TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 260, Number 2, August 1980

PRODUCTS IN THE ATIYAH-HIRZEBRUCH SPECTRAL SEQUENCE AND THE CALCULATION OF MSO_*

BY

BRAYTON GRAY

ABSTRACT. It is possible to put a multiplicative structure in the Atiyah-Hirzebruch spectral sequence in certain cases even though the spectra involved are not both ring spectra. As a special case, an easy calculation of the homotopy of MSO is obtained.

It is the purpose of this paper to give a calculation of the oriented bordism ring MSO_* based on some rather elementary considerations. In particular, we recover Wall's result [11] from arguments with the Atiyah-Hirzebruch spectral sequence and the Hurewicz homomorphism for MO_*.

We introduce products in the Atiyah-Hirzebruch homology spectral sequence under rather mild conditions. These are weak enough to include the case of $MSO \wedge X \simeq MO$ where X is a spectrum with $SX \simeq RP^\infty$. (In particular, X is not a ring spectrum.)

We will operate in a suitable category of spectra (e.g., [1] or [6]), which we will not make explicit. All ring spectra will be assumed to be homotopy commutative and homotopy associative and the unit and product maps will be assumed to be cellular.

1. We begin by discussing multiplicative structure in the Atiyah-Hirzebruch spectral sequence [1]. Recall that for given spectra X and Y, there is a spectral sequence $\{E^r_{p,q}\}$ defined with

$$E^1_{p,q} = \pi^S_{p+q}(X^p \wedge Y, X^{p-1} \wedge Y), \quad D_{p,q} = \pi^S_{p+q}(X^p \wedge Y)$$

satisfying

(a) $E^2_{p,q} \cong H_p(X; \pi^S_q(Y))$,

(b) $d^r_{p,q}\colon E^r_{p,q} \to E^r_{p-r,q+r-1}$, $H(E^r_{p,q}, d^r_{p,q}) \cong E^{r+1}_{p,q}$,

(c) $\pi^S_{p+q}(X \wedge Y)$ is filtered by subgroups

$$F^p = \operatorname{im}\left\{\pi^S_{p+q}(X^p \wedge Y) \to \pi^S_{p+q}(X \wedge Y)\right\} \quad \text{and} \quad F^p/F^{p-1} \cong E^\infty_{p,*}.$$

We *assume* that X and Y are connective spectra. We will now further *assume* that

1. Y and $X \wedge Y$ are ring spectra.

Received by the editors June 5, 1979 and, in revised form, October 2, 1979.
1980 *Mathematics Subject Classification*. Primary 57R75.

2. There is given a map $\mathfrak{S}: S^0 \to X$ so that the diagram

$$\begin{array}{ccc} X \wedge (Y \wedge Y) & \equiv & (X \wedge Y) \wedge (S^0 \wedge Y) \\ \downarrow 1 \wedge \bar{\mu} & & \downarrow 1 \wedge \varepsilon \wedge 1 \\ X \wedge Y & \overset{\mu}{\leftarrow} & (X \wedge Y) \wedge (X \wedge Y) \end{array}$$

homotopy commutes (i.e., the product in $X \wedge Y$ extends the module action of Y).

Condition 2 is valid, for example, if X is also a homotopy commutative and associative ring spectrum and the multiplication in $X \wedge Y$ is the induced one; this will not be the case, however, in our applications.

Consider now the composition H: $X \wedge X \to (X \wedge Y) \wedge (X \wedge Y) \overset{\mu}{\to} X \wedge Y$, defined via the unit of Y. Since H is cellular the image of $X^m \wedge X^n$ lies in $(X \wedge Y)^{m+n} \subset X^{m+n} \wedge Y$. We have a homotopy commutative diagram

$$\begin{array}{ccc} (X \wedge Y) \wedge (X \wedge Y) & \overset{\mu}{\to} & X \wedge Y \\ & & \uparrow 1 \wedge \bar{\mu} \\ \downarrow T & & X \wedge Y \wedge Y \\ & & \uparrow 1 \wedge \bar{\mu} \\ X \wedge X \wedge Y \wedge Y & \overset{H \wedge 1}{\to} & (X \wedge Y) \wedge (Y \wedge Y) \end{array}$$

so replacing μ by the composition around the square, we have $\mu(X^m \wedge Y, X^n \wedge Y) \subset X^{m+n} \wedge Y$. This consequently induces a product in the spectral sequence

$$D_{p,q} \otimes D_{p',q'} \to D_{p+p',q+q'},$$
$$E^1_{p,q} \otimes E^1_{p',q'} \to E^1_{p+p',q+q'}$$

and we immediately have

(A) $E^r_{p,q}$ is a bigraded ring,

(B) $d^r_{p,q}$ is a graded derivation and the isomorphism in (b) above is a ring isomorphism,

(C) $F^p \cdot F^{p'} \subset F^{p+p'}$ and the isomorphism $F^p/F^{p-1} \cong E^\infty_{p,*}$ is an isomorphism of rings.

