TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 259, Number 1, May 1980

# ON THE HARDY-LITTLEWOOD MAXIMAL FUNCTION AND SOME APPLICATIONS

BY

C. J. NEUGEBAUER

ABSTRACT. With a monotone family $F = \{S_\alpha\}$, $S_\alpha \subset \mathbf{R}^n$, we associate the Hardy-Littlewood maximal function $M_F f(x) = \sup_\alpha (1/|S_\alpha|) \int_{S_\alpha + x} |f|$. In general, $M_F$ is not weak type (1.1). However, if we replace in the denominator $S_\alpha$ by $S_\alpha^* = \{x - y: x, y \in S_\alpha\}$, and denote the resulting maximal function by $M_F^*$, then $M_F^*$ is weak type (1, 1) with weak type constant 1.

**1.** Let $F = \{S_\alpha\}$, $\alpha \in \Gamma$, be a family of measurable sets in $\mathbf{R}^n$, $0 < |S_\alpha| < \infty$, and for $f$: $\mathbf{R}^n \to \mathbf{R}$ measurable, let $M_F f(x) = \sup_\alpha (1/|S_\alpha|) \int_{S_\alpha + x} |f(t)|\,dt$, the Hardy-Littlewood maximal function relative to $F$. For many important families $F$, $M_F$ is weak type (1, 1) and consequently strong type $(p, p)$, $1 < p \leq \infty$. This is true if $S_\alpha = \{|x| \leq \alpha\}$, or if $S_\alpha$ is the cube of side length $\alpha$ centered at the origin, or if $\{S_\alpha\}$ is a nested family of rectangles with sides parallel to the coordinate axes (see [7]). However, one cannot go much beyond families of this kind and still expect a weak type result. If, for example, $F$ is the collection of all oriented rectangles, $M_F$ is not weak type (1, 1) (see [6]), even though it is strong type $(p, p)$, $1 < p \leq \infty$, and if $F$ is the collection of all rectangles, not even this holds (see [4]).

It is well known that the maximal function is an important tool (i) in obtaining pointwise bounds for convolutions [7], (ii) in problems of convergence a.e. and (iii) in the theory of differentiability [4]. It is thus of interest to study $M_F$ for relatively arbitrary $F$.

For the study of $M_F$, as we have seen, one needs to restrict $F = \{S_\alpha\}$, and a natural minimal condition is that $\alpha \leq \beta$ implies $S_\alpha \subset S_\beta$. We will give, however, an example (due to R. Hunt) which shows that $M_F$ need not be weak type. This forces us to introduce a modified maximal function $M_F^*$ which is equivalent with $M_F$ whenever $F$ is one of the classical families, and which is weak type (1, 1). This is done in the next section, and the remaining sections deal with some applications and R. Hunt's example.

**2.** For $E \subset \mathbf{R}^n$, let $E^* = E - E = \{x - y: x, y \in E\}$. We need the following lemma.

LEMMA. *Let $E \subset \mathbf{R}^n$ be measurable with $0 < |E| < \infty$. If every point of $E$ is a point of density of $E$, then $E^*$ is open.*

---

Received by the editors November 6, 1978.

*AMS (MOS) subject classifications* (1970). Primary 42A96; Secondary 28A15.

*Key words and phrases*. Hardy-Littlewood maximal function, monotone family, weak type.


