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ON OSCILLATORY ELLIPTIC EQUATIONS ON MANIFOLDS

BY

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ABSTRACT. In this note we investigate the possibility of oscillatory behavior for a second-order selfadjoint elliptic operators on noncompact Riemannian manifolds (M, g). Let A be such an operator which is semibounded below and symmetric on $C_0^\infty(M) \subseteq L^2(M, d\mu)$ where $d\mu$ is a volume element on M. If φ is a C^∞ function such that $A\varphi = \lambda\varphi$, we would naively say that φ is oscillatory (and by extension λ is oscillatory if it possesses such an eigenfunction φ) if $M - \varphi^{-1}(0)$ has an infinite number of bounded connected components. For technical reasons this is not quite adequate for a definition. However, in §1 we give the usual definition of oscillatory which is a slight generalization of the one above. Let Λ_0 be the number below which this phenomenon cannot occur; Λ_0 is the oscillatory constant for the operator A. In that A is semibounded and symmetric on $C_0^\infty(M) \subseteq L^2(M, d\mu)$, A has a Friedrichs extension. Let Λ_c be the bottom of the continuous spectrum of the Friedrichs extension of A. Our main result is $\Lambda_0 = \Lambda_c$.

This is the exact analogue of a theorem first proved by Hartman and Putnam [6], when M is the real line. More recently, John Piepenbrink [7] generalized [6] to $\mathbf{R}^n$, requiring in the process, a technical condition (see [7, (3.1)]), which is, roughly, a restriction on the growth at infinity of the coefficients of the principal part of A. For example, if

$$A = -\sum \frac{\partial}{\partial x_i}\left[\left(1 + |x|^2\right)^{1/2}\right]^\alpha \frac{\partial}{\partial x_i},$$

then Piepenbrink's condition (3.1) holds if and only if $\alpha \leq 2$. In our generalization to arbitrary noncompact manifolds, no restrictions are placed upon the behavior of A at large distances. More significantly, the equality $\Lambda_0 = \Lambda_c$ is valid regardless of the "amount of topology" that M may possess. In [7] this type of difficulty is obviated by the fact that $\mathbf{R}^n$ is contractible.

The proof depends upon two theorems, Theorem A and Theorem B. Theorem B is a geometric characterization of Λ_c due to one of the authors in [1], while Theorem A is a deformation of domain argument in the spirit of [3], [8], [9]. The paper is arranged in the following manner. In §1 we give the necessary definitions, state Theorems A and B and show how the equality follows from Theorems A and B. The remainder of the paper is a proof of Theorem A. §2 is the analytic part of the proof of Theorem A. The core of the argument is the "crushed ice" estimates of Rauch and Taylor [8], [9], with which we prove Theorem A, provided we have a fine enough "even spacing lemma". This "even spacing lemma", is the geometric-

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topological part of the argument, as well as the most technical, and is exposed in §3. An interesting aspect of this lemma, is the need to use a triangulation of the manifold, which is a good approximation to the Riemannian structure.

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1. Definitions and the statements of the results. Let M be a manifold possibly with boundary. Let g be a Riemannian metric on M, and let μ be a smooth volume element on M, i.e., $\mu = f\,dv$ where f is a smooth positive function, and where dv is the volume element g induces on M. For each smooth vector field v on M, we define the smooth functions $\operatorname{div}_\mu v$, the divergence of v with respect to μ, by the formula $d(i_v\mu) = (\operatorname{div}_\mu v)\mu$. If f is a C^∞ function defined on M, then

$$\operatorname{div}_\mu fv = f \operatorname{div}_\mu v + df(v).$$

Finally, if f is a smooth function we define the elliptic operator

$$\Delta_\mu f = -\operatorname{div}_\mu \operatorname{grad} f,$$

which is symmetric because

$$\int_M \varphi \Delta_\mu \psi \, d\mu = \int_M \Delta_\mu \varphi \psi \, d\mu = \int_M \operatorname{grad} \varphi \cdot \operatorname{grad} \psi \, d\mu,$$

where $\varphi, \psi \in C_0^\infty(\operatorname{Int} M)$ are smooth functions with compact support in the interior of M, and $\cdot$ is the fiberwise inner product given by g.

