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A REPRESENTATION THEOREM AND APPLICATIONS TO TOPOLOGICAL GROUPS

BY

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ABSTRACT. We show that, given a set S dense in a compact Hausdorff space X and a complex-valued bounded linear functional Λ on the space $C(X)$ of continuous complex-valued functions on X with uniform norm, there exist an algebra $\mathcal{Q}$ of sets in S and a unique bounded finitely additive set function $\mu: \mathcal{Q} \to \mathbf{C}$ which is inner and outer regular with respect to the zero and cozero sets respectively and such that $\int_S f|S\, d\mu$ exists and is equal to $\Lambda(f)$ for all $f \in C(X)$. In the context of topological groups, this theorem permits us to obtain (1) a concrete representation theorem for bounded complex-valued linear functionals on the space of almost periodic functions with uniform norm, (2) a representation theorem for (not necessarily continuous) positive definite functions, (3) a characterization of the space B of finite linear combinations of positive definite functions, and (4) a necessary and sufficient condition to have a linear transformation from B to B.

1. Introduction.

1.1. In a paper of Hewitt [4] an attempt was made at obtaining a concrete representation theorem for bounded complex-valued linear functionals defined on the space of continuous almost periodic functions on the real line. Though the problem was not solved satisfactorily (see the footnote on p. 379 in [2]), we were inspired by the methods of that paper to obtain a representation theorem for bounded linear functionals on subspaces of continuous functions defined on sets more general than locally compact Abelian groups. Though a general approach to representation theorems has been developed in recent years (see [3], [10], and [16]), we have found that, to obtain a complex-valued finitely additive set function representing the linear functional, which is inner and outer regular with respect to the zero and cozero sets respectively, the method of Loomis [9, p. 169] to generate an algebra of sets (on which is defined the set function) by means of the linear functional in question was our best approach. In the special case of a topological group, it is this algebra, which depends on the linear functional defined on the almost periodic functions, that makes the difference between our results and those of Hewitt.

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2. A representation theorem on spaces of continuous functions.

2.1. Let X be a topological space. By a measure on the Borel subsets $\mathfrak{B}(X)$ of X we mean a complex-valued (and so finite) countably additive set function on $\mathfrak{B}(X)$. Let $C(X)$ and $C_b(X)$ denote the continuous and the bounded continuous complex-valued functions on X respectively. Clearly, if μ is a measure on $\mathfrak{B}(X)$, then any $f \in C_b(X)$ is integrable with respect to μ and so the set $\{\omega \in \mathbf{C}: \mu(f^{-1}(\omega)) \neq 0\}$ is at most countable. Let $|\mu|$ denote the total variation measure on $\mathfrak{B}(X)$ associated with μ. Given an algebra of sets $\mathcal{Q} \subset \mathfrak{B}(X)$, let $\mathcal{Q}_\mu$ be the algebra of sets $E \in \mathcal{Q}$ such that $|\mu|(\bar{E} \setminus \mathring{E}) = 0$. If $A, B \in \mathcal{Q}_\mu$ and if $A \cap S = B \cap S$ for some set S dense in X, then $|\mu|(A \triangle B) = 0$. This is easily shown from the inclusion $A \setminus B \subset (A \setminus \mathring{A}) \cup (\mathring{A} \setminus \bar{B}) \cup (\bar{B} \setminus B)$. If for any algebra $\mathfrak{S} \subset \mathfrak{B}(X)$ we write $\mathfrak{S} \cap S$ for $\{E \cap S: E \in \mathfrak{S}\}$ then $\mathcal{Q}_\mu \cap S$ is an algebra of subsets of S such that the set function $\mu_S: \mathcal{Q}_\mu \cap S \to \mathbf{C}$ given by $\mu_S(E \cap S) = \mu(E)$ $(E \in \mathcal{Q}_\mu)$ is a well-defined bounded finitely additive set function. If $|\mu_S|$ is the total variation of μ_S then clearly $|\mu|(E) = |\mu_S|(E \cap S) = |\mu|_S(E \cap S)$ for all $E \in \mathcal{Q}_\mu$.

2.2 Definition. Given a set Y and an algebra $\mathfrak{S}$ of subsets of Y, a function f: $Y \to \mathbf{C}$ is said to be $\mathfrak{S}$-continuous if, given $\varepsilon > 0$, there exists a finite partition of $\mathbf{C}$ into rectangles $(E_1, \ldots, E_n)$ with $f^{-1}(E_i) \in \mathfrak{S}$ and $|x - y| < \varepsilon$ for all $x, y \in E_i \cap f(Y)$ $(i = 1, \ldots, n)$. We write $C(Y, \mathfrak{S})$ for the class of $\mathfrak{S}$-continuous functions on Y.

2.3 Remarks. (1) In the context of the definition above, a necessary condition for a function f: $Y \to \mathbf{C}$ to be $\mathfrak{S}$-continuous is that $\int_Y f\, d\nu$ exists for every bounded finitely additive set function ν on $\mathfrak{S}$ [**1**, p. 293], where the integral is defined by the usual Moore-Smith method ([**11**, pp. 183–191], [**15**, pp. 401–404]) or equivalently [**7**] by the Dunford-Schwartz method [**2**, pp. 101–125]. Note that our definition of $\mathfrak{S}$-continuity is more restrictive than the one usually given [**1**, p. 293].

(2) Let $\mathcal{Q} \subset \mathfrak{B}(X)$ be an algebra of subsets of the topological space X such that each $f \in C_b(X)$ is $\mathcal{Q}$-continuous. For example, $\mathcal{Q} = \mathfrak{B}(X)$ is such an algebra. Given a complex-valued measure μ on $\mathcal{Q}$ and a set S dense in X such that $f|S$ is $\mathcal{Q}_\mu \cap S$-continuous for all $f \in C_b(X)$, then $\int_S f|S\, d\mu_S = \int_X f\, d\mu$.

2.4. Let $MR(\mathfrak{B}(X))$ denote the space of complex-valued measures ν on $\mathfrak{B}(X)$ which are such that, given $\varepsilon > 0$ and $E \in \mathfrak{B}(X)$, there is a closed set C and an open set U in X such that $C \subset E \subset U$ and $|\nu|(U \setminus C) < \varepsilon$. Given an algebra $\mathfrak{S} \subset \mathfrak{B}(X)$, a finitely additive set function μ: $\mathfrak{S} \to \mathbf{C}$ will be said to be regular on $\mathfrak{S}$ if, given $\varepsilon > 0$ and $E \in \mathfrak{S}$, there exist constants $\alpha, \beta > 0$ and real-valued functions $f, g \in C_b(X)$ such that the sets $K = \{x \in X: f(x) \leqslant \alpha\}$ and $V = \{x \in X: g(x) < \beta\}$ have the properties (i) $K, V \in \mathfrak{S}$, (ii) $K \subset E \subset V$ and (iii) $|\mu|(V \setminus K) < \varepsilon$.