Now the isomorphism

$$E^1_{p,q} \cong \pi^S_p(X^p, X^{p-1}) \otimes \pi^S_q(Y)$$

is an $E^1_{0,*}$ module isomorphism. The multiplication in $E^1_{*,0}$ is induced from H by homomorphism

$$\pi^S_p(X^p \wedge Y, X^{p-1} \wedge Y) \otimes \pi^S_{p'}(X^{p'} \wedge Y, X^{p'-1} \wedge Y)$$
$$\to \pi^S_{p+p'}(X^{p+p'} \wedge Y, X^{p+p'-1} \wedge Y).$$

Since $\pi^S_m(X^m \wedge Y, X^{m-1} \wedge Y) \cong \pi^S_m(X^m, X^{m-1}) \otimes \pi^S_0(Y)$, this is determined by

$$\pi^S_p(X^p, X^{p-1}) \otimes \pi^S_{p'}(X^{p'}, X^{p'-1}) \to \pi^S_{p+p'}(X^{p+p'}, X^{p+p'-1}) \otimes \pi^S_0(Y).$$

We now further *assume*[1]

3. $\pi^S_0(Y) \cong Z$.

[1]This is done for simplicity. Otherwise the product of elements of $H_*(X)$ lies in $H_*(X; \pi^S_0(Y))$.

Thus the multiplication in $E^1_{p,q}$ is determined by that in $\pi^S_q(Y)$ and a multiplication in the chain complex of X. Our assumption in fact determines a multiplication in $H_*(X)$. Such an isomorphism corresponds to a map $\varepsilon\colon Y \to HZ$ (the Eilenberg-Mac Lane spectrum for integral cohomology) which is multiplicative. Thus we have (for homology with the coefficients in a ring R)

$$H_p(X) \otimes H_{p'}(X) \to H_{p+p'}(X \wedge X) \to H_{p+p'}(X \wedge Y) \xrightarrow{(1\wedge\varepsilon_*)} H_{p+p'}(X \wedge HZ) \xrightarrow{\alpha} H_{p+p'}(X)$$

where α is defined from the map $HZ \wedge HR \to HR$. This is compatible with the map defined above on the chain level and we have

THEOREM 1.1 *Assuming* 1–3, *the Atiyah-Hirzebruch spectral sequence has products satisfying* (A)–(C). *Furthermore the isomorphism* $E^2_{p,q} \cong H_p(X; \pi^S_q(Y))$ *is a ring isomorphism and the diagram*

$$\begin{array}{ccccc} E^2_{p,0} \cong H_p(X) & \xleftarrow{\alpha} & H_p(X \wedge HZ) & \xleftarrow{(1\wedge\varepsilon)_*} & H_p(X \wedge Y) \\ \uparrow & & & & \uparrow \\ E^\infty_{p,0} & & \longleftarrow & & \pi^S_p(X \wedge Y) \end{array}$$

commutes.

2. Suppose now that we are given a cellular map $B \xrightarrow{f} BO$. This defines a spectrum Tf called the Thom spectra associated with f. If B is a homotopy associative and homotopy commutative H-space with a cellular multiplication, and f is an H-map, Tf becomes a ring spectrum.

Consider now the map $g\colon RP^\infty = BO(1) \to BO$, and let $X = Tg$. Since $RP^\infty \times BSO \xrightarrow{g\times i} BO \times BO \to BO$ is a homotopy equivalence, $X \wedge MSO \cong MO$. Let $Y = MSO$. Conditions 1 and 3 of §1 are clear. The commutativity of the diagram

$$\begin{array}{ccc} X \wedge MSO \wedge MSO & \equiv & (X \wedge MSO) \wedge (S^0 \wedge MSO) \\ \downarrow 1 \wedge \bar{\mu} & & \downarrow \\ X \wedge MSO & \xleftarrow{\mu} & (X \wedge MSO) \wedge (X \wedge MSO) \\ \downarrow \cong & & \downarrow \cong \\ MO & \xleftarrow{\mu} & MO \wedge MO \end{array}$$

follows from the corresponding diagram

$$\begin{array}{ccc} RP^\infty \times BSO(m) \times BSO(n) & & \\ \downarrow & \searrow & \\ RP^\infty \times BSO(m+n) & & BO(m+1) \times BSO(n) \\ \downarrow & \swarrow & \\ BO(m+n+1) & & \end{array}$$

of classifying spaces.

