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NONEXISTENCE OF CONTINUOUS SELECTIONS OF THE METRIC PROJECTION FOR A CLASS OF WEAK CHEBYSHEV SPACES

BY

MANFRED SOMMER

ABSTRACT. The class $\mathfrak{W}$ of all those n-dimensional weak Chebyshev subspaces of $C[a, b]$ whose elements have no zero intervals is considered. It is shown that there does not exist any continuous selection of the metric projection for G if there is a nonzero g in G having at least $n + 1$ distinct zeros. Together with a recent result of Nürnberger-Sommer, the following characterization of continuous selections for $\mathfrak{W}$ is valid: There exists a continuous selection of the metric projection for G in $\mathfrak{W}$ if and only if each nonzero g in G has at most n distinct zeros.

If G is a nonempty subset of a normed linear space E, then for each f in E we define $P(f) := \{g_0 \in G \mid \|f - g_0\| = \inf\{\|f - g\| \mid g \in G\}\}$. P defines a set-valued mapping of E into 2^G which in the literature is called *the metric projection* onto G. A continuous mapping s of E onto G is called a *continuous selection for the metric projection P* (or, more briefly, continuous selection) if $s(f)$ is in $P(f)$ for each f in E. In this paper we treat the problem of the existence of continuous selections for n-dimensional subspaces G of $C[a, b]$, with $C[a, b]$, as usual, the Banach space of real-valued continuous functions on $[a, b]$ under the uniform norm.

A. Lazar, P. Morris and D. Wulbert [3] have characterized the 1-dimensional subspaces of $C(X)$ with X compact Hausdorff, which admit a continuous selection. They have raised the problem of characterizing the corresponding n-dimensional subspaces.

Using the kind of selection established by Lazar-Morris-Wulbert, it does not seem possible to get a general theorem for n-dimensional subspaces of $C[a, b]$. With new methods, however, and in the setting of weak Chebyshev subspaces, Nürnberger-Sommer [4], [5] and Sommer [7], [8] have given both sufficient conditions for the existence of continuous selections and characterization theorems of the existence of continuous selections for several classes of n-dimensional weak Chebyshev subspaces of $C[a, b]$ (see also Nürnberger [6]).

In the following we refer to a result in [4].

Nürnberger-Sommer have shown that for those weak Chebyshev spaces G whose elements g ($g \not\equiv 0$) have at most n distinct zeros on $[a, b]$, there exists exactly one continuous selection.

Here we show that for those weak Chebyshev spaces G which have no elements

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vanishing on intervals but which have elements g ($g \not\equiv 0$) with at least $n + 1$ distinct zeros, there does not exist any continuous selection. To prove this result we apply a fundamental lemma of Lazar-Morris-Wulbert.

Hence we have the following characterization of the existence of continuous selections for those n-dimensional weak Chebyshev subspaces G of $C[a, b]$ whose elements do not vanish on intervals.

> There exists a continuous selection for G if and only if each g in G has at most n distinct zeros.

In the following let G be an n-dimensional subspace of $C[a, b]$.

1. DEFINITION. G is called weak Chebyshev if each g in G has at most $n - 1$ changes of sign, i.e. there do not exist points $a \leqslant x_0 < x_1 < \cdots < x_n \leqslant b$ such that $g(x_i) \cdot g(x_{i+1}) < 0$, $i = 0, \ldots, n - 1$.

R. C. Jones and L. A. Karlovitz have characterized these spaces. For this characterization we need the following definition.

2. DEFINITION. If f is in $C[a, b]$, then g in $P(f)$ is called an *alternation element* (AE) of f if there exist $n + 1$ distinct points $a \leqslant x_0 < x_1 < \cdots < x_n \leqslant b$ such that $\varepsilon(-1)^i(f - g)(x_i) = \|f - g\|$, $i = 0, \ldots, n$, $\varepsilon = \pm 1$. The points $x_0, \ldots, x_n$ are called *alternating extreme points* of $f - g$.

Jones and Karlovitz [1] have proved the following theorems.

3. THEOREM. *G is weak Chebyshev if and only if for each f in $C[a, b]$ there exists at least one AE in $P(f)$.*

4. THEOREM. *G is weak Chebyshev if and only if given $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ there exists a g in G, $g \not\equiv 0$, such that*

$$(-1)^{i+1}g(x) \geqslant 0, \qquad x_{i-1} < x < x_i, \; i = 1, \ldots, n.$$

In order to show the nonexistence of a continuous selection for a class of weak Chebyshev spaces, we need the following standard definition.