2.5 Lemma. *Given a normal space X and $\mu \in MR(\mathfrak{B}(X))$, let $\mathcal{Q} \subset \mathfrak{B}(X)$ be an algebra for which each $f \in C_b(X)$ is $\mathcal{Q}_\mu$-continuous. If S is dense in X, then μ_S is regular on $\mathcal{Q}_\mu \cap S$ and $\mathcal{Q}_\mu \cap S$ consists of all $E \in \mathcal{Q} \cap S$ such that*

$$\inf\left\{ \int_S (h|S - g|S)\, d|\mu_S| : g|S \leqslant \chi_E \leqslant h|S;\, g, h \in C_b(X) \right\} = 0$$

where χ_E is the characteristic function for E.

PROOF. Since $\mu \in MR(\mathfrak{B}(X))$, given $\varepsilon > 0$ and $E \in \mathcal{Q}_\mu$, there exist a closed set C and an open set U in X such that $C \subset \mathring{E} \subset E \subset \bar{E} \subset U$, $|\mu|(\mathring{E} \setminus C) < \frac{1}{2}\varepsilon$ and $|\mu|(U \setminus \bar{E}) < \frac{1}{2}\varepsilon$. By Urysohn's Lemma, there exist continuous functions φ, ψ: $X \to [0, 1]$ such that $\varphi | C = 0$, $\varphi | (X \setminus \mathring{E}) = 1$, and $\psi | \bar{E} = 0$, $\psi | (X \setminus U) = 1$. Since φ and ψ are $\mathcal{Q}_\mu$-continuous, there exist constants $\alpha, \beta \in (0, 1)$ for which $|\mu|(\varphi^{-1}(\alpha)) = 0$ and $|\mu|(\psi^{-1}(\beta)) = 0$ and if $K = \{x \in X: \varphi(x) \leqslant \alpha\}$ and $V = \{x \in X: \psi(x) < \beta\}$, then $K, V \in \mathcal{Q}_\mu$; $C \subset K \subset \mathring{E} \subset E \subset \bar{E} \subset V \subset U$. So from the inclusion $V \setminus K \subset (U \setminus \bar{E}) \cup (\bar{E} \setminus \mathring{E}) \cup (\mathring{E} \setminus C)$ follows that $|\mu|(V \setminus K) < \varepsilon$. This proves that μ is regular on $\mathcal{Q}_\mu$. It now follows that μ_S is regular on $\mathcal{Q}_\mu \cap S$.

Since $\int_S f | S d|\mu_S| = \int_X f\, d|\mu|$ for all $f \in C_b(X)$, to identify $\mathcal{Q}_\mu \cap S$ it is sufficient to show that $\mathcal{Q}_\mu$ consists of all $F \in \mathcal{Q}$ such that

$$\inf\left\{ \int_X (f_2 - f_1)\, d|\mu| : f_1 \leqslant \chi_F \leqslant f_2; f_1, f_2 \in C_b(X) \right\} = 0. \tag{2.5.1}$$

Given $F \in \mathcal{Q}_\mu$, then $\mathring{F}, \bar{F} \in \mathcal{Q}_\mu$. By regularity on $\mathcal{Q}_\mu$, given $\varepsilon > 0$ there is a closed set $K \in \mathcal{Q}_\mu$ and an open set $V \in \mathcal{Q}_\mu$ such that $K \subset \mathring{F} \subset F \subset \bar{F} \subset V$, $|\mu|(K \setminus \mathring{F}) < \frac{1}{2}\varepsilon$, and $|\mu|(V \setminus \bar{F}) < \frac{1}{2}\varepsilon$. By Urysohn's Lemma, there exist continuous functions φ, ψ: $X \to [0, 1]$ such that $\varphi | K = 1$, $\varphi | (X \setminus \mathring{F}) = 0$, and $\psi | \bar{F} = 1$, $\psi | (X \setminus V) = 0$. Hence

$$\begin{aligned} \int_X \varphi\, d|\mu| &\leqslant \sup\left\{ \int_X f\, d|\mu| : f \leqslant \chi_F; f \in C_b(X) \right\} \\ &\leqslant \inf\left\{ \int_X f\, d|\mu| : \chi_F \leqslant f; f \in C_b(X) \right\} \leqslant \int_X \psi\, d|\mu| \end{aligned}$$

and

$$\begin{aligned} \int_X \psi\, d|\mu| - \int_X \varphi\, d|\mu| &= \int_X (\psi - \varphi)\, d|\mu| = \int_{V \setminus K} (\psi - \varphi)\, d|\mu| \\ &\leqslant |\mu|(V \setminus K) = |\mu|(V \setminus \bar{F}) + |\mu|(\bar{F} \setminus \mathring{F}) + |\mu|(\mathring{F} \setminus K) < \varepsilon \end{aligned}$$

imply equation (2.5.1).

Conversely, if $F \notin \mathcal{Q}_\mu$, then $|\mu|(\bar{F} \setminus \mathring{F}) > 0$. Choose $\varepsilon > 0$ such that $|\mu|(\bar{F} \setminus \mathring{F}) > \varepsilon$. Given any two continuous functions φ, ψ: $X \to (-\infty, \infty)$ such that $\varphi \leqslant \chi_F \leqslant \psi$, then $\varphi | (\bar{F} \setminus \mathring{F}) \leqslant 0$, $\psi | (\bar{F} \setminus \mathring{F}) \geqslant 1$ and so $\int_X \psi\, d|\mu| - \int_X \varphi\, d|\mu| \geqslant |\mu|(\bar{F} \setminus \mathring{F}) > \varepsilon$. Thus, if $F \notin \mathcal{Q}_\mu$, then equation (2.5.1) does not hold. This completes the proof of the lemma.

2.6. Given a topological space Y and a complex-valued linear functional Λ on a subspace B of the Banach space $C_b(Y)$ with uniform norm, we write $\|\Lambda\|$ for the norm of Λ:

$$\|\Lambda\| = \sup\{ |\Lambda(f)| : f \in B, |f| \leqslant 1 \}.$$

If $\|\Lambda\| < \infty$, then we say that Λ is bounded on B. We are now in a position to prove the following crucial result.

