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A CHARACTERIZATION OF PERIODIC AUTOMORPHISMS OF A FREE GROUP

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ABSTRACT. Let θ be an automorphism of finite order of a free group X. We characterise the action of θ on X by showing that X has a free basis which is the disjoint union of finite subsets S_j, where if $S_j = \{u_0, u_1, \ldots, u_k\}$ then $u_i\theta = u_{i+1}$ $(0 < 1 < k)$ and $u_k\theta = A_j u_0^\varepsilon B_j$ for some A_j, B_j in X and $\varepsilon = \pm 1$. As an application of this result, we obtain a list of the conjugacy classes of periodic automorphisms of the free group of rank three.

1. Introduction. In [2], by using the structure theorem for finite extensions of free groups obtained by Karrass, Pietrowski and Solitar [5], Cohen [1] and Scott [7], Dyer and Scott were able to give an explicit description of the way in which an automorphism of prime order p can act on a free group X. The aim of the present work is to extend this result to give a description of the way in which an automorphism θ of finite order n can act on a free group X. Specifically, we shall show that X has a free basis which is the disjoint union of finite subsets S_j, where if $S_j = \{u_0, u_1, \ldots, u_k\}$ then

$$u_i\theta = u_{i+1} \qquad (0 < i < k)$$

and

$$u_k\theta = A_j u_0^\varepsilon B_j$$

for some A_j, B_j in X and $\varepsilon = \pm 1$.

Dyer and Scott also showed in [2] that the fixed point subgroup X^π of any finite group π of automorphisms of X is a free factor of X. We show, in the case $\pi = \langle\theta\rangle$, that $X^{\langle\theta\rangle}$ has a free basis consisting of the union of all S_j above such that $|S_j| = 1$, $A_j = B_j = 1$ and $\varepsilon = 1$. We also show, as in the case $n = p$, that $X = X^{\langle\theta\rangle} * H$, where $(H)\theta = H$.

As an application of our main result, we obtain a list of the conjugacy classes of periodic automorphisms of a free group F of rank three. Such a list, in the case where the rank of F is two, has been obtained by Meskin [6], using different techniques.

2. Statement of the result. In order to facilitate the statement of the main theorem, we now introduce the notion of an n, n_1-based tree. These trees occur as maximal subtrees of certain graphs which arise naturally in the theory of finite extensions of free groups. The reader might find it helpful to examine the second

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half of the proof of the theorem in order to understand the motivation for the introduction of this concept.

Let $\mathfrak{T}$ be a tree with a distinguished vertex a. By the *level* $l(c)$ of a vertex c of $\mathfrak{T}$, we mean the number of edges in the unique reduced path of $\mathfrak{T}$ from a to c. Let n be an integer with $n \geqslant 2$, and suppose that:

(1) For each vertex b of $\mathfrak{T}$, there are assigned positive integers γ_b, τ_b with $\gamma_b > 1$, $\gamma_b \tau_b = n$ and $\tau_a = 1$.

(2) For each pair of vertices c, d joined by an edge of $\mathfrak{T}$, with $l(c) = l(d) - 1$, there are assigned positive integers $\alpha_{c,d}$, β_d with $1 < \alpha_{c,d}$, $1 < \beta_d$, $\alpha_{c,d} | \gamma_c$, $\beta_d | \gamma_d$ and $\gamma_c \beta_d = \alpha_{c,d} \gamma_d$. We note that $\alpha_{c,d}$ and β_d depend only on d.

(3) For each unordered pair $\{e, f\}$ of vertices with $e \neq f$, a specific order e, f is chosen, satisfying $l(e) \leqslant l(f)$, and to each such ordered pair (e, f), and to each pair (e, e), there is assigned a family (possibly empty) of pairs of positive integers $\delta_{e,f,i}$, $\rho_{e,f,i}$, indexed by a set $I_{e,f}$, with $\delta_{e,f,i} | \gamma_e$, $\rho_{e,f,i} | \gamma_f$, $\gamma_e \rho_{e,f,i} = \gamma_f \delta_{e,f,i}$, and, in addition, $1 < \rho_{e,f,i}$, $1 < \delta_{e,f,i}$ if $e \neq f$. Also, for each $i \in I_{e,f}$, there is assigned an integer $\eta_{e,f,i}$ satisfying $0 \leqslant \eta_{e,f,i} < (\tau_e, \tau_f)$, where (τ_e, τ_f) is the greatest common divisor of τ_e and τ_f.

Then $\mathfrak{T}$, together with the above assignments, is said to be a *based* n, n_1-*tree* (with base a) if the least common multiple of the set of all $\beta_d \tau_d$, $\delta_{e,f,i} \tau_e$ is the integer n_1. We note that n_1 is a divisor of n.