The general operator we want to consider is of the form $A = \Delta_\mu + c$ where c is a continuous function, such that $(A\varphi, \varphi) \geqslant l(\varphi, \varphi)$ for some $l > -\infty$, where $(\ ,\ )$ denotes the inner product on $L^2(M, d\mu)$, and $\varphi \in C_0^\infty(M)$. Thus A is semibounded and symmetric.

Let U be an open set in M and let $C_0^\infty(U)$ be the C^∞ functions with compact support in U. We then define

$$\lambda_A(U) = \inf_{\varphi \in C_0^\infty(U),\, \varphi \not\equiv 0} \frac{(A\varphi, \varphi)}{(\varphi, \varphi)}.$$

Note that if $U \subset V$, then $\lambda_A(U) \geqslant \lambda_A(V)$.

THEOREM A. *Let M_0 be a compact manifold with boundary $\partial M_0 = N$. Let g be a Riemannian metric on M_0 and $d\mu$ be a volume element on M_0. Let c be a continuous function on M_0, and set $A = -\operatorname{div}_\mu \operatorname{grad} + c$. Let $\Lambda = \lambda_A(\operatorname{int} M_0)$, and pick any number $l > \Lambda$. Then we can find $M_1 \subseteq M_0$, a compact submanifold with boundary, and a smooth* 1*-parameter family of diffeomorphism $h_t\colon M_1 \to M_0$ such that*

(i) $\lambda_A(\operatorname{int} M_1) > l$,

(ii) $h_0 = \operatorname{id}$, $h_1(M_1) = M_0$,

(iii) $h_t(M_1) \subseteq h_{t'}(M_1)$ *if* $t \leqslant t'$, *and thus* $\lambda(t) = \lambda_A(\operatorname{int}(h_{1-t}(M_1)))$ *is monotone increasing and continuous.*

Let M be a smooth manifold without boundary. Let g, $d\mu$ and A be as above. Then A defines a symmetric linear map on $C_0^\infty(M) \subseteq L^2(M, d\mu)$ which is semibounded below. We can then form the Friedrichs extension [**10**, p. 329] which we will also denote by A. The Friedrichs extension is a selfadjoint operator on its domain D_A and possesses a spectral decomposition. Thus we can speak of the discrete and the continuous part of the spectrum. In particular, let $\lambda_c(A)$ denote the bottom of the continuous spectrum of A, i.e., $\lambda_c(A) = \inf\{\lambda\colon \lambda \text{ is in the continuous spectrum of } A\}$. Baider **[1]** has given the following geometric characterization of $\lambda_c(A)$.

THEOREM B. $\lambda_c(A) = \sup_{K \text{ compact}} \lambda_A(M - K)$.

We will now begin to discuss oscillatory behavior. Let M be a smooth manifold, $\partial M = \emptyset$. Let g, $d\mu$ and A be as before. An open connected set in M will be referred to as a *domain*. If the boundary of a domain Ω, $\partial\Omega = \overline{\Omega} - \Omega$ is a smooth $n - 1$ dimensional manifold, then we call Ω a *regular domain*. A domain is *bounded* if $\overline{\Omega}$ is compact.

DEFINITION. Let λ be a real number. $\Omega \subseteq M$ a domain. We call Ω a *λ-domain for A* if Ω is bounded and if $\lambda_A(\Omega) = \lambda$. Notice that if Ω is a regular λ-domain (this clearly can be weakened some), then $\lambda_A(\Omega)$ is the lowest eigenvalue for A, with Dirichlet boundary data on $\partial\Omega$.

PROPOSITION. *Let $\lambda \in \mathbf{R}$. The following two statements are equivalent.*

(a) *For each compact subset $K \subseteq M$, we can find a regular λ-domain Ω such that $\overline{\Omega} \subseteq M - K$.*

(b) *For each compact subset $K \subseteq M$, we can find a $\tilde{\lambda}$-domain $\tilde{\Omega}$, such that $\overline{\tilde{\Omega}} \subseteq M - K$ and $\tilde{\lambda} \leqslant \lambda$.*

PROOF. It is trivial to see that (a) implies (b).