2.7 THEOREM. *Let S be a dense subset of a compact Hausdorff space X, $\mathcal{A} \subset \mathcal{B}(X)$ an algebra such that $f|S$ is $\mathcal{A}_\mu \cap S$-continuous for all $f \in C(X)$ and all $\mu \in MR(\mathcal{B}(X))$, and Λ a complex-valued bounded linear functional on $C(X)$. The class $\mathcal{E}_\Lambda$ of sets $E \in \mathcal{A} \cap S$ such that*

$$\inf_{f_1, f_2} \sup_{|g|=1} \{|\Lambda((f_2 - f_1)g)| : f_1|S \leq \chi_E \leq f_2|S; f_1, f_2, g \in C(X)\} = 0$$

is an algebra of sets for which $f|S$ is $\mathcal{E}_\Lambda$-continuous for all $f \in C(X)$, and there exists a unique regular bounded finitely additive set function λ: $\mathcal{E}_\Lambda \to \mathbf{C}$ with

$$\Lambda(f) = \int_S f|S\, d\lambda \qquad (f \in C(X)).$$

Moreover,

$$\|\Lambda\| = |\lambda|(S).$$

PROOF. *Step* 1. By the Riesz representation theorem, there exists a unique $\mu \in MR(\mathcal{B}(X))$ for which

$$\Lambda(f) = \int_X f\, d\mu \qquad (f \in C(X)).$$

Let $\lambda = \mu_S$. By the second remark in 2.3, if $f \in C(X)$, then

$$\Lambda(f) = \int_S f|S\, d\lambda.$$

Since a compact Hausdorff space is normal, by Lemma 2.5, λ is regular on $\mathcal{A}_\mu \cap S$ and $\mathcal{A}_\mu \cap S$ consists of all $E \in \mathcal{A} \cap S$ such that

$$\inf\left\{ \int_X (f_2 - f_1)\, d|\mu| : f_1|S \leq \chi_E \leq f_2|S; f_1, f_2 \in C(X) \right\} = 0.$$

But $d|\mu| = g\, d\mu$ for some Borel measurable function g such that $|g| = 1$ [**12**, p. 126]. Hence $\mathcal{A}_\mu \cap S$ consists of all $E \in \mathcal{A} \cap S$ such that

$$\inf\left\{ \int_X (f_2 - f_1) g\, d\mu : f_1|S \leq \chi_E \leq f_2|S; f_1, f_2 \in C(X) \right\} = 0.$$

But, by Lusin's theorem, any complex Borel measurable function on X can be approximated in μ-measure by functions in $C(X)$. Hence, $\mathcal{A}_\mu \cap S$ consists of all $E \in \mathcal{A} \cap S$ such that

$$\inf_{f_1, f_2} \sup_{|g|=1} \left\{ \left| \int_X (f_2 - f_1) g\, d\mu \right| : f_1|S \leq \chi_E \leq f_2|S; f_1, f_2, g \in C(X) \right\} = 0$$

i.e. such that

$$\inf_{f_1, f_2} \sup_{|g|=1} \{|\Lambda((f_2 - f_1)g)| : f_1|S \leq \chi_E \leq f_2|S; f_1, f_2, g \in C(X)\} = 0.$$

Let $\mathcal{E}_\Lambda = \mathcal{A}_\mu \cap S$.

Step 2. In this step we establish the uniqueness of λ on $\mathcal{E}_\Lambda$. Clearly, the total variation of the difference $\nu = \lambda - \lambda'$ of two bounded finitely additive set functions regular on $\mathcal{E}_\Lambda$ is also a bounded finitely additive set function regular on $\mathcal{E}_\Lambda$. Thus it is enough to show that, given a nontrivial bounded finitely additive nonnegative set function ν regular on $\mathfrak{S}_\Lambda$, then $\int_S h|S\, d\nu \neq 0$ for some $h \in C(X)$. Since $\nu \neq 0$, there exists $E \in \mathfrak{S}_\Lambda$ such that $\nu(E) > 0$. By regularity, given $\varepsilon < \frac{1}{2}\nu(E)$, there exist continuous functions $f, g\colon S \to [0, \infty)$ and $\alpha, \beta > 0$ such that if $K = \{x \in S\colon f(x) \leq \alpha\}$ and if $V = \{x \in S\colon f(x) < \beta\}$ then (i) $K, V \in \mathfrak{S}_\Lambda$, (ii) $K \subset E \subset V$, and (iii) $\nu(V \setminus K) < \varepsilon$. Now if $\bar{K}$ and $\mathring{\bar{V}}$ are respectively the closed and open subsets of X for which $K = \bar{K} \cap S$ and $V = \mathring{\bar{V}} \cap S$, then by Urysohn's Lemma, there exists a continuous function $h\colon X \to [0, 1]$ such that $h|\bar{K} = 1$ and $h|(X \setminus \mathring{\bar{V}}) = 0$. Thus

$$\int_S h|S\, d\nu = \int_K h|S\, d\nu + \int_{V\setminus K} h|S\, d\nu = \nu(K) + \int_{V\setminus K} h|S\, d\nu$$
$$= \nu(E) - \nu(E\setminus K) + \int_{V\setminus K} h|S\, d\nu \geq \nu(E) - \nu(E\setminus K) - \int_{V\setminus K} h|S\, d\nu$$
$$\geq \nu(E) - 2\nu(V\setminus K) \geq \nu(E) - 2\varepsilon > 0.$$

Step 3. This final step consists of showing that $\|\Lambda\| = |\lambda|(S)$. By the Riesz representation theorem

$$\|\Lambda\| = |\mu|(X) = \sup\left\{\left|\int_X f\, d\mu\right| : f \in C(X), |f| \leq 1\right\}.$$

But if $L(X, \mu)$ is the Banach space of μ-integrable functions with norm $\int_X |f|\, d\mu < \infty$, then $C(X) \subset C(X, \mathfrak{A}_\mu) \subset L(X, \mu)$ and, by Lusin's theorem, $C(X)$ is dense in $L(X, \mu)$. Hence

$$\|\Lambda\| = \sup\left\{\left|\int_X f\, d\mu\right| : f \in C(X, \mathfrak{A}_\mu), |f| \leq 1\right\}$$
$$= \sup\left\{\left|\int_S g\, d\lambda\right| : g \in C(S, \mathfrak{S}_\Lambda), |g| \leq 1\right\} = |\lambda|(S).$$

The theorem is proved.

