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ON THE TOPOLOGY OF SIMPLY-CONNECTED ALGEBRAIC SURFACES

BY

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ABSTRACT. Suppose X is a smooth simply-connected compact 4-manifold. Let $P = \mathbf{C}P^2$ and $Q = -\mathbf{C}P^2$ be the complex projective plane with orientation opposite to the usual. We shall say that X is completely decomposable if there exist integers a, b such that X is diffeomorphic to $aP \# bQ$.

By a result of Wall [W1] there always exists an integer k such that $X \# (k+1)P \# kQ$ is completely decomposable. If $X \# P$ is completely decomposable we shall say that X is almost completely decomposable. In [MM] we demonstrated that any nonsingular hypersurface of $\mathbf{C}P^3$ is almost completely decomposable. In this paper we generalize this result in two directions as follows:

THEOREM 3.5. *Suppose W is a simply-connected nonsingular complex projective 3-fold. Then there exists an integer $m_0 > 1$ such that any hypersurface section V_m of W of degree $m > m_0$ which is nonsingular will be almost completely decomposable.*

THEOREM 5.3. *Let V be a nonsingular complex algebraic surface which is a complete intersection. Then V is almost completely decomposable.*

Introduction. Suppose X is a simply-connected compact 4-manifold. Let $P = \mathbf{C}P^2$ and $Q = -\mathbf{C}P^2$ be the complex projective plane with orientation opposite to the usual. We shall say that X is completely decomposable if there exist integers a, b such that $X \approx aP \# bQ$. (Read '$\approx$' as 'is diffeomorphic to'.) By a result of Wall [W1], [W2] there always exists an integer k such that $X \# (k+1)P \# kQ$ is completely decomposable. If $X \# P$ is completely decomposable we shall say that X is almost completely decomposable. In [MM] we demonstrated that any nonsingular hypersurface of $\mathbf{C}P^3$ is almost completely decomposable. There are several possible ways of generalizing these results. One can consider hypersurface sections of a simply-connected algebraic 3-fold W instead of those of $\mathbf{C}P^3$, or one can consider nonsingular algebraic surfaces defined by the intersection of k hypersurfaces of $\mathbf{C}P^{k+2}$ (so-called complete intersections). For these two possible generalizations we obtain the following results.

THEOREM 3.5. *Suppose W is a simply-connected nonsingular complex projective 3-fold. Then there exists an integer $m_0 \geqslant 1$ such that any hypersurface section V_m of degree $m \geqslant m_0$ which is nonsingular will be almost completely decomposable.*

THEOREM 5.3. *Let V be a nonsingular compact complex algebraic surface which is a complete intersection. Then V is almost completely decomposable.*

To introduce our other results we must first establish some more terminology. We recall that the field F is called an algebraic function field of two variables over

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$\mathbf{C}$ if F is a finitely generated extension field of $\mathbf{C}$ of transcendence degree two. It is well known that for such a field there exists a nonsingular algebraic surface whose field of meromorphic functions is F **[Z]**. We call any such nonsingular surface a model for F. It is then easy to see that given any two such models V_1, V_2 for F their fundamental groups $\pi_1(V_i)$ are isomorphic. Thus we can define the fundamental group of F, $\pi_1(F)$, as $\pi_1(V)$ for any model V for F. In particular the notion of a simply-connected such field is well defined. We shall let $\mathfrak{F} = \{F | F$ is an algebraic function field of two variables over $\mathbf{C}\}$ and $\mathfrak{F}_0 = \{F \in \mathfrak{F} | \pi_1(F) = 0\}$. For $F \in \mathfrak{F}$ with $\pi_1(F) = 0$ define $\mu(F)$ to be $\inf\{k \in \mathbf{Z} |$ there exists a model V for F such that $V \# kP$ is completely decomposable$\}$. Now suppose V is any model for F. Then as previously mentioned there always exists some integer k such that $V \# (k + 1)P \# kQ$ is completely decomposable. Then blowing V up at k-points to get $V' = V \# kQ$ we observe that V' is also a model for F with $V' \# (k + 1)P$ completely decomposable. Thus we have $0 \leqslant \mu(F) \leqslant k + 1$ and so $\mu(F)$ is always a finite nonnegative integer. If F is a pure transcendental extension of $\mathbf{C}$ of degree 2 then any model V for F is a rational algebraic surface. But then by classical results **[Z]**, **[Sf2]** it immediately follows that $\mu(F) = 0$. We expect that for all other simply-connected $F \in \mathfrak{F}$ we will have $\mu(F) > 0$ (see the conjectures in **[MM]**). If $\mu(F) \leqslant 1$ we shall call F a *topologically normal* field. A consequence of Theorem 4.2 of this paper is that any simply-connected $F \in \mathfrak{F}$ has a simply-connected quadratic extension which is topologically normal.

Suppose $L, K \in \mathfrak{F}$. Then we shall say that L is a *flexible cyclic extension* of K if there exist models V_L, V_K for L resp. K, and a morphism F: $V_L \to V_K$ with discrete fibers whose ramification locus R_F is a nonsingular flexible curve in V_K (where flexible is defined in the beginning of the Appendix). Corollary A.2 of our Appendix then says that if L is such an extension of K and if K is simply-connected then so is L and L is a cyclic field extension of K. Thus flexible cyclic extensions are cyclic extensions which preserve simple-connectivity of fields.

A slightly stronger concept of cyclic extension is then the following:

DEFINITION. Let $L, K \in \mathfrak{F}$. Then we shall say that L is a *satisfactory cyclic extension* of K if there exist models V_L, V_K for L, resp. K, and a morphism Φ: $V_L \to V_K$ with discrete fibers whose ramification locus R_Φ is a nonsingular hypersurface section E of V_K with deg E being a multiple of deg Φ.

A reformulation of Theorem 5.2 then gives

THEOREM 5.2′. *Let $K \in \mathfrak{F}$ with K simply-connected. Then K has a satisfactory cyclic extension L of degree* 2 *over K which is topologically normal.*

In **[M]** it is further shown that if K itself is topologically normal then so is any satisfactory cyclic extension. We use these two results to motivate a partial order in $\mathfrak{F}_0$ defined as follows.

For $L, K \in \mathfrak{F}_0$ we shall say that L is a satisfactorily resolvable extension of K if there exist a finite sequence $L_0, \ldots, L_n \in \mathfrak{F}$ with $L_0 = K$, L_{i+1} a satisfactory cyclic extension of L_i and $L_n = L$. We write $K < L$ if L is a satisfactorily resolvable extension of K. Then $<$ induces a partial ordering on $\mathfrak{F}_0$. The two results quoted above then say that in terms of this partial ordering we have that every sufficiently 'large' field L is topologically normal.

The problems associated with decomposability can also be topologically recast in terms of 'singular knots'. Suppose V is an almost completely decomposable 4-manifold. Then V blown up by means of a $\bar{\sigma}$-process (see [MM] or §1 of this paper for a definition) at some point $p \in V$ gives us a completely decomposable 4-manifold $X \approx V \# P$. Since X is completely decomposable it can be considered as arising from S^4 by blowing up a finite number of points by means of σ- and $\bar{\sigma}$-processes. Then there will exist a blowing down map $X \to S^4$ which we denote by Φ_X. Now let L be the preimage in X of the point $p \in V$ at which V was blown up to get X. Let $S = \Phi_X(L)$. Without loss of generality we can always arrange that S will be an immersed 2-sphere in S^4. Topologically the pair (S^4, S) is a 'singular knotting' of S^2 in S^4. If V is itself indecomposable (see [MM, §6] for a definition) then the 'singular knot' S contains in some sense the code for the topological structure of V. If our expectation that $\mu(L) > 0$ for irrational L is correct then it would be of great interest to understand which 'singular knots' correspond to the minimal models of irrational fields. Currently we do not even know a good description of those knots corresponding to irrational hypersurfaces in $\mathbf{C}P^3$. In particular we do not even have a simple description of the knot corresponding to the K-3 surface $V_4 \subset \mathbf{C}P^3$.

The bulk of this work was done at Institut des Hautes Études Scientifiques and we would like to express our appreciation to them for their hospitality during our visits there.

1. Notation and terminology. We work entirely in the smooth category and all our spaces and maps are assumed to be smooth unless something to the contrary is indicated. When speaking about a tubular neighborhood of a submanifold X of a manifold W we shall assume that W has a fixed Riemannian metric and the tubular neighborhood T_X of X in W is a disc bundle $T_X \to X$ embedded in W such that for any $p \in X$ the fiber T_p of T_x at p consists of all points on the normal geodesics through p whose distance from p along the geodesic is less than or equal to some constant ε. When speaking about cross-sections of vector bundles we shall always identify the cross-section with its image. Thus if $V \to X$ is a vector bundle over X with cross-sections P_1, P_2 then $P_1 + \lambda P_2$ will denote a submanifold of V as well as the appropriate cross-section of $V \to X$. Furthermore if $V \to X$ is a vector bundle over X we shall always identify X with the zero-section of $V \to X$ (doing the same with projective bundles $P \to X$ obtained by projective completion of vector bundles). In particular then $P_1 \cap X$ or $(P_1 + \lambda P_2) \cap X$ will simply represent the subset of X given by the zero-locus of the cross-section P_1, resp. $P_1 + \lambda P_2$ of X.

Now suppose X is a differentiable manifold and Y_1, Y_2 are subspaces of X. Let $x \in X$. We shall say Y_1 and Y_2 intersect transversely at x if there exists a neighborhood U of x in X such that $U \cap Y_i$ is a differentiable submanifold of X and $U \cap Y_1 \cap Y_2$ is a differentiable submanifold of $Y_1 \cap Y_2$ with $T_x(Y_1) + T_x(Y_2) = T_x(X)$. We say Y_1 intersects Y_2 transversely along Z iff Y_1 and Y_2 intersect transversely at all $x \in Z$. In particular note that if Y_1, Y_2 are analytic subspaces of some complex manifold M then if Y_1 intersects Y_2 transversely along Z then $Z \cap Y_1 \cap Y_2$ is a submanifold of M and the Y_i are nonsingular at $Z \cap Y_1 \cap Y_2$.

We need the following generalization of the above. Suppose M is a complex manifold and $Y_1, \ldots, Y_k$ are complex subvarieties of M. Let $x \in M$. Then we shall say that $Y_1, \ldots, Y_k$ cross normally at x if there exists a coordinate neighborhood U of x in M with local coordinates $(Z_1, \ldots, Z_n)$ such that $Z_i(x) = 0$ and $U \cap Y_i = \{Z_i\} = 0$ for $i = 1, \ldots, k$. We shall say that $Y_1, \ldots, Y_k$ have normal crossing in W if whenever $x \in Y_{i_1} \cap \cdots \cap Y_{i_j}$ for some subsequence i_l of $\{1, \ldots, k\}$ then $Y_{i_1}, \ldots, Y_{i_j}$ cross normally at x. In particular if Y_1, Y_2, Y_3 have normal crossing in M then Y_i intersects Y_j transversely along $Y_i \cap Y_j$, $1 \leqslant i < j \leqslant 3$, in M and $Y_1 \cap Y_2$ intersects $Y_1 \cap Y_3$ transversely in Y_1. We shall also use the terminology of normal crossing whenever M is an arbitrary smooth manifold and the Y_i are subspaces of codimension two. Since normal crossing is a local concept given any $x \in M$ we can always assume that a coordinate neighborhood U of x can be split as $U \approx U_1 \times U_2 \subset \mathbf{C}(Z_1, \ldots, Z_l) \times R^{n-2l}(y)$ where $\dim M = n$ and $n - 2l \geqslant 0$. Thus if the Y_i are codimension two subspaces we shall say they cross normally at X if the coordinate neighborhood U of x can be picked so that U has local coordinate $\{Z_1, \ldots, Z_n, Y\}$ as above with $Y_i \cap U = \{Z_i = 0\}$. Our other terminology regarding normal crossings will then be used without change for this situation also.

We now introduce the notion of blowing up a manifold along a submanifold. To begin with suppose X is a manifold and $E \xrightarrow{\pi} X$ a complex vector bundle over X. Let $\mathcal{P}(E) \xrightarrow{p} X$ be the complex projective bundle over X obtained by projectivizing $E \xrightarrow{\pi} X$ and suppose $L \xrightarrow{\rho} \mathcal{P}(E)$ is the tautological complex line bundle over $\mathcal{P}(E)$ (where we recall that if (x, l) is a point of $\mathcal{P}(E)$, where l is a complex line through the origin in the complex vector space $\pi^{-1}(x)$ then l is the fiber of L over (x, l)). Then there exists a canonical map ϕ: $L \to E$ (simply send the point $((x, l), t) \in \rho^{-1}(x, l)$, where $t \in l$, to $(x, t) \in \pi^{-1}(x)$), which is clearly a diffeomorphism outside $\phi^{-1}(X)$ and such that $\phi^{-1}(X)$ is a complex projective bundle over X. We call L or ϕ: $L \to E$, E blown up along X. Now suppose W is a manifold and X a submanifold of W. Consider the normal bundle $\nu(X, W)$ of X in W and recall that there exists a diffeomorphism e of a neighborhood $B(X) = \{(x, w) \in \nu(X, W) | \, \|w\| \leqslant 1\}$ of X in $\nu(X, W)$ onto a tubular neighborhood $T(X)$ of X in W (where the norm $\| \;\|$ on $\nu(X, W)$ is induced via some underlying Riemannian metric on W). Suppose the structure group of $\nu(X, W)$ can be reduced to the complex linear group. Thus $\nu(X, W)$ can be thought of as a complex vector bundle over X with fiber $\mathbf{C}^k$ for some $k > 0$. (Clearly we must have codim(X, W) even.) Then we can blow $\nu(X, W)$ up along X to obtain $\hat{\phi}$: $\hat{V} \to \nu(X, W)$. Let $\hat{B} = \hat{\phi}^{-1}(B(X))$ and set $\phi = \hat{\phi} | \hat{B}$. Note that ϕ is a diffeomorphism outside $\phi^{-1}(X) = \hat{\phi}^{-1}(X)$. In particular the map $\Psi = e \circ \phi$ maps $\hat{B}$ onto $T(X)$ and is a diffeomorphism outside $\Psi^{-1}(X)$. Thus if we set $\hat{W} = W - T(X) \cup_{\Psi | \partial \hat{B}} \hat{B}$ we see that $\hat{W}$ is a smooth manifold and there exists a map σ: $\hat{W} \to W$ induced by the identity on $W - T(X)$ and ϕ on $\hat{B}$ which is a diffeomorphism outside $\sigma^{-1}(X)$ and such that

$\sigma^{-1}(X) \to X$ is a $\mathbf{C}P^{k-1}$ bundle over X. It can be verified that $\hat{W}$, up to diffeomorphism, depends only on the choice of reduction of the structure group of $\nu(X, W)$ to the complex linear group. We call $\hat{W}$ or σ: $\hat{W} \to W$ the blowing up of W along X or the blowing up of W with center X (relative to the given choice of reduction of $\nu(X, W)$). In terms of local coordinates we can describe $\hat{W}$ as follows:

Since $\nu(X, W)$ is reducible to a $\mathbf{C}^k$-bundle over X there exists a coordinate covering $\{U_\alpha\}$, $\{U_\beta\}$ of W such that

(1) $X \cap U_\beta = \varnothing$ for $U_\beta \in \{U_\beta\}$.