0002-9947/80/0000-0205/$02.75

PROOF. We give the short proof (which is standard) for the sake of completeness. We denote by $d(x, E) = \lim|E \cap Q|\,|Q|^{-1}$, as $|Q| \to 0$, where $Q$ is a cube, $x \in Q$. The hypothesis says that $d(x, E) = 1$, $x \in E$. We consider the continuous function $\psi(z) = \int_{\mathbf{R}^n} \chi_E(z + t)\chi_E(t)\, dt = \chi_E * \chi_E(z)$, and we need to show that for $z_0 \in E^*$, $\psi(z_0) > 0$. If $A = \{\tau\colon x_0 + \tau \in E\}$, $B = \{\tau\colon y_0 + \tau \in E\}$, where $z_0 = x_0 - y_0$, then $d(0, A) = d(0, B) = 1$, and hence $d(0, A \cap B) = 1$. For $\tau \in A \cap B$, and $t = y_0 + \tau$ we get $z_0 + t = x_0 + \tau \in E$ and hence $\psi(z_0) \geqslant \int_{A \cap B + y_0} \chi_E(z_0 + t)\chi_E(t)\, dt > 0$.

For $F = \{T_\gamma\}$, $\gamma \in \Gamma$, a family of measurable sets in $\mathbf{R}^n$ with (i) $0 < |T_\gamma| < \infty$, let

$$M_F^* f(x) = \sup_\gamma \frac{1}{|T_\gamma^*|} \int_{T_\gamma + x} |f(t)|\, dt.$$

By the above lemma we may assume that $T_\gamma^*$ is open, and we shall do so throughout the paper. In addition to (i) above, we need to assume that there is a map $\sigma\colon \Gamma \to (0, \infty]$ such that (ii) $\sigma(\gamma_1) \leqslant \sigma(\gamma_2)$ implies $T_{\gamma_1} \subset T_{\gamma_2}$, (iii) $\sigma(\gamma_i)\uparrow\sigma$ implies there exists $\gamma \in \Gamma$ such that $\sigma = \sigma(\gamma)$ and $T_{\gamma_i}\uparrow T_\gamma$.

REMARK. If $T_\gamma$ is convex, then $|T_\gamma^*| = 2^n|T_\gamma|$, and hence $M_F^*$ is equivalent with $M_F$.

THEOREM 1. *Under the above conditions on $F$,*

$$|\{x\colon M_F^* f(x) > y\}| \leqslant \|f\|_1/y, \qquad y > 0, \textit{ i.e., } M_F^* \textit{ is weak type } (1, 1).$$

REMARK. The reader will notice that the proof of Theorem 1 is a refinement of the covering and weak type theorems in §2 of [5].

PROOF. We will first assume that $T_\gamma \subset \{|x| \leqslant N\}$, $\gamma \in \Gamma$. We fix $y > 0$ and let $E = \{x\colon M_F^* f(x) > y\}$. For $r > 0$, let $E_r = E \cap \{|x| \leqslant r\}$. We will show that $|E_r| \leqslant \|f\|_1/y$, and this will prove the theorem under the hypothesis of boundedness of $T_\gamma$.

We may clearly assume that the map $\sigma$ is bounded on $\Gamma$; otherwise, the map $\sigma'(\gamma) = \tan^{-1}\sigma(\gamma)$ produces a reindexing with the required property.

We note that $E_r$ is contained in the set $A_0$ of all $x$ for which $|x| \leqslant r$ and, for some $\gamma \in \Gamma$, $\int_{T_\gamma + x}|f(t)|\, dt \geqslant y|T_\gamma^*|$. For $x \in A_0$, let $\tau(x) = \sup\{\sigma(\gamma)\colon \int_{T_\gamma + x}|f(t)|\, dt \geqslant y|T_\gamma^*|\}$. The number $\tau(x) < \infty$, and there is a sequence $\{\gamma_j\}$ such that $\sigma(\gamma_j)\uparrow\tau(x)$, and $\int_{T_{\gamma_j} + x}|f(t)|\, dt \geqslant y|T_{\gamma_j}^*|$. By hypothesis, there is $\gamma \in \Gamma$ such that $\tau(x) = \sigma(\gamma)$ and $T_{\gamma_j}\uparrow T_\gamma$, from which $\int_{T_\gamma + x}|f(t)|\, dt \geqslant y|T_\gamma^*|$.