5. DEFINITION. A zero x_0 of f in $C[a, b]$ is said to be a *simple zero* if f changes sign at x_0 or if $x_0 = a$ or $x_0 = b$. A zero x_0 of f in $C[a, b]$ is said to be a *double zero* if f does not change sign at x_0 and x_0 is in (a, b). Let $Z(f) := \{x \in [a, b] \mid f(x) = 0\}$ the set of zeros of f.

We denote by $|Z(f)|$ the number of distinct zeros of f and by $|Z^*(f)|$ the number of zeros of f, counting simple zeros as one zero and double zeros as two zeros.

6. DEFINITION. Let x_1, x_2 be zeros of f in $C[a, b]$. These zeros are said to be *separated* if there is a y_1 in $[a, b]$ with

$$x_1 < y_1 < x_2, \qquad f(y_1) \neq 0.$$

A zero x_0 of f in G is said to be an *essential* zero with respect to G, if there is a g in G with $g(x_0) \neq 0$.

Nürnberger-Sommer [4] have given the following sufficient condition for the existence of a continuous selection.

7. THEOREM. *Let G be weak Chebyshev. Let $|Z(g)| \leqslant n$ for each g in G, $g \not\equiv 0$. Then there exists exactly one continuous selection.*

Since $|Z(g)| \leqslant n$ for each g in G, $g \not\equiv 0$, no g in G has a zero interval. We will show that for those weak Chebyshev spaces G which have no elements with zero intervals but which have elements g with $|Z(g)| \geqslant n + 1$, there does not exist any continuous selection. For this we need the following lemmas.

8. LEMMA (LAZAR-MORRIS-WULBERT [3]). *If s is a continuous selection of $C[a, b]$ onto G and f is in $C[a, b]$, $\|f\| = 1$ and 0 is in $P(f)$, then there is a g_0 in $P(f)$ such that*

(1) *for every $x \in \operatorname{bd} Z(P(f)) \cap f^{-1}(1)$ and every g in $P(f)$ there is a neighborhood U of x for which $g_0 \geqslant g$ on U and*

(2) *for every $x \in \operatorname{bd} Z(P(f)) \cap f^{-1}(-1)$ and every g in $P(f)$ there is a neighborhood V of x for which $g_0 \leqslant g$ on V.*

Here let $Z(P(f)) := \{x \in [a, b] \mid g(x) = 0 \text{ for each } g \text{ in } P(f)\}$ and $\operatorname{bd} Z(P(f))$ is the set of boundary points of $Z(P(f))$ under the topology of $[a, b]$.

9. LEMMA (STOCKENBERG [9]). *Let G be weak Chebyshev. Then no g in G has more than n separated, essential zeros and if there is a g in G with n separated, essential zeros $x_1 < \cdots < x_n$, then $g(x) = 0$ for all x in $[a, x_1] \cup [x_n, b]$.*

Now we are able to prove the nonexistence of a continuous selection for a class of weak Chebyshev spaces.

10. THEOREM. *Let G be weak Chebyshev. Let no g in G ($g \not\equiv 0$) have a zero interval but let a nontrivial function $\tilde{g}_0$ be in G such that $|Z(\tilde{g}_0)| \geqslant n + 1$. Then there does not exist any continuous selection.*

PROOF. Let $a \leqslant z_0 < z_1 < \cdots < z_n \leqslant b$ be $n + 1$ distinct zeros of $\tilde{g}_0$.

First case: $a \leqslant z_0$, $z_n < b$. We will construct a function g_0 having exactly $n - 1$ zeros with changes of sign and two further zeros on $[a, b)$. Since $z_0, \ldots, z_n$ are separated zeros of $\tilde{g}_0$, by Lemma 9, there are two points $z_i, z_j \in \{z_0, \ldots, z_n\}$, $i < j$, such that

$$g(z_i) = g(z_j) = 0 \quad \text{for all } g \in G.$$

We choose $n - 1$ distinct points

$$z_n < t_1 < \cdots < t_{n-1} < b.$$

By Theorem 4 there exists a $g_0 \in G$, $g_0 \not\equiv 0$, such that

$$(-1)^{i+1} g_0(x) \geqslant 0, \qquad t_{i-1} < x < t_i, i = 1, \ldots, n, t_0 = a, t_n = b.$$

Moreover $g_0(z_i) = g_0(z_j) = 0$.