(2) $U_\alpha \approx U'_\alpha \times U''_\alpha \subset \mathbf{R}^{N-2k}(X_1, \ldots, X_{N-2k}) \times \mathbf{C}^k(Z_1, \ldots, Z_k)$ with $X \cap U_\alpha = \{Z_1 = \cdots = Z_k = 0\} \approx U'_\alpha$ for $U_\alpha \in \{U_\alpha\}$.

(3) The collection $\{U'_\alpha\}$ forms a trivializing cover for $T(X)$ with $T(X)|_{U'_\alpha} \subset U_\alpha$.

Then let

$$W_\alpha = \{((X, Z), [t_1, \ldots, t_k]) \in U_\alpha \times \mathbf{C}P^{k-1} | Z_i t_j - Z_j t_i = 0 \text{ for } 1 < i, j < k\}.$$

Note that the projection map restricted to W_α gives a map ϕ_α: $W_\alpha \to U_\alpha$ which is the identity outside $X \cap U_\alpha$ and such that $\phi_\alpha^{-1}(X \cap U_\alpha) \approx U'_\alpha \times \mathbf{C}P^{k-1}$. Then $\hat{W}$ is the manifold constructed via the coordinate covering $\{W_\alpha, U_\beta\}$ and will have a map ϕ: $\hat{W} \to W$ with the requisite properties.

Now suppose σ: $\hat{W} \to W$ is W blown up along X. Corresponding to the blowing up there exists a unique (up to isomorphism) line bundle $L \to \hat{W}$ which we shall now describe.

We recall, using the notation above, that $\hat{W} = \overline{W - T(X)} \cup_\psi \hat{B}$. Set $E = \nu(X, W)$ and let $T \xrightarrow{\rho'} \mathcal{P}(E)$ be a tubular neighborhood of $\mathcal{P}(E)$ in $\hat{W}$ which we suppose without loss of generality is entirely contained inside of $\hat{B}$. Let $\{\tilde{U}_\alpha\}$ be a trivializing cover for $T \to \mathcal{P}(E)$ and set $U_\alpha = \rho'^{-1}(\tilde{U}_\alpha)$. Let ξ_α be the fiber coordinate over $\tilde{U}_\alpha$ so that $\mathcal{P}(E) \cap \tilde{U}_\alpha = \{\xi_\alpha = 0\}$. Now let $\{W_\gamma\} = \{\nu_\alpha, \nu'_\beta\}$ be a coordinate cover for $\hat{W}$ such that $\nu'_\beta \cap \psi(\mathcal{P}(E)) = \varnothing$ and $\nu_\alpha = \psi(U_\alpha)$, where ψ: $\hat{B} \to \hat{W}$ is the obvious inclusion. Let

$$S_\alpha = \begin{cases} \xi_\alpha \circ \psi^{-1} & \text{if } W_\gamma = \nu_\alpha, \\ 1 \quad \text{if } W_\gamma = \nu'_\beta. \end{cases}$$

Set $f_{\alpha\beta} = S_\alpha S_\beta^{-1}$ in $W_\alpha \cap W_\beta$. Then the collection $\{f_{\alpha\beta}\}$ defines a line bundle $L = [\sigma^{-1}(X)]$ which we call the line bundle corresponding to the blowing up of W along X (relative to the fixed reduction of $\nu(X, W)$). It can be verified that up to isomorphism this bundle also depends only on our choice of reduction of $\nu(X, W)$. For further use we call the covering $\{W_\gamma, S_\gamma\}$ a standard covering of $\hat{W}$ defining $[\sigma^{-1}(X)]$.

In a number of cases $\nu(X, W)$ will have preferable reductions of structure group.

Suppose $L \to W$ is a complex line bundle over W and Ψ is a nonsingular cross-section intersecting W transversely. We note that $\nu(\Psi, L)$ and $\nu(W, L)$ have canonical orientations induced by the orientation of the fiber ($= \mathbf{C}$) of $L \to W$ and the fact that cross-sections are transversal to fibers. Clearly then $\nu(\Psi, L)$ and $\nu(W, L)$ have canonical structures as $\mathbf{C}$-bundles. By transversality we also find that there exists a canonical isomorphism of $\nu(\Psi \cap W, W)$ with $\nu(\Psi, L)|_{\Psi \cap W}$ inducing on the former bundle a canonical structure as a **C**-bundle. We call this structure the standard reduction of $\nu(\Psi \cap W, W)$. More generally suppose $\Psi_1, \ldots, \Psi_k$ are k cross-sections of $L \to X$ crossing normally in L. Let $C = \bigcap_{1 \leq i \leq k} \Psi_i$, let $S_j = \bigcap_{1 \leq i \leq k;\, i \neq j} \Psi_i$ and let $V_j = \bigcap_{i=1}^{i=j} \Psi_i$ with $V_0 = L$. Then we note that using the normal crossing of the cross-sections we can establish that $\nu(C, S_j) \approx \nu(\Psi_i, L)$ for $i \neq j$. In particular then each $\nu(C, S_j)$ has a canonical structure as a $\mathbf{C}^1$-bundle over **C**. But $\nu(C, V_j) \approx \bigoplus_{i=j+1}^{k} \nu(C, S_i)$. Thus $\nu(C, V_j)$ always has an induced structure as a $\mathbf{C}^{k-j}$-bundle over C. We call this structure a standard reduction of $\nu(C, V_j)$ or a standard structure on $\nu(C, V_j)$.

Lastly suppose p is a point in W. Then if W is even-dimensional any coordinate neighborhood of p in W has the structure of a subset of a $\mathbf{C}^n$. Thus even-dimensional manifolds can always be blown up along points. If W is an oriented manifold and U is a small neighborhood of p then as a manifold U either has the same orientation as W or the opposite orientation. If U has the same orientation we refer to the blowing up of W along p using the vector space structure of U as a σ-process on W at p. If U has the opposite orientation we call it a $\bar{\sigma}$-process at p. (Compare [**MM**, Introduction].) We note that if W is 4-dimensional and $\hat{W}$ is W blown up at p then if the blowing up was by a σ-process we have that $\hat{W} \approx W \# Q$; while if it was by means of a $\bar{\sigma}$-process we have that $\hat{W} \approx W \# P$.

If W is a complex manifold to begin with then all the normal bundles in question will have canonical structures as complex line bundles. Using these structures our notion of blowing up along $X \subset W$ is equivalent to the concept of a monoidal transformation of W with center X. For more details on these see [**KM**], [**Sf1**].

In general if we can blow W up along X to get a new manifold $\hat{W}$ and if Y is a subspace of $\hat{W}$ then by the strict image Y' of Y in $\hat{W}$ we shall mean the closure of $\phi^{-1}(Y - Y \cap X)$ in W. Clearly if $Y \cap X = \varnothing$ then $Y' = \phi^{-1}(Y) \approx Y$. Lastly as far as special notation goes we abbreviate $\mathbf{C}P^2$ by P and $-\mathbf{C}P^2 = \{$complex projective plane with opposite orientation$\}$ by Q. We reserve D for a disc in **C** about the origin and write D_ε for $\{z | z \in \mathbf{C} |\, |z| < \varepsilon\}$ and set $D_\varepsilon^* = D_\varepsilon - \{0\}$. In general if U is a disc or manifold then U^* will represent the punctured disc or the manifold with a point removed unless some other interpretation is specifically indicated. We let I be the closed unit interval [0, 1]. If M is a complex manifold and $\rho \in M$ then $\mathcal{O}_\rho$ will be the local ring of analytic functions at p. For further terminology and definitions of algebraic geometric terms such as ample divisor, etc. see [**H**], [**Sf1**]. For more information on the quadratic forms associated to 4-manifolds and their relation to the homotopy of these manifolds see the introduction to [**MM**] and the references quoted there.

2. Deformation theorems. A common approach used in algebraic geometry in studying invariants of algebraic manifolds is to see how these invariants behave under degeneration. For example if V is an algebraic manifold then Griffiths [**Gf**] analyzed the period matrix A of V by considering a family $W \xrightarrow{\phi} D^2$ of algebraic varieties having the property that $\phi^{-1}(t)$ was a deformation of V for all $t \neq 0$ while $\phi^{-1}(0)$ was allowed to acquire singularities. Then the period matrix A_t of $\phi^{-1}(t)$ was analyzed by allowing t to go to zero and seeing what happens.

We wish to adopt a similar approach in studying the topology of an arbitrary C^∞ manifold. That is, given a manifold X^n we will try to embed X^n as a nonsingular fiber of a 'family' $W \xrightarrow{\phi} D^2$ where W is a C^∞ manifold of dimension $n+2$ and ϕ has no critical values on $D^2 - \{0\}$. We will next assume that the 'critical' fiber $X_0 = \phi^{-1}(0)$ of $W \xrightarrow{\phi} D^2$ can be realized as the union of two submanifolds A^n, B^n of W intersecting transversely. Our goal will be to show that if ϕ satisfies an additional mild technical requirement then

$$X \simeq \overline{A - T_A(A \cap B)} \cup_\psi \overline{B - T_B(A \cap B)}$$

where $T_A = T_A(A \cap B)$, $T_B = T_B(A \cap B)$ are tubular neighborhoods of $A \cap B$ in A and B respectively and ψ is some fiber preserving diffeomorphism of ∂T_A onto ∂T_B. Thus we will be able to study the topology of X in terms of the hopefully easier topology of A and B.

The basic key to our arguments will be repeated use of transversality and to do this successfully we begin with some technical definitions and a technical lemma.

DEFINITION 2.1. Let $w = F(z_1, z_2, x)$ be a smooth map of a domain U in $\mathbf{C}^2 \times \mathbf{R}^n$ into $\mathbf{C}$.

Then we shall say that F is a *WL* (Weierstrass-like) function if and only if

(1) for some $K_U > 0$ (depending only on U and F)

$$\left|z_i^{-1}F(z_1, z_2, x)\right| \leq K_U\left(|z_1|^2 + |z_2|^2\right) \quad \text{for } i = 1, 2;$$

(2) any first order derivative $D_\beta F$ satisfies either

$$\left|z_i^{-1}D_\beta F(z_1, z_2, x)\right| \leq K_U\left(|z_1|^2 + |z_2|^2\right) \quad \text{or} \quad \left|z_2^{-1}D_\beta F(z_1, z_2, x)\right| \leq K_U\left(|z_1|^2 + |z_2|^2\right).$$

DEFINITION 2.2. $W^n \xrightarrow{\phi} D$ is a nicely 2-degenerating family of manifolds if and only if ϕ is a proper smooth map of the real n-dimensional manifold W onto the open disc D about the origin in $\mathbf{C}$ such that

(1) ϕ has a critical value only at $0 \in D$.

(2) If $V_\lambda = \phi^{-1}(\lambda)$ for $\lambda \in D$ then V_0 is the union of two submanifolds A_1, A_2 of W which intersect in a compact connected manifold S.

(3) $\{p \mid p$ is a critical point of $\phi\} = S$ and for any $p \in S$ there exists a neighborhood U_p of p in W with $U_p \approx V_1 \times V_2 \subset \mathbf{C}^2\{z_1, z_2\} \times \mathbf{R}^{n-4}(x)$ such that as a function in the local coordinates (z_1, z_2, x) we have

$$\phi(z_1, z_2, x) = z_1 z_2 + F(z, x)$$

where $A_i \cap U_p = \{(z_1, z_2, x) \in U_p \mid z_i = 0\}$, $i = 1, 2$, and F is a *WL* function.

LEMMA 2.3. *Let $W \xrightarrow{\phi} D$ be a nicely 2-degenerating family of manifolds.*

For $\lambda \in D$ let $V_\lambda = \phi^{-1}(\lambda)$ and suppose V_0 is the union of the submanifolds A_1, A_2 with $S = A_1 \cap A_2$.

Then for any sufficiently small tubular neighborhood $T \xrightarrow{\pi} S$ of S in W there exists $\delta > 0$ such that $\lambda \in D$ and $0 < |\lambda| < \delta$ implies

(1) *If $H = \partial T$ and $X_\lambda = V_\lambda \cap T$ and $p = \pi|X_\lambda$, then $X_\lambda \xrightarrow{p} S$ is an $S^1 \times I$ bundle over S with $\partial X_\lambda = V_\lambda \cap H$ having two components X'_λ, X''_λ each an S^1-bundle over S with projection maps $p' = p|X'_\lambda$ and $p'' = p|X''_\lambda$ respectively.*

(2) *There exists a diffeomorphism* $\omega\colon X_\lambda \to X'_\lambda \times I$ *such that*

$$\begin{tikzcd}
X_\lambda \arrow[r, "\omega"] \arrow[ddr, "p"'] & X'_\lambda \times I \arrow[d, "\pi_1"] \\
 & X'_\lambda \arrow[d, "p'"] \\
 & S
\end{tikzcd}$$

commutes (*where* π_1 *is projection on the first factor*).