We let now $\tau_1 = \sup\{\tau(x)\colon x \in A_0\}$. Then $\tau_1 < \infty$ and we have a sequence $\{x_j\} \subset A_0$ such that $\tau(x_j)\uparrow\tau_1$ and $x_j \to x_1$. As above, $\tau(x_j) = \sigma(\gamma_j)$, and there is $\gamma_1 \in \Gamma$ such that $\tau_1 = \sigma(\gamma_1)$, $T_{\gamma_j}\uparrow T_{\gamma_1}$, and $\int_{T_{\gamma_1} + x_1}|f(t)|\, dt \geqslant y|T_{\gamma_1}^*|$. Thus $x_1 \in A_0$. Let now $A_1 = A_0 \setminus (T_{\gamma_1}^* + x_1)$, and let $\tau_2 = \sup\{\tau(x)\colon x \in A_1\}$. Then $\tau_1 \geqslant \tau_2$, and as before there is $\{x_j\} \subset A_1$ such that $x_j \to x_2$, $\tau(x_j)\uparrow\tau_2$. Arguing as above there is $\gamma_2 \in \Gamma$ such that $\tau_2 = \sigma(\gamma_2)$ and $\int_{T_{\gamma_2} + x_2}|f(t)|\, dt \geqslant y|T_{\gamma_2}^*|$. Hence $x_2 \in A_0$, and since $A_1$ is closed in $A_0$ (recall $T_{\gamma_1}^*$ is open), $x_2 \in A_1$. We also note that $T_{\gamma_2} \subset T_{\gamma_1}$.

We continue this process and obtain sequences $\{x_j\}$, $\{T_{\gamma_j}\}$, $\{A_j\}$ such that

(1) $j < k$ implies $T_{\gamma_j} \supset T_{\gamma_k}$,

(2) $x_{j+1} \in A_j$ and $A_j = A_{j-1} \setminus (T^*_{\gamma_j} + x_j)$,

(3) $\int_{T_{\gamma_j}+x_j} |f(t)|\, dt > y|T^*_{\gamma_j}|$.

We claim now that for $j \neq k$, $(x_j + T_{\gamma_j}) \cap (x_k + T_{\gamma_k}) = \varnothing$. Assume that $j < k$, and $x_j + t_j = x_k + t_k$, $t_j \in T_{\gamma_j}$, $t_k \in T_{\gamma_k}$. Then $x_k \in x_j + (T_{\gamma_j} - T_{\gamma_k}) \subset x_j + T^*_{\gamma_j}$ by (1). But by (2), $x_k \in A_{k-1}$ and $A_{k-1} \cap (T^*_{\gamma_j} + x_j) = \varnothing$.

We assert next that $\bigcap A_j = \varnothing$. Assume that $z \in \bigcap A_j$. Then $\tau(z) > 0$ (see the beginning of the proof for the definition of $\tau(z)$), and as observed there $\tau(z) \doteq \sigma(\gamma)$ and $\int_{T_\gamma + z}|f(z)|\, dt > y|T^*_\gamma|$. Since $\tau_j \geq \tau(z)$, we have $T_{\gamma_j} \supset T_\gamma$, and thus, if $F = \bigcup (x_j + T_{\gamma_j})$, $|F| = \infty$. Also $T_{\gamma_j} \subset T_{\gamma_1}$ so that $\bigcup T_{\gamma_j} \subset T_{\gamma_1}$. Since $|x_j| \leq r$ we see that $F \subset T_{\gamma_1} + \{|x| \leq r\} \subset \{|x| \leq N\} + \{|x| \leq r\}$, contradicting $|F| = \infty$.