Since $t_1, \ldots, t_{n-1}$ are zeros with changes of sign of g_0 and G is weak Chebyshev of dimension n, the function g_0 has no further change of sign on (a, b) and, therefore, no change of sign at z_i and z_j.

Let $\|g_0\| \leqslant 1$. We choose $n + 1$ distinct points $\{v_l\}_{l=0}^{n}$ satisfying

$$z_i < v_0 < z_j < v_1 < t_1 < v_2 < \cdots < t_{n-1} < v_n < b.$$

We choose $\varepsilon > 0$ such that

$$\{z_i, z_j, t_1, \ldots, t_{n-1}, b\} \cap [v_l - \varepsilon, v_l + \varepsilon] = \varnothing, \qquad l = 0, \ldots, n.$$

Now we construct an $f \in C[a, b]$ as follows:

(a)

$$f(z_i) = 1, \qquad f(z_j) = -1,$$
$$f(x) = 1 \quad \text{for all } x \in [v_0 - \varepsilon, v_0 + \varepsilon],$$
$$f(x) = (-1)^{l+1} \quad \text{for all } x \in [v_l - \varepsilon, v_l + \varepsilon], l = 1, \ldots, n.$$

(b)

$$\max\{-1 + g_0(x), -1\} \leqslant f(x) \leqslant \min\{1 + g_0(x), 1\} \quad \text{for all } x \in [a, b].$$

Then $\|f - 0\| = \|f - g_0\| = 1$. Because of $g(z_i) = 0$ for all $g \in G$, we always get $f(z_i) - g(z_i) = 1$ and, therefore, 0 and g_0 are elements of $P(f)$.

Now let $g \in P(f)$, $g \not\equiv 0$. For each $l \in \{1, \ldots, n\}$ there exists a $y_l \in [v_l - \varepsilon, v_l + \varepsilon]$ such that $(-1)^{l+1}g(y_l) > 0$. Hence the function g has at least $n - 1$ changes of sign on $(v_1 - \varepsilon, b)$. Then $g \geqslant 0$ on $[a, v_1 - \varepsilon]$.

Therefore the function g has a double zero in z_j and also in z_i, if $z_i > a$.

Since no $g \in G$ has a zero interval, it follows that $z_i, z_j \in \operatorname{bd} Z(P(f))$. Now we apply Lemma 8.

If there exists a continuous selection, then there exists a $\tilde{g} \in P(f)$ such that

(i) for z_i and g_0 there is a neighborhood U of z_i for which $\tilde{g} \gtrdot g_0$ on U, and

(ii) for z_j and 0 there is a neighborhood V of z_j for which $\tilde{g} \lessdot 0$ on V.

Since $\tilde{g} \geqslant g_0$ on U, $\tilde{g} \not\equiv 0$.

Moreover, $\tilde{g} \geqslant 0$ on $[a, v_1 - \varepsilon]$. Therefore, in every neighborhood V of z_j there is a point $\tilde{x}$ such that $\tilde{g}(\tilde{x}) > 0$. But this is a contradiction to Lemma 8.

Second case: $a < z_0$, $z_n \leqslant b$. We can treat this case analogously.

Third case: Let $|Z(g)| \leqslant n$ on $[a, b)$ and on $(a, b]$ for all $g \in G$. By hypothesis there is a $\tilde{g}_0 \in G$, $\tilde{g}_0 \not\equiv 0$, with exactly $n + 1$ distinct zeros $a = z_0 < z_1 < \cdots < z_n = b$. Let $a < t_1 < t_2 < \cdots < t_k < b$ be all zeros with changes of sign and $a < y_1 < y_2 < \cdots < y_{n-k-1} < b$ be all double zeros of $\tilde{g}_0$.

First we show that $k = n - 1$ or $k = n - 2$. No other possibilities are allowed.

We assume that $k \leqslant n - 3$. We choose $n - k - 1$ points $z_{n-1} < t_{k+1} < \cdots < t_{n-1} < b$. By Theorem 4 there exists a $g_0 \in G$, $g_0 \not\equiv 0$, such that

$$(-1)^{i+1} g_0(x) \geqslant 0, \qquad t_{i-1} < x < t_i, i = 1, \ldots, n, t_0 = a, t_n = b.$$

We may assume that $g_0 \cdot \tilde{g}_0 \geqslant 0$ on $[a, t_{k+1}]$. Since $n - k \gtrdot 3$, $\tilde{g}_0$ has at least $n - k - 1 \geqslant 2$ double zeros on (a, b) and therefore $|Z^*(\tilde{g}_0)| \gtrdot n + 1$ on (a, b). If $g_0(y_i) \neq 0$ for all $i \in \{1, \ldots, n - k - 1\}$, then for sufficiently small $c > 0$ the function $\tilde{g}_0 - cg_0$ has at least $n + 1$ changes of sign. This is a contradiction of the hypothesis on G.