PROOF. We begin by claiming that if $T \to S$ is any sufficiently small tubular neighborhood of S in W we can pick $\delta > 0$ such that $0 \leqslant |\lambda| < \delta$ implies

(a) If $H = \partial T$ with projection $\pi' = \pi | H$ then V_λ is transversal to $H \xrightarrow{\pi'} S$ and its fibers.

(b) If in addition $|\lambda| \neq 0$ then V_λ is transversal to any fiber of $T \xrightarrow{\pi} S$. (*)

Suppose (*) is true. Then X_λ and each of its fibers are manifolds. Furthermore a straightforward calculation then shows that if D is a sufficiently small 2-disc on S then $p^{-1}(D) \approx D \times \{\text{Closed Annulus}\}$. Thus $X_\lambda \xrightarrow{p} S$ is an $S^1 \times I$ bundle over S. Furthermore by our hypothesis ∂X_0 has exactly two components and thus if $T \xrightarrow{\pi} S$ is sufficiently small then ∂X_λ will also have exactly two components for $|\lambda| < \delta$. Thus the bundle $X_\lambda \xrightarrow{\phi} S$ is an orientable $S^1 \times I$ bundle with structure group $\mathrm{DIFF}^+(S^1 \times I)$ (where DIFF^+ is the group of orientation preserving diffeomorphisms). Now using [E], [S] we obtain that the natural embedding $\mathrm{DIFF}^+(S^1) \hookrightarrow \mathrm{DIFF}^+(S^1 \times I)$ is in fact a homotopy equivalence and so we can reduce the structure group of $X_\lambda \xrightarrow{\phi} S$ to $\mathrm{DIFF}^+(S^1)$. In particular this means that there exists a diffeomorphism $\omega\colon X_\lambda \to X'_\lambda \times I$ such that

$$\begin{tikzcd}
X_\lambda \arrow[r, "\omega"] \arrow[dr, "\phi"'] & X_\lambda \times I \arrow[r, "p_1"] & X' \arrow[dl, "\phi'"] \\
 & S &
\end{tikzcd}$$

is a commutative diagram thus concluding our proof modulo our transversality assertions. It thus suffices to prove the statements (*).

Since S is compact it in fact suffices to show that for any $p \in S$ there exist a neighborhood $\mathfrak{N}_p$ of p in W and constants $\varepsilon > 0$, $\delta > 0$ such that if T is any tubular neighborhood of S consisting of geodesic discs of radius $r < \varepsilon$ relative to some fixed Riemannian metric g on W and $0 \leqslant |\lambda| < \delta$ then V_λ is transversal to $H \xrightarrow{\pi'} S$ and its fibers in $\mathfrak{N}_p$ and if $\lambda > 0$, V_λ is transversal to the fibers of $T \xrightarrow{\pi} S$ in $\mathfrak{N}_p$.

Pick $p \in S$ and let U_p be a neighborhood of p as specified in Definition 2.2. We can thus write $\phi(z_1, z_2, x) = z_1 z_2 + F(z, x)$ as in that definition. We may without loss of generality suppose that $\exists \varepsilon > 0$ such that $T = \{(z_1, z_2, x) | F(0, x) = 0$ and $|z_1|^2 + |z_2|^2 < r\}$ for some $0 < r < \varepsilon < 1$.

We clearly see that for $0 < r < \varepsilon$ we have that V_0 is transversal to H and its fibers in U_p. Thus $\exists\delta > 0$ such that $0 \leqslant |\lambda| < \delta$ implies V_λ is transversal to H and its fibers in U_p. We thus simply must show that V_λ for $0 < |\lambda| < \delta$ is transversal to the fibers of T in U_p.

Let $a \in V_\lambda \cap T \cap U_p$ with local coordinates (ϕ_1, ϕ_2, ξ) so that $\pi(a) = (0, 0, \xi)$ and the fiber of T containing a is F_ξ. We can identify the tangent space to U_p at a with $\mathbf{C}^2(dz_1, dz_2) \times \mathbf{R}^{n-4}(dx_1, \ldots, dx_{n-4})$. Then the tangent space $TF|_a$ of F_ξ at a can clearly be identified with the subspace $dx = (dx_1, \ldots, dx_{n-4}) = 0$. The tangent space $TV_\lambda|_a$ of V_λ at a can now be identified with the subspace

$$\phi_1 dz_2 + \phi_2 dz_1 + dF = 0.$$

Now since F was a WL function we can write $dF = dF^{(1)} + dF^{(2)}$ where $dF^{(i)}$ consists of those forms $D_\beta F d\beta$ satisfying $|z_i^{-1} D_\beta F| \leqslant K_{U_p}(|z_1|^2 + |z_2|^2)$.

Let L_a be the subspace of $TV_\lambda|_a$ given by the equations:

$$\begin{aligned} &(1) \quad \phi_1 dz_2 + dF^{(1)} = 0 \quad \text{if } \phi_1 \neq 0; \qquad dz_2 = 0 \quad \text{if } \phi_1 = 0; \\ &(2) \quad \phi_2 dz_1 + dF^{(2)} = 0 \quad \text{if } \phi_2 \neq 0; \qquad dz_1 = 0 \quad \text{if } \phi_2 = 0. \end{aligned} \qquad (**)$$

We note that $\dim L_a = n - 4$ and $\dim TF_\xi|_a = 4$. Furthermore $L_a + TF_\xi|_a \subseteq TV_\lambda|_a + TF_\xi|_a$; so to prove our transversality assertion it suffices to show that $L_\alpha \cap TF_\xi|_a = 0$. So let $\omega \in L_a \cap TF_\xi|_a$. Thus $\omega = (dz_1(\omega), dz_2(\omega), 0)$. Since $\omega \in L_a \cap TF_\xi|_a$ we must have $t\omega \in L_a \cap TF_\xi|_a$ for any $t > 0$. Then

$$|dz_1(t\omega)|^2 + |dz_2(t\omega)|^2 \leqslant \left|\frac{dF^{(1)}}{\phi_1}\right|^2 + \left|\frac{dF^{(2)}}{\phi_2}\right|^2 \leqslant n^2 K_{U_p}^2.$$

But

$$|dz_1(t\omega)|^2 + |dz_2(t\omega)|^2 = t^2\left(|dz_1(\omega)|^2 + |dz_2(\omega)|^2\right).$$

This is possible only if $\omega = 0$. Q.E.D.

THEOREM 2.4. *Let $W \xrightarrow{\phi} D$ be a nicely 2-degenerating family and suppose ϕ is a proper map. Set $V_\lambda = \phi^{-1}(\lambda)$ and $V_0 = A_2 \cup A_1$ with $A_1 \cap A_2 = S$. Suppose $T \to S$ is a tubular neighborhood of S in W sufficiently small so that $T_i = T \cap A_i$, $i = 1, 2$, are tubular neighborhoods of S in A_i and let $H_i = \partial T_i$.*

Then there exists a bundle isomorphism η: $H_1 \to H_2$ which reverses orientation on fibers such that for any $x \in D - \{0\}$ V_λ is diffeomorphic to $\overline{A_1 - T_1} \cup_\eta \overline{A_2 - T_2}$.

PROOF. We first note that T can always be taken sufficiently small to guarantee the existence of the requisite tubular neighborhoods. Furthermore restricting T still further we can by (*) in Lemma 2.3 pick $\delta > 0$ so that $0 < |\lambda| < \delta$ guarantees that V is transversal to $\partial T \to S$ and its fibers. In particular we note that it is possible to restrict δ even further if necessary and obtain that for $0 < |\lambda| < \delta$ there exists a diffeomorphism G: $(\overline{V_\lambda - V_\lambda \cap T}, V_\lambda \cap \partial T) \tilde{\to} (\overline{V_0 - V_0 \cap T}, V_0 \cap \partial T)$ such that if p_λ: $V_\lambda \cap \partial T \to S$ are the obvious projection maps then $p_\lambda = p_0 \circ G | V_\lambda \cap \partial T$.

Then using Lemma 2.3 we obtain upon setting $X_\lambda = V_\lambda \cap T, g = G|V_\lambda \cap \partial T = G|\partial X_\lambda$ that

$$V_\lambda = \overline{V_\lambda - V_\lambda \cap T} \cup X_\lambda \simeq \overline{V_0 - V_0 \cap T} \cup_g X_\lambda.$$

But

$$\overline{V_0 - V_0 \cap T} = \overline{A_1 - T_1} \amalg \overline{A_2 - T_2} \quad \text{and} \quad X_\lambda \simeq^\omega X'_\lambda \times I \simeq \partial\,\overline{(A_1 - T_1)} \times I.$$

Thus

$$\overline{V_0 - V_0 \cap T} \cup X_\lambda \simeq \overline{A_1 - T_1} \cup_\eta \overline{A_2 - T_2}$$

for some diffeomorphism η: $\partial(\overline{A_1 - T_1}) \to \partial(\overline{A_2 - T_2})$ which is fiber-preserving as a consequence of 2.3(2). Q.E.D.

Having established that in a nicely degenerating family we can relate the topology of the singular and nonsingular fibers we pause to examine an example. Thus let F_n, G_p, H_{n+p} be homogeneous forms of degree n, resp. p, resp. $n + p$, in $k + 1$ variables. Suppose $X_1 = \{Z \in \mathbf{C}P^k | F_n(z) = 0\}$, $X_2 = \{z \in \mathbf{C}P^k | G_p(z) = 0\}$ and $V = \{z \in \mathbf{C}P^k | H_{n+p}(z) = 0\}$ are nonsingular and intersect normally. How can we relate the topology of V with that of X_1 and X_2? We can begin by constructing the pencil $L_{[\lambda:\mu]}$: $\lambda F_n G_p + \mu H_{n+p} = 0$. The problem is then to go from the pencil L to a nice 2-degenerating family W whose nonsingular fiber will be diffeomorphic to V and whose singular fiber will be related in a simple fashion to $X_1 \cup X_2$. We can try to obtain this pencil by thinking of L as sitting in $\mathbf{C}P^k \times \mathbf{C}P^1$. Unfortunately L will then have singularities at the points of intersection $A = X_1 \cap X_2 \cap V$. To get around this difficulty we must first blow up $\mathbf{C}P^k$ along $X_1 \cap V$ and then blow up again along the strict image of $X_2 \cap V$. This will result in a map Φ: $\tilde{\mathbf{C}}P^k \to \mathbf{C}P^1$ whose general fiber is diffeomorphic to V and which has a special fiber equal to $\sigma_A X_1 \cup X_2$, where $\sigma_A X_1$ is X_1 blown up at A.

In the following corollary we generalize this procedure for constructing nicely degenerating families out of linear pencils. We first note that in a C^∞ manifold there is no natural notion of a linear pencil of divisors. However we recall that any divisor on a complex manifold can be realized as a cross-section of a complex line bundle over that manifold. Since we can speak of complex line bundles over any manifold we phrase our corollary in that language.

COROLLARY 2.5 (SEE DIAGRAM 1). *Suppose W is a compact manifold. Let $L_i \to W$, $i = 1, 2$, be complex line bundles over W. Let Ψ_1, Ψ_2 be distinct cross-sections of $L_i \to W$, $i = 1, 2$, each intersecting W transversely and suppose $(\Psi_1 \cap W)$ is transversal to $(\Psi_2 \cap W)$ in W. Let $L_3 = L_1 \otimes L_2$ be the 2-fold tensor product bundle over W. Let $\Phi_1 = \Psi_1\Psi_2$ considered as a cross-section of $L_3 \to W$ and suppose Φ_3 is a cross-section of $L_3 \to W$ intersecting W transversely such that $\Phi_3 \cap W$, $\Psi_1 \cap W$, $\Psi_2 \cap W$ have normal crossing in W. Let $X_i = \Psi_i \cap W$, $i = 1, 2$; $V = \Phi_3 \cap W$; $S = X_1 \cap X_2$ and $C = X_1 \cap X_2 \cap V$.*

Let $\nu(C, X_2)$ be the normal bundle of C in X_2 with its standard structure as a $\mathbf{C}^2$-bundle over C. Let X'_2 be X_2 blown up along C using the standard complex structure of $\nu(C, X_2)$. Let S' denote the strict image of S in X'_2 and suppose $T'_2 \xrightarrow{\pi'} S'$ is *a tubular neighborhood of S' in X_2' with boundary bundle $H_2' \xrightarrow{\rho'} S'$. Let $T_1 \xrightarrow{\pi} S$ be a tubular neighborhood of S in X_1 with boundary bundle $H_1 \xrightarrow{p} S$.*

Then there exist a bundle isomorphism η: $H_2' \to H_1$ and a constant $\varepsilon > 0$ such that if V_λ is the intersection of the cross-section $\Phi_1 + \lambda\Phi_3$ with W then for $\lambda \in D_\varepsilon^*$, V_λ is a manifold diffeomorphic to $\overline{X_1 - T_1} \cup \overline{X_2' - T_2'}$.