Finally, since $E_r \subset \bigcup (x_j + T^*_{\gamma_j})$, we obtain

$$|E_r| \leq \Sigma |T^*_{\gamma_j}| \leq \frac{1}{y} \Sigma \int_{T_{\gamma_j}+x_j} |f(t)|\, dt \leq \frac{\|f\|_1}{y}.$$

We remove now the restriction that each $T_\gamma \subset \{|x| \leq N\}$. We let $F_N = \{T^N_\gamma\}$, where $T^N_\gamma = T_\gamma \cap \{|x| < N\}$. Then

$$M_N f(x) \equiv \sup_\gamma \frac{1}{|T^*_\gamma|} \int_{T^N_\gamma + x} |f(t)|\, dt \leq \sup_\gamma \frac{1}{|T^{N*}_\gamma|} \int_{T^N_\gamma + x} |f(t)|\, dt,$$

and since the first part of the proof applies to $F_N$, $|\{x\colon M_N f(x) > y\}| \leq \|f\|_1/y$. Since $M_N f(x) \uparrow M^*_F f(x)$, we see that $|\{x\colon M^*_F f(x) > y\}| = \lim_{N\to\infty} |\{x\colon M_N f(x) > y\}| \leq \|f\|_1/y$. This completes the proof of Theorem 1.

COROLLARY 1. $\|M^*_F f\|_p \leq A_p \|f\|_p$, $1 < p \leq \infty$, *where $A_p$ depends only on $p$. In particular, $A_p$ does not depend on $F, f$.*

COROLLARY 2. $|\{x\colon M^*_F f(x) > y\}| \leq (2/y) \int_{|f(t)|>y/2} |f(t)|\, dt$.

PROOF. Let $f = f_1 + f_2$, where $f_1(x) = f(x)$, whenever $|f(x)| > y/2$, and $f_1(x) = 0$ elsewhere. Then $|f(x)| \leq |f_1(x)| + y/2$. Hence $\{x\colon M^*_F f(x) > y\} \subset \{x\colon M^*_F f_1(x) > y/2\}$, and thus

$$|\{x\colon M^*_F f(x) > y\}| \leq \frac{2}{y}\|f_1\|_1 = \frac{2}{y}\int_{|f(t)|>y/2} |f(t)|\, dt.$$

We give now an important special example of Theorem 1. We let $F = \{S_\alpha\}$, $\alpha \in \Gamma$, $\Gamma \subset (0, \infty)$, such that (i) $0 < |S_\alpha| < \infty$, (ii) $\alpha \leq \beta$ implies $S_\alpha \subset S_\beta$.

THEOREM 2. $|\{x\colon M^*_F f(x) > y\}| \leq \|f\|_1/y$.

PROOF. Let $\Gamma' = \{\tau\colon \exists \{\alpha_j\} \subset \Gamma$ such that $\alpha_j \uparrow \tau\}$. If $\tau \in \Gamma' \setminus \Gamma$, let $S_\tau = \bigcup_{\alpha<\tau} S_\alpha$, $\alpha \in \Gamma$. We let $\tilde{\Gamma} = \Gamma' \times [0, 1]$, and for $\gamma = (\alpha, \beta)$ in $\tilde{\Gamma}$ we define $T_\gamma$ as follows. If $\beta = 0$, $T_\gamma = \bigcup_{\tau<\alpha} S_\tau$, and if $0 < \beta \leq 1$, $T_\gamma = S_\alpha$. We set $\tilde{F} = \{T_\gamma\}$, and we define $\sigma\colon \tilde{\Gamma} \to (0, \infty)$ by $\sigma(\alpha, \beta) = \alpha$. It is easily checked that Theorem 1 applies to $\tilde{F}$ and hence

$$|\{x\colon M^*_F f(x) > y\}| \leq |\{x\colon M^*_{\tilde{F}} f(x) > y\}| \leq \|f\|_1/y.$$

**3.** The hypothesis in Theorem 2 that the family $F = \{S_\alpha\}$ be monotone can be somewhat weakened. For $E \subset \mathbf{R}^n$ and $k = (k_1, \ldots, k_n)$, $k_i > 0$, let $kE = \{(k_1x_1, \ldots, k_nx_n): (x_1, \ldots, x_n) \in E\}$.