It remains to calculate $H_*(X)$. $X \cong S^{-1}RP^\infty$, so

$$H_r(X) = \begin{cases} 0, & r \text{ odd}, \\ Z_2, & r \text{ even}. \end{cases}$$

To find the multiplication, note that $H_r(X; Z_2) \cong H_r(RP^\infty; Z_2)$ by the Thom isomorphism theorem. Now the following diagram commutes.

$$\begin{array}{ccc} H_p(X; Z_2) \oplus H_{p'}(X; Z_2) & \to & H_{p+p'}(X \wedge X; Z_2) \\ \downarrow \cong & & \downarrow \cong \\ H_p(RP^\infty; Z_2) \oplus H_{p'}(RP^\infty; Z_2) & \to & H_{p+p'}(RP^\infty \times RP^\infty; Z_2) \end{array}$$

$$\begin{array}{ccc} \xrightarrow{H_*} H_{p+p'}(MO; Z_2) & \to & H_{p+p'}(X) \\ \downarrow \cong & & \downarrow \cong \\ \to H_{p+p'}(BO; Z_2) & \xrightarrow{(w_1)_*} & H_{p+p'}(RP^\infty) \end{array}$$

The last square commutes because the upper homomorphism is

$$H_{p+p'}(MO; Z_2) \cong H_{p+p'}(X \wedge MSO; Z_2) \xrightarrow{(1 \wedge \varepsilon)_*} H_{p+p'}(X \wedge HZ; Z_2)$$
$$\to H_{p+p'}(X; Z_2).$$

Now

$$RP^\infty \times RP^\infty \to BO \xrightarrow{w_1} RP^\infty$$

is just the H space structure in RP^∞. Thus the isomorphism $H_*(X; Z_2) \cong H_*(RP^\infty; Z_2)$ is a ring isomorphism.

The only primitive elements in $H^r(RP^\infty; Z_2)$ occur when $r = 2^s$, $s \geqslant 0$, so the only indecomposables occur in $H_{2^s}(RP^\infty, Z_2)$. Let $x_r \in H_r(RP^\infty; Z_2)$ be the nonzero element. $(x_{2^s})^2 = 0$ since the only nonzero element is indecomposable. Thus

PROPOSITION 2.1.

$$H_*(X; Z_2) \cong \bigwedge_{n>0} (x_{2^n}), \qquad H_*(X) \cong \bigwedge_{n>1} (x_{2^n})$$

(*where* $\bigwedge(a_i)$ *represents an exterior algebra on the* a_i *over* Z_2).

3.

LEMMA 3.1.

$$E^2_{2p,q} \cong E^2_{2p,0} \otimes E^2_{0,q}, \qquad E^2_{2p+1,q} \cong E^2_{2p,0} \otimes E^2_{1,q}.$$

PROOF. This follows immediately from the isomorphisms

$$E^2_{2p,q} \cong H_{2p}\big(X; \pi_q^S(Y)\big) \cong H_{2p}(X) \otimes \pi_q^S(Y),$$
$$E^2_{2p+1,q} \cong H_{2p+1}\big(X; \pi_q^S(Y)\big) \cong H_{2p}(X) * \pi_q^S(Y).$$

Thus all indecomposables lie in $E^2_{0,q}$, $E^2_{1,q}$ or $E^2_{p,0}$.

LEMMA 3.2.

$$E^2_{p,q} \cong E^\infty_{p,q}.$$

PROOF. It is only necessary to show that $x_{2^r} \in E^2_{2^r,0}$ is an infinite cycle. By Theorem 1.1 it is sufficient to show that x_{2^r} is the image of the homomorphism

$$\pi_{2^r}(MO) \to H_{2^r}(MO) \to H_{2^r}(X).$$

By commutativity of the square

$$\begin{array}{ccc} H_{2^r}(MO) & \to & H_{2^r}(X) \\ \downarrow & & \downarrow \\ H_{2^r}(MO; Z_2) & \to & H_{2^r}(X; Z_2) \\ \downarrow \cong & & \downarrow \cong \\ H_{2^r}(BO; Z_2) & \to & H_{2^r}(RP^\infty; Z_2) \end{array}$$

(see the discussion before Proposition 2.1), it is sufficient to find a manifold M^{2^r} such that $w_1^{2^r}([M]) \neq 0$. RP^{2^r} will do.

