psi'}$, etc., and the macro-forms $\{\phi\psi, \phi'\psi', \ldots\}$ are distinct.

Now notice that if $b_k \circ b_l \circ b_m$ has macro-form $\phi\psi$, then $b_k \circ b_l \circ b_m * t^p$ has macro-form $\phi\psi$ for all $p \in Z$ (see Lemma 8) and the elements of B_r are linearly independent over Z. So it follows that

$$\sum \bar{b}_k \circ \bar{b}_l \circ b_m * f_\alpha = \sum \bar{b}_u \circ \bar{b}_v \circ b_w * f_\beta$$
$$= \cdots = \sum \bar{b}_x \circ \bar{b}_y \circ b_z * f_\gamma = 0.$$

So the lemma now follows immediately from Lemma 10.

We have now shown that the set S_r forms a free set of generators for the ZT-module $\gamma_r F$ and hence completed the proof of Theorem 2'.

5. Consequences. We shall derive some consequences to the above theorems. To begin we state, without proof, the following:

LEMMA 12. *Let H be a two-generator group, say $H = gp(a, b)$. Then H is free if and only if the set $\{b_j = a^{-j}ba^{-j} | j \in Z\}$ freely generates a free group.*

Now we prove

THEOREM 3. *Let $1 \to F \to G \to T \to 1$ be a short exact sequence where F is parafree and $T = \langle t \rangle$ is infinite cyclic. If F/F' is a free ZT-module, then every two-generator subgroup of G is free.*

PROOF. Let H be any two-generator subgroup of G, say $H = gp(a, b)$. If $a \in F$, $b \in F$ then H belongs to F and hence is free. We divide the remainder of the proof into two cases.

Case 1. $a \in G \setminus F, b \in F$.

(i) $a = t^n, b \in F$.
We may assume that $b \in \gamma_r F$ but $b \notin \gamma_{r+1} F$ where $r \geqslant 1$. But by Theorem 2, $\gamma_r F / \gamma_{r+1} F$ is a free ZT-module. Hence modulo $\gamma_{r+1} F$, we have

$$b * \sum_{\text{finite}} e_i t^i = 0 \text{ if and only if } e_i = 0 \quad \text{for all } e_i$$

i.e. $\prod (t^{-i}bt^i)^{e_i} \equiv 1$ if and only if $e_i = 0$ for all e_i i.e., modulo $\gamma_{r+1} F$, the set $\{b_i = t^{-i}bt^i, i \in Z\}$ is independent over Z. This implies that $gp(b_i = t^{-i}bt^i | i \in Z)$ is free on the b_i's. Hence by Lemma 12, the group $H = gp(a, b)$ is free.

(ii) $a = t^n f, b \in F$ where $1 \neq f \in F$.
Observe that modulo $\gamma_{r+1} F$ we have $(t^n f)^{-1} b (t^n f) \equiv t^{-n} b t^n$ since $b \in \gamma_r F$ by assumption. So this subcase can be reduced to the previous subcase.

Case 2. $a \in G \setminus F$, $b \in G \setminus F$.

We shall reduce this case to Case 1. Let $a = t^{n_1}f_1$, $b = t^{n_2}f_2$ where f_1, $f_2 \in F$, n_1, $n_2 \in Z$. We can further assume $n_1 > 0$, $n_2 > 0$. Then

$$H = gp(a, b) = gp(t^{n_1}f_1, t^{n_2}f_2).$$

If $n_1 = n_2$, then

$$H = gp\left(t^{n_1}f_1, t^{n_2}f_2(t^{n_1}f_1)^{-1}\right) = gp(t^{n_1}f_1, f)$$

where $f \in F$ and we are in Case 1. If $n_1 < n_2$, then choose k such that $n_2 = kn_1 + n_3$ with $0 \leqslant n_3 < n_1$. Then

$$H = gp\left(t^{n_1}f_1, t^{n_2}f_2(t^{n_1}f_1)^{-k}\right) = gp(t^{n_1}f_1, t^{n_3}f)$$

where $f \in F$. Now repeating this process we find

$$H = gp(t^n g, h), \qquad g, h \in F,$$

and we are again in Case 1. This completes the proof of the theorem.

Similarly we have

THEOREM 4. *Let* $1 \to F \to G \to T \to 1$ *be a short exact sequence where F is parafree and $T = \langle t \rangle$ is infinite cyclic. If F/F' is a finitely generated free ZT-module freely generated by $\{cF', dF', \ldots, eF'\}$, then the group $H = gp(t, c, d, \ldots, e)$ is free.*

PROOF. Notice that the set $\{t^{-i}ct^i, t^{-i}dt^i, \ldots, t^{-i}et^i; i \in Z\}$ freely generates a free group and hence the theorem follows.

We next show

THEOREM 5. *Let* $1 \to F \to G \to T \to 1$ *be a short exact sequence where F is parafree and $T = \langle t \rangle$ is infinite cyclic. If F/F' is a free ZT-module and N is a finitely generated normal subgroup of G contained in F, i.e., $N \lhd G$, $N \leqslant F$, then N is trivial, i.e., $N = 1$.*

PROOF. Suppose $N \neq 1$. Then there exists an $i \geqslant 1$ such that $N \leqslant \gamma_i F$, $N \nleqslant \gamma_{i+1}F$. Then $N\gamma_{i+1}F/\gamma_{i+1}F$ is a submodule of $\gamma_i F/\gamma_{i+1}F$ and $N\gamma_{i+1}F/\gamma_{i+1}F$ is a finitely generated abelian group since $N\gamma_{i+1}F/\gamma_{i+1}F \cong N/N \cap \gamma_{i+1}F$.

Now let $a \in N$, $a \notin \gamma_{i+1}F$. Then working modulo $\gamma_{i+1}F$, the conjugates of a by the powers of t are independent since $\gamma_i F/\gamma_{i+1}F$ is a free ZT-module. Hence $N\gamma_{i+1}F/\gamma_{i+1}F$ contains a free abelian group of infinite rank. This is a contradiction and hence $N = 1$.

6. An example. We prove the following example (see G. Baumslag [7]).

THEOREM 6. *Let i and j be nonzero integers and*

$$G = \langle a, b, c; a = [c^i, a][c^j, b] \rangle.$$

Then (i) *G is parafree.*

(ii) *G is the third term of a short exact sequence $1 \to F \to G \to T \to 1$ where F is free, T is infinite cyclic and F/F' is a free ZT-module.*

(iii) *G is not free.*

PROOF. Let $F = gp_G(a, b)$. By the Reidemeister-Schreier rewriting procedure (see W. Magnus, A. Karrass and D. Solitar [21]),

$$F = \langle \ldots, a_{-1}, a_0, a_1, \ldots, \ldots, b_{-1}, b_0, b_1, \ldots ;$$

$$\ldots, a_n = a_{n+i}^{-1} a_n b_{n+j}^{-1} b_n, \ldots \rangle$$

where $a_k = c^{-k} a c^k$, $b_k = c^{-k} b c^k$, $k \in Z$. We may assume $j > 0$ and it follows that F is a free group freely generated by the elements $\ldots, a_{-1}, a_0, a_1, \ldots, b_0, b_1, \ldots, b_{j-1}$. Obviously G/F is free cyclic on cF'.

Again from the defining relations of F, it follows that F/F' is a free abelian group freely generated by $\ldots, b_{-1}, b_0, b_1, \ldots$. Hence F/F' is a free ZT-module where $T = \langle c \rangle$.

By Theorem 1, G is parafree and, by G. Baumslag [5], G is not free.

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