THEOREM 3. *Let $F = \{S_\alpha\}$, $\alpha \in \Gamma$, have the property that there is $k$ such that $\alpha', \alpha'' < \beta$ in $\Gamma$ implies $S_{\alpha'} - S_{\alpha''} \subset kS_\beta^*$. Then*

$$|\{x: M_F^* f(x) > y\}| < \frac{k_1 \cdots k_n}{y} \|f\|_1.$$

PROOF. Let $T_\beta = \bigcup_{\alpha < \beta} S_\alpha$. Since we may assume that every point of $S_\alpha$ is a point of density of $S_\alpha$, the set $T_\beta$ is measurable. Let $F' = \{T_\beta\}$. It is clear that $F'$ is monotone, and $T_\beta^* \subset \bigcup_{\alpha',\alpha'' < \beta}(S_{\alpha'} - S_{\alpha''}) \subset kS_\beta^*$. From this we get $M_F^* f(x) < k_1 \cdots k_n M_{F'}^* f(x)$, and Theorem 2 completes the proof.

COROLLARY. *If $F = \{S_\alpha\}$ has the property that there is $k$ so that $\alpha < \beta$ implies $kS_\beta \supset S_\alpha$, then the above weak type inequality holds.*

PROOF. Simply observe that $\alpha', \alpha'' < \beta$ implies $S_{\alpha'} - S_{\alpha''} \subset kS_\beta^*$.

REMARK. The special case of the above corollary with $F$ a family of oriented rectangles centered at the origin can be found in [1] with a different weak type constant.

**4.** The following example, due to R. Hunt, shows that for a nested family $F = \{S_\alpha\}$ (as in Theorem 2) the maximal function $M_F f(x) = \sup_\alpha (1/|S_\alpha|) \int_{S_\alpha + x} |f(t)|\, dt$ need not be weak type (1, 1).

Let $S_N = \bigcup_{j=N}^{\infty} (2^{-j}, 2^{-j} + 2^{-2j})$ in $\mathbf{R}$, so that $S_{N+1} \subset S_N$. Let $F = \{S_N\}$, and note that $|S_N| < 2^{-2N+1}$. Let $f_N$ be the characteristic function of $(0, 2^{-2N-1})$. Then $\|f\|_1 = 2^{-2N-1}$. It is readily checked that

$$x \in (-2^{-N-1}, -2^{-N-1} + 2^{-2N-2}) \Rightarrow M_F f_N(x) > 2^{-2N-2}/|S_{N+1}| > 1/2,$$
$$x \in (-2^{-N-2}, -2^{-N-2} + 2^{-2N-2}) \Rightarrow M_F f_N(x) > 2^{-2(N+2)}/|S_{N+2}| > 1/2,$$
$$\vdots$$
$$x \in (-2^{-(N+N+1)}, 0) \Rightarrow M_F f_N(x) > 1/2.$$

Hence $|\{x: M_F f_N(x) > 1/2\}| > N \cdot 2^{-2N-2} = N\|f_N\|_1/2$, and $M_F$ is not weak type (1, 1). Incidentally, $M_F$ is not weak type $(p, p)$, $1 < p < \infty$.

The above example raises the question for which class of functions is $M_F f$ weak type (1, 1). For $F = \{S_\alpha\}$, $\alpha \in \Gamma$, a monotone family, and $\mu > 0$ a measure on $\mathbf{R}^n$, let us define $\mu_F(t) = \sup \mu(S_\alpha^* + x)/|S_\alpha|$, where the sup is taken over all $\alpha$, $x$ for which $t \in S_\alpha + x$. Since $S_\alpha^*$ is open (recall that we always assume that $x \in S_\alpha$ is a point of density of $S_\alpha$), it follows easily that $\mu_F$ is lower semicontinuous and hence measurable.