2.8 Remark. If Λ is a positive linear functional on $C(X)$, then the associated regular set function $\lambda\colon \mathfrak{S}_\Lambda \to \mathbf{C}$ of Theorem 2.7 is nonnegative. One way to see this is to apply the Riesz representation theorem for positive linear functionals in Step 1 of the proof. Furthermore, $\mathfrak{S}_\Lambda$ can be chosen to consist of all $E \in \mathfrak{A} \cap S$ such that

$$\inf\{\Lambda(f_2 - f_1)\colon f_1|S \leq \chi_E \leq f_2|S; f_1, f_2 \in C(X)\} = 0.$$

2.9 Corollary. *Given a completely regular (Hausdorff) space Y and a bounded linear functional L on $C_b(Y)$, let $\mathfrak{S}_L$ be the class of sets $E \in \mathfrak{B}(Y)$ such that*

$$\inf_{f_1, f_2} \sup_{|g| = 1} \{|L((f_2 - f_1)g)| : f_1 \leq \chi_E \leq f_2; f_1, f_2, g \in C(X)\} = 0.$$

$\mathfrak{S}_L$ is an algebra of sets for which (i) *f is $\mathfrak{S}_L$-continuous for all $f \in C_b(Y)$, and* (ii) *there exists a unique regular bounded finitely additive complex-valued set function λ on $\mathfrak{S}_L$ with*

$$L(f) = \int_Y f\, d\lambda \qquad (f \in C_b(Y)).$$

Moreover,

$$\|L\| = |\lambda|(Y).$$

PROOF. Let X be the Stone-Čech compactification of Y. Then L admits a unique (norm-preserving) bounded linear extension $\Lambda: C(X) \to \mathbf{C}$. Taking $S = Y$ and $\mathcal{A} = \mathfrak{B}(X)$ in Theorem 2.7, and noting that, for any $\mu \in MR(\mathfrak{B}(X))$ and any $f \in C(X)$, $f|S$ is $\mathcal{A}_\mu \cap S$-continuous since the set $\{\omega \in \mathbf{C}: \mu(f^{-1}(\omega)) \neq 0\}$ is at most countable, we get the corollary.

2.10 REMARK. In [6, pp. 207, 210–211] sufficient conditions are given for a linear functional on $C(Y)$ (Y completely regular (Hausdorff)) to admit an integral representation by means of a countably additive measure on Y.

2.11. Given a uniform space Y, let $UC_b(Y)$ denote the Banach space of complex-valued uniformly continuous bounded functions on Y with uniform norm. We obtain a compactification for Y by the following well-known construction. Let $\{\varphi_y: y \in Y\}$ be those linear functionals on $UC_b(Y)$ given by $\varphi_y(f) = f(y)$. Then the weak closure X of $\{\varphi_y: y \in Y\}$ in the dual of $UC_b(Y)$ is weakly compact and we can view Y as a dense subset of X such that $UC_b(Y) = \{f|Y: f \in C(X)\}$. Thus, Theorem 2.7 yields the following.

2.12 COROLLARY. *Given a uniform space Y and a bounded linear functional L on $UC_b(Y)$, let $\mathcal{E}_L$ be the class of sets $E \in \mathfrak{B}(Y)$ such that*

$$\inf_{f_1, f_2} \sup_{|g|=1} \{|L((f_2 - f_1)g)|: f_1 < \chi_E < f_2; f_1, f_2, g \in UC_b(Y)\} = 0.$$

$\mathcal{E}_L$ is an algebra of sets for which (i) *f is $\mathcal{E}_L$-continuous for all $f \in UC_b(Y)$ and* (ii) *there exists a unique regular bounded finitely additive complex-valued set function λ on $\mathcal{E}_L$ with*

$$L(f) = \int_Y f\, d\lambda \qquad (f \in UC_b(Y)).$$

Moreover,

$$\|L\| = |\lambda|(Y).$$

3. Linear functionals on almost periodic functions.

3.1. Let G be an Abelian locally compact group with dual group $G^\wedge$ and Bohr compactification $\overline{G}$ ([8, p. 137], [5, §24], [13, p. 30]). The value of an element $\hat{z} \in G^\wedge$ at the point $z \in G$ will be denoted by $\langle z, \hat{z} \rangle$ or $\hat{z}(z)$. Let $AP(G)$ be the space of continuous almost periodic functions on G with uniform norm. By a well-known theorem of harmonic analysis [8, p. 168] every element of $AP(G)$ is the restriction on G of a unique element of $C(\overline{G})$. Since $G^\wedge \subset AP(G)$, any given $\hat{z} \in G^\wedge$ can be extended to a unique continuous function on $\overline{G}$ with values in $T = \{\omega \in \mathbf{C}: |\omega| = 1\}$. This extension shall be denoted by the same symbol, i.e. by $\hat{z}$. Let m denote the normalized Haar measure on $\mathfrak{B}(\overline{G})$.

3.2 LEMMA. *If $\mathcal{A}$ is the algebra of sets in $\mathfrak{B}(\overline{G})$ generated by sets of the form $\{z \in \overline{G}: \hat{z}(z) \in T_0\}$ for $\hat{z} \in G^\wedge$ and T_0 an arc on T with $m(\{\hat{z}^{-1}(\overline{T}_0 \setminus \mathring{T}_0)\}) = 0$, then $f|G$ is $\mathcal{A}_\mu \cap G$-continuous for all $f \in C(\overline{G})$ and all $\mu \in MR(\mathfrak{B}(\overline{G}))$.*

PROOF. Given $\varepsilon > 0$, by the Stone-Weierstrass theorem there exists a polynomial $p(z)$ on $\overline{G}$ such that $|f(z) - p(z)| < \varepsilon/3$ for all $z \in \overline{G}$. Since every finite linear combination of $\mathcal{Q}_\mu \cap G$-continuous functions is $\mathcal{Q}_\mu \cap G$-continuous and since $p(z)$ is a finite linear combination of characters, which are clearly $\mathcal{Q}_\mu \cap G$-continuous, it follows that there exists a finite $\mathcal{Q}_\mu \cap G$-measurable partition Π of G such that, for all $E \in \Pi$, we have $|p(z) - p(z')| < \varepsilon/3$ for all $z, z' \in E$. So from the inequality $|f(z) - f(z')| \leq |f(z) - p(z)| + |p(z) - p(z')| + |p(z') - f(z')|$ follows that $|f(z) - f(z')| < \varepsilon$ $(z, z' \in E)$. The lemma is proved.

This lemma now permits us to generalize a concrete representation theorem due to Hewitt [**4**, pp. 307–308] for bounded linear functionals on $AP(G)$. We note that our algebra is smaller than his and depends on the linear functional in question.

3.3 THEOREM. *Given a locally compact Abelian group G with dual group $G^\wedge$ and given a bounded linear functional L: $AP(G) \to \mathbf{C}$, let $\mathbf{A}$ be the algebra of subsets of G generated by sets of the form $\{z \in G$: $\hat{z}(z) \in T_0\}$ for $\hat{z} \in G^\wedge$ and T_0 an arc on T with $m(\{z \in \bar{G}$: $\hat{z}(z) \in \bar{T}_0 \setminus \mathring{T}_0\}) = 0$ and let $\mathfrak{S}_L$ be the class of sets $E \in \mathbf{A}$ such that*

$$\inf_{f_1, f_2} \sup_{|g| = 1} \{|L((f_2 - f_1)g)| : f_1 \leq \chi_E \leq f_2; f_1, f_2, g \in AP(G)\} = 0.$$

$\mathfrak{S}_L$ is an algebra for which (i) *f is $\mathfrak{S}_L$-continuous for all $f \in AP(G)$ (or equivalently, $\hat{z}$ is $\mathfrak{S}_L$-continuous for all $\hat{z} \in G^\wedge$) and* (ii) *there exists a unique regular bounded finitely additive complex-valued set function λ on $\mathfrak{S}_L$ with*

$$L(f) = \int_G f \, d\lambda \qquad (f \in AP(G)).$$

Moreover,

$$\|L\| = |\lambda|(G).$$

PROOF. Recall that a function lies in $AP(G)$ if and only if it is the restriction on G of a (unique) function in $C(\bar{G})$ [**8**, p. 168]. Hence, there exists a bounded linear functional Λ on $C(\bar{G})$ given by $\Lambda(h) = L(h|G)$. Let $\mathcal{Q}$ be the algebra of sets of Lemma 3.2. Then $\mathbf{A} = \mathcal{Q} \cap G$. To complete the proof, apply Theorem 2.7 for $X = \bar{G}$, $S = G$, and $\mathfrak{S}_\Lambda = \mathfrak{S}_L$.