Our main result is

THEOREM. *Let $\mathfrak{T}$ be a based n, n_1-tree with base a, as above. Let X be a free group with free generating set S, where S is the disjoint union of subsets $S(d)$ (d ranging over the vertices of $\mathfrak{T}$ other than a) and subsets $S(e, f, i)$ (e, f, i as in* (3) *above), with*

$$S(d) = \{D_0, D_1, \ldots, D_{(\beta_d - 1)\tau_d - 1}\}$$

and

$$S(e, f, i) = \{T_{e,f,i,0}, T_{e,f,i,1}, \ldots, T_{e,f,i,\rho_{e,f,i}\tau_f - 1}\}.$$

We shall use the convention that if d is a vertex of $\mathfrak{T}$ and k, j are integers with $j \geqslant 0$ and $k > 0$, then $\mathfrak{D}_j(k)$ shall denote the product $D_j D_{j+\tau_d} \cdots D_{j+(k-1)\tau_d}$.

Let θ be the endomorphism of X defined inductively as follows: firstly,

$$D_k \theta = D_{k+1} \quad \textit{if } 0 \leqslant k < (\beta_d - 1)\tau_d - 1,$$

$$T_{e,f,i,k}\theta = T_{e,f,i,k+1} \quad \textit{if } 0 \leqslant k < \rho_{e,f,i}\tau_f - 1;$$

next, for each vertex d of $\mathfrak{T}$ with $d \neq a$, let c be the vertex with $l(c) = l(d) - 1$ which is joined to d by an edge of $\mathfrak{T}$; we define

$$D_{(\beta_d - 1)\tau_d - 1}\theta = \mathfrak{D}_0^{-1}(\beta_d - 1)\mathfrak{C}_0(\alpha_{c,d}); \tag{2.1}$$

here we have assumed, inductively, that $C_i = C_0\theta^i$ has been defined for all nonnegative integers i, with the convention that, in case $c = a$, we put $A_i = 1$ for all i. This clearly defines $D_i = D_0\theta^i$ for all nonnegative i. Finally, we define

$$T_{e,f,i,\rho_{e,f,i}\tau_f - 1}\theta = \mathfrak{F}_0^{-1}(\rho_{e,f,i}) T_{e,f,i,0} \mathfrak{E}_{\eta_{e,f,i}}(\delta_{e,f,i}). \tag{2.2}$$

Then θ is an automorphism of X of order n_1.

Conversely, if θ is an automorphism of order n of a free group X, then there is a based n, n-tree $\mathfrak{T}$ so that X, θ are defined in terms of $\mathfrak{T}$ as above.

3. Proof of the Theorem. Let $\mathfrak{T}$ be a based n, n_1-tree, and let X, θ be defined in terms of $\mathfrak{T}$ as above. It is easy to see that $(S)\theta$ is a free generating set of X, so that θ is an automorphism of X. We now show that θ has order n_1.

We note firstly that if c, d are the vertices of an edge of $\mathfrak{T}$, with $l(c) = l(d) - 1$, then from $\gamma_c\tau_c = \gamma_d\tau_d = n$ and $\gamma_c\beta_d = \alpha_{c,d}\gamma_d$, it follows that $\gamma_c\tau_c\alpha_{c,d} = \gamma_c\beta_d\tau_d$, so that $\alpha_{c,d}\tau_c = \beta_d\tau_d$. Similarly we see that $\delta_{e,f,i}\tau_e = \rho_{e,f,i}\tau_f$, for all appropriate e, f, i.

Next we note that (2.1) may be written as

$$\mathfrak{D}_0(\beta_d) = \mathcal{C}_0(\alpha_{c,d}). \tag{3.1}$$

Now suppose that d is a vertex with $l(d) = 1$. Then (3.1) becomes $\mathfrak{D}_0(\beta_d) = 1$. We remark here that if c is a vertex such that $\mathcal{C}_0(k) = 1$ for some $k > 0$, and k_1 is the least positive integer such that $\mathcal{C}_0(k_1) = 1$, then it is easy to see that k is a multiple of k_1. Now from $\mathfrak{D}_0(\beta_d) = \mathfrak{D}_{\beta_d}(\beta_d) = 1$, it follows that $D_0 = D_{\tau_d\beta_d}$. Defining the *order* m_d of θ on D_0 to be the order of θ on the subgroup $\langle D_0, D_1, \ldots \rangle$ of X generated by all the D_i, we see that $m_d | \tau_d\beta_d$. Now from the definition of θ we have $m_d > (\beta_d - 1)\tau_d$. Since $\beta_d \geqslant 2$, it follows that $m_d = \beta_d\tau_d$. We note that in this case m_d may be described as $\lambda_d\tau_d$, where λ_d is the least positive integer such that $\mathfrak{D}_0(\lambda_d) = 1$.

Now let $a, b, \ldots, c, d$ be a reduced path in $\mathfrak{T}$ beginning at a. Then the subgroup G of X generated by $S(b) \cup \cdots \cup S(c) \cup S(d)$ is clearly invariant under θ. We show

The order of θ on G is the least common multiple of the set $\{\beta_b\tau_b, \ldots, \beta_d\tau_d\}$. (3.2)

The order m_d of θ on $\mathfrak{D}_0$ is of the form $\lambda_d\tau_d$, where $\beta_d | \lambda_d$, $\lambda_d | \gamma_d$, and λ_d is the least positive integer such that $\mathfrak{D}_0(\lambda_d) = 1$. (3.3)

The order of θ on G is equal to m_d. (3.4)

We shall prove these results by induction on the number q of vertices in the path under consideration. Since the results are known if $q = 2$, we suppose that $q > 2$.