THEOREM 4. *Under the above conditions*

$$\mu\{x: M_F f(x) > y\} < \frac{1}{y} \int_{\mathbf{R}^n} |f(t)| \mu_F(t)\, dt.$$

PROOF. As in Theorem 2 we extend $F$ to a family $\tilde{F} = \{T_\gamma\}$, $\gamma \in \tilde{\Gamma}$, for which there is a mapping $\sigma: \tilde{\Gamma} \to (0, \infty)$ so that the hypotheses of Theorem 1 hold. Then as in the proof of Theorem 1 we select $\{T_{\gamma_j}\}$, $\{x_j\}$ and obtain

$$\begin{aligned}\frac{1}{y}\int_{\mathbf{R}^n}|f(t)|\mu_F(t)\,dt &\geq \frac{1}{y}\sum\int_{T_{\gamma_j}+x_j}|f(t)|\mu_F(t)\,dt\\ &\geq \frac{1}{y}\sum\frac{\mu(T^*_{\gamma_j}+x_j)}{|T_{\gamma_j}|}\int_{T_{\gamma_j}+x_j}|f(t)|\,dt \geq \sum\mu(T^*_{\gamma_j}+x_j)\\ &\geq \mu\{x: M_Ff(x) > y\}.\end{aligned}$$

In the second inequality we used the fact that $\mu_F(t) = \mu_{\tilde{F}}(t)$ which follows easily from the definition of $\tilde{F}$ in the proof of Theorem 2.

**5.** In this section we will present some applications of Theorem 2 concerning pointwise boundedness of convolutions and approximate identities.

The problem about convolutions which we will discuss can be put in the following way. Suppose $\lim \phi_n * g(x)$ exists for every $x$ and $g \in C_c(\mathbf{R}^n)$. Under what conditions does it follow that $\lim \phi_n * f(x)$ exists for a.e. $x$ and $f \in L^p(\mathbf{R}^n)$ (see e.g. [3]). The hypothesis on $C_c(\mathbf{R}^n)$ implies that $\|\phi_n\|_1 \leq K < \infty$ (by the Banach-Steinhaus theorem). Our condition will be a more stringent requirement on the norm.

Let $F = \{S_\alpha\}$, $\alpha \in \Gamma \subset (0, \infty)$, be a family of measurable sets satisfying (i) $0 < |S_\alpha| < \infty$, (ii) $\alpha < \beta$ implies $S_\alpha \subset S_\beta$. Let $\sigma(F)$ be the collection of all simple functions $s \geq 0$ which can be written as $s = \sum c_j\chi_{A_j}$, $c_j \geq 0$, and $A_j \subset -S_{\alpha_j}$ for some $\alpha_j \in \Gamma$, where $-E = \{x: -x \in E\}$. We set $\|s\| = \inf \sum c_j|S^*_{\alpha_j}|$, where the inf is extended over all representations $s = \sum c_j\chi_{A_j}$ and all sequences $\{S_{\alpha_j}\} \subset F$ with $A_j \subset -S_{\alpha_j}$. We note that $\|s\| \geq \|s\|_1$. Denote now by $L(F)$ all functions $\phi \geq 0$ for which there is a sequence $\{s_n\} \subset \sigma(F)$ such that $s_n\uparrow\phi$, and let $\|\phi\|_F = \inf \liminf\|s_n\|$, where the inf is extended over all such $\{s_n\}$. Again, $\|\phi\|_F \geq \|\phi\|_1$.

As an example, let $\phi \geq 0$, $\phi \in L^1(\mathbf{R}^n)$, and let $S_\alpha = \{x: \phi(x) > \alpha^{-1}\}$. It is easily verified that $\phi \in L(F)$, where $F = \{S_\alpha\}$.