3.4. Given $f \in C(\bar{G})$, write $\hat{f}$: $G^\wedge \to \mathbf{C}$ for its Fourier transform

$$\hat{f}(\hat{z}) = \int_{\bar{G}} f(z) \overline{\langle z, \hat{z} \rangle} \, dm(z).$$

Let $F(G)$ denote the family of finite linear combinations of characters. By Bohr's fundamental theorem (or equivalently, the Stone-Weierstrass theorem on the Bohr compactification), $F(G)$ is dense in $AP(G)$ with respect to the uniform norm.

3.5 DEFINITION. A complex-valued function $p(\hat{z})$ defined on $G^\wedge$ is said to be positive definite if

$$\sum_{i,j=1}^{n} p(\hat{z}_i - \hat{z}_j) c_i \bar{c}_j \geq 0$$

for all $\hat{z}_1, \ldots, \hat{z}_n \in G^\wedge$ and for all $c_1, \ldots, c_n \in \mathbf{C}$.

The following is a generalization of Hewitt's representation of positive definite functions [**4**, pp. 310–311].

3.6 THEOREM. *Given a locally compact Abelian group G with dual group $G\hat{}$ and given a complex-valued function $p(\hat{z})$ on $G\hat{}$, let $\mathbf{A}$ be the algebra of subsets of G generated by sets of the form $\{z \in G\colon \hat{z}(z) \in T_0\}$ for $\hat{z} \in G\hat{}$ and T_0 an arc on T with $m(\{z \in \overline{G}\colon \hat{z}(z) \in \overline{T}_0 \setminus \mathring{T}_0\}) = 0$, and let $\mathfrak{S}_p$ be the class of sets $E \in \mathbf{A}$ such that*

$$\inf\left\{\left|\sum_{\hat{z}\in G\hat{}} (\hat{f}_2(\hat{z}) - \hat{f}_1(\hat{z}))p(\hat{z})\right| : f_1 \leq \chi_E \leq f_2; f_1, f_2 \in F(G)\right\} = 0. \quad (3.6.1)$$

The function $p(\hat{z})$ is positive definite on $G\hat{}$ if and only if (i) *$\mathfrak{S}_p$ is an algebra such that f is $\mathfrak{S}_p$-continuous for all $f \in AP(G)$ (or equivalently, $\hat{z}$ is $\mathfrak{S}_p$-continuous for all $\hat{z} \in G\hat{}$) and* (ii) *there exists a unique regular bounded finitely additive nonnegative set function λ on $\mathfrak{S}_p$ such that*

$$p(\hat{z}) = \int_G \langle z, \hat{z}\rangle \, d\lambda(z)$$

for all $\hat{z} \in G\hat{}$.

PROOF. Sufficiency is easily shown. Necessity can be established by repeating the ideas behind the proof of Bochner's theorem as found in [**13**, pp. 19–21]. The proof that we now provide is somewhat shorter.

Clearly, a function lies in $F(G)$ if and only if it is the restriction on G of a (unique) function in $F(\overline{G})$. Also, since $|p(z)| \leq p(0)$ [**13**, p. 19], $p(z)$ is uniformly bounded. Assume $p(0) = 1$. On $F(G)$, define the linear functional Λ by

$$\Lambda(f|G) = \sum_{\hat{z}\in G\hat{}} \hat{f}(\hat{z})p(\hat{z}) \qquad (f \in F(\overline{G})).$$

Let $\hat{F}(\overline{G}) = \{\hat{f}\colon f \in F(\overline{G})\}$ and given $f, g \in F(\overline{G})$, put

$$(\hat{f}, \hat{g}) = \Lambda(f\overline{g}|G).$$

This defines an inner product on $\hat{F}(\overline{G})$ and so $|\Lambda(f\overline{g}|G)|^2 \leq \Lambda(|f|^2|G)\Lambda(|g|^2|G)$. Thus,

$$\begin{aligned}|\Lambda(f|G)|^2 &\leq \Lambda(|f|^2|G) \leq \Lambda\left(|f^2|^2|G\right)^{1/2} \leq \cdots \leq \Lambda\left(|f^2|^{2^n}|G\right)^{2^{-n}} = \Lambda\left(|f^{2^n}|^2|G\right)^{2^{-n}} \\ &= \left(\sum_{\hat{z}} \sum_{\hat{w}} (f^{2^n})\hat{}(\hat{z})\overline{(f^{2^n})\hat{}}(\hat{w})p(\hat{z} - \hat{w})\right)^{2^{-n}} \\ &\leq \left(\sum_{\hat{z}} |(f^{2^n})\hat{}(\hat{z})|\right)^{2^{-n+1}} = \left(\int_{\overline{G}} |f^{2^{n-1}}|^2 \, dm\right)^{2^{-n+1}} \\ &\to \sup\left\{|f(z)|^2\colon z \in \overline{G}\right\} \qquad (n \to \infty).\end{aligned}$$

Thus Λ is a bounded linear functional on $F(G)$ and so can be extended to a unique bounded linear functional Λ' on the closure $AP(G)$ of $F(G)$ with respect to the uniform norm. Note that $\|\Lambda'\| = 1$. Thus, by Theorem 3.3, there exist an algebra $\mathfrak{S} \subset \mathbf{A}$ and a unique regular finitely additive complex-valued set function λ on $\mathfrak{S}$ such that

$$\Lambda'(f) = \int_G f \, d\lambda \qquad (f \in AP(G))$$

and

$$\|\Lambda'\| = |\lambda|(G).$$

In particular, if $f(z) = \langle z, \hat{z}\rangle$, then

$$p(\hat{z}) = \int_G \langle z, \hat{z}\rangle \, d\lambda(z)$$

for all $\hat{z} \in G\hat{}$. Furthermore, since $1 = p(0) = \int_G d\lambda = \lambda(G) \leqslant |\lambda|(G) = \|\Lambda'\| = 1$, then λ is nonnegative on $\mathfrak{S}$ and by Remark 2.8, $\mathfrak{S}$ consists of all $E \in \mathbf{A}$ with

$$\inf\{\Lambda'(f_2 - f_1): f_1 \leqslant \chi_E \leqslant f_2; f_1, f_2 \in AP(G)\} = 0.$$

Hence, since $F(G)$ is dense in $AP(G)$ with respect to the uniform norm, equation (3.6.1) identifies $\mathfrak{S}$. The theorem is proved.