Let H be the subgroup of G generated by $S(b) \cup \cdots \cup S(c)$. Since both H and G are θ invariant, θ induces a homomorphism θ_1 on G modulo the normal closure of H. Applying the argument of the case $l(d) = 1$ above to θ_1 shows that m_d is a multiple of $\tau_d\beta_d$.

Let μ be the least common multiple of the pair $\alpha_{c,d}$, λ_c. We write $\mu = u\alpha_{c,d} = v\lambda_c$, and note that $\mu | \gamma_c$. We have, using the induction hypothesis,

$$\begin{aligned} 1 &= \mathcal{C}_0(\lambda_c)\mathcal{C}_{\lambda_c\tau_c}(\lambda_c) \cdots \mathcal{C}_{(v-1)\lambda_c\tau_c}(\lambda_c) \\ &= \mathcal{C}_0(u\alpha_{c,d}) \\ &= \mathcal{C}_0(\alpha_{c,d})\mathcal{C}_{\alpha_{c,d}\tau_c}(\alpha_{c,d}) \cdots \mathcal{C}_{(u-1)\alpha_{c,d}\tau_c}(\alpha_{c,d}). \end{aligned}$$

Using (3.1) and the fact that $\alpha_{c,d}\tau_c = \beta_d\tau_d$, this becomes

$$1 = \mathfrak{D}_0(\beta_d)\mathfrak{D}_{\beta_d\tau_d}(\beta_d) \cdots \mathfrak{D}_{(u-1)\beta_d\tau_d}(\beta_d) = \mathfrak{D}_0(u\beta_d), \tag{3.5}$$

and it follows that $m_d | u\beta_d\tau_d$.

Now using $\mathfrak{D}_0(\beta_d) = \mathcal{C}_0(\alpha_{c,d})$ and $\mathfrak{D}_{\tau_d}(\beta_d) = \mathcal{C}_{\tau_d}(\alpha_{c,d})$ we see that

$$D_{\beta_d\tau_d} = \mathcal{C}_0^{-1}(\alpha_{c,d})D_0\mathcal{C}_{\tau_d}(\alpha_{c,d}). \tag{3.6}$$

Applying $\theta^{(h-1)\beta_d\tau_d}$ to (3.6), and using induction on h, we see that for any positive integer h we have

$$D_{h\beta_d\tau_d} = \mathcal{C}_0^{-1}(h\alpha_{c,d})D_0\mathcal{C}_{\tau_d}(h\alpha_{c,d}). \tag{3.7}$$

Now let $m_d = h\beta_d\tau_d$. Since the subgroup of X generated by $S(d)$ and $\langle C_0, C_1, \dots \rangle$ is the free product of these two subgroups, it follows from (3.7) that $\mathcal{C}_0(h\alpha_{c,d}) = 1$, so that, by the induction hypothesis, $m_c = \lambda_c\tau_c$ divides $h\alpha_{c,d}\tau_c$. Hence $\lambda_c | h\alpha_{c,d}$. However, since $\mu = u\alpha_{c,d}$ is the least common multiple of λ_c and $\alpha_{c,d}$, it follows that $u|h$. Now $m_d | u\beta_d\tau_d$, so that $u = h$, $m_d = u\beta_d\tau_d$, and (3.3) follows easily, using (3.5).

To prove (3.2) and (3.4) we note that, by the induction hypothesis, $m_c = \operatorname{lcm}\{\beta_b\tau_b, \dots, \beta_c\tau_c\} = \lambda_c\tau_c$. Now

$$\begin{aligned} \operatorname{lcm}\{\beta_b\tau_b, \dots, \beta_d\tau_d\} &= \operatorname{lcm}\{l_c\tau_c, \beta_d\tau_d\} = \operatorname{lcm}\{\lambda_c\tau_c, \alpha_{c,d}\tau_c\} \\ &= \tau_c \operatorname{lcm}\{\lambda_c, \alpha_{c,d}\} = \tau_c u\alpha_{c,d} \\ &= u\beta_d\tau_d = m_d, \end{aligned}$$

as required.