LEMMA. *Let $\phi \in L(F)$. Then $|\phi * f(x)| \leq \|\phi\|_F M^*_F f(x)$.*

PROOF. Let $s \in \sigma(F)$, $s \leq \phi$. Choose a representation $s = \sum c_j\chi_{A_j}$, $c_j \geq 0$, and $A_j \subset -S_{\alpha_j}$. Then $|s * f(x)| \leq \sum c_j\int|f(t)|\chi_{A_j}(x - t)\,dt \leq \sum c_j\int_{S_{\alpha_j}+x}|f(t)|\,dt \leq \sum c_j|S^*_{\alpha_j}|M^*_Ff(x)$. Thus $|s * f(x)| \leq \|s\|M^*_Ff(x)$, and by the monotone convergence theorem the lemma follows.

Let $\{\phi_n\}$, $\phi_n \geq 0$, be a sequence of measurable functions. We say that a family $F = \{S_\alpha\}$, satisfying (i), (ii) above, dominates $\{\phi_n\}$ if $\{\phi_n\} \subset L(F)$ and $\|\phi_n\|_F \leq K < \infty$, $n = 1, 2, \ldots$.

THEOREM 5. *If $F = \{S_\alpha\}$ dominates $\{\phi_n\}$ and if* $\lim \phi_n * g(x)$ *exists for a.e. $x$ and $g \in C_c(\mathbf{R}^n)$, then* $\lim \phi_n * f(x)$ *exists for a.e. $x$ if $f \in L^p(\mathbf{R}^n)$, $1 \leq p < \infty$.*

PROOF. From the lemma, $\sup_n|\phi_n * f(x)| \leq KM_F^*f(x)$. For $g \in C_c(\mathbf{R}^n)$ we have

$$\begin{aligned}|\{x: \limsup \phi_n * f(x) - \liminf \phi_n * f(x) > \eta\}| \\ = |\{x: \limsup \phi_n * (f-g)(x) - \liminf \phi_n * (f-g)(x) > \eta\}| \\ \leq |\{x: M_F^*(f-g)(x) > \eta/2K\}|.\end{aligned}$$

By Theorem 2, $M_F^*$ is weak type $(p,p)$ and hence the measure of the above set $\leq A_p\|f-g\|_p^p/\eta^p$, which can be made as small as we wish.

REMARK. (1) If $\lim \phi_n * g(x) = g(x)$, for a.e. $x$ and $g \in C_c$, then $\lim \phi_n^* f(x) = f(x)$, a.e. $x$ and $f \in L^p$, $1 \leq p < \infty$.

(2) As an example illustrating Theorem 3, let $\{\phi_j\}$ be a sequence of nonnegative functions such that $\phi_j(x) \leq \psi(x)$, $x \in \mathbf{R}^n$, $j = 1, 2, \ldots$, $\psi(x) \in L^1(\mathbf{R}^n)$, $\psi(x) = \psi_0(|x|)$ and $\psi_0(r)$ is nonincreasing. Then the family $F$ of all spheres centered at 0 dominates $\{\phi_n\}$. This covers the classical case (see e.g. [7, p. 62]).

For the study of approximate identities, let $\phi \geq 0$ on $\mathbf{R}^n$, $\int_{\mathbf{R}^n} \phi(t)\,dt = 1$ and let $\phi_\varepsilon(t) = \varepsilon^{-n}\phi(t/\varepsilon)$. Then $\int \phi_\varepsilon\,dt = 1$.

Let $F = \{S_\alpha\}$ be a nested family as above and satisfying $\varepsilon F \subset F$, $0 < \varepsilon \leq 1$, and $\phi \in L(F)$. Here $\varepsilon F \subset F$ means $\varepsilon S_\alpha \in F$ for $S_\alpha \in F$.

THEOREM 6. *If* $\|\phi\|_F < \infty$, *then* $\phi_\varepsilon * f(x) \to f(x)$ *as* $\varepsilon \to 0$ *for a.e.* $x$, $f \in L^p(\mathbf{R}^n)$, $1 \leq p < \infty$.