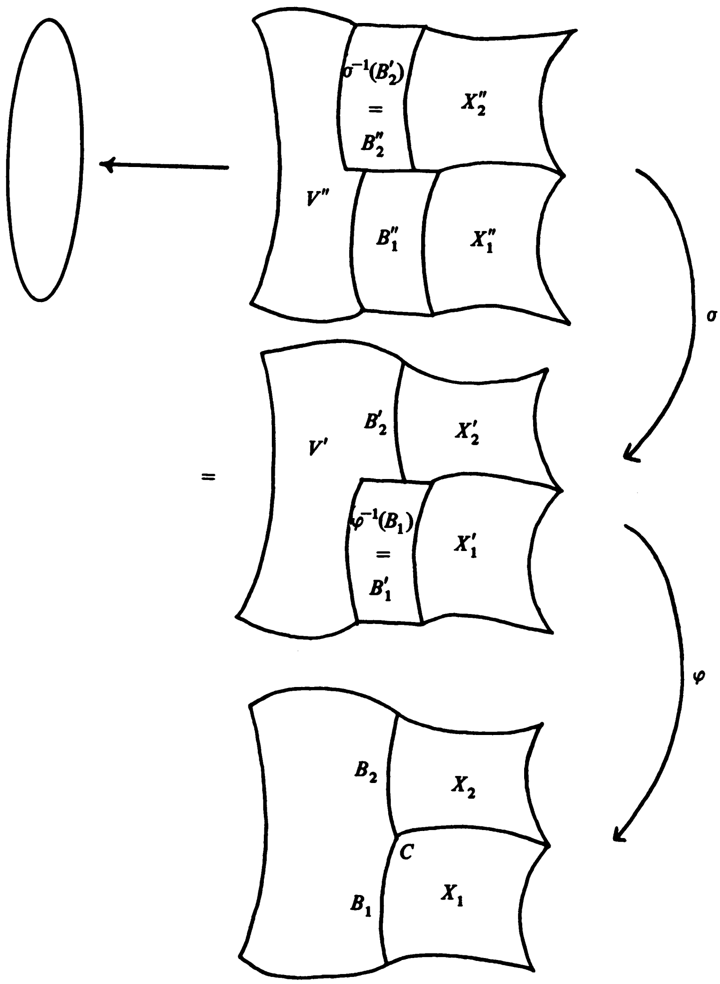

DIAGRAM 1

PROOF. Let $B_i = X_i \cap V$, $i = 1, 2$. Then by transversality the B_i are submanifolds of W and the normal bundle $\nu(B_1, W)$ of B_1 in W has a standard structure as a C^2-bundle over B_1. Blow W up along B_1 using this structure to get the manifold W' with μ: $W' \to W$ the blowing-down map. Let $\tilde{B}_1 \to W'$ be the complex line bundle over W' corresponding to this blowing-up and let $L_i' \to W'$, $i = 1, 2, 3$, the unique complex line bundles over W' such that $\tilde{B}_1 \otimes L_i' = \mu^* L_i$, $i = 1, 2, 3$. Let $\Psi_2' = \mu^*\Psi_2$ and let $X_2' = \Psi_2' \cap W'$. Since $\nu(C, X_2) = \nu(B_1, W)|_{C = B_1 \cap X_2}$ we see that X_2' is simply X_2 blown up along C using the standard structure of $\nu(C, X_2)$.

Now suppose $\{\mathcal{W}_\alpha, S_\alpha\}$ is a standard covering of W' defining the bundle B_1 with the open sets $\mathcal{W}_\alpha$ sufficiently small so that $\mu^* L_i|\mathcal{W}_\alpha$, $i = 1, 2, 3$, are product bundles. Let $\tilde{\Psi}_1$, $\tilde{\Phi}_3$ be the cross-sections of $\mu^* L_1 \to W'$, respectively $\mu^* L_3 \to W'$, induced by Ψ_1, resp. Φ_3. Let $\tilde{\psi}_{1\alpha}$, resp. $\tilde{\phi}_{3\alpha}$, be the complex-valued functions on $\mathcal{W}_\alpha$ representing $\tilde{\psi}_1|\mathcal{W}_\alpha$, resp. $\tilde{\phi}_3|\mathcal{W}_\alpha$. Then it can be easily verified that $\tilde{\psi}_{1\alpha}$, $\tilde{\phi}_{3\alpha}$ are divisible by S_α and that there thus exist cross-sections Ψ_1' of $L_1' \to W'$ resp. Φ_3' of $L_3' \to W'$ such that if $\psi_{1\alpha}'$, $\phi_{3\alpha}'$ are the functions representing Ψ_1', resp. Φ_3' over $\mathcal{W}_\alpha$ then $\tilde{\psi}_{1\alpha} = S_\alpha \psi_{1\alpha}'$ and $\tilde{\phi}_{3\alpha} = S_\alpha \phi_{3\alpha}'$. Let $X_1' = \Psi_1' \cap W'$ and $V' = \Phi_3' \cap W'$. Let $\Phi_1' = \Psi_1' \Psi_2'$ considered as a cross-section of $\phi^* L_1 \otimes L_2' = L_3'$ and let V_λ' be the zero-locus of $\Phi_1' + \lambda \Phi_3' = (\Phi_1' + \lambda \Phi_3') \cap W'$. Then it can be readily verified that X_1' is diffeomorphic to X_1, V' is diffeomorphic to V and V_λ' is diffeomorphic to V_λ.

Now set $S' = X_1' \cap X_2'$. Then it can be readily verified that X_1' intersects X_2' transversely and that S' is in fact simply the strict image of S in the blowing up $X_2' \to X_2$ of X_2 along C. Note further that as a result of our blowing up we now have that $X_1' \cap V' = \varnothing$ and $X_1' \cap V_\lambda' = \varnothing$.

We then also note that $\Phi_1' \cap W' = X_1' \cup X_2'$ so that $V' \cap (\Phi_1' \cap W') = V' \cap X_2'$. Furthermore Φ_1' is transversal to $W' \subset L_3'$ at any point of intersection and $(\Phi_1' \cap W')$ is transversal to $(\Phi_3' \cap W')$ along $V' \cap X_2'$. Thus $B_2' = V' \cap X_2'$ is a submanifold of W' and $\nu(B_2', W')$ has a standard structure as a $\mathbf{C}^2$-bundle over B_2'. This standard structure can be used to blow W' up along B_2' to get the manifold W'' with $\sigma\colon W'' \to W'$ the blowing down map. Let $B_2 \to W''$ be the complex line bundle over W'' corresponding to this blowing up and let $L_i'' \to W''$, $i = 1, 2, 3$, be the unique complex line bundles over W'' such that $B_2 \otimes L_i'' = \sigma^* L_i'$, $i = 1, 2, 3$. Let $\Psi_1'' = \sigma^* \Psi_1'$ and $X_1'' = \Psi_1'' \cap W''$. Note that $X_1'' \approx X_1' \approx X_1$. By the same arguments as before if $\{U_\alpha, t_\alpha\}$ is a standard covering of W'' defining B_2 and trivializing $\sigma^* L_i'$, $i = 1, 2, 3$, then there exist cross-sections Φ_3'' of L_3'' and Ψ_2'' of L_2'' such that locally $t_\alpha \phi_{3\alpha}'' = \tilde{\phi}_{3\alpha}'$ and $t_\alpha \psi_{2\alpha}'' = \tilde{\psi}_{2\alpha}'$ where $\tilde{\phi}_3'$, $\tilde{\psi}_2'$ are the cross-sections of $\sigma^* L_3'$ resp. $\sigma^* L_2'$ induced by ϕ_3', resp. ψ_2'.

Let $X_2'' = \Psi_2'' \cap W''$ and $V'' = \Phi_2'' \cap W''$. Then it can be verified that $X_2'' \approx X_2'$, $V'' \approx V' \approx V$, $V'' \cap X_2'' = \varnothing$, $V'' \cap X_1'' = \varnothing$ and that $X_1'' \cap X_2'' = S''$ is diffeomorphic to S'. Furthermore if $\Phi_1'' = \Psi_1'' \Psi_2''$ considered as a cross-section of L_3'' with $V_\lambda'' = (\Phi_1'' + \lambda \Phi_2'') \cap W''$ then $V_\lambda'' \approx V_\lambda' \approx V_\lambda$.

Now $(\Phi_1'' \cap W'') \cap (\Phi_2'' \cap W'') = (X_1'' \cup X_2'') \cap V'' = \varnothing$. Thus the map $\tilde{f}(w) = [\Phi_1''(w), \Phi_3''(w)] = [t_0, t_1]$ is a well-defined map of W'' onto $\mathbf{C}P^1$. Let us examine $\tilde{f}$ in a neighborhood of its zero-fiber, $\tilde{f}^{-1}[0, 1]$. Suppose $p \in \tilde{f}^{-1}[0, 1]$. Then $\Phi_1''(p) = 0$ so $p \in X_1'' \cup X_2''$. Suppose $p \notin S''$ so we may assume without loss of generality that $p \in X_1'' - X_2''$. Then there exists a sufficiently small neighborhood U_α of p in W'' such that if $\psi_{1\alpha}''$ is the complex-valued function corresponding to $\psi_1''|U_\alpha$, with $\psi_{2\alpha}''$, $\phi_{3\alpha}''$ the corresponding representatives of ψ_2'', ϕ_3'', then on U_α we have $f^{-1}[0, 1] \cap U_\alpha = \{\psi_{1\alpha}'' = 0\}$. However since ψ_1'' is transversal to W'' we can thus choose local coordinates $(x_1, \ldots, x_N)$ in U_α (shrinking it if necessary) such that $x_1 = \operatorname{Re} \psi_{1\alpha}''$ and $x_2 = \operatorname{Im} \psi_{1\alpha}''$. Now at $p \in U_\alpha$ we have that $\psi_{2\alpha}''(p) \neq 0$ and $\phi_{3\alpha}''(p) \neq 0$. Thus we can shrink U_α even further so that $\psi_{2\alpha}''$, $\phi_{3\alpha}''$ do not vanish on it. Set $z_\alpha(q) = \psi_{1\alpha}''(q)\psi_{2\alpha}''(q)/\phi_{3\alpha}''(q)$ with $\xi_1 = \operatorname{Re} z_\alpha$ and $\xi_2 = \operatorname{Im} z_\alpha$. Then

$(\xi_1, \xi_2, x_3 \ldots x_N)$ provide a new set of local coordinates on U_α in terms of which f coincides with projection on the first two coordinates. In particular f has no critical points in U_α.

Now suppose $p \in S''$. Again there is a neighborhood U_α of p such that $\tilde{f}^{-1}[0, 1] \cap U_\alpha = \{\psi''_{1\alpha} = \psi''_{2\alpha} = 0\}$. Using the transversality of $(\psi''_1 \cap W'')$ with $(\Psi''_1 \cap W'')$ we can choose local coordinates $(x_1, \ldots, x_N)$ on U_α with $x_{2i-1} =$ Re $\psi''_{i\alpha}$ and $x_{2i} =$ Im $\psi''_{i\alpha}$. Now let $z_1 = \psi''_{i\alpha}/\phi''_{3\alpha}$ and $z_2 = \psi''_{2\alpha}$. We can shrink U_α further so that ϕ''_3 is nonzero on U_α and we can choose new local coordinates $(u_1, v_1, u_2, v_2, x_5 \ldots x_N)$ on U_α such that $z_j = u_j + iv_j$. Since $\tilde{f}$ is a proper map we can choose an $\varepsilon > 0$ such that $Z = \tilde{f}^{-1}(D_\varepsilon[0, 1])$ is contained in $\bigcup U_\alpha$. Then setting $f(z) = \psi_{1\alpha}\psi_{2\alpha}(z)/\phi_{3\alpha}$ we note that f is a well-defined proper map onto $D_\varepsilon \subset \mathbf{C}$ and can be written locally as $f(z) = z_1 z_2$ in some neighborhood of any point of S''. Furthermore we clearly have $f^{-1}(\lambda) = V''_\lambda$ for $\lambda \in D_\varepsilon$. Thus by Theorem 2.4 we can conclude that $V'' \approx \overline{X''_1 - T''_1} \cup_{\eta''} \overline{X''_2 - T''_2}$ for appropriate tubular neighborhoods T''_1, T''_2 of S''. But we have shown that $V'' \approx V$; $X''_1 \approx X_1$, $X''_2 \approx X'_2$ and $S'' \approx S' \approx S$ so our corollary follows.

In the case W is a complex manifold and f is holomorphic the statements of the preceding corollary and theorem can be simplified considerably. We then have:

Corollary 2.6. *Let f: $W \to \Delta$ be a nonconstant, proper holomorphic mapping having a critical value only at zero. Suppose the zero divisor Z_f of f consists of two nonsingular irreducible components A_1, A_2 of multiplicity 1 crossing normally in a nonsingular connected submanifold S. Suppose $T \to S$ is a tubular neighborhood of S in W sufficiently small so that $T_i = T \cap A_i$, $i = 1, 2$, are tubular neighborhoods of S in A_i and let $H_i = \partial T_i$.*

Then there exists a bundle isomorphism η: $H_1 \to H_2$ which reverses orientation on fibers such that for any regular value $\lambda \in \Delta$ of f, $f^{-1}(\lambda) = V_\lambda$ is diffeomorphic to $\overline{A_1 - T_1} \cup_\eta \overline{A_2 - T_2}$.

Proof. Since Z_f consists of two nonsingular irreducible components of multiplicity one crossing normally we can choose for any $p \in S$ a coordinate neighborhood U_p around p in W with local coordinates $\{z'_1, \ldots, z'_n\}$ so that $z'(p) = 0$ and $A_i \cap U_p = \{z'_i = 0\}$. Then as a function of the z'_i we see that on U_p, $u = f/z'_1 z'_2$ is a unit in $\mathcal{O}_p$ **[GR]**.

Thus we can pick new coordinates $\{z_1, \ldots, z_n\}$ for U_p with $z_i = z'_i$ for $i \neq 1$ and $z_1 = uz'_1$. In terms of these new coordinates we have $f = z_1 z_2$ and so the hypotheses of Theorem 2.4 on f are satisfied. Furthermore since holomorphic maps have isolated critical values we can shrink U_p until zero is the only critical value of f in U_p. Then we may apply Theorem 2.4 to conclude. $\square$

If we begin with a linear pencil in a complex manifold we have

Corollary 2.7. *Suppose W is a compact complex manifold and V, X_1, Y_2 are closed complex submanifolds with normal crossing in W. Let $S = X_1 \cap X_2$ and $C = V \cap S$. Suppose as divisors V is linearly equivalent to $X_1 + X_2$. Let σ: $X'_2 \to X_2$ be the monoidal transformation of X_2 with center C. Let S' be the strict image of S in*

X_2' and let $T_2' \xrightarrow{\pi'} S'$, $T_1 \xrightarrow{\pi} S$ be tubular neighborhoods of S' in X_2' and S in X_1 respectively. Denote the S^1-bundles $\partial T_2' \xrightarrow{p'} S'$, $\partial T_1 \xrightarrow{p} S'$ by H_2' resp. H_1.

Then there exists a bundle isomorphism η: $H_2' \to H_2$ which reverses orientation on fibers such that

$$V \approx \overline{X_2' - T_2'} \cup_\eta \overline{X_1 - T_1}.$$

PROOF. Using the well-known correspondence **[KM]**, **[Sf1]** between divisors of a complex manifold and cross-sections of the corresponding line bundles on W defined by those divisors we can rephrase our hypothesis so as to be in the situation covered by Corollary 2.5. Furthermore if P is the pencil of divisors given by $\mu(X_1 + X_2) + \lambda V$ then we can use Bertini's theorem to conclude that V is diffeomorphic to the generic regular element V_λ of this pencil. But using Corollary 2.5 we have that $V_\lambda \approx \overline{X_1 - T} \cup \overline{X_2' - T_2'}$ and thus the same is true for V. Q.E.D.