We now investigate the order $m = m_{e,f,i}$ of θ on the generator $T_{e,f,i,0}$. For ease of notation, we shall omit the subscripts e, f, i from $T_{e,f,i,j}$, $\delta_{e,f,i}$, $\rho_{e,f,i}$, and $\eta_{e,f,i}$. We have $S(e, f, i) = \{T_0, T_1, \dots, T_{\rho\tau_f-1}\}$, and, writing T_j for $T_0\theta^j$ for all nonnegative j, (2.2) becomes

$$T_{\rho\tau_f} = \mathfrak{F}_0^{-1}(\rho)T_0\mathfrak{E}_\eta(\delta). \tag{3.8}$$

From this, it is easily shown that for any $k \geqslant 1$ we have

$$T_{k\rho\tau_f} = \mathfrak{F}_0^{-1}(k\rho)T_0\mathfrak{E}_\eta(k\delta). \tag{3.9}$$

Now factoring out the subgroup of X generated by the union of the sets $S(d)$ shows immediately that m is a multiple of $\rho\tau_f$. It then follows from (3.9) that $m = k\rho\tau_f$, where k is the least positive integer such that $\mathfrak{F}_0(k\rho) = 1$ and $\mathfrak{E}_0(k\delta) = 1$. Thus k is the least positive integer such that $\lambda_f | k\rho$ and $\lambda_e | k\delta$, i.e. such that $m_f | k\rho\tau_f$ and $m_e | k\delta\tau_e$. Hence $m = \operatorname{lcm}\{m_e, m_f, \rho\tau_f\}$.

It now follows immediately that the order of θ on X is the least common multiple of the set of all $\beta_d\tau_d$ and $\delta_{e,f,i}\tau_e$, i.e. $|\theta| = n_1$. This proves the first part of the Theorem.

Now we suppose that θ is an automorphism of order n of a free group X. We show that there is a based n, n_1-tree $\mathfrak{T}$ so that X, θ are described in terms of $\mathfrak{T}$ as in the statement of the Theorem. The approach used is similar to that of **[2]**, and as in **[2]**, we shall use the terminology of **[1]** and **[8]**.

We denote by Y the split extension of X by Z_n, a cycle of order n, formed using the action of θ on X, and by π the natural projection from Y to Z_n with kernel X. The fundamental theorem of Karrass et al. (**[5]**, **[1]**, **[7]**) tells us that Y is an HNN extension whose base is a tree product of finite groups, and whose associated subgroups lie in vertex groups of the base. In the language of [**8**], this states that Y is the fundamental group of a connected graph of groups $\mathcal{Y}$, which is such that every vertex group Y_d of $\mathcal{Y}$ is finite.

Since X is torsion-free, π provides an injection of each finite subgroup of Y into Z_n. Thus each finite subgroup of Y is cyclic, and of order a divisor of n.

From the construction of Y, there exists $A \in Y$ such that $\langle A \rangle$ has order n, and $AxA^{-1} = (x)\theta$, for all $x \in X$. By conjugating $\mathcal{Y}$ (see [**1**]) if necessary, we may assume that A is in a vertex group Y_a of $\mathcal{Y}$. It then follows that $Y_a = \langle A \rangle$.

We now observe that by applying suitable contractions (see [**1**]) to $\mathcal{Y}$, we may assume, without loss of generality, that for each edge v of $\mathcal{Y}$ which is not a loop, the associated subgroup of v is (embedded as) a proper subgroup of each of the two corresponding vertex groups.

Now let $\mathcal{T}$ be a maximal subtree of (the graph of) $\mathcal{Y}$, regarded as being based at a. For each pair of vertices e, f with $e \neq f$ we choose an order e, f as in part (3) of the definition of a based n-tree. For such an ordered pair e, f the (directed) edges of $\mathcal{Y}$, not in $\mathcal{T}$, joining f to e will be called *free edges* of $\mathcal{Y}$, and are regarded as being indexed by a set $I_{e,f}$. Also, for each vertex e and each (undirected) edge beginning and ending at e, we select one of the two corresponding directed edges. The edges so selected are also called free edges of $\mathcal{Y}$, and those at vertex e are regarded as being indexed by the set $I_{e,e}$. It now follows (see [**8**]) that Y has presentation $\mathcal{P} = (\mathcal{S}\,;\, \mathcal{R})$, where $\mathcal{S}$ consists of one generator D for each vertex d of $\mathcal{Y}$ (D corresponds to a generator of the vertex subgroup Y_d of Y) and one generator $T_{e,f,i}$ for each $i \in I_{e,f}$, where (e, f) ranges over the selected pairs of edges (the T generators correspond to the free part of the HNN extension Y). Here $\mathcal{R}$ consists of three sets of relations; firstly the set $\mathcal{R}_1$, consisting of one relation $D^{\gamma_d} = 1$ for each vertex generator, where γ_d is the order of the vertex subgroup Y_d, so that

$$1 < \gamma_d \leqslant n, \qquad \gamma_d | n, \qquad \gamma_a = n. \tag{3.10}$$

Next, $\mathcal{R}_2$ consists of one relation $C^{\alpha_{c,d}} = D^{\beta_d}$ for each edge v of $\mathcal{T}$, where c, d are the vertices of v, and $l(c) = l(d) - 1$. This relation corresponds to the embedding of the associated subgroup of v as a subgroup of Y_c and Y_d, so that

$$1 < \alpha_{c,d}, \qquad 1 < \beta_d, \qquad \gamma_c/(\alpha_{c,d}, \gamma_c) = \gamma_d/(\beta_d, \gamma_d). \tag{3.11}$$

In addition, we may assume that

$$\alpha_{c,d} | \gamma_c, \tag{3.12}$$

since the subgroup $\langle C^{\alpha_{c,d}} \rangle$ has a generator C^k with $k | \gamma_c$, and it is sufficient to identify C^k with an appropriate power of D.