Let $M = \bigcup_{i=1}^{\infty} U_i$, U_i open, regular bounded domains where $\overline{U_i} \subseteq U_{i+1}$. For each compact set K, we can find $i(K)$ such that $K \subseteq \overline{U_{i(K)}}$ and hence $M - \overline{U_{i(K)}} \subseteq M - K$. If (b) holds, we can find a $\tilde{\lambda}$-domain Ω such that $\overline{\Omega} \subseteq V \subseteq M - \overline{U_{i(K)}}$, where V is the connected component of $M - \overline{U_{i(K)}}$ containing $\overline{\Omega}$ and $\tilde{\lambda} < \lambda$. If V is bounded, it is therefore a regular bounded domain, such that $\partial V \cap \overline{\Omega} = \emptyset$. Thus we can find a bounded, regular domain W, $\overline{\Omega} \subseteq W \subseteq \overline{W} \subseteq V$. Therefore, W is a regular λ'-domain, $\lambda' \leqslant \lambda$. (a) then follows from Theorem A. Finally if V is not bounded, we can clearly find a regular bounded domain W, such that $\overline{\Omega} \subseteq W \subseteq \overline{W} \subseteq V$. (a) then follows as above.

DEFINITION. We say λ is *oscillatory* for A if λ satisfies (a) or (b) of the proposition. If λ is oscillatory and if $\lambda \leqslant \mu$, then (b) of the proposition shows that μ is also oscillatory. Thus we can define the *oscillation constant* $\Lambda_0(A)$ of A to be

$$\Lambda_0(A) = \inf\{\lambda\colon \lambda \text{ is oscillatory}\}.$$

THEOREM. $\Lambda_c(A) = \Lambda_0(A)$.

PROOF. Let $M = \bigcup_{i=1}^{\infty} U_i$, U_i all regular bounded domains, such that $\overline{U_i} \subseteq U_{i+1}$. Thus $\lambda_A(M - \overline{U_i}) \leqslant \lambda_A(M - \overline{U_{i+1}})$ and it follows easily from Theorem B that

$$\Lambda_c(A) = \sup \lambda_A\big(M - \overline{U_i}\big) = \lim_{i\to\infty} \lambda_A\big(M - \overline{U_i}\big).$$

Assume λ is oscillatory. For each i, we can find a regular λ-domain in $M - \overline{U_i}$ and thus a regular λ-domain in $U_k - \overline{U}_i$ for some $k > i$. Hence $\lambda_A(M - \overline{U}_i) = \inf_{l > k}(U_l - \overline{U}_i) \leqslant \lambda$. which implies that $\lambda_c(A) \leqslant \lambda_0(A)$. Conversely, let $\lambda < \lambda_0(A)$. Choose an $\varepsilon > 0$ such that $\lambda + \varepsilon < \lambda_0(A)$. We can then find some U_i such that $M - \overline{U}_i$ contains no $\tilde{\lambda}$-domains, $\tilde{\lambda} \leqslant \lambda + \varepsilon$. Hence $\lambda_A(M - \overline{U}_i) \geqslant \lambda$ which implies $\lambda_c(A) \geqslant \lambda_0(A)$.

2. Crushed ice. For the remainder of the note let (M, g) be a compact, connected m-dimensional Riemannian manifold and let $M_0 \subseteq M$ be a compact m-dimensional manifold with $\partial M_0 = N$. Let $d\mu$ be a volume element on M, let c be a continuous function on M_0, let $A = -\operatorname{div}_\mu \operatorname{grad} + c$, and let $\Delta_g = -\operatorname{div}_g \operatorname{grad}$ be the usual Laplacian. $d\mu = f\, dv$ where $f > 0$ and dv is the Riemannian volume element. Then we can find a real number $K > 0$ such that $(1/K)d\mu \leqslant dv < K\, d\mu$ and let $\alpha = \inf c(x)$. If ψ is a smooth function on M_0, then $\int \|\operatorname{grad} \psi\|^2 d\mu \geqslant (1/K)\int \|\operatorname{grad} \psi\|^2 dv$, and $\int \psi^2 d\mu \leqslant K \int \psi^2 dv$. Thus if ψ is the first nontrivial eigenfunction for the Dirichlet problem for $M_0 \subseteq M$, we see that

$$\lambda_A(M_0) \geqslant \frac{1}{K^2} \frac{\int \|\operatorname{grad} \psi\|^2 dv}{\int \psi^2 dv}.$$

Therefore, $\lambda_A(M_0) \geqslant (1/K^2)\lambda_\Delta(M_0) + \alpha$, and it suffices to prove Theorem A in the case where the operator $A = \Delta_g = \Delta$. We will denote $\lambda_A(\ )$ by $\lambda(\ )$, and furthermore denote by $B(x, r)$, $x \in M$ and r a positive real, the open metric ball centered at x of radius r.