We are now in a position to show that in many cases we can obtain a diffeomorphism between a manifold V and a union $A_1 - T_1 \cup_\eta A_2 - T_2$ of manifolds with boundary. Furthermore if this diffeomorphism was obtained as in Corollary 2.5 or 2.7 we know that A_1 is obtained by blowing up some other manifold B_1. If V is a 4-manifold then $A_1 \cap A_2$ is a surface, and a union of the form $A_1 - T_1 \cup A_2 - T_2$ is called an irrational connected sum (see **[M]**). In such a case we can, by surgering V, obtain the following additional topological information on its structure.

THEOREM 2.8 (SEE **[M]**). *Suppose V, M_1, M_2' are oriented simply-connected compact 4-manifolds and suppose S_1, S_2' are compact 2-submanifolds of M_1, M_2' with tubular neighborhoods T_1, T_2' respectively. Let $k = \operatorname{rk} H_1(S_1^*; \mathbf{Z})$, where $S_1^* = S_1 - \{pt\}$.*

Suppose

V is diffeomorphic to $\overline{M_1 - T_1} \cup_\eta \overline{M_2' - T_2'}$ for some bundle isomorphism η: $\partial(M_1 - T_1) \to \partial(M_2' - T_2')$. Then

(1) *either $V \# S^2 \times S^2 \approx M_1 \# M_2' \# k(S^2 \times S^2)$ or $V \# P \# Q \approx M_1 \# M_2' \# k(P \# Q)$, with the second alternative holding if either V or M_1 or M_2' is not a spin manifold.*

(2) *If M_2' is obtained by blowing up the 4-manifold M_2 at a point P in a compact 2-submanifold S_2 in M_2 whose strict image is S_2' then*

(a) *$M_2' = M_2 \# Q$ implies $V \# P = M_1 \# M_2 \# k(P \# Q)$ and*

(b) *$M_2' = M_2 \# P$ implies $V \# Q = M_1 \# M_2 \# k(P \# Q)$.*

3. Resolving numbers of 4-manifolds. Let M be a simply-connected compact 4-manifold. Then as a consequence of **[W1]**, **[W2]** there exists an integer $k > 0$ such that either $M \# (k+1)P \# kQ$ or $M \# (k+1)(P \# Q)$ is completely decomposable. The minimum such integer k will be called the resolving number for M and denoted by $k(M)$.

A straightforward computation then gives us

LEMMA 3.1. *Suppose X, X_1, X_2 are simply-connected compact 4-manifolds with $k_1 = k(X_1) < k_2 = k(X_2)$. Let B_i, σ_i denote the 2nd betti number and signature respectively of X_i, $i = 1, 2$, and set $c_i = \frac{1}{2}(B_i - |\sigma_i|)$. Then if either*

$$X \# P \# Q \approx X_1 \# X_2 \# m(P \# Q) \quad \textit{or} \quad X \# P \approx X_1 \# X_2 \# m(P \# Q)$$

then

$$k(X) \leq \max\{k_1 - m, k_2 - m - c_1, -1\} + 1.$$

THEOREM 3.2. *Suppose W, A_1, A_2, V are as in Theorem 2.4. Suppose further V_λ, A_1, A_2 are simply-connected and of dimension 4 with $k_1 = k(A_1) \leq k_2 = k(A_2)$ and either A_1 or A_2 not spin. Set $m = \operatorname{rk} H_1(S^*; \mathbf{Z})$ where $S^* = S - pt$.*

Then for any $\lambda \in D^*$,

$k(V_\lambda) \leq \max\{k_1 - m, k_2 - m - c_1, -1\} + 1$ *where* $2c_1 =$ $\operatorname{rk} H_2(A; \mathbf{Z}) - |\sigma(A_1)|$ ($\sigma(X)$ = *signature of the 4-manifold X*). (*)

Thus in particular if $k_1 = 0$, $k_2 = 0$, $m > 0$ then $k(V_\lambda) = 0$.

PROOF. By Theorem 2.4 we obtain that V_λ is diffeomorphic to $\overline{A_1 - T_1} \cup_\eta \overline{A_2 - T_2}$ where T_i is a tubular neighborhood of S in A_i and η: $\partial T_1 \to \partial T_2$ is a bundle isomorphism reversing orientation on the fibers. Furthermore V_λ is simply-connected so we can apply Theorem 2.8(1) to obtain that

$$V_\lambda \# P \# Q \approx A_1 \# A_2 \# m(P \# Q).$$

Our result now follows from Lemma 3.1.

COROLLARY 3.3. *Let V, V_λ, X_1, X_2, S be as in Corollary 2.5 and suppose X_1, X_2 are simply-connected and 4-dimensional. Suppose also $C = X_1 \cap X_2 \cap V \neq 0$ and $k(X_1) \leq k(X_2)$. Set $n = \operatorname{card} C$, $c = \frac{1}{2}(\dim H_2(X_1; \mathbf{Z}) - \sigma(X_1))$, and*

$$m = \operatorname{rk} H_1(S - \{pt\}, \mathbf{Z}).$$

Then there exists $\varepsilon > 0$ such that $0 < |\lambda| < \varepsilon$ implies

(1) *$V_\lambda \# P \approx X_1 \# (n-1)Q \# m(P \# Q)$, and*

(2) *if $k = \max(k(X_1) - m, k(X_2) - m - c, -1) + 1$ then $V_\lambda \# (k+1)P \# kQ$ is completely decomposable.*

PROOF. We first note that transversality and compactness insure that C is simply a finite set of points. Then using Corollary 2.5 and Theorem 2.8 gives us (1). (We note that the appearance of Q's in the term $\# (n-1)Q$ rather than P's follows directly from the fact that the standard reduction of the normal bundle of a point in X_2 corresponds precisely to choosing a neighborhood of the point y with orientation induced by the orientation of X_2. That is, such a choice makes blowing up into a standard σ-process at y and so is the same as taking a connected sum with Q.)

(2) follows immediately from Lemma 3.1.

In the complex case we note that S must be an orientable 2-manifold and so m must be even. We can then immediately obtain

COROLLARY 3.4. *Suppose W is a compact complex 3-manifold and V, X_1, X_2 are closed simply-connected complex submanifolds with normal crossing in W. Let $S = X_1 \cap X_2$ and $C = V \cap S$. Suppose as divisors V is linearly equivalent to $X_1 + X_2$ and that $C \neq \varnothing$. Set $n = \operatorname{card} C$ and $g = \operatorname{genus}(S)$. Then*

(1) $V \# P = X_1 \# X_2 \# (n-1)Q \# 2g(P \# Q)$,

(2) $g > 0$ *and* $\max(k(X_1), k(X_2)) \leqslant 1$ *imply that* $k(V) = 0$ *and* $V \# P$ *is completely decomposable.*

EXAMPLES. (1) Suppose V_1, V_1' are planes in $\mathbf{C}P^3$ and V_2 is a nonsingular quadric in $\mathbf{C}P^3$ such that V_1, V_1', V_2 have normal crossing. Then $S = V_1 \cap V_1'$ is a 2-sphere and $C = V_1 \cap V_1' \cap V_2$ is simply 2 points.

We thus get that since as a divisor V_2 is linearly equivalent to $V_1 + V_1'$ then $V_2 \# P \approx V_1 \# V_1' \# Q \approx 2P \# Q$. Since $V_2 \approx S^2 \times S^2$ we have simply recovered the classical fact that $(S^2 \times S^2) \# P \approx 2P \# Q$.

An entirely similar calculation shows that if V_3 is a nonsingular cubic with V_1, V_2, V_3 crossing normally then since the divisor V_3 is linearly equivalent to $V_1 + V_2$ we obtain:

$$V_3 \# P \approx V_1 \# V_2 \# 5Q \approx (S^2 \times S^2) \# P \# 5Q \approx 2P \# 6Q.$$

It is easy to see that if V_n is any nonsingular hypersurface of degree n in $\mathbf{C}P^3$ there will always exist nonsingular hypersurfaces V_{n-1}, V_1 of degree $n-1$ and 1 respectively such that V_1, V_{n-1}, V_n have normal crossing. In particular then our examples show that any nonsingular hypersurface of degree 2 or 3 is almost completely decomposable. Similarly an inductive argument then shows that any nonsingular V_n is almost completely decomposable, which is the main theorem of [**MM**]. In Theorem 5.3 of this paper we generalize this to nonsingular complete intersections. As an example of such an intersection we consider:

(2) Suppose $V(2, 2)$ is a nonsingular intersection of two quadrics W, Y in $\mathbf{C}P^4$ with W nonsingular. Then there exist nonsingular complete intersections $X(2, 1)$, $X'(2, 1)$ of the quadric W and hyperplanes H, H' in $\mathbf{C}P^4$ such that as divisors on the analytic 3-fold W we have that $V(2, 2)$ is linearly equivalent to $X(2, 1) + X'(2, 1)$ and that $V(2, 2)$, $X(2, 1)$, $X'(2, 1)$ have normal crossing in W. Then since $X(2, 1) \cap X'(2, 1)$ is a 2-sphere and $C = V(2, 2) \cap X(2, 1) \cap X'(2, 1)$ is four distinct points we obtain that since $X(2, 1) \approx X'(2, 1) \approx S^2 \times S^2$ that

$$V(2, 2) \# P \approx X(2, 1) \# X'(2, 1) \# 3Q \approx (S^2 \times S^2) \# (S^2 \times S^2) \# 3Q \approx 2P \# 5Q.$$

Furthermore as a consequence of Corollary 5.2 any nonsingular complete intersection of quadrics T, T' in $\mathbf{C}P^4$ can be obtained as the intersection of quadrics W, Y in $\mathbf{C}P^4$ with W nonsingular. Thus our example shows us that any nonsingular $V(2, 2)$ is almost completely decomposable. (Actually $V(2, 2)$ is one of the classical Fano surfaces [**SR**, VII§5] and is known to be simply P blown up at 5 points. Thus $V(2, 2)$ is in fact completely decomposable. However, no such information is available for complete intersections of higher degree to which our methods are still applicable.)

THEOREM 3.5. *Let W be a simply-connected compact complex submanifold of* $\mathbf{C}P^N$ *of complex dimension* 3.

Let H denote a hyperplane section of W and for any $m \geqslant 1$ *suppose* $V_m \in |mH|$ *is nonsingular. Set* $k_m = k(V_m)$ *and* $b = H^3$. *Then*

(1) $k_{m+1} \leqslant \max\{k_1 - \frac{1}{3}(m-1)^3 b, 0\}$.

(2) $m > (3b^{-1}k_1)^{1/3}$ *implies* $V_{m+1} \# P$ *is completely decomposable.*

(3) $k_1 = 0$ *implies* $k_m = 0$ *for all* $m > 1$.

Proof. We first note that:

(a) For any $m > 0$ there exists a nonsingular $V_m \in |mH|$.

(b) If $V_m, V_m' \in |mH|$ are nonsingular then they are diffeomorphic.

(c) For any $m > 0$ there exist V_1, V_m, V_{m+1} all nonsingular and having normal crossing.

Now set $S_m = V_1 \cap V_m$, $g_m = \text{genus}(S_m)$ and note that $\text{card}(V_1 \cap V_m \cap V_{m+1}) = m(m+1)V^3 = m(m+1)b$. Then using (c) and Corollary 3.4 we obtain that

$$V_{m+1} \# P \approx V_1 \# V_m \# \left[b(m^2+m) - 1\right]Q \# 2g_m(P \# Q). \qquad (**)$$

Furthermore using the adjunction formula [Sf2] we find that $2g_m - 2 = (K_{V_1} + S_m) \cdot S_m$ where K_{V_1} is the canonical divisor on V_1. But again by use of the adjunction formula we obtain $K_{V_1} = (K_W + V_1) \cdot V_1$, where K_W is the canonical divisor on W. Then setting $a = K_W \cdot V_1^2$ we get

$$\begin{aligned} 2g_m - 2 &= (K_W \cdot V_1 + V_1 \cdot V_1 + V_1 \cdot V_m) \cdot S \\ &= K_W \cdot V_1 \cdot V_m + V_1 \cdot V_1 \cdot V_m + V_1 \cdot V_m \cdot V_m \\ &= mK_W \cdot V_1^2 + mV_1^3 + m^2 V_1^3 \\ &= ma + m(m+1)b. \end{aligned}$$

Thus $2g_m = ma + m(m+1)b + 2 = (m-1)(mb-2) + 2mg_1$.