Now let $W_* = \operatorname{im}\{\pi_*(X^1 \wedge MSO) \to \pi_*(MO)\}$ (see **[11]** for an equivalent definition). By Lemma 3.2, $W_* \cong \pi_*(X^1 \wedge MSO)$, and there is a short exact sequence

$$0 \to E^\infty_{0,q} \to W_q \to E^\infty_{1,q-1} \to 0.$$

W_* is a subalgebra of MO_*, since $(E^\infty_{1,q}) \cdot (E^\infty_{1,q'}) = 0$.

PROPOSITION 3.3. *The Poincaré series of* W_* *is*

$$\prod_{\substack{r \neq 2^s \\ r \neq 2^s - 1}} \frac{1}{1 - t^r} \cdot \prod_{\substack{r = 2^s \\ s > 1}} \frac{1}{1 - t^{2r}}.$$

PROOF. Since the exact sequence above is one of Z_2 modules, the Poincaré series of W_* is the same as that for the subalgebra $\{E^\infty_{\varepsilon,*}, \varepsilon = 0, 1\}$ of E^∞. Call this series Π. By Lemmas 3.1 and 3.2, the Poincaré series for E^∞ is $\Pi \cdot \Pi_{r>1}(1 + t^{2^r})$. This, of course, is the Poincaré series for MO_*, so we have

$$\prod_{r \neq 2^s - 1} \frac{1}{1 - t^r} = \Pi \cdot \prod_{r > 1} (1 + t^{2^r}).$$

The result now follows since $(1 + t^{2^r})(1 - t^{2^r}) = (1 - t^{2 \cdot 2^r})$.

4. We have now determined the size of the subalgebra W_*. It is the object of this section to find its algebraic structure.

Let $x_n \in H_n(RP^\infty; Z_2)$ be the nonzero element and also its image in $H_n(BO; Z_2)$ under the inclusion of $RP^\infty = BO(1)$. It is well known **[5]** that $H_*(BO; Z_2) \cong Z_2[x_n; n \geq 1]$, a Z_2 polynomial algebra. We now study the image of $H_*(BSO; Z_2)$ in $H_*(BO; Z_2)$.

LEMMA 4.1. $\alpha \in H_r(BO; Z_2)$ *is in the image of* $H_r(BSO; Z_2)$ *iff every element of* $H^r(BO; Z_2)$ *that contains* w_1 *as a factor is* 0 *on* α.

PROOF. Let $I = \{\alpha | \langle \xi \cup w_1, \alpha \rangle = 0$ for all $\xi\}$. Certainly the image of $H_*(BSO; Z_2)$ is contained in I. By checking ranks one sees that this is an equality.

Let d: $H_r(BO; Z_2) \to H_{r-1}(BO; Z_2)$ be defined by $d(u) = w_1 \cap u$.

PROPOSITION 4.2. d *is a derivation,* $d(x_n) = x_{n-1}$ *and the sequence*

$$0 \to H_*(BSO; Z_2) \to H_*(BO; Z_2) \xrightarrow{d} H_*(BO; Z_2) \to 0$$

is exact.

PROOF. Since w_1 is primitive, it is easy to see that d is a derivation.

Since $w_1 \cap x_n = x_{n-1}$ in RP^∞, it holds also in BO. $\langle \xi \cup w_1, \alpha \rangle = 0$ for all ξ iff $\langle \xi, w_1 \cap \alpha \rangle = 0$ for all ξ. Hence there is exactness in the middle. The first map is a monomorphism since $BO \cong BSO \times RP^\infty$. Given $\alpha \in H_*(BO; Z_2)$, define $\beta \in H_*(BO; Z_2)$ by $\langle \xi, \beta \rangle = \langle \xi w_1, \alpha \rangle$. Then $d\beta = \alpha$.

It is easy to see, for example, that the image of $H_*(BSO; Z_2)$ is the subalgebra generated in the first few dimensions by x_1^2, $x_1^3 + x_1x_2 + x_3$, x_2^2, $x_5 + x_1$, $x_4 + x_2x_3 + x_2^2x_1$, $x_2^3 + x_2x_4 + x_1x_2x_3 + x_1^2x_4 + x_1x_5 + x_6$, and that for each $r \geqslant 1$

$$x_{2r+1} + x_1x_{2r} + \dots + x_rx_{r+1} + x_r^2x_1$$

is in the image. We will not use this information.

Note that $X^1 \wedge MSO = (S^0 \cup_2 e^1) \wedge MSO$, i.e., MSO with Z_2 coefficients. Let β: $X^1 \wedge MSO \to S(X^1 \wedge MSO)$ be the Bockstein. This induces a homomorphism which we also call β, β: $W_n \to W_{n-1}$.