PROPOSITION 2.1. *Let (M, g), $M_0 \subseteq M$, $\partial M_0 = N_0$ be as before and let $S = (x_1, \ldots, x_K)$ be a finite set of points in M_0 and let $0 < r_0 < R_0$ be real numbers such that $2r_0 < R_0$ and*

(i) *$B(x_i, r_0) \subseteq \operatorname{Int} M_0$,*

(ii) *$\{B(x_i, R_0)\}$ cover M_0,*

(iii) *$R_0 <$ the injectivity radius of M,*

(iv) *no point of M_0 can be in more than N of sets $B(x_i, R_0)$.*

Set $M_1 = M_0 - \bigcup_{i=1}^{K} B(x_i, r_0)$.

Then we can find a constant $c > 0$ which depends only on (M, g) such that

$$\lambda(M_1) \geqslant \begin{cases} \dfrac{cKr_0^{m-2}}{Nd\operatorname{vol}(M_0)} & \text{if } m = \dim M > 2, \\[2ex] \dfrac{cK}{Nd\operatorname{vol}(M_0)|\log r_0|} & \text{if } \dim M = 2, \end{cases}$$

where d is a number such that $KR_0^m = d\operatorname{vol}(M_0)$.

NOTE. The expression "crushed ice" for this sort of situation is due to Rauch and Taylor **[8]**, **[9]**, where they view the $B(x_i, r_0)$ as K little coolers (round ice cubes) and λ represents the "rate" of cooling. This part of the proof is essentially theirs, our contribution being the refined "even spacing" lemma.

PROOF. Let $\psi\colon M_1 \to \mathbf{R}$ be such that $\psi|\partial M_1 = 0$ and $\Delta\psi = \lambda(M_1)\psi$. Let $A_i = \{x \in M | r_0 \leqslant d(x, x_i) < R_0\}$ and let $M_2 = \cup A_i$. Extend ψ to a continuous function with square integrable weak differential on M_2, by making it zero on $M_2 - M_1$. Denote this also by ψ.

$$\int_{M_1} |\text{grad } \psi|^2 = \int_{M_2} |\text{grad } \psi|^2 \geqslant \frac{1}{N} \sum_i \int_{A_i} |\text{grad } \psi|^2.$$

If on the set A_i we introduce Riemann normal coordinates centered at x_i, and if r is the radial coordinate, we see

$$\int_{A_i} |\text{grad } \psi|^2 \geqslant \int_{A_i} \left(\frac{\partial\psi}{\partial r}\right)^2 dv.$$

Let $V = \{v \in TM| \, \|v\| \leqslant R_0\}$ and let $E\colon V \to M \times M$ be $E(v) = (p(v), \exp_{p(v)} v)$, where p is the bundle projection. If dv_E is the Euclidean volume element on the tangent spaces and if dv is the Riemannian volume element, then

$$E^*(dv(E(x, v))) = f(x, v)\, dv_E, \qquad v \in T_xM.$$

Thus f is a smooth positive function defined on a compact set and we can find some number $c > 0$ which depends on (M, g) alone such that

$$\int_{A_i} |\text{grad } \psi|^2 \geqslant c \int_{A_i} \left(\frac{\partial\psi}{\partial r}\right)^2 r^{m-1}\, dr\, d\Sigma,$$

where $d\Sigma$ is the volume element of the unit $m - 1$ sphere in $\mathbf{R}^m$. We can then apply the argument in [**9**, p. 45] to get the estimate

$$\int_{A_i} |\text{grad } \psi|^2 \geqslant c' \left(\int_{A_i} \psi^2\, dv\right) \frac{r_0^{m-2}}{R_0^m} \quad \text{for } m > 2,$$

and a slightly different estimate involving logs for $m = 2$. Therefore,

$$\int_{M_1} |\text{grad } \psi|^2 \geqslant \frac{c'}{N} \frac{r_0^{m-2}}{R_0^m} \sum_i \int_{A_i} \psi^2 \geqslant \frac{c' K r_0^{m-2}}{R_0^m K} \int_{M_1} \psi^2,$$

and hence, $\lambda(M_1) \geqslant c' r_0^{m-2}/Nd \operatorname{vol}(M_0)$ because $R_0^m K = d \operatorname{vol}(M_0)$. The case $m = 2$ follows in a similar way.