Furthermore if $g_1 = 0$ then V_1 must be rational. But it is classical that for rational surfaces we must have $k_1 = 0$. Furthermore if $g_1 = 0$ we obtain that $V_2 \# P \approx V_1 \# V_1 \# (2b-1)Q$. But if V_1 is rational $V_1 \# V_1 \# (2b-1)Q$ will always be completely decomposable so that $k_2 = 0$. Also for V_3 we obtain that $V_3 \# P \approx V_1 \# V_2 \# (6b-1)Q$. Since V_1 is rational we have $V_1 \# (6b-1)Q \approx P \# (r + 6b - 1)Q$ for some integer $r \geqslant 0$. Thus

$$\begin{aligned} V_3 \# P &\approx V_2 \# P \# (r + 6b - 1)Q \approx V_1 \# V_1 \# (r + 8b - 2)Q \\ &\approx 2P \# (2r + 8b - 2)Q \end{aligned}$$

and so k_3 is zero. Now g_3 is always positive so that we always obtain that $k_{m+1} \leqslant \max(k_m, k_1)$ for $m \geqslant 3$. But $k_1 = k_2 = k_3 = 0$ if $g_1 = 0$ so that all the k_m equal zero in this case and our assertions are trivially fulfilled. Thus henceforth we may assume that $g_1 > 0$. But then $k_{m+1} \leqslant \max(k_m, k_1)$ and so by induction we always have $k_m \leqslant k_1$. Then by Corollary 3.3 we obtain

$$k_{m+1} \leqslant \max\{k_m - 2g_m, k_1 - 2g_m - c_m, -1\} + 1. \qquad (\dagger)$$

We can compute c_m as follows:

Using the decomposition $(**)$ above we see that if $B_m = \text{rk}\, H_2(V_m)$ then

$$\begin{aligned} B_{m+1} &= B_m + B_1 + 2\{ma + m(m+1)b + 1\} + m(m+1)b, \\ \sigma_{m+1} &= \sigma_m + \sigma_1 - m(m+1)b. \end{aligned}$$

Using these recursive formulas we find $B_m = mB_1 + m(m-1)a + m(m^2-1)b + 2(m-1)$, $\sigma_m = m\sigma_1 - \frac{1}{3}m(m^2-1)b$,

$$c_m = \begin{cases} m(B_1 + \sigma_1) + m(m-1)a + \frac{2}{3}m(m^2-1)b + 2(m-1) & \text{if } \sigma_m < 0, \\ m(B_1 - \sigma_1) + m(m-1)a + \frac{4}{3}m(m^2-1)b + 2(m-1) & \text{if } \sigma_m > 0. \end{cases}$$

Now our recursion relation (†) implies that

$$k_{m+1} \leq \max_{0<l<m-2} \left\{ k_1 - 2\sum_{j=0}^{l} g_{m-j} - c_{m-l} + l + 1, k_1 - 2\sum_{j=0}^{m} g_j + m, 0 \right\}.$$

But an explicit calculation using our formulas for c_m and $2g_m$ then shows that

$$\begin{aligned} k_{m+1} &\leq \max\left\{ k_1 - 2\sum_{j=1}^{m} g_j + m, 0 \right\} \\ &= \max\left\{ k_1 - \tfrac{1}{3}m(m^2-1)b - m(m+1)g_1 + m^2, 0 \right\}. \end{aligned}$$

But $g_1 > 0$ so that

$$\begin{aligned} &k_1 - \tfrac{1}{3}m(m^2-1)b - m(m+1)g_1 + m^2 \\ &\qquad \leq k_1 - \tfrac{1}{3}m(m^2-1)b < k_1 - \tfrac{1}{3}(m-1)^3 b \end{aligned}$$

as desired.

Thus if $m > \sqrt[3]{3k_1/b}$ we have $k_{m+1} = 0$ as desired. Clearly now if $k_1 = 0$ so do all the k_m. Q.E.D.

We restate Theorem 3.5 in a particularly attractive fashion as follows:

THEOREM 3.5′. *Suppose W is a simply-connected nonsingular complex projective 3-fold. Then there exists an integer $m_0 \geq 1$ such that any hypersurface section V_m of W of degree $m \geq m_0$ which is nonsingular will be almost completely decomposable.*

We now apply the results of this section to the study of ramified covers of 4-manifolds in §4 and to complete intersections in §5.

4. Ramified covering manifolds. Let W^n be a compact manifold. Then by a theorem of Alexander [A] we can always realize W^n as a finite-sheeted branched covering of S^n. In general one always has many realizations of any given manifold as a branched covering of other manifolds. If W is complex and can be realized as a cyclic branched covering of some V, we can by [Wv] embed W as a section of some line bundle over V. Using this and the fact that the estimates in the last section imply that the resolving number of a section of high degree is less than that of a section of low degree we shall analyze cyclic branched coverings in some detail.

Our goal will be to show that the structure of a cyclic branched covering manifold is in some sense simpler than that of the manifold it is covering. To this end we shall show how given any simply-connected compact 4-manifold M and any integral second homology class $\mathfrak{T}$ on M which is divisible by m, we can construct an almost completely decomposable cyclic branched covering of M whose ramification locus is in $\mathfrak{T}$. More precisely we have

THEOREM 4.1. *Suppose X is a simply-connected compact 4-manifold. Let $\mathfrak{T} \in H_2(X, \mathbf{Z})$ with $\mathfrak{T}^2 \neq 0$ and $\mathfrak{T}$ divisible by some integer $m > 1$. Then there exist a compact simply-connected 4-manifold X and a map ϕ: $\tilde{X} \to X$ exhibiting $\tilde{X}$ as an m-fold branched cover over X whose ramification locus R is a nonsingular representative of $\mathfrak{T}$ such that*

(1) *if $\mathfrak{T}^2 > 0$ then $\tilde{X} \# P$ is completely decomposdble,*

(2) *if $\mathfrak{T}^2 < 0$ then $\tilde{X} \# Q$ is completely decomposable.*

REMARK. If $\mathfrak{T}^2 = 0$ our proof works only under the additional supposition that there exist a special representative R of $\mathfrak{T}$ such that the m-fold cover of X ramified over $\mathfrak{T}$ is simply-connected. If $H_2(X) = 0$ then it can be directly seen that there will exist m-fold ramified covers of X satisfying the conclusions of both (1) and (2).

PROOF. We shall only consider the case $m = 2$. The general case is taken care of by an induction patterned on the proof of the $m = 2$ case. Furthermore by reversing the orientation of X if necessary we may always assume that $\mathfrak{T}^2 > 0$. Thus it suffices to prove (1).

Since $\mathfrak{T}$ is divisible by 2 there exists a homology class $\alpha \in H_2(X)$ with $\mathfrak{T} = 2\alpha$. Let $n = \alpha^2 > 0$, let $k = k(X)$ and let S be an oriented surface of genus $g > \frac{1}{2}(k + 1)$ embedded in X and representing α. Such a surface will clearly always exist [T]. We shall now construct $\tilde{X}$ as a 2-sheeted branched cover of X with ramification locus S. To prove that $\tilde{X} \# P$ is completely decomposable we shall construct a nicely 2-degenerating family whose general fiber is $\tilde{X}$ and whose singular fiber consists of a union $A_1 \cup A_2$ with $A_1 \approx X$, $A_2 \approx X \# rQ$ with $r > 0$ and $A_1 \cap A_2 = X$. Then applying 3.2 will give us $\tilde{X} \# P = X \# X \# (r - 1)Q \# k + 1(P \# Q)$ and since $k = k(X)$ we obtain that $\tilde{X} \# P$ is completely decomposable.

To construct our family we will apply the constructions of Corollary 2.5 and begin by constructing a 6-manifold W and a complex line bundle $E \to W$ as in the hypothesis of that corollary.

We begin by letting $N = \nu(S, X)$ be the normal bundle of S in X. Since S is of codimension two in X, N has a canonical structure as a $\mathbf{C}$-bundle over X. We can now clearly produce a covering $\tilde{\mathfrak{U}} = \{\tilde{U}_0, U_1\}$ of S such that $\tilde{U}_0$ is a disc on S, $N_{|\tilde{U}_0}$ and $N_{|U_1}$ are trivial. We may suppose that $\tilde{U}_0 = \{z | \, |z| < 1 + \varepsilon\}$ some ε, with $0 < \varepsilon \ll 1$ and $\tilde{U}_0 \cap U_1 = \{z | \, 1 - \varepsilon < |z| < 1 + \varepsilon\}$. Let U_0 be the subdisc of $\tilde{U}_0$ given by $U_0 = \{z | \, |z| < 1\}$. Clearly $\mathfrak{U} = \{U_0, U_1\}$ is again a trivializing cover of S for N. Clearly the transition function ϕ_{01} for N relative to the cover $\mathfrak{U}$ is given by $\phi_{01} = z^n$ on $U_0 \cap U_1$. Now let η_i represent the fiber coordinate on U_i. We construct submanifolds p_1, p_2 of N as follows:

Over $N_{|U_0}$ let p_1 be given by

$$p_1\colon \eta_0 - \tfrac{1}{4}(z^n + \delta) = 0 \quad \text{some } \delta, \text{ with } 0 < \delta \ll 1.$$

Now let ρ be a smooth function on S into $[0, 1]$ with supp $\rho \subset \tilde{U}_0$ and $\rho | U_0 \equiv 1$.

Over $N_{|U_1}$ let p_1 be given by

$$p_1\colon \eta_1 - \tfrac{1}{4}(1 + \delta p(z) z^{-n}) = 0.$$

It is immediately verifiable that p_1 is a well-defined submanifold of N. (In fact it is a cross-section of N.)

We now shall construct p_2.

On $N_{|U_0}$, p_2: $\eta_0^2 - \frac{1}{16}(\mu - z^{2n}) = 0$, $0 < \mu \ll 1$; $\mu \neq \delta$, on $N_{|U_1}$, p_2: $\eta_1^2 - \frac{1}{16}(\mu\rho(z)z^{-2n} - 1) = 0$.

Again it is clear that p_2 is a submanifold of N. Now let $T \xrightarrow{\pi_T} S$ be an open tubular neighborhood of the zero-section S in N such that each fiber of T contains $|\eta_i| < 1$. Identify T with a tubular neighborhood of S in X. Clearly we can choose ε, μ, δ sufficiently small so that $p_1, p_2 \subset T \subset X$.

We now construct a line bundle $L \to X$ as follows:

Let $V_i \subset X$ be the subset of $\pi_T^{-1}(U_i)$ given by $|\eta_i| < 1$, $i = 0, 1$.

Let $V_2 = X - S$. Then $\upsilon = \{V_0, V_1, V_2\}$ is an open covering of X. Define transition functions for $L \to X$ relative to υ by

$$\psi_{01} = \pi_T^* z^n = \psi_{10}^{-1} \quad \text{and} \quad \psi_{i2} = \eta_i = \psi_{2i}^{-1}, \qquad i = 0, 1.$$

As $\pi_T^* z^n = \eta_0/\eta_1$ on $V_0 \cap V_1$ we see that the collection $\{\psi_{ij}\}$ defines a bundle $L \to X$ as desired. Let ξ_i be the fiber coordinate in V_i. We now construct submanifolds Q_0, P_1, P_2 of $L \to X$ as follows: Firstly we construct Q_0 so that the defining equations for Q_0 are

$$\text{On } L_{|V_i}, \ \xi_i = \eta_i, \quad i = 0, 1, \quad \text{on } L_{|V_2}, \ \xi_2 = 1.$$

Q_0 is clearly well defined (Q_0 is in fact a cross-section of $L \to X$), and we note that $Q_0 \cap X = S$. We recall that on V_i, $i = 0, 1$, p_1 is given by an equation of the form

$$\eta_i - \pi_T^* A_i(X) = 0 \quad \text{for } x \in S.$$

We then define P_1 on $L_{|V_i}$, $i = 1, 2$, by

$$P_1': \xi_i = \beta(\eta_i - \pi_T^* A_i(x)) \quad \text{with } 0 < \beta \ll 1.$$

Now let ρ' be a smooth function on L into $[0, 1]$ with $\operatorname{supp} \rho' \subset T$ and $\rho'|T \cap \{\eta_i| \, |\eta_i| < 1\} \equiv 1$. Then on $L|V_2$ we define

$$P_1: \xi_2 = \beta\big(1 - \rho'(x)\eta_i^{-1}A_i(x)\big) \quad \text{on } V_2 \cap \pi_T^{-1}(U_i), i = 0, 1,$$
$$\xi_2 = \beta \quad \text{on } V_2 - \big(\pi_T^{-1}(U_0) \cup \pi_T^{-1}(U_1)\big).$$

Again a straightforward verification shows p_1 to be well defined with $P_1 \cap X = p_1$.

Lastly we construct P_2. We recall that on V_i, $i = 0, 1$, p_2 is given by an equation of the form $\eta_i^2 = \pi_T^* B_i(X)$. We now define P_2 by the following equations:

On $L|V_i$, $\xi_i^2 = \pi_i^* B_i(x) - \eta_i^2$, for $i = 0, 1$.

On $L|V_2$:

$$\xi_2^2 = \rho'(x)\eta_i^{-2}\pi_T^* B_i(X) - 1 \quad \text{on } V_2 \cap \pi_T^{-1}(U_i), i = 0, 1,$$
$$\xi_2^2 = -1 \quad \text{on } V_2 - \big(\pi_T^{-1}(U_0) \cup \pi_T^{-1}(U_1)\big).$$

P_2 is also a submanifold of L with $P_2 \cap X = p_2$. Let W be the projective closure of the line bundle $L \xrightarrow{\pi_L} X$ with projection map $W \xrightarrow{\pi_W} X$ such that $\pi_W|L = \pi_L$. We now construct a complex line bundle $E \to W$ as follows:

Let $W_i = \pi_L^{-1}(V_i) \subset W$ for $i = 0, 1, 2$. $W_3 = W - X$ (where $X \hookrightarrow L \hookrightarrow W$).

Then $\mathfrak{W} = \{W_0, W_1, W_2, W_3\}$ is an open covering of W and we may define a line bundle $E \to W$ by means of transition function f_{kl} relative to the covering $\mathfrak{W}$ where $f_{kl} = f_k/f_l$ on $W_k \cap W_l$ and f_j on W_j is given by $f_j = \xi_j$, $j = 0, 1, 2$; $f_3 \equiv 1$. Denote the fiber coordinate of $E \to W$ by ϕ_j. Let $E_2 = E^{\otimes 2} = E \otimes E$ be the 2-fold tensor product bundle of E over W. We now shall construct cross-sections $\tilde{P}_1$, $\tilde{P}'$ of $E \to W$ and $\tilde{P}_2$ of $E_2 \to W$ as follows. Identifying W_i with $L|V_i$ for $i = 0, 1, 2$ we note that P_1, Q_0, P_2 are given by equations of the form $\xi_k = \pi_W^* D_k(y)$; $\xi_k = \pi_W^* C_k(y)$, $\xi_k = \pi_W^* F_k(y)$, $y \in X$, respectively on W_k, $k = 0, 1, 2$. Then let $\tilde{P}_1$ be given by:

$$\phi_k = \xi_k - \pi_W^* D_k(y) \quad \text{on } W_k, k = 0, 1, 2,$$

$$\phi_3 = 1 - \xi_k' \pi_W^* D_k(y) \quad \text{on } W_3 \cap \pi_W^{-1}(V_k), k = 0, 1, 2,$$

where $\xi_k' = \xi_k^{-1}$ is well defined on such intersections.

Note also our definition is compatible with triple intersections. Let $\tilde{P}'$ be given by equations identical in form to the above ones only replacing $D_k(y)$ by $\beta C_k(y) + D_k(y)$.