PROOF. It is easily verified that $\phi_\varepsilon * g(x) \to g(x)$, $x \in \mathbf{R}^n$, $g \in C_c(\mathbf{R}^n)$. By the lemma, $|\phi_\varepsilon * f(x)| \leq \|\phi_\varepsilon\|_F M_F^*(x)$. One readily verifies that $\|\phi_\varepsilon\|_F = \|\phi\|_F$, and thus Theorem 3 completes the proof.

**6.** In this section we give an application to the differentiability of the integral. We let $0 < |B| < \infty$ and $B_\varepsilon = \varepsilon \cdot B$, $0 < \varepsilon \leq 1$. Assume that (i) $B \subset \bigcup B_k$, (ii) for each $k$, the family $F_k = \{\varepsilon B_k\}$, $0 < \varepsilon \leq 1$, is nested, i.e., $\varepsilon_1 \leq \varepsilon_2$ implies $\varepsilon_1 B_k \leq \varepsilon_2 B_k$, (iii) $\Sigma|B_k^*| < \infty$. If for example $B$ is starlike and symmetric about 0, then $B_k$ can be chosen to be a rectangle so that the above conditions hold (see [2]).

THEOREM 7. *Under the above conditions, if* $f^*(x) = \sup_\varepsilon(1/|B_\varepsilon|)\int_{B_\varepsilon + x}|f(t)|\,dt$, *then* $\|f^*\|_p \leq A_p\|f\|_p$, $1 < p \leq \infty$.

PROOF. Let $\varepsilon B_k = B_{k\varepsilon}$ and $M_k^* f(x) = \sup_\varepsilon(1/|B_{k\varepsilon}^*|)\int_{B_{k\varepsilon}+x}|f(t)|\,dt$. Then

$$\frac{1}{|B_\varepsilon|}\int_{B_\varepsilon + x}|f(t)|\,dt \leq \frac{1}{|B_\varepsilon|}\Sigma \int_{B_{k\varepsilon}+x}|f(t)|\,dt$$
$$\leq \frac{1}{|B_\varepsilon|}\Sigma\,|B_{k\varepsilon}^*|M_k^* f(x) = \frac{1}{|B|}\Sigma\,|B_k^*|M_k^* f(x).$$

Hence $f^*(x) \leq (1/|B|)\Sigma\,|B_k^*|M_k^* f(x)$, and Corollary 1 completes the proof.

REMARK. (1) The above result generalizes part of Theorem A in [1].

(2) If the condition $\Sigma\,|B_k^*| < \infty$ is replaced by $|B_k^*| \to 0$ and $\Sigma\,|B_k^*|\,|\log|B_k^*|| < \infty$, then by [4, p. 145], $f^*$ is also weak type (1, 1).

## REFERENCES

1. L. A. Cafarelli and C. P. Calderón, *Weak type estimates for the Hardy-Littlewood maximal function*, Studia Math. **49** (1974), 217–223.

2. C. P. Calderón, *Differentiation through star-like sets in* $\mathbf{R}^n$, Studia Math. **48** (1973), 1–13.

3. R. E. Edwards and E. Hewitt, *Pointwise limit for sequences of convolution operators*, Acta Math. **113** (1965), 181–218.

4. M. Guzmán, *Differentiation of integrals in* $\mathbf{R}^n$, Lecture Notes in Math., vol. 481, Springer-Verlag, Berlin and New York, 1975.

5. N. M. Rivière, *Singular integrals and multiplier operators*, Ark. Mat. **9** (1971), 243–278.

6. S. Saks, *Remark on the differentiability of the Lebesgue indefinite integral*, Fund. Math. **22** (1934), 257–261.

7. E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Univ. Press, Princeton, N. J., 1970.

DIVISION OF MATHEMATICAL SCIENCES, PURDUE UNIVERSITY, WEST LAFAYETTE, INDIANA 47907