Proposition 2.1 is our main analytic lemma; we now need the even spacing notions.

DEFINITION 2.2. Let (M, g) be a Riemannian manifold possibly with boundary and let d be the induced metric. We say a sequence of finite subsets of points $S_n = \{x_1, \ldots, x_{K_n}\}$, $x_i \in \operatorname{Int} M$, are *evenly spaced* if for each n we can find a number $R(n)$ such that the balls $B(x_i, R(n))$, $1 \leqslant i \leqslant K_n$, cover M and such that we can find a number N independent of n, such that each point of M is in no more than N of the $B(x_i, R(n))$ for each n.

THEOREM 2.3 (EVEN SPACING LEMMA). *Let M be a compact connected m-dimensional Riemannian manifold and let $M_0 \subseteq M$ be a compact connected m-dimensional submanifold with boundary $\partial M_0 = N$. Then we can find sets of points $S_n = \{x_1, \ldots, x_{K_n}\}$ in M_0 which are evenly spaced in M_0. Let $R(n)$ be the radius in the* *definition. We can also find numbers* $r(n) > 0$, $r(n) < R(n)/2$ *such that the sets* $B(x_i, r(n))$ *satisfy*

(a) $B(x_i, r(n)) \subseteq \operatorname{Int}(M_0)$, $1 \leqslant i \leqslant K_n$,

(b) $B(x_i, r(n))$ *and* $B(x_i, R(n))$ *are diffeomorphic to smooth Euclidean balls,*

(c) $B(x_i, r(n))$ *are pairwise disjoint,*

(d) $K_n = K_0 2^{mn}$, $R(n) = R_0/2^n$ *and* $r(n) = r_0/2^n$.

We can now apply the Even Spacing Lemma to Proposition 2.1. We can assume the number R_0 in 2.3 is less than the injectivity radius of M. Let $M_n = M_0 - \bigcup_{i=1}^{K_n} B(x_i, r(n))$ and we see

$$\lambda(M_n) \geqslant \begin{cases} \dfrac{CK_n r(n)^{m-2}}{Nd_n \operatorname{vol}(M_0)}, & m > 2, \\ \dfrac{C}{Nd_n \operatorname{vol}(M_0)} \cdot \dfrac{K_n}{|\log r(n)|}, \end{cases}$$

where $d_n \operatorname{vol}(M_0) = R(n)^m K_n = K_0 2^{mn} R_0^m / 2^{mn} = K_0 R_0^m = d_0 \operatorname{vol}(M_0)$ and is independent of n. Hence $\lambda(M_n) \to \infty$ as $n \to \infty$ and we get the following theorem.

THEOREM 2.4. *Let* M, M_0 *and* $\partial M_0 = N$ *be as above. Then for each number* $K > 0$ *we can find* $r_0 > 0$ *and a finite set of points* $x_1, \dots, x_s$, $x_i \in M_0$, *such that*

(i) $B(x_i, r_0) \subseteq \operatorname{Int} M_0$;

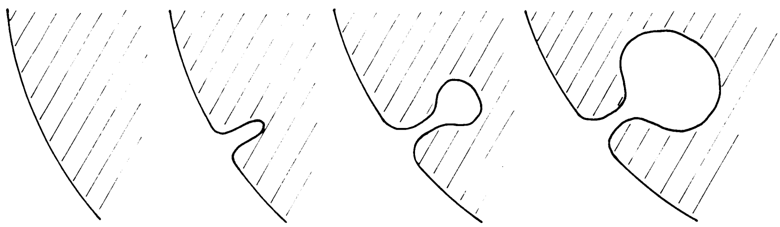
(ii) *the* $B(x_i, r_0)$ *are pairwise disjoint*;

(iii) *if* $M_1 = M_0 - \bigcup_{i=1}^{s} B(x_i, r_0)$ *then* $\lambda(M_1) \geqslant K$.

REMARK. This discussion also holds for the case where we have Neumann boundary data on $N = \partial M_0$ and zero boundary data on the boundaries of the $B(x_i, r_0)$. One follows [8], [9] as we just did and uses the Lions' reflection to extend ψ from M_0 to all of M without significantly changing its H_1 norm.

COROLLARY 2.5. *Theorem* A *for the Laplacian follows from Theorem* 2.4.