Finally, $\mathcal{R}_3$ consists of one relation

$$T_{e,f,i} E^{\delta_{e,f,i}} T_{e,f,i}^{-1} = F^{\rho_{e,f,i}}$$

for each 'free generator' $T_{e,f,i}$ of Y. These are the HNN relations of Y, and correspondingly we have

$$1 \leqslant \delta_{e,f,i}, \qquad 1 \leqslant \rho_{e,f,i} \tag{3.13}$$

with strict inequality if $e \neq f$, and

$$\gamma_e/(\gamma_e, \delta_{e,f,i}) = \gamma_f/(\gamma_f, \rho_{e,f,i}). \tag{3.14}$$

In addition, we may assume that

$$\delta_{e,f,i} | \gamma_e. \tag{3.15}$$

Now from the choice of A, it follows that for each D generator there is a positive integer τ_d such that $D\pi = (A\pi)^{\tau_d}$, i.e. that $DA^{-\tau_d} \in X$. Since $A\pi$ has order n, it follows that $\gamma_d = n/(n, \tau_d)$. Let r, s be integers such that $r\tau_d + sn = (n, \tau_d)$. Then $(r, \gamma_d) = 1$, so that $\langle D \rangle = \langle D^r \rangle$. We note that $D^r\pi = (A\pi)^{\tau_d r} = (A\pi)^{(n,\tau_d)}$. Clearly we could have selected the generator D^r originally, in place of D, when obtaining the presentation $\mathcal{P}$. Thus we may assume that D was chosen so that

$$\gamma_d \tau_d = n. \tag{3.16}$$

The conditions (3.10) to (3.15) continue to hold with this choice of D.

We now show that β_d may be chosen so that $\beta_d | \gamma_d$. Applying π to the relations in $\mathcal{R}_2$ we obtain, since $(Y)\pi$ is abelian, that

$$(A\pi)^{\tau_c \alpha_{c,d}} = (A\pi)^{\tau_d \beta_d},$$

so that

$$\tau_c \alpha_{c,d} \equiv \tau_d \beta_d \pmod{n}. \tag{3.17}$$

From (3.10) and (3.11) we have $\alpha_{c,d} = \gamma_c(\beta_d, \gamma_d)/\gamma_d$, and substitution in (3.17) yields

$$\tau_c \gamma_c(\beta_d, \gamma_d)/\gamma_d \equiv \tau_d \beta_d \pmod{n}. \tag{3.18}$$

Now from (3.16) we have $\tau_c \gamma_c = \tau_d \gamma_d$, so that (3.18) may be written as

$$\tau_d(\beta_d, \gamma_d) \equiv \tau_d \beta_d \pmod{n},$$

and therefore

$$(\beta_d, \gamma_d) \equiv \beta_d \pmod{n/\tau_d = \gamma_d}.$$

Since D has order γ_d, it follows that $D^{\beta_d} = D^{(\beta_d, \gamma_d)}$. Thus we may assume $\beta_d | \gamma_d$.

Now by applying π to $\mathcal{R}_3$, we see that

$$\tau_e \delta_{e,f,i} \equiv \tau_f \rho_{e,f,i} \pmod{n}.$$

It is now easy to see, by an argument similar to that used in the case for $\mathcal{R}_2$, that $\rho_{e,f,i} | \gamma_f$.

For each free generator $T_{e,f,i}$ we have $(T_{e,f,i})\pi = (A\pi)^{\eta_{e,f,i}}$, for some integer $\eta_{e,f,i}$ satisfying $0 \leqslant \eta < n$. Since $T_{e,f,i}$ occurs in only one relation of $\mathcal{P}$, we may, in view of the form of that relation, replace $T_{e,f,i}$ by a new generator $T'_{e,f,i}$, where $T'_{e,f,i} = F^r T_{e,f,i} E^s$ for some integers r, s, and the presentation obtained is just $\mathcal{P}$ again, if we now denote $T'_{e,f,i}$ by $T_{e,f,i}$. We then have

$$(T_{e,f,i})\pi = (A\pi)^{r\tau_f + s\tau_e + \eta_{e,f,i}}.$$

We choose r, s so that $r\tau_f + s\tau_e + \eta_{e,f,i}$ is the remainder on division of $\eta_{e,f,i}$ by (τ_e, τ_f). Thus we may suppose that $0 \leqslant \eta_{e,f,i} < (\tau_e, \tau_f)$.

It follows from the above that $\mathcal{T}$ is a based n, n_1-tree, where n_1 is the least common multiple of the set of all $\beta_d \tau_d$, $\delta_{e,f,i} \tau_e$.