4. Fourier-Stieltjes transforms.

4.1. Given a locally compact Abelian group G, let $M(G)$ denote the family of tuples $(\mu, \mathcal{Q})$ where $\mathcal{Q}$ is an algebra of sets in $\mathfrak{B}(G)$ such that each $f \in AP(G)$ (or equivalently, each character $\hat{z} \in G\hat{}$) is $\mathcal{Q}$-continuous and μ is a bounded finitely additive complex-valued set function on $\mathcal{Q}$. Recall that the Fourier-Stieltjes transform $\hat{\mu}$ of $(\mu, \mathcal{Q})$ is given by $\hat{\mu}(\hat{z}) = \int_G \langle z, \hat{z}\rangle \, d\mu(z)$. Write $B(G\hat{})$ for the space of all finite linear combinations of positive definite functions on $G\hat{}$. By the Pontryagin duality theorem, the group G can be identified with $G\hat{\hat{}}$ and so, if $z \in G$, the character $\hat{z} \to \langle z, \hat{z}\rangle$ lies in $B(G\hat{})$. Also, given $(\mu, \mathcal{Q}) \in M(G)$, then $\mu = (\mu_1^+ - \mu_1^-) + i(\mu_2^+ - \mu_2^-)$ where $\mu_j^{\pm}$ $(j = 1, 2)$ are nonnegative bounded set functions on $\mathcal{Q}$ [**15**, p. 401]. So, it is easy to see that $\hat{\mu} \in B(G\hat{})$. In fact, we have the following.

4.2 Lemma. *A complex-valued function on $G\hat{}$ lies in $B(G\hat{})$ if and only if it is the Fourier-Stieltjes transform of an element of $M(G)$.*

Proof. It remains to prove necessity. Let $p \in B(G\hat{})$. Then $p(\hat{z}) = \sum_k p_k(\hat{z})$ where the sum is finite and p_k is a scalar multiple of a positive definite function on $G\hat{}$. So, by Theorem 3.6, there exist $(\mu_k, \mathcal{Q}_k) \in M(G)$ for all k, such that, $p_k(\hat{z}) = \int_G \langle z, \hat{z}\rangle \, d\mu_k(z)$. Let Λ_k be those bounded linear functionals on $AP(G)$ given by $\Lambda_k(f) = \int_G f \, d\mu_k$ and put $\Lambda = \sum_k \Lambda_k$ on $AP(G)$. Then, by Theorem 3.3, there exists $(\mu, \mathfrak{S}) \in M(G)$ such that $\Lambda(f) = \int_G f \, d\mu$ $(f \in AP(G))$ and so, if $g(z) = \langle z, \hat{z}\rangle$, then

$$\hat{\mu}(\hat{z}) = \Lambda(g) = \sum_k \Lambda_k(g) = \sum_k p_k(\hat{z}) = p(\hat{z}).$$

The lemma is proved.

4.3. Since a function lies in $B(G\hat{})$ if and only if it is the Fourier-Stieltjes transform $\hat{\mu}$ of a particular $(\mu, \mathcal{Q}) \in M(G)$ given uniquely by Theorem 3.6, we define a norm on $B(G\hat{})$ by $\|\hat{\mu}\| = |\mu|(G)$. We can now prove the following analog for $B(G\hat{})$ of Eberlein's characterization of $B(G\hat{}) \cap C(G\hat{})$ (see [**13**, pp. 32–34]).

4.4 Theorem. *A function p lies in $B(G\hat{})$ if and only if*

$$\left|\sum_{G\hat{}} \hat{f}(\hat{z}) p(\hat{z})\right| \leqslant \|p\| \sup\{|f(z)|: z \in G\} \tag{4.4.1}$$

for all $f \in F(G)$.

PROOF. Suppose $p \in B(G^{\wedge})$. By Theorem 3.6, there exists $(\lambda, \mathfrak{S}) \in M(G)$ such that $p(\hat{z}) = \int_G \langle z, \hat{z} \rangle \, d\lambda(z)$. Hence $\|p\| = |\lambda|(G)$. Let Λ be the linear operator on $AP(G)$ given by $\Lambda(f) = \int_G f \, d\lambda$. Then $\Lambda(f) = \Sigma_{G^{\wedge}} \hat{f}(\hat{z}) p(\hat{z})$ for all $f \in F(G)$. Thus, since $\|\Lambda\| = |\lambda|(G) = \|p\|$ and $|\Lambda(f)| \leq \|\Lambda\| \sup\{|f(z)|: z \in G\}$, then (4.4.1) holds for all $f \in F(G)$.

Conversely, suppose that (4.4.1) holds for all $f \in F(G)$. Let Λ be the linear functional on $F(G)$ defined by $\Lambda(f) = \Sigma_{G^{\wedge}} \hat{f}(\hat{z}) p(\hat{z})$. Then, by (4.4.1), Λ is bounded by $\|p\|$ on $F(G)$. Let Λ' be the (unique) linear functional extending Λ to the closure $AP(G)$ of $F(G)$ with respect to the uniform norm. Then $\|\Lambda'\| \leq \|p\|$. By Theorem 3.3, there exists a tuple $(\lambda, \mathfrak{S}) \in M(G)$ such that $\Lambda'(f) = \int_G f \, d\lambda$ for all $f \in AP(G)$. So, given $\hat{z} \in G^{\wedge}$, if $g(z) = \langle z, \hat{z} \rangle$, then $\hat{\lambda}(\hat{z}) = \int_G g \, d\lambda = \Lambda'(g) = \Lambda(g) = p(\hat{z})$. Since $\hat{\lambda} \in B(G^{\wedge})$, then $p \in B(G^{\wedge})$ and this completes the proof of the theorem.

4.5 COROLLARY. *Given a pointwise convergent net in $B(G^{\wedge})$ uniformly bounded in norm by some constant, the limit also lies in $B(G^{\wedge})$ and is bounded in norm by the same constant.*

PROOF. This follows immediately from (4.4.1).