It is readily verifiable that $\tilde{P}_1$, $\tilde{P}'$ are smooth cross-sections of $E \to W$ with $\tilde{P}_1 \cap W = P_1$ and $(\tilde{P}' - \tilde{P}_1) \cap W = Q_0$.

Now let K_j be the fiber coordinate on E_2. Define $\tilde{P}_2$ by

$$K_j = \xi_k^2 - \pi_W^* F(y) \quad \text{on } W_j, j = 0, 1, 2,$$

$$K_j = 1 - \xi_k'^2 \pi_W^* F(y) \quad \text{on } W_3 \cap \pi_W^{-1}(V_k).$$

$\tilde{P}_2$ is then a smooth cross-section of E_2 with $\tilde{P}_2 \cap W = P_2$.

Now let $\tilde{V}_\lambda$ be the cross-section $\tilde{P}_1\tilde{P}' + \lambda\tilde{P}_2$ of E_2 and set V_λ equal to the zero-locus of $\tilde{V}_\lambda = \tilde{V}_\lambda \cap W$. Let $\pi_\lambda = \pi_W|V_\lambda : V_\lambda : V_\lambda \to X$.

Then it is straightforward to verify that $V_\lambda \xrightarrow{\pi_\lambda} X$ is a double covering of X with branch locus a nonsingular oriented surface R_λ in X representing $2\alpha = \mathfrak{T}$. Furthermore we note that V_λ, $\tilde{P}_1 \cap W$, $\tilde{P}' \cap W$ satisfy the hypotheses of Corollary 3.3 provided λ is sufficiently small and that $\tilde{P}_1 \cap \tilde{P}' \cap W = Q_0 \cap X = S$ is an oriented surface of genus $g \geqslant \frac{1}{2}(k + 1)$. Then by Corollary 3.3 we obtain that for λ sufficiently small

$$V_\lambda \# P \approx X \# X \# (n - 1)Q \# 2g(P \# Q)$$

and since $X \# 2g(P \# Q)$ is completely decomposable so is $V_\lambda \# P$. Now simply set $\tilde{X} = V_{\lambda_0}$ for some appropriate λ_0.

We get similar but stronger results in the algebraic category. We have

THEOREM 4.2. *Let V be a simply-connected nonsingular algebraic surface. Let D be an ample-divisor on V.*

Then for any integer $p \geqslant 2$ there exists an integer $m_p > 0$ such that if $m \geqslant m_p$ then

(1) *There exists a nonsingular curve $E_{pm} \in |pmD|$.*

(2) *If E_{pm} is any nonsingular curve in $|pmD|$ there exists an algebraic p-fold covering X_{pm} of V with ramification locus E_{pm} such that $X_{pm} \# P$ is completely decomposable.*

(3) *If* $X \xrightarrow{f} V$ *is any cyclic p-fold branched cover of V with nonsingular ramification locus* $R \in |pmD|$ *with* $m \geqslant m_p$ *then* $X \# P$ *is completely decomposable.*

PROOF. As in Theorem 4.1 we shall consider only the case of $p = 2$. The general case is proved in an entirely analogous fashion and we leave it to the reader. As mentioned in the introduction our theorem implies that any simply-connected algebraic function field of 2 variables has a satisfactory cyclic extension of degree 2 which is topologically normal. We also note that since V is projective it always admits ample divisors.

Now let $k = k(V)$ and let n_0 be an integer such that nD is very ample for $n > n_0$. By Bertini's theorem there then exists a nonsingular curve $E_n \in |nD|$ for any $n \geqslant n_0$. Furthermore by the adjunction formula $2g(E_n) = (K_V + E_n) \cdot E_n + 2$ where K_V is the canonical divisor on V. Then $2g(E_n) = (K_V + nD) \cdot nD + 2 = nK_V \cdot D + n^2D^2 + 2$. Since D is ample, $D^2 > 0$ and there thus exists $m_0 > n_0$ such that for any $m \geqslant m_0$ we obtain $2g(E_m) \geqslant k + 1$. Pick $m \geqslant m_0$. Then since $m > n_0$ Bertini's theorem gives us the existence of nonsingular $E_{2m} \in |2mD|$. Let E_{2m} be such a nonsingular curve. By Theorem A.1 of our Appendix we have $\pi_1(X - E_{2m}) = \mathbf{Z}_{2N}$ for some multiple N of m. Thus there exists a unique subgroup $G \subset \mathbf{Z}_{2N}$ of index 2 and so a unique unramified 2-fold covering of $X - E_{2m}$. But then noting the proof of Corollary A.2 in the Appendix we see that there exists a unique completion X_{2m} which is a simply-connected 2-fold cover of V with ramification locus E_{2m}. We have thus demonstrated existence in (1) and both the existence and *uniqueness* of the 2-fold covering X_{2m}. We note that since X_{2m} is *unique*, if $X_{2m} \# P$ is completely decomposable then assertions (2) and (3) are true. We thus proceed to show that $X_{2m} \# P$ is completely decomposable. To this end let $[E_m]$ be the unique line bundle associated to the divisor mD and suppose $\{U_\alpha\}$ is a trivializing cover for $[E_m]$ with transition functions $\{\phi_{\alpha\beta}\}$. Thus $\{\phi^2_{\alpha\beta}\}$ gives us transition functions for the line bundle $[E_{2m}]$ associated to $2mD$ relative to the cover $\{U_\alpha\}$. Let ξ_k, η_k be fiber coordinates for $[E_m]$, $[E_{2m}]$ respectively. Let e_{2m} be the cross-section of $[E_{2m}]$ corresponding to the nonsingulr curve E_{2m} in V and suppose e_{2m} is given locally by $\eta_k = F_k(y)$ where $y \in U_k$. Let P_2 be the submanifold of $[E_m]$ given locally by $\xi_k^2 = F_k^2(y)$ and suppose P_1, Q_0 are cross-sections of $[E_m]$ with zero-locus E_m, $\hat{E}_m$ respectively. We can clearly choose P_1, Q_0, P_2 so that E_m, $\hat{E}_m$ are nonsingular and E_m, $\hat{E}_m$, E_{2m} have normal crossing in V. Let W be the projective closure of $[E_m]$ with V considered as the zero-section of W. Let $E = [V]$ be the line bundle on W corresponding to the divisor V of W. Then define cross-sections $\tilde{P}_1$, $\tilde{P}'$ of $E \to W$ and $\tilde{P}_2$ of $E^{\otimes 2} \to W$ exactly as in Theorem 4.1. Similarly let $V_\lambda = (\tilde{P}\,\tilde{P}' + \lambda\tilde{P}_2) \cap W$ as in Theorem 4.1. Thus $V_\lambda \xrightarrow{\pi} V$ is a double covering of V with $V_\lambda \# P$ decomposable. However V_λ is linearly equivalent to P_2 in W and both are nonsingular so V_λ is diffeomorphic to P_2. Set $X_{2m} = P_2$. Then X_{2m} is a 2-fold cover of V ramified over E_{2m} with $V \# \mathbf{C}P^2$ completely decomposable.

EXAMPLE. Suppose V is a nonsingular minimal algebraic surface satisfying $C_1^2[V] = 2P_g[V] - 4$ or $C_1^2[V] = 2P_g[V] - 3$. (By the results in **[Hor]** if (x, y) are any pair of integers with $x \geqslant 3$ and $y = 2x - 4$ or $x > 2$ and $y = 2x - 3$ there exists a minimal algebraic surface $V_{(x,y)}$ which is simply-connected and satisfies

$C_1^2[V] = y$ and $P_g[V] = x$.) Then by **[Hor]** V is simply-connected. Now by **[Msh]** we have that $k(V) \leqslant l(P_g)$ where l is the cubic polynomial on p. 67 of **[Msh]**. An explicit calculation using Theorem 4.2 and the adjunction formula then shows that if X is any 2-fold covering of V whose ramification locus R satisfies $R \in |2D|$ for some very ample divisor D, then X is almost completely decomposable.

Similarly if W is any 3-fold such that V is a very ample divisor on W then any nonsingular multiple of V is almost completely decomposable.

5. Complete intersections.

DEFINITION. Let X be an algebraic subvariety of $\mathbf{C}P^N$ of complex dimension k. Then X will be called a complete intersection if its ideal $\mathcal{I}(X)$ is generated by $N - k$ elements. (Recall that $\mathcal{I}(X) = \{w | w$ is an $(N + 1)$ form on $\mathbf{C}P^N$ vanishing on $X\}$.)

If X is a nonsingular complete intersection we have some degree of freedom in specifying $N - k$ generators for $\mathcal{I}(X)$. To see this precisely we need

LEMMA 5.1. *Let V be a nonsingular projective subvariety of $\mathbf{C}P^N$. Let $F_1, \ldots, F_l$ be homogeneous polynomials in $N + 1$ variables of degree $n_1, \ldots, n_l$ respectively with $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_l$ and let H_i denote the zero-locus of the polynomials F_i. Let $X = V \cap (\bigcap_{i=1}^l H_i)$ and suppose that V is not contained in any H_i and V is transversal to H_i at all $x \in X$. Then there exists a homogeneous polynomial $\tilde{F}_1$ in $N + 1$ variables and of degree n_1 such that if $\tilde{H}_j = H_j$, $j > 1$, and $\tilde{H}_1$ is the zero-locus of $\tilde{F}_1$ then*

(1) $X = V \cap (\bigcap_{i=1}^l \tilde{H}_i)$ and

(2) V is transversal to $\tilde{H}_1$ at all points of their intersection.

PROOF. Let $\mathfrak{L}$ be the linear system on V generated by the restriction to V of the polynomials $\{F_1, X_j^{n_1 - n_k} F_k\}$, $j = 0, \ldots, N$, $k = 2, \ldots, l$. The base set of this system is clearly $V \cap (\bigcap_{i=1}^l H_i) = X$. By Bertini's theorem the singularities of a generic element of $\mathfrak{L}$ lie in X. However, the element of $\mathfrak{L}$ corresponding to F_1 is nonsingular along X and thus so is the generic element. Let $\tilde{F}_1$ be a polynomial corresponding to the generic element of $\mathfrak{L}$. Then if $\tilde{H}_1$ is the zero-locus of $\tilde{F}_1$ and $\tilde{H}_j = H_j, j \geqslant 2$, we clearly have that $X = V \cap (\bigcap_{i=1}^l \tilde{H}_i)$ and that $\tilde{H}_1$ intersects V transversely in a nonsingular subvariety $V \cap \tilde{H}_1$.

We then immediately have by induction

COROLLARY 5.2. *Let X^k be a nonsingular k-dimensional complete intersection in $\mathbf{C}P^N$. Then there exist hypersurfaces $H_1, \ldots, H_{N-k}$ in $\mathbf{C}P^N$ with $X^k = \bigcap_{j=1}^{N-k} H_j$ such that if $Y_j = \bigcap_{i=1}^{j} H_i$ then Y_j is nonsingular for $1 \leqslant j \leqslant N - k$ and Y_j intersects H_{j+1} transversely along Y_{j+1} for $j = 1, \ldots, N - k - 1$.*

We can now state

THEOREM 5.3. *Let X be a nonsingular compact complex algebraic surface which is a complete intersection. Then $X \# P$ is completely decomposable.*

PROOF. Since X is a nonsingular algebraic surface it is projectively embedded in some $\mathbf{C}P^N$ in which we assume it is a complete intersection. We proceed by induction on the dimension N above. If $N = 2$ then $X = \mathbf{C}P^N$ and we are done. Thus suppose our theorem is true for all nonsingular 2-dimensional complete intersections $X \subset \mathbf{C}P^N$ where $N \leqslant M$ for some integer $M \geqslant 2$. We then must show that every nonsingular 2-dimensional complete intersection $X \subset \mathbf{C}P^{M+1}$ is also almost completely decomposable. If X is such a subvariety of $\mathbf{C}P^{M+1}$ let $m(X)$ be the minimum degree of the $(M + 1) - 2 = M - 1$ equations defining X and set $m(X) = \infty$ if X is not a nonsingular 2-dimensional complete intersection. We now use induction on $m(X)$. If $m(X) = 1$ then X can be defined as a nonsingular 2-dimensional complete intersection in $\mathbf{C}P^N$ and thus satisfies our conclusion by induction. Thus suppose the theorem is true for all $X \subset \mathbf{C}P^{M+1}$ with $m(X) \leqslant l$ some integers $l \geqslant 1$. Let $X \subset \mathbf{C}P^{M+1}$ with $m(X) = l + 1$.

Let $F_1, \ldots, F_{M-1}$ be homogeneous polynomials on $\mathbf{C}P^{M+1}$ with $\deg F_i = n_i$ and $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_{M-1} = l + 1$ such that $X = \bigcap_{i=1}^{M-1} H_i$ where H_i is the zero-locus of F_i. By Corollary 5.2 we may assume that $W = \bigcap_{i=1}^{M-2} H_i$ is a nonsingular 3-fold and that W and H_{M-1} intersect transversely along X. Now let L, G be polynomials of degree l, 1 respectively whose zero-locus V_L, V_G cut out nonsingular divisors D_L, D_G on W such that X, D_L, D_G intersect normally in W. (Clearly such polynomials will always exist.) Let $C = D_L \cap D_G$ and note that C is a nonsingular 1-dimensional complete intersection of homogeneous polynomials $g_1, \ldots, g_{M-1}$ on $\mathbf{C}P^m$ with $d_i = \deg(g_i) = n_i$ for $i = 1, \ldots, M - 2$ and $d_{M-1} = n_{M-1} - 1 = l$. Then by the classical adjunction formula we have that

$$2g(C) - 2 = \prod_{i=1}^{M-1} d_i\left(\sum_{i=1}^{M-1} d_i - M - 1\right).$$

Now D_L is a nonsingular complete intersection in $\mathbf{C}P^{M+1}$ with $m(D_L) = l$ and D_G is a nonsingular complete intersection in $\mathbf{C}P^M$. Now by Corollary 3.3 we have that $X \# P \approx D_L \# D_G \# (q - 1)Q \# 2g(C)(P \# Q)$ where $q = \operatorname{card}(D_L \cap D_G \cap X) = D_L \cdot D_G \cdot X$. If $g(C) > 0$ we are, using our inductive hypothesis, finished. So suppose $g(C) = 0$. This can occur if and only if $\sum_{i=1}^{M-1} d_i \leqslant M$. That is if either all the d_i equal 1 or one d_j is equal to 2 and the rest of the $d_j = 1$. But it is then easy to see that $g(C) = 0$ if and only if X is one of the following three surfaces

(I) X is a nonsingular quadric in $\mathbf{C}P^3$,

(II) X is a nonsingular cubic in $\mathbf{C}P^3$,

(III) X is a nonsingular intersection of two quadrics in $\mathbf{C}P^4$.