PROOF. Pick $K > 0$. Let us apply Theorem 2.4 and find a finite number of points $x_1, \dots, x_s$ and a number $r > 0$, $r <$ injectivity radius of M such that the geodesic balls $B(x_i, r) = B_i$, $1 \leqslant i \leqslant s$, are pairwise disjoint in $\operatorname{Int} M_0$ and $\lambda(M_0 - \bigcup B(x_i, r)) > K$. For each B_i let γ_i be a smooth curve joining ∂B_i to ∂M_0, which meets these boundaries perpendicularly. Thicken γ_i a bit and smooth the edges at the boundaries to form a tube denoted by τ_i. Let $O_i = B_i \cup \tau_i$ and we can assume we have chosen the γ_i and τ_i such that the O_i are pairwise disjoint. Then $\lambda(M_0 - \bigcup O_i) \geqslant \lambda(M_0 - \bigcup B_i) \geqslant K$. We can now define our diffeomorphism h_t by eliminating the O_i's one at a time, by first shrinking the ball and then withdrawing the tube as in the pictures.



3. The proof of Theorem 2.3 (the Even Spacing Lemma). We will first discuss the case where $M_0 = M$ and $\partial M_0 = \varnothing$. We will then show how to alter the proof for the general case.

Let us isometrically embed M in $\mathbf{R}^s$ for s sufficiently large. Let $\pi\colon U_\eta \to M$ be the tubular neighborhood of radius η of M in $\mathbf{R}^s$ ($U_\eta = (x \in \mathbf{R}^s|\ \|x - M\| \leqslant \eta$ and π is the projection back onto M)). Following Whitney [**12**], for any $\eta > 0$ we can find a simplicial complex K, $K \subseteq U_\eta$ (geometrically in $\mathbf{R}^s$, not just abstract) with the following properties.

(a) K is itself a topological m-dimensional manifold.

(b) $\pi|K\colon K \to M$ is a homeomorphism of K onto M.

(c) Each m-simplex $\sigma \subseteq K$ is almost parallel to the tangent plane $T_{\pi(x)}(M)$ for each $x \in \sigma$. More precisely for each m-simplex $\sigma \subseteq K$, there exists a constant $c > 0$, such that $|\det(T_\pi|\sigma)| \geqslant c > 0$.

The locally affine structure in K allows us to speak of a distance d_L between two points (the inf of the arclengths of continuous piecewise affine paths joining two points) and an m-volume v_L of measurable subsets of K. Let d and v be the corresponding distance and volume defined by the Riemannian structure on M. It is easy to see from the properties of our triangulation and the compactness of M that we can find positive numbers c_1, c_2, d_1 and d_2 such that

(A) for $x, y \in K$, $c_1 d(\pi(x), \pi(y)) \leqslant d_L(x, y) \leqslant c_2 d(\pi(x), \pi(y))$;

(B) for any measurable $S \subseteq K$, $d_1 v(\pi(S)) \leqslant v_L(S) \leqslant d_2 v(\pi(S))$.

Let σ be an m-simplex of K, $\sigma = p_0, \ldots, p_m$. Set $v_i = p_i - p_0$. Then σ is congruent to an m-simplex with 0 as a vertex and $v_1, \ldots, v_m$ as edges. For each σ we can then form $2^m - 1$ other m-simplices $\sigma(i_1, \ldots, i_m)$ where 0 is a vertex and the edges are $i_j v_j$, $i_j = \pm 1$ $(\sigma \rightleftarrows \sigma(1, \ldots, 1))$.

In each $\sigma \subseteq K$, there is a point $x_\sigma \in \sigma$ which is the center of the largest inscribed ball $B_L(x_\sigma, r_\sigma)$ of radius $r(\sigma)$. Let $r_\sigma(i, \ldots, i_m)$ be the radius of the corresponding inscribed ball in $\sigma(i_1, \ldots, i_m)$, and set

$$\tilde{r}_0 < \inf_\sigma (r_\sigma(i_1, \ldots, i_n)/2).$$

Let $p_\sigma = \pi(x_\sigma)$, pick $r_0 < \tilde{r}_0/c_2$ and note that the ball

$$B(p_\sigma, r_0) = \{y|d(y, p_\sigma) < r_0\} \leqslant \pi(B_L(x_\sigma, \tilde{r}_0)).$$

Hence the balls $B(p_\sigma, r_0)$ are disjoint.