We now apply the Reidemeister-Schreier rewriting process (cf. [5]) to obtain a presentation for X. We use the coset representative system $1, A, \ldots, A^{n-1}$ for X in Y. Writing D_i for the word $A^iDA^{-(\tau_d+i)'}$, where $0 \leq i < n$ and $(\tau_d + i)'$ is the remainder of $\tau_d + i$ on division by n, and similarly writing $T_{e,f,i,j}$ for $A^jT_{e,f,i}A^{-(j+\eta_{e,f,i})'}$, where $0 \leq j < n$, we find that X is generated by all the D_i and $T_{e,f,i,j}$, together with $W = A^n$. Rewriting the relations of Y in terms of these generators, we firstly obtain $W = 1$ from the relation $A^n = 1$. We shall therefore delete all occurrences of W in what follows.

Rewriting $A^iD^{\gamma_d}A^{-i}$ $(d \neq a)$ we have, if $i = k\tau + s$, where $\tau = \tau_d$, $0 \leq s < \tau$ and $\gamma = \gamma_d$, that

$$A^iD^{\gamma_d}A^{-i} = \{A^{k\tau+s}DA^{-\tau(k+1)-s}\}\{A^{\tau(k+1)+s}DA^{-\tau(k+2)-s}\}\ldots$$
$$\{A^{\tau(\gamma-1)+s}DA^{-s}\}\{A^sDA^{-\tau-s}\}\cdots\{A^{\tau(k-1)}DA^{-\tau k-s}\}.$$

Thus we obtain from $\mathfrak{R}_1$ the following set of relations of X:

$$D_sD_{s+\tau_d}\cdots D_{s+(\gamma_d-1)\tau_d} = 1,$$

where $0 \leq s < \tau_d$. In our previous notation, these relations may be written as

$$\mathfrak{D}_s(\gamma_d) = 1 \tag{3.19}$$

for $0 \leq s < \tau_d$.

A similar argument shows that the relations $\mathfrak{R}_2$ yield the following set of relations for X, where $0 \leq j < n - 1$:

$$\mathfrak{C}_j(\alpha_{c,d}) = \mathfrak{D}_j(\beta_d), \tag{3.20}$$

with the understanding that the subscripts in the written out forms of $\mathfrak{C}_j(\alpha_{c,d})$ and $\mathfrak{D}_j(\beta_d)$ are to be reduced modulo n, and in case $c = a$ we put $A_j = 1$ for all j.

Now we rewrite the word $TE^\delta T^{-1}F^{-\beta}$ corresponding to the relation $\mathfrak{R}_3$, where we have omitted the subscripts e, f, i. We may assume that $\delta\tau_e < n$, since otherwise the relation is a consequence of $\mathfrak{R}_1$. We have

$$TE^\delta T^{-1}F^{-\rho} = (TA^{-\eta})(A^\eta EA^{-\eta-\tau_e})\cdots(A^{\eta+(s-1)\tau_e}EA^{-\eta-\delta\tau_e})(A^{\eta+s\tau_e}T^{-1}F^{-\rho}).$$

We note that $\eta + \delta\tau_e < n$, since $\delta\tau_e | n$, $\delta\tau_e < n$ and $\eta < \tau_e$. Rewriting $F^\rho TA^{\eta+\delta\tau_e}$ we have, since $\delta\tau_e = \rho\tau_f$,

$$F^\rho TA^{-\eta-\delta\tau_e} = (FA^{-\tau_f})(A^{\tau_f}FA^{-2\tau_f})\cdots(A^{(\rho-1)\tau_f}FA^{-\rho\tau_f})(A^{\rho\tau_f}TA^{-\eta-\rho\tau_f}).$$

Thus the corresponding relation of X is

$$T_0E_\eta E_{\eta+\tau_e}\cdots E_{\eta+(s-1)\tau_e} = F_0F_{\tau_f}\cdots F_{(\rho-1)\tau_f}T_{\rho\tau_f},$$

which may be written, with the same understanding as in the case of $\mathfrak{R}_2$, in the form $T_0\mathfrak{E}_\eta(\delta) = \mathfrak{F}_0(\rho)T_{\rho\tau_f}$. A similar argument shows that the relation of X corresponding to the rewritten word $A^jTE^\delta T^{-1}F^{-\rho}A^{-j}$, where $0 \leq j < n$, is

$$T_j\mathfrak{E}_{j+\eta}(\delta) = \mathfrak{F}_j(\rho)T_{j+\rho\tau_f}, \tag{3.21}$$

and this is therefore the set of relations of X corresponding to the relations $\mathfrak{R}_3$, where we have omitted the subscripts e, f, i.