4.6. Let X and Y be arbitrary sets and let $f: X \times Y \to \mathbf{C}$. We say that f satisfies the double limit condition if, whenever (x_i) and (y_i) are sequences in X and Y respectively such that the iterated limits $\alpha = \lim_{i \to \infty} \lim_{j \to \infty} f(x_i, y_j)$ and $\beta = \lim_{j \to \infty} \lim_{i \to \infty} f(x_i, y_j)$ exist, then $\alpha = \beta$. In particular, for a locally compact Abelian group G, the function $f: G \times G^{\wedge} \to T$ given by $f(z, \hat{z}) = \langle z, \hat{z} \rangle$ satisfies the double limit condition. To see this, we first note that the restriction to G of an element of $\overline{(\overline{G})^{\wedge}}$ (also denoted $G^{-\wedge-}$) belongs to $\overline{G^{\wedge}}$ (also denoted $G^{\wedge-}$). From this follows that the restriction to $G^{-\wedge}$ of an element of $G^{-\wedge-\wedge-}$ belongs to G^{-}. Note also that G and $G^{\wedge}$ with discrete topology are topological subgroups of $G^{-\wedge-\wedge}$ and $G^{-\wedge-\wedge-\wedge}$ respectively. Hence, if (z_i) and $(\hat{z}_j)$ are sequences in G and $G^{\wedge}$ respectively, then there are subsequences (z_{im}) and $(\hat{z}_{jn})$ of (z_i) and $(\hat{z}_j)$ respectively such that $\lim_{m \to \infty} \lim_{n \to \infty} \langle z_{im}, \hat{z}_{jn} \rangle = \lim_{n \to \infty} \lim_{m \to \infty} \langle z_{im}, \hat{z}_{jn} \rangle$. From this follows that the function $f(z, \hat{z}) = \langle z, \hat{z} \rangle$ satisfies the double limit condition.

An algebra $\mathfrak{A}$ of sets in X is said to separate points on X if, whenever $x, y \in X$, $x \neq y$, there are disjoint sets $A, B \in \mathfrak{A}$ such that $x \in A$, and $y \in B$. Clearly, if $\mathfrak{A}$ is a subalgebra of $\mathfrak{B}(G)$ such that each $f \in AP(G)$ is $\mathfrak{A}$-continuous, then by Urysohn's Lemma and the Stone-Weierstrass theorem on $\overline{G}$, it follows that $\mathfrak{A}$ separates points on G. Now, by a theorem of Sinclair [**14**, pp. 363, 364], if $\mathfrak{A}$ and $\mathfrak{S}$ are algebras of sets which separate points on X and Y respectively and if f: $X \times Y \to \mathbf{C}$ is a bounded function for which (i) $f(\cdot, y)$ is $\mathfrak{A}$-continuous for all $y \in Y$, (ii) $f(x, \cdot)$ is $\mathfrak{S}$-continuous for all $x \in X$, and (iii) f satisfies the double limit condition, then (1) $\int_X f(x, y) \, d\mu(x)$ and $\int_Y f(x, y) \, d\lambda(y)$ are integrable with respect to μ and λ respectively and (2) $\int_X \int_Y f \, d\lambda \, d\mu = \int_Y \int_X f \, d\mu \, d\lambda$, for all finitely additive bounded complex-valued set functions μ and λ on $\mathfrak{A}$ and $\mathfrak{S}$ respectively. Thus, if $(\mu, \mathfrak{A}) \in M(G)$ and if $(\lambda, \mathfrak{S}) \in M(G^{\wedge})$, then (1) the Fourier-Stieltjes transforms $\hat{\mu}(\hat{z}) = \int_G \langle z, \hat{z} \rangle \, d\mu(z)$ and $\hat{\lambda}(z) = \int_{G^{\wedge}} \langle z, \hat{z} \rangle \, d\lambda(\hat{z})$ are integrable with respect to λ and μ respectively, and (2) $\int_G \int_{G^\wedge} \langle z, \hat{z}\rangle\, d\lambda(\hat{z})\, d\mu(z) = \int_{G^\wedge} \int_G \langle z, \hat{z}\rangle\, d\mu(z)\, d\lambda(\hat{z})$.

4.7 REMARK. Let $L(G^\wedge)$ denote the space of complex-valued functions on $G^\wedge$ which are integrable with respect to the Haar measure $d\hat{z}$ on $\mathfrak{B}(G^\wedge)$. Given $h \in L(G^\wedge)$, write $\tilde{h}$ for the continuous complex-valued function on G given by the Fourier transform $\tilde{h}(z) = \int_{G^\wedge} \langle z, \hat{z}\rangle h(\hat{z})\, d\hat{z}$. If $p \in {}^{\cdot}B(G^\wedge)$, then

$$\left|\int_{G^\wedge} h(\hat{z})p(\hat{z})\, d\hat{z}\right| \leq \|p\| \sup\{|\tilde{h}(z)| : z \in G\}$$

for all $h \in L(G^\wedge)$. This well-known property of Fourier-Stieltjes transforms (see [5, §33.20]) can be proved as follows. Suppose that $p \in B(G^\wedge)$. Then, there exists $(\lambda, \mathcal{E}) \in M(G)$ such that $p(\hat{z}) = \int_G \langle z, \hat{z}\rangle\, d\lambda(z)$. Also, for any $h \in L(G^\wedge)$, $h(\hat{z})\, d\hat{z}$ is a finitely additive set function, say μ, on $\mathfrak{B}(G^\wedge)$. So, by Sinclair's theorem,

$$\begin{aligned}\left|\int_{G^\wedge} h(\hat{z})p(\hat{z})\, d\hat{z}\right| &= \left|\int_{G^\wedge} p(\hat{z})\, d\mu(\hat{z})\right| = \left|\int_{G^\wedge} \int_G \langle z, \hat{z}\rangle\, d\lambda(z)\, d\mu(\hat{z})\right| \\ &= \left|\int_G \int_{G^\wedge} \langle z, \hat{z}\rangle\, d\mu(\hat{z})\, d\lambda(z)\right| = \left|\int_G \int_{G^\wedge} \langle z, \hat{z}\rangle h(\hat{z})\, d\hat{z}\, d\lambda(z)\right| \\ &= \left|\int_G \tilde{h}(z)\, d\lambda(z)\right| \leq |\lambda|(G) \sup\{|\tilde{h}(z)| : z \in G\} \\ &= \|p\| \sup\{|\tilde{h}(z)| : z \in G\} \quad \text{for all } h \in L(G^\wedge).\end{aligned}$$

4.8. Recall that if $G = T$, then $G^\wedge = Z$ where $Z = \{0, \pm 1, \pm 2, \ldots\}$. Let $\{a_{k,n}: k, n \in Z\}$ be a matrix of complex numbers such that $\sum_n |a_{k,n}| < \infty$ for all $k \in Z$. Then the sums $\sum_n p(n)a_{k,n}$ are well defined for all $p \in B(Z)$. We wish to give a criterion for the matrix to have the property that, for all $p \in B(Z)$, $\sum_n p(n)a_{k,n}$ lies in $B(Z)$ as a function of k. As the following theorem shows, such a condition is given by $\sum_n z^n a_{k,n} \in B(Z)$ for all $z \in T$.