If there are $i_1, i_2 \in \{1, \ldots, n - k - 1\}$ such that $g_0(y_{i_1}) = g_0(y_{i_2}) = 0$, then g_0 has $n + 1$ distinct zeros $t_1, \ldots, t_{n-1}, y_{i_1}, y_{i_2}$ on (a, b). This is also a contradiction of the hypothesis.

Therefore there is exactly one double zero y_{i_0} of $\tilde{g}_0$ such that $g_0(y_{i_0}) = 0$. Then g_0 has n distinct zeros $t_1, \ldots, t_{n-1}, y_{i_0}$ on (a, b). Then for sufficiently small $c > 0$ the function $\tilde{g}_0 - cg_0$ has at least $k + 2(n - k - 2) = n + n - k - 4 \geqslant n - 1$ changes of sign on (a, b) because g_0 does not vanish on exactly $n - k - 2$ double zeros of $\tilde{g}_0$. Moreover $\tilde{g}_0 - cg_0$ has a further zero in y_{i_0} and also a further zero on a neighborhood of a, because $g_0 \cdot \tilde{g}_0 \geqslant 0$ on $[a, t_{k+1}]$.

Hence $\tilde{g}_0 - cg_0$ has at least $n + 1$ distinct zeros on $[a, b)$. This is a contradiction of the hypothesis of this case. Hence we have shown that $n - k = 1$ or $n - k = 2$.

We distinguish these two cases:

(i) $n - k = 1$. Therefore $\tilde{g}_0$ has exactly $n - 1$ changes of sign on (a, b). Let $g \in G$, $g \not\equiv 0$. Then $g(a) = g(b) = 0$, because otherwise the function $\tilde{g}_0 - cg$ has n changes of sign for sufficiently small c.

Therefore $g(a) = g(b)$ for all $g \in G$.

Now we proceed as we did in the first case. We choose n distinct points $\{v_l\}_{l=1}^n$ satisfying

$$a < v_1 < z_1 < v_2 < z_2 < \cdots < v_{n-1} < z_{n-1} < v_n < b.$$

Let $\|\tilde{g}_0\| \leqslant 1$ and $\tilde{g}_0 \geqslant 0$ on $[a, z_1]$. We choose $\varepsilon > 0$ such that

$$\{z_0, \ldots, z_n\} \cap [v_l - \varepsilon, v_l + \varepsilon] = \varnothing, \qquad l = 1, \ldots, n.$$

We construct an $f \in C[a, b]$ as follows:

(a)

$$\begin{aligned} f(a) &= 1, \\ f(x) &= (-1)^{l-1} \quad \text{for all } x \in [v_l - \varepsilon, v_l + \varepsilon], l = 1, \ldots, n, \\ f(b) &= (-1)^n; \end{aligned}$$

(b)

$$\max\{-1 + \tilde{g}_0(x), -1\} \leqslant f(x) \leqslant \min\{1 + \tilde{g}_0(x), 1\} \quad \text{for all } x \in [a, b].$$

Then $\|f - 0\| = \|f - \tilde{g}_0\| = 1$ and $0, \tilde{g}_0 \in P(f)$.

It is easy to show that each $g \in P(f)$ has exactly $n - 1$ changes of sign. Moreover $g(a) = g(b) = 0$ for all $g \in P(f)$. Therefore $a,b \in \operatorname{bd} Z(P(f))$. Applying Lemma 8 to the point a and $\tilde{g}_0 \in P(f)$ and to the point b and $0 \in P(f)$ we get a contradiction of the hypothesis that there exists a continuous selection.

(ii) $n - k = 2$. Therefore, $\tilde{g}_0$ has exactly $n - 2$ zeros with changes of sign and exactly one double zero z_i on (a, b).

We choose $n - 1$ distinct points $\{v_l\}_{l=1}^{n-1}$ satisfying

$$a < z_1 < v_1 < z_2 < v_2 < \cdots < v_{n-2} < z_{n-1} < v_{n-1} < b.$$

Let $\|\tilde{g}_0\| \leqslant 1$ and $\tilde{g}_0 \geqslant 0$ on $[a, z_1]$. We choose $\varepsilon > 0$ such that

$$\{z_0, \ldots, z_n\} \cap [v_l - \varepsilon, v_l + \varepsilon] = \varnothing, \qquad l = 1, \ldots, n - 1$$

and $a + \varepsilon < z_1$.