Let K_1 be the standard subdivision of K. For all the properties of K_1 and its definition see [**12**]; we will need the following facts. Each m-simplex σ is subdivided into 2^m m-simplices of K_1. Furthermore if $\sigma = p_0, \ldots, p_m$ is an m-simplex of K, form the congruent m-simplex with edges $v_i = p_i - p_0$ and a vertex at 0. Then each of the m-simplices σ^1 of K_1 into which σ has been subdivided is congruent to one of the simplices with a vertex 0 and edges $i_j v_j$, $i_j = \pm\frac{1}{2}$. Thus if $x_{\sigma(1)}$ are the centers of the inscribed balls of the m-simplices $\sigma(1)$ of K_1, then the balls $B_L(x_{\sigma(1)}, \tilde{r}_0/2)$ are disjoint, as are

$$B(p_{\sigma(1)}, r_0/2) = \{y \in M | d(p_{\sigma(1)}, y) < r_0/2\} \quad \text{where } p_{\sigma(1)} = \pi(x_{\sigma(1)}).$$

Let R_σ be the radius of the ball centered at x_σ which contains σ, $\sigma \subseteq K$ on m-simplex and let $R_\sigma(i_1, \ldots, i_m)$ be the corresponding radius for the simplex $\sigma(i_1, \ldots, i_m)$. Set $\tilde{R}_0 = \sup R_\sigma(i, \ldots, i_n)$. The balls $B_L(p_\sigma, \tilde{R}_0)$ cover K and if we choose $R_0 \geqslant c_1 \tilde{R}_0$ and $R_0 > 2r_0$, we see that the balls $B(p_\sigma, R_0)$ cover M. Let us pass to the standard subdivision K_1. If $\sigma(1)$ is an m-simplex in K_1, then the balls $B(p_{\sigma(1)}, R_0/2)$ must cover K.

Let $\tilde{\delta}_0$ be the supremum of the diameters of m-simplices $\sigma(i_1, \ldots, i_m)$ as σ runs through the m-simplices of K. Let $\delta_0 = \tilde{\delta}_0/c_1$. Hence diameter $(\pi(\sigma)) < \delta_0$.

If we replace K by K_1, then $\tilde{\delta}_0$ and δ_0 are replaced by $\tilde{\delta}_0/2$ and $\delta_0/2$. Let $d(p)$ = distance from p to its first cut point and let $\gamma = \min d(p)$. Replacing K by $K_{(n)}$ (nth standard subdivision) if need be, we can assume $2R_0 + \delta_0 < \gamma$.

Let us collect what we have been able to show. Let us pick K as above, r_0 and R_0 such that $r_0 < 2R_0$ and $2R_0 + \delta_0 < \gamma$. Let $K_{(n)}$ be the nth standard subdivision of K. Let $\{\sigma(n)\}$ be the set of m-simplices of $K_{(n)}$. Let $\{x(\sigma(n))\}$ be the "centers" of the $\{\sigma(n)\}$ as constructed above and let $p(\sigma(n)) = \pi(x(\sigma(n)))$. If we set $S_n = \{(p(\sigma(n)))\}$, $r(n) = r_0/2^n$ and $R(n) = R_0/2^n$, we see that

(i) the balls $B(p(\sigma(n)), R(n))$ cover M;

(ii) the balls $B(p(\sigma(n)), r(n))$ are disjoint;

(iii) these balls are diffeomorphic to smooth euclidean balls, and

(iv) $r(n) < R(n)/2$.

If $L(n)$ is the number of points in S_n, we then see that $L(n) = L(0) \cdot 2^{mn}$ and therefore

$$L(n)r(n)^{m-2} = L(0)2^{mn}\left(\frac{r_0^{m-2}}{2^n}\right)^{m-2} = \frac{L(0)r_0^{m-2}}{4^n}.$$

Thus it remains to show the even spacing. Let N_σ = the number of p_σ in $B(p_\sigma, 2R_0)$. Let $N_0 = \sup_{\sigma \subseteq K} N_\sigma$ where the σ's are m-simplices. Then $N_0 >$ largest number of sets $B(p_\sigma, R_0)$ any point in M may touch. We wish to estimate N_0. We know if $p_{\sigma'} \in B(p_\sigma, 2R_0)$ then $\pi(\sigma') \subseteq B(p_\sigma, 2R_0 + \delta_0)$. Hence

$$v(B(p_\sigma, 2R_0 + \delta_0)) > \sum_{p_{\sigma'} \in B(p_\sigma, 2R_0)} v(\pi(\sigma')).$$