Then, either, using the classical facts [**SR**, VII§5] that (I) is $S^2 \times S^2$, (II) is $\mathbf{C}P^2$ blown up at 6 points, (III) is $\mathbf{C}P^2$ blown up at 5 points, or using the examples we considered following Corollary 3.3 we see that $X \# P$ is completely decomposable in these three cases also.

Thus $X \# P$ will be completely decomposable for all $X \subset \mathbf{C}P^{M+1}$ with $m(X) = l + 1$ and by induction our proof is concluded.

Appendix.

DEFINITION. Let V be an algebraic variety and suppose D is a nonsingular divisor on V. We shall say D is flexible if there exists a nonsingular $D' \in |D|$ such that D' intersects D transversely in a nonempty subvariety of D.

THEOREM A.1. *Suppose V is a nonsingular connected algebraic surface. Let E be a flexible irreducible nonsingular curve on V. Then $\pi_1(V - E)$ is a finite cyclic group of order d = index of imprimitivity d of E (where the index of imprimitivity d of W is the maximum integer such that there exists an $\alpha \in H_2(V_j, \mathbf{Z})$ with E homologous to $d\alpha$).*

PROOF. (SEE FIGURE 2). Since E is flexible there exists a meromorphic function f on V such that $(f) = E - E'$ with E' nonsingular and having transverse intersection with E in a nonempty set of points $K = E \cap E' = \{p_1, \ldots, p_k\}$. Let us blow V up along K. We then get a manifold $\tilde{V} \xrightarrow{\phi} V$ and a holomorphic map $\tilde{f}$: $\tilde{V} \to \mathbf{C}P^1$ induced by f. Let $S_1, \ldots, S_k$ be the preimages of $p_1, \ldots, p_k$ in $\tilde{V}$ and let $\tilde{E}$, $\tilde{E}'$ be the strict images of E, E' in $\tilde{V}$. Let T be a tubular neighborhood of E in V. Replacing E' by a linear combination $E + \lambda E'$ if necessary we may without loss of generality suppose that $E' \subset T$. Let $\tilde{T} = \phi^{-1}(T) \subset \tilde{V}$.

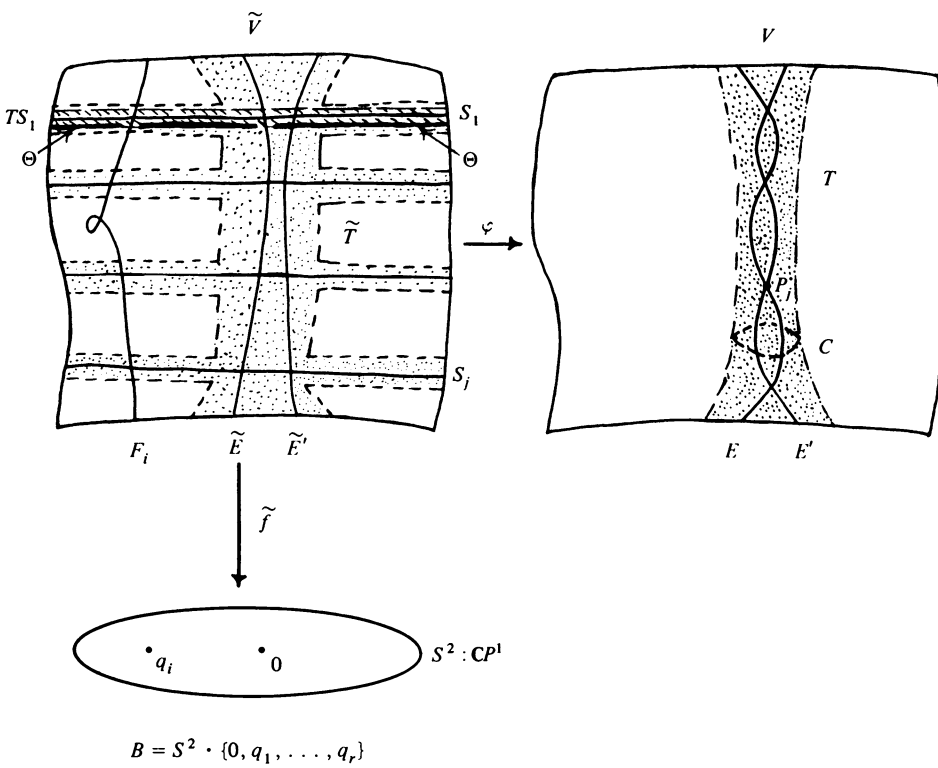

FIGURE 2

Since $\tilde{f}$ is a holomorphic map of $\tilde{V}$ onto $\mathbf{C}P^1$ it has a finite number of critical values $q_1, \ldots, q_r$. We assume $\tilde{f}(\tilde{E}) = 0$ and note that since $\tilde{E}$ is nonsingular $0 \notin \{q_1, \ldots, q_r\}$. Let $F_i = \tilde{f}^{-1}(q_i)$ be the fibers in $\tilde{V}$ corresponding to the q_i. As a consequence of the flexibility of E we see that the F_i, $i = 1, \ldots, r$, intersects the S_j, $j = 1, \ldots, k$, transversely and outside the critical points of $\tilde{f}$ on F_i. Let $B = \mathbf{C}P^1 - \{0, q_1, \ldots, q_r\}$. Let $Y = \tilde{V} - \bigcup_{j=1}^{k} S_j - \bigcup_{i=1}^{r} F_i - \tilde{E}$ and let $\pi = \tilde{f}|Y$. Then since all the critical points of $\tilde{f}$ were within $\cup F_i$ and all the intersections $F_i \cap S_j$ were transverse we immediately obtain that π: $Y \to B$ is a fiber bundle over B with typical fiber E^* diffeomorphic to $\tilde{E}' - (E' \cap \cup_{j=1}^{k} S_j)$. Using the transversality of S_1 to all fibers of $\tilde{f}$ we can find a tubular neighborhood $TS_1 \xrightarrow{\rho} S_1$ of S_1 in $\tilde{V}$ entirely contained in $\tilde{T}$ such that $\rho^{-1}(x) = TS_1 \cap \tilde{f}^{-1}(\tilde{f}(x))$ for all $x \in S_1$. Let $HS_1 = \partial TS_1$ be the corresponding circle bundle over S_1 and $S' = S_1 - (S_1 \cap (\tilde{E} \cup \cup_{i=1}^{r} F_i))$ with $H' = HS_{1|S'}$. Now S is homotopically a 1-complex so there exists a cross-section θ of H' over S'. Clearly θ is then also a cross-section of $Y \xrightarrow{\pi} B$. From the homotopy exact sequence of the bundle $Y \xrightarrow{\pi} B$ we see that $\pi_1(Y)$ is generated by the images of $\pi_1(\theta)$ and $\pi_1(E^*)$ in $\pi_1(Y)$. But $\theta \subset \tilde{T} \cap Y$ and $E^* \subset \tilde{T} \cap Y$ so in particular $\pi_1(\tilde{T} \cap Y) \to \pi_1(Y)$ is surjective. Furthermore $\pi_1(Y) \to \pi_1(\tilde{V} - \tilde{E} - \cup_{j=1}^{k} S_j)$ is also surjective since $Y = (\tilde{V} - \tilde{E} - \cup S_j) - (\cup F_i)$ and $\cup F_i$ has codimension 2 in $\tilde{V} - \tilde{E} - \cup S_j$. We thus obtain the commutative diagram

$$\begin{array}{ccc} & \pi_1(\tilde{T} \cap Y) & \\ {}^{\alpha}\swarrow & & \searrow{}^{\gamma} \\ \pi_1(\tilde{T} - E - \cup S_j) & \xrightarrow{\beta} & \pi_1(\tilde{V} - \tilde{E} - \cup S_j) \end{array}$$

and since α and γ are surjective β: $\pi_1(\tilde{T} - \tilde{E} - \cup S_j) \to \pi_1(\tilde{V} - \tilde{E} - \cup S_j)$ is also surjective. But $T - E \approx \tilde{T} - \tilde{E} - \cup S_j$ and $V - E \approx \tilde{V} - \tilde{E} - \cup S_j$ so we have that $\pi_1(T - E) \to \pi_1(V - E)$ is surjective.

Now suppose C is a fiber of $\partial T \to E$. We consider C as an element of $\pi_1(V - E)$ and suppose $\delta \in \pi_1(V - E)$ is represented by a loop which we shall also call δ. Since V is simply-connected there exists a disc D immersed in V which spans δ. Without loss of generality we may assume D is transversal to E and intersects ∂T in a finite number of fibers $C_1, \dots, C_n$ each homotopically equivalent to C. But this implies that $\delta = \prod r_i C r_i^{-1}$ for some loops $r_i \in \pi_1(V - E)$. We claim that as an element of $\pi_1(T - E) \approx \pi_1(\partial T)$ C is central (i.e. in the center of $\pi_1(T - E)$). So suppose that $\lambda \in \pi_1(\partial T)$ is represented by the loop $\lambda \subset \partial T$. Now since the fibers of ∂T have codimension two in ∂T we can always suppose that λ is outside some fiber of $\partial T \to E$. Thus λ is a loop in $\partial T_{|E-pt}$. However $E - pt$ is homotopically just a 1-complex and since E and ∂T are orientable $\partial T_{|E-pt}$ is just a direct product. Thus $\pi_1(\partial T_{|E-pt})$ is just a direct product of $\pi_1(E - pt)$ and $\pi_1(C)$. In particular λ must commute with C on $\pi_1(\partial T_{|E-pt})$ and thus in $\pi_1(\partial T)$ and so C is central. But $\pi_1(T - E) \to \pi_1(V - E)$ is surjective so C also lies in the center of $\pi_1(V - E)$. But then any $\delta \in \pi_1(V - E)$ simply equals C^n for some integer n and so $\pi_1(V - E)$ is a cyclic group and thus we have $\pi_1(V - E) \approx H_1(V - E)$. Thus to conclude our proof it suffices to show that $H_1(V - E)$ is cyclic of order d. Consider then the Gysin exact sequence of the pair $(V, V - E)$. We have

$$H_2(V) \xrightarrow{E} H_0(E) \to H_1(V - E) \to H_1(V). \qquad (*)$$

Now $H_1(V) = 0$ since V is simply-connected and $H_0(E) \simeq \mathbf{Z}$ since E is connected. Then $H_1(V - E) \approx \mathbf{Z}/\mathrm{Im}\,\psi$ where ψ: $H_2(V) \to H_0(E)$ is the map $\psi(N) = N \cdot E$ for 2-cycles $N \in H_2(V)$.

We claim $\operatorname{Im}\psi = d\mathbf{Z}$. So let $e_1, \ldots, e_M$ be a basis of $H_1(V - E)$ and look at the ideal I in $\mathbf{Z}$ generated by $\psi(e_i) = e_i \cdot E$ for $i = 1, \ldots, M$. But all ideals of $\mathbf{Z}$ are principal and thus $I = (d')$ where $d' = \text{g.c.d.}(e_1 \cdot E, \ldots, e_M \cdot E)$. But by definition then d' is the index of imprimitivity of E and so $\operatorname{Im}\psi = d\mathbf{Z}$ as claimed. Q.E.D.

COROLLARY A.2. *Suppose in the above theorem, that $W \xrightarrow{\pi} V$ is a nonsingular m-fold ramified cover of V with E as ramification locus. Then W is a simply-connected cyclic cover of V.*

PROOF. Let $x \in E$. We claim that there exists a coordinate neighborhood U of x in V with coordinates (z_1, z_2) such that $E \cap U = \{z_1 = 0\}$, $\pi^{-1}(U) \approx \{(z_1, z_2, w) \in U \times \mathbf{C}^1(w) | z_1 = w_1^m\}$ and in terms of these local coordinates $\pi|\pi^{-1}(U) \to U$ is just projection on the first two coordinates. To see this consider the commutative diagram

$$\begin{array}{ccc} \pi^{-1}(U - E) & \xrightarrow{\tilde{j}} & W - \pi^{-1}(E) \\ \downarrow p = \pi|\pi^{-1}(U - E) & & \downarrow \pi \\ U - E & \xrightarrow{j} & V - E \end{array}$$

and let G_m be the subgroup of index m of $\pi_1(V - E) \simeq \mathbf{Z}_d$ corresponding to the connected unramified covering $W - \pi^{-1}(E) \xrightarrow{\pi} V - E$.

Now $j_{\#}\colon \pi_1(U - E) \to \pi_1(V - E)$ is clearly onto and thus we obtain that $\pi^{-1}(U - E) \xrightarrow{p} U - E$ is an unramified covering of $U - E$ corresponding to $G_u = j_{\#}^{-1}(G_m)$. But since $\pi^{-1}(U - E)$ is connected $\pi^{-1}(U \cap E) \approx U \cap E$. Then our assertion follows by standard considerations of the extendability of local analytic maps (see, for example, [**GR**]). Note that $\pi^{-1}(E) \approx E$ and $\pi_1(W - \pi^{-1}(E))$ is a subgroup of index m in $\pi_1(V - E)$ so that by [**Wv**] W is clearly a cyclic cover of V.

Now $\pi_1(W - \pi^{-1}(E)) \to \pi_1(W)$ is onto since $\pi^{-1}(E) \approx E$ is of codimension two in W. But $\pi_1(W - \pi^{-1}(E)) = G_m$ is generated by a fiber of $\partial T(\pi^{-1}(E)) \to \pi^{-1}(E)$ in W. Call a representative of this fiber C. But in W clearly C is null homotopic bounding the corresponding fiber of $T(\pi^{-1}(E)) \to \pi^{-1}(E)$ and since C generates $\pi_1(W)$ we have that W is simply-connected as desired.

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