We construct an $f \in C[a, b]$ as follows:

(a)

$$f(x) = 1 \quad \text{for all } x \in [a, a + \varepsilon],$$
$$f(x) = (-1)^l \quad \text{for all } x \in [v_l - \varepsilon, v_l + \varepsilon], l = 1, \ldots, i - 1,$$
$$f(z_i) = (-1)^i,$$
$$f(x) = (-1)^{l+1} \quad \text{for all } x \in [v_l - \varepsilon, v_l + \varepsilon], l = i, \ldots, n - 1,$$
$$f(b) = (-1)^n;$$

(b)

$$\max\{-1 + \tilde{g}_0(x), -1\} \leqslant f(x) \leqslant \min\{1 + \tilde{g}_0(x), 1\} \quad \text{for all } x \in [a, b].$$

Then $\|f - 0\| = \|f - \tilde{g}_0\| = 1$ and $0, \tilde{g}_0 \in P(f)$, since $f - 0$ has $n + 1$ alternating extreme points.

Let $g \in P(f)$, $g \not\equiv 0$. Then it is easy to show that g has at least $n - 2$ changes of sign and a double zero at z_i, since otherwise g has n changes of sign. This would be a contradiction of the hypothesis on G.

Since $g(a) \geqslant 0$, $(-1)^n g(b) \geqslant 0$, for sufficiently small $c > 0$ the function $\tilde{g}_0 - cg$ has $n - 2$ changes of sign, a double zero at z_i and two further zeros on neighborhoods of a and b. Since by hypothesis $|Z(\tilde{g}_0 - cg)| \leqslant n$ on $[a, b)$ and on $(a, b]$, the function $\tilde{g}_0 - cg$ has two zeros at a and b. Therefore $g(a) = g(b) = 0$ for all $g \in P(f)$ and $a, z_i, b \in \operatorname{bd} Z(P(f))$.

Applying Lemma 8 to the point a and $\tilde{g}_0 \in P(f)$ and to the point z_i and $0 \in P(f)$ we get a contradiction of the hypothesis that there exists a continuous selection.

Now we give two examples showing that it is necessary to distinguish the two cases $n - k = 1$ and $n - k = 2$ in the third part of the above proof.

EXAMPLE 1. $G := \langle \sin \frac{1}{2}x, \sin x \rangle \subset C[0, 2\pi]$. Here $|Z(g)| \leqslant 2$ for all $g \in G$ on $[0, 2\pi)$ and on $(0, 2\pi]$. The function $\tilde{g}_0(x) = \sin x$ has exactly the distinct zeros 0, π, 2π such that $n - k = 2 - 1 = 1$.

EXAMPLE 2. $G := \langle x^3, |x|(1 - |x|) \rangle \subset C[-1, 1]$. Here $|Z(g)| \leqslant 2$ for all $g \in G$ on $[-1, 1)$ and on $(-1, 1]$. There is no $g \in G$ with three zeros $-1 = z_0 < z_1 < z_2 = 1$ such that z_1 is a zero with change of sign of g but the function $\tilde{g}_0(x) = |x|(1 - |x|)$ has exactly the distinct zeros $-1, 0, 1$ where 0 is a double zero of $\tilde{g}_0$. Therefore $n - k = 2 - 0 = 2$.

Last we give a class of weak Chebyshev subspaces G of $C[a, b]$ satisfying the additional condition that no g in G, $g \not\equiv 0$, has a zero interval. Let g_0 be a nonnegative function in $C[a, b]$ having no zero interval, but at least two distinct zeros on $[a, b]$. Then for any g_0 having these properties, the space G spanned by the functions $g_0(x), xg_0(x), \ldots, x^{n-2}g_0(x), x^{n-1}g_0(x)$ is a weak Chebyshev space in $C[a, b]$, since each g in G has the representation $g(x) = g_0(x)\sum_{i=0}^{n-1} a_i x^i$ and since the function $\sum_{i=0}^{n-1} a_i x^i$ has at most $n - 1$ changes of sign on (a, b). Since g_0 has no zero interval, no g in G, $g \not\equiv 0$, has a zero interval.

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Institut fur Angewandte Mathematik der Universität Erlangen-Nürnberg, Martensstrasse 3, 8520 Erlangen, Germany