Let $\Sigma = \inf\{v_L(\sigma(i_1, \ldots, i_m)) | \sigma \text{ an } m\text{-simplex} \subseteq K\}$. We can find a constant e depending on the metric g above such that

$$N_0 < d_2 e(2R_0 + \delta_0)^m/\Sigma'.$$

If we replace K by K_1 the standard subdivision, we are interested in the number N_1 of $p_{\sigma'(1)}$ in a ball radius R_0 about a given $p_{\sigma(1)}$. Here δ_0 is replaced by $\delta_1 = \delta_0/2$. d_2

and e remain unchanged. Each m-simplex σ splits into 2^m m-simplices of equal volume, so the number Σ' in the denominator is replaced by $\Sigma'/2^m$ and we see that

$$N_1 \leqslant \frac{d_2 e(R_0 + \delta_0/2)^m}{(1/2)^m \Sigma'} = N_0,$$

which completes our proof in the case where $M = M_0$.

Before we proceed to the case of a manifold with boundary we need a definition.

DEFINITION. Let $x \in \partial M_0$, let $v \in T(M_0)_x$ be a unit vector perpendicular to ∂M_0 which points into M_0 and let $\gamma_x(t)$ $(t \geqslant 0)$ be the unique geodesic, parametrized by arclength such that $\gamma_x(0) = x$ and $\dot{\gamma}_x(0) = v$. For small $t > 0$, $\gamma_x[0, t] \subseteq M_0$ and x is the unique point on ∂M_0 which is closest to $\gamma_x(t)$. Let t_x be the supremum of all $t > 0$ such that the above properties hold. Thus the paths $\gamma_x[0, t]$, $t > t_x$, no longer minimize the distance from $\gamma_x(t)$ to ∂M_0. The point $\gamma_x(t_x)$ is called the *focal cut point* of ∂M_0 along γ_x and the set of points $\{\gamma_x(t_x) | x \in \partial M_0\}$ is called the *focal cut locus* of ∂M_0.

We now proceed to the general case. Let δ_0 and γ be as before. Let K be the complex. Let

$$A_n = \{m\text{-simplices in } K_n \text{ such that } \pi(\sigma) \subseteq \operatorname{Int} M_0\},$$
$$B_n = \{m\text{-simplices in } K_n \text{ such that } \pi(\sigma) \cap N = \partial M_0 \neq \varnothing\},$$
$$C_n = \{m\text{-simplices in } K_n \text{ such that } \pi(\sigma) \subseteq M - M_0\},$$

where K_n is the nth standard subdivision of K. We assume that our initial triangulation K and our initial R_0 are so chosen that

(i) $R_0 > 3\delta_0$,

(ii) $2R_0 + \delta_0 < \gamma$,

(iii) A_0, B_0 and C_0 are all nonempty,

(iv) $2\delta_0 <$ distance from ∂M_0 to its focal cut locus.

By successively subdividing K if necessary we can clearly arrange this situation. Let $S_n = \{p_{\sigma(n)} | B(p_{\sigma(n)}, r(n)) \subseteq \operatorname{Int} M_0\}$. The only thing left to show is that $\{B(p(\sigma(n)), R(n))\}$ cover M. If $x \in \pi(\sigma(n))$, $\sigma(n) \in A_n$, then $x \in B(p(\sigma(n)), R(n))$. Say $x \in \pi(\sigma(b)) \cap M_0$, $\sigma(n) \in B_n$. Then we can find $y \in N$ such that $d(x, y) < \delta_0/2^n$. Let $T_n = \{z \in M_0 | d(Z, 2M_0) = \delta_0/2^n\}$. It is easy to see that each $x \in T_n$ is in $\pi(\sigma(n))$ for some $\sigma(n) \in A_n$. Hence we can find some $\sigma(n) \in A_n$ such that $d(y, p(\sigma(n))) < 2\delta_0/2^n$ and thus $d(x, p(\sigma(n))) < 3\delta_n/2^n < R_0/2^n$.

REMARK. A careful accounting of this construction shows the following result. For each $\varepsilon > 0$ we can choose the original r_0 so small that $\operatorname{vol}(M_0 - M_n) < \varepsilon$ for all n, where $M_n = M_0 - \bigcup_{\sigma(n) \in K_n} B(x_{\sigma(n)}, r_0/2^n)$.

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