Now let X_1, θ_1 be the free group and corresponding automorphism constructed from the based n, n_1-tree $\mathfrak{T}$ as in the statement of the Theorem. We regard X_1 as being generated by the set S_1 of all $D_0, D_1, \ldots, D_{n-1}$ and $T_0, T_1, \ldots, T_{n-1}$. We note that, since $n_1|n$, the elements of S_1 continue to satisfy the relations given by (3.1) and (3.8), when the subscripts in these are reduced modulo n. These relations are clearly sufficient to express each element of S_1 as a word on the free generating set S of X_1, and hence constitute a set of defining relations for X_1 on the generating set S_1. In addition, it follows from (3.3) that the set of relations $\mathfrak{D}_0(\gamma_d) = 1$, with reduced subscripts, are satisfied by the D-generators of X_1. Comparing with (3.19), (3.20) and (3.21), we see that X and X_1 have the same presentation on the set S_1, and that the actions of θ and θ_1 are the same. It follows that $n = n_1$ and that X, θ have the required form. This proves the Theorem.

4. Remarks on the Theorem. Let X, θ be defined as in the statement of the Theorem, and let $\mathcal{Y}$ be the corresponding graph of groups. We have:

(1) If X is finitely generated then it follows immediately from the definition of X that its rank $r(X)$ is given by

$$r(X) = \sum_d (\beta_d - 1)\tau_d + \sum_{e,f,i} \rho_{e,f,i}\tau_f. \tag{4.1}$$

(2) It is easy to see, from the conditions on the various parameters used, that the description of the action of θ on X reduces to that of [2] in the case n is a prime.

(3) We now find the fixed point subgroup $X^{\langle\theta\rangle}$ of θ. We note that if K is the centralizer of A in Y, then $X^{\langle\theta\rangle}$ is the intersection of K and X.

Let $T = T_{a,a,i}$ be a 'free generator' of Y such that $\rho_{a,a,i} = 1$. Then $T \in K$, and if K_1 is the group generated by A and all such T, we shall show that $K = K_1$.

To prove this, we use the normal form theorem for the elements of the fundamental group of a graph of groups which was given in [3]. We begin with some notation. Let r be a (directed) edge of $\mathcal{Y}$, with initial vertex e and terminal vertex f. We denote by $L(r)$ the subgroup of Y_e associated with r, and by $M(r)$ the corresponding subgroup of Y_f. Of course, if $\bar{r}$ denotes the edge opposite to r, then $L(\bar{r}) = M(r)$. We now choose, for each r, a coset representative system $\mathfrak{L}(r)$ for $L(r)$ in Y_e, so that $Y_e = \mathfrak{L}(r)L(r)$, and $1 \in \mathfrak{L}(r)$. We now define an element t_r of Y for each edge r. If $r \in \mathfrak{T}$, then we put $t_r = 1$; otherwise, we put $t_r = T_{e,f,i}^{-1}$ or $t_r = T_{e,f,i}$, according as r or $\bar{r}$ is the ith free edge corresponding to the pair of vertices e, f (as in the construction of $\mathfrak{T}$ in the second part of the proof of the Theorem).

The normal form theorem of [3] (see also [9]) states that every nonidentity element y of Y has a unique expression

$$y = \gamma_1 t_{r_1} \gamma_2 t_{r_2} \cdots \gamma_k t_{r_k} h, \tag{4.2}$$

where $k \geqslant 0$, $r_1, r_2, \ldots, r_k$ are the successive edges of a path in $\mathcal{Y}$ beginning and ending at a, $\gamma_j \in \mathfrak{L}(r_j)$, $1 \leqslant j \leqslant k$, $h \in Y_a$ and $r_{j-1} \neq \bar{r}_j$ if $\gamma_j = 1$ $(2 \leqslant j \leqslant k)$.

We note that if the edge r has initial vertex a, then we may suppose that the transversal $\mathfrak{L}(r)$ contains A, unless r is a loop at a with corresponding $\rho = 1$, since if r is not such a loop then the subgroup $L(r)$ is a proper subgroup of $\langle A \rangle$.

Let $y \in Y$, with normal form (4.2), be an element of K. We show that $y \in K_1$. We may suppose that $k \geqslant 1$, since otherwise $y = h \in K_1$. We note that $\gamma_1 \in K_1$, and that replacing γ_1 by 1 in the right side of (4.2) again yields a normal form expression. Thus we may suppose that $\gamma_1 = 1$. Next, if t_{r_1} is a loop at a with corresponding $\rho = 1$, then $t_{r_1} \in K_1$, and the expression obtained from the right side of (4.2) by deleting $\gamma_1 t_{r_1}$ is again a normal form expression. Thus we may suppose, also, that r_1 is not such a loop. We show that, with these assumptions, we cannot have $k > 1$. Suppose otherwise. Then, from the conditions satisfied by γ_1 and r_1, it follows immediately that both $At_{r_1}\gamma_2 \cdots t_{r_k}h$ and $\gamma_1 t_{r_1} \cdots t_{r_k}(hA)$ are normal form expressions. Since these expressions are both equal to y, they must be identical. This is clearly not the case, and hence $k = 0$. This proves that $K = K_1$.

It follows that $X^{\langle\theta\rangle}$ is the subgroup of X generated by all $T_{a,a,i,0}$ with $\rho_{a,a,i} = 1$. Visibly, $X^{\langle\theta\rangle}$ is a free factor of X, and $X = X^{\langle\theta\rangle} * H$, where $H\theta = H$.