4.9 THEOREM. *Let G be a locally compact Abelian group and let $\{L_{\hat{z}}: \hat{z} \in G^\wedge\}$ be a family of complex-valued linear functionals on $B(G^\wedge)$ bounded with respect to the uniform norm. In order that, for all $(\mu, \mathcal{Q}) \in M(G)$, $\hat{z} \to L_{\hat{z}}(\hat{\mu})$ be the Fourier-Stieltjes transform of an element of $M(G)$, it is necessary and sufficient that $L_{\hat{z}}(z)$ lie in $B(G^\wedge)$ for each function $z \in G^{\wedge\wedge}$.*

PROOF. By Theorem 3.3, for each $\hat{z} \in G^\wedge$, there exists $(\lambda_{\hat{z}}, \mathcal{Q}_{\hat{z}}) \in M(G^\wedge)$ such that $L_{\hat{z}}(g) = \int_{G^\wedge} g(\hat{w})\, d\lambda_{\hat{z}}(\hat{w})$ for all $g \in F(G^\wedge)$. Hence the theorem reduces to showing the following: given $\{(\lambda_{\hat{z}}, \mathcal{Q}_{\hat{z}}): \hat{z} \in G^\wedge\} \subset M(G^\wedge)$, then in order that, for all $(\mu, \mathcal{Q}) \in M(G)$, $\int_{G^\wedge} \hat{\mu}(\hat{w})\, d\lambda_{\hat{z}}(\hat{w})$ be the Fourier-Stieltjes transform of an element of $M(G)$, it is necessary and sufficient that $\hat{\lambda}_{\hat{z}}(z) \in B(G^\wedge)$ for each $z \in G^{\wedge\wedge}$.

To prove necessity, suppose that, for any given $w \in G$, $(\mu, \mathcal{Q}) \in M(G)$ is such that for any $E \in \mathcal{Q}$, $\mu(E) = 1$ if $w \in E$ and $\mu(E) = 0$ otherwise. Then, by hypothesis, there exists $(\nu, \mathcal{E}) \in M(G)$ such that

$$\hat{\nu}(\hat{z}) = \int_{G^\wedge} \hat{\mu}(\hat{w})\, d\lambda_{\hat{z}}(\hat{w}) = \int_{G^\wedge} \int_G \langle z, \hat{w} \rangle\, d\mu(z)\, d\lambda_{\hat{z}}(\hat{w})$$

$$= \int_{G^\wedge} \langle w, \hat{w} \rangle\, d\lambda_{\hat{z}}(\hat{w}) = \hat{\lambda}_{\hat{z}}(w).$$

But $\nu = (\nu_1^+ - \nu_1^-) + i(\nu_2^+ - \nu_2^-)$ where $\nu_j^\pm$ $(j = 1, 2)$ are nonnegative bounded finitely additive set functions on $\mathfrak{S}$ [**15**, p. 401]. So, $\hat{\nu} = (\hat{\nu}_1^+ - \hat{\nu}_1^-) + i(\hat{\nu}_2^+ - \hat{\nu}_2^-)$ where, by Theorem 3.6, $\hat{\nu}_j^\pm$ $(j = 1, 2)$ are positive definite on $G^\wedge$. Hence, for any given $w \in G$, $\hat{\lambda}_{\hat{z}}(w)$ is a linear combination of four positive definite functions in $\hat{z}$.

We now prove sufficiency. Given $(\mu, \mathfrak{A}) \in M(G)$, there exist bounded nonnegative finitely additive set functions $\mu_j^\pm$ $(j = 1, 2)$ on $\mathfrak{A}$ such that $\mu = (\mu_1^+ - \mu_1^-) + i(\mu_2^+ - \mu_2^-)$. Hence, for $j = 1, 2$,

$$\int_{G^\wedge} \hat{\mu}_j^\pm(\hat{w})\, d\lambda_{\hat{z}}(\hat{w}) = \int_{G^\wedge} \int_G \langle z, \hat{w} \rangle\, d\mu_j^\pm(z)\, d\lambda_{\hat{z}}(\hat{w})$$

$$= \int_G \int_{G^\wedge} \langle z, \hat{w} \rangle\, d\lambda_{\hat{z}}(\hat{w})\, d\mu_j^\pm(z) = \int_G \hat{\lambda}_{\hat{z}}(z)\, d\mu_j^\pm(z).$$

Thus, by hypothesis on $\hat{\lambda}_{\hat{z}}(z)$, $\int_{G^\wedge} \hat{\mu}_j^\pm(\hat{w})\, d\lambda_{\hat{z}}(\hat{w})$ lies in $B(G^\wedge)$. So, by Lemma 4.2, there exist elements $(\nu_j^\pm, \mathfrak{A}_j^\pm) \in M(G)$ $(j = 1, 2)$ such that $\hat{\nu}_j^\pm(\hat{z}) = \int_{G^\wedge} \hat{\mu}_j^\pm(\hat{w})\, d\lambda_{\hat{z}}(\hat{w})$. Let $\Lambda_j^\pm$ $(j = 1, 2)$ be those bounded complex-valued linear functionals on $AP(G)$ given by $\Lambda_j^\pm(f) = \int_G f(z)\, d\nu_j^\pm(z)$ and, on $AP(G)$, put $\Lambda = (\Lambda_1^+ - \Lambda_1^-) + i(\Lambda_2^+ - \Lambda_2^-)$. Then, by Theorem 3.3, there exists $(\nu, \mathfrak{S}) \in M(G)$ such that $\Lambda(f) = \int_G f(z)\, d\nu(z)$ for all $f \in AP(G)$, and so $\hat{\nu}(\hat{z}) = \Lambda(\hat{z}) = \int_{G^\wedge} \hat{\mu}(\hat{w})\, d\lambda_{\hat{z}}(\hat{w})$. The theorem is proved.

4.10 Corollary. *If* $(\lambda, \mathfrak{S}) \in M(G^\wedge)$, *then, for all* $(\mu, \mathfrak{A}) \in M(G)$,

$$\int_{G^\wedge} \hat{\mu}(\hat{z} - \hat{w})\, d\lambda(\hat{w})$$

is the Fourier-Stieltjes transform of an element of $M(G)$.

Proof. Let $\{L_{\hat{z}} : \hat{z} \in G^\wedge\}$ be that family of bounded complex-valued linear functionals on $B(G^\wedge)$ given by $L_{\hat{z}}(f) = \int_{G^\wedge} f(\hat{z} - \hat{w})\, d\lambda(\hat{w})$. Then, for $g(\hat{z}) = \langle z, \hat{z} \rangle$ where z is given in G, $L_{\hat{z}}(g) = \int_{G^\wedge} \langle z, \hat{z} - \hat{w} \rangle\, d\lambda(\hat{w}) = \langle z, \hat{z} \rangle \hat{\lambda}(-z) \in B(G^\wedge)$. So, the corollary follows from Theorem 4.9.

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