5. The periodic automorphisms of the free group of rank three. The Theorem can be used to verify that each periodic automorphism of a free group of rank three is conjugate to exactly one of the following: First, we have eight automorphisms of order two

$$B_0 \to B_0^{-1}, \quad C_0 \to C_0^{-1}, \quad D_0 \to D_0^{-1}, \tag{5.1}$$
$$B_0 \to B_0^{-1}, \quad C_0 \to C_0^{-1}, \quad T_0 \to T_0, \tag{5.2}$$
$$B_0 \to B_0^{-1}, \quad T_0 \to T_0, \quad J_0 \to J_0, \tag{5.3}$$
$$B_0 \to B_0^{-1}, \quad C_0 \to C_0^{-1}, \quad T_0 \to B_0^{-1}T_0B_0, \tag{5.4}$$
$$B_0 \to B_0^{-1}, \quad T_0 \to T_1 \to T_0, \tag{5.5}$$
$$B_0 \to B_0^{-1}, \quad T_0 \to B_0^{-1}T_0B_0, \quad J_0 \to J_0, \tag{5.6}$$
$$B_0 \to B_0^{-1}, \quad T_0 \to B_0^{-1}T_0B_0, \quad J_0 \to B_0^{-1}J_0B_0, \tag{5.7}$$
$$T_0 \to T_1 \to T_0, \quad J_0 \to J_0. \tag{5.8}$$

Next, we have three automorphisms of order three

$$B_0 \to B_1 \to B_1^{-1}B_0^{-1}, \quad T_0 \to T_0, \tag{5.9}$$
$$B_0 \to B_1 \to B_1^{-1}B_0^{-1}, \quad T_0 \to B_0^{-1}T_0B_0, \tag{5.10}$$
$$T_0 \to T_1 \to T_2 \to T_0. \tag{5.11}$$

Next, we have three automorphisms of order four

$$B_0 \to B_1 \to B_0^{-1}, \quad C_0 \to C_0^{-1}, \tag{5.12}$$
$$B_0 \to B_1 \to B_0^{-1}, \quad T_0 \to T_0, \tag{5.13}$$
$$B_0 \to B_1 \to B_2 \to B_2^{-1}B_1^{-1}B_0^{-1}. \tag{5.14}$$

Finally, we have four of order six.

$$B_0 \to B_1 \to B_1^{-1}B_0^{-1}, \quad C_0 \to C_0^{-1}, \tag{5.15}$$
$$B_0 \to B_1 \to B_1^{-1}B_0^{-1}, \quad C_0 \to C_0^{-1}B_0B_1, \tag{5.16}$$
$$B_0 \to B_1 \to B_1^{-1}B_0^{-1}C_0, \quad C_0 \to C_0^{-1}, \tag{5.17}$$
$$B_0 \to B_1 \to B_2 \to B_0^{-1}. \tag{5.18}$$

We indicate briefly how the list is obtained. Let X be a free group of rank three, θ a periodic automorphism of X, $\mathcal{Y}$ the corresponding graph of groups and $\mathcal{T}$ a maximal tree in $\mathcal{Y}$. Since each $\beta_d \geqslant 2$, it follows from (4.1) that the rank of X is at least $\sum_c \tau_c$, where c ranges over the vertices of $\mathcal{Y}$ other than a. Thus $\mathcal{Y}$ has at most four vertices. It is now a simple matter to list all possible graphs $\mathcal{Y}$ and compute the corresponding automorphisms. To illustrate this, suppose that $\mathcal{Y}$ has three vertices a, b, c. From (4.1) we see that there are two possibilities for the pair τ_b, τ_c, namely $\tau_b = \tau_c = 1$ and $\tau_b = 2, \tau_c = 1$. We consider only the first case (the second gives rise to the automorphism (5.12)). For the case $\tau_b = \tau_c = 1$ we may have, again from (4.1), either $\beta_b = \beta_c = 2$ and $\mathcal{Y}$ has one free edge with $\rho = 1$ (and the free edge must therefore be a loop, with η value zero since all τ values are one) or $\beta_b = 3, \beta_c = 2$ and $\mathcal{Y}$ has no free edges. We consider only the latter possibility (the former gives rise to the automorphisms (5.2) and (5.4)).

Thus we have vertices a, b, c with $\tau_b = \tau_c = 1$, $\beta_b = 3$, $\beta_c = 2$. Hence $|\theta| = 6$, and all γ values are 6. There are three possible labellings of the vertices of the tree $\mathcal{Y}$, from which we obtain three (isomorphic) groups, namely $\langle A, B, C;\, A^6 = B^6 = C^6 = 1,\, A^3 = B^3,\, A^2 = C^2\rangle$, and the groups obtained from this by, firstly, interchanging A and B, and, secondly, interchanging A and C. The α values can now be read from this, and the automorphisms obtained are, respectively, (5.15), (5.16) and (5.17).

On obtaining the list of possible θ's, as given, some further arguments are required to show that no two of the elements listed are conjugate. Details are available from the author.

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