THEOREM 4.3. *There are polynomial generators* $u_n \in MO_*$, $n \neq 2^s - 1$ *such that*

(a) $u_n \in W_n$, $n \neq 2^s$,

(b) $\beta u_{2n} = u_{2n-1}$, $\beta u_{2n-1} = 0$, $n \neq 2^s$,

(c) $u_{2n-1} \in$ image MSO_*,

(d) $(u_{2^r})^2 \in$ image MSO_*.

COROLLARY 4.4. $W_* \cong Z_2[u_{2n}, u_{2n-1}, (u_{2^r})^2; n \neq 2^s, r \geqslant 0]$.

PROOF OF COROLLARY 4.4. By 4.3, W_* contains this subalgebra of MO_*. But by 3.3 they are equal.

PROOF OF THEOREM 4.3. (d) is immediate since u_{2^r} is represented by RP^{2^r} and $RP^n \times RP^n$ is cobordant to CP^n by an easy Stiefel-Whitney number argument [**11**]. Suppose now that $n \neq 2^s$ $[RP^{2n}]$ is indecomposable in MO_{2n} [**8**]. In fact $h([RP^{2n}]) \in H_{2n}(MO; Z_2)$ is indecomposable since the primitive in $H^{2n}(BO; Z_2)$ is nonzero on it and the Thom isomorphism $H_*(MO; Z_2) \cong H_*(BO; Z_2)$ is a ring isomorphism.

Claim 1. There is an indecomposable $u_{2n} \in W_{2n}$ such that $h(u_{2n}) \in H_{2n}(MO; Z_2)$ is indecomposable ($n \neq 2^s$). This follows since in dimension $2n$ ($n \neq 2^s$!) there are no indecomposables in $E^\infty_{p,2n-p}$ for $p > 1$. Thus by adding decomposables to $[RP^{2n}]$ we can lower the filtration to 1.

Claim 2. $h(u_{2n}) = \bar{x}_{2n} + f(\bar{x}_1, \dots, \bar{x}_{2n-2})$, where $U \cap \bar{x}_r = x_r$, $\bar{x}_r \in H_r(MO; Z_2)$, U = Thom class.

For this we need

LEMMA 4.5. $\beta\bar{x}_n = n\bar{x}_{n-1}$ *where* β: $H_r(\ ; Z_2) \to H_{r-1}(\ ; Z_2)$ *is the* Z_2 *Bockstein.*

PROOF. Since $\beta x_n = (n-1)x_{n-1}$ in RP^∞, this also holds in BO. Since $\beta = \mathrm{Sq}_1$, we have

$$\begin{aligned}(n-1)x_{n-1} = \beta x_n &= \beta(U \cap \bar{x}_n) = \mathrm{Sq}_1(U \cap \bar{x}_n)\\ &= (\mathrm{Sq}^1 U) \cap \bar{x}_n + U \cap \mathrm{Sq}_1\bar{x}_n\\ &= w_1 \cap (U \cap \bar{x}_n) + U \cap \mathrm{Sq}_1\bar{x}_n\\ &= x_{n-1} + U \cap (\mathrm{Sq}_1\bar{x}_n).\end{aligned}$$

Since $U \cap$ is an isomorphism, the result follows. To prove Claim 2, write

$$h(u_{2n}) = \bar{x}_{2n} + \alpha \bar{x}_1 \bar{x}_{2n-1} + f(\bar{x}_1, \ldots, \bar{x}_{2n-2}).$$

Such a formula exists since $h(u_{2n})$ is indecomposable. But $\beta h(u_{2n}) = 0$, so $\alpha = 0$.

Claim 3. $u_{2n} \notin$ image MSO_*.

If so $h(u_{2n}) \in$ image $H_*(MSO; Z_2)$. But

$$dh(u_{2n}) = \bar{x}_{2n-1} + f(\bar{x}_1, \ldots, \bar{x}_{2n-2}) \neq 0.$$

Claim 4. $H_*(X^1 \wedge MSO; Z_2) \cong \operatorname{im}\{H_*(X^1 \wedge MSO; Z_2) \to H_*(MO; Z_2)\}$ is a subalgebra and consists of all elements which can be written in the form $a + b\bar{x}_1$ with $d(a) = d(b) = 0$. This follows from the decomposition

$$H_*(MO; Z_2) \cong H_*(MSO; Z_2) \otimes H_*(X; Z_2),$$

$$H_*(X^1 \wedge MSO; Z_2) \cong H_*(MSO; Z_2) \otimes H_*(X^1; Z_2).$$

It is a subalgebra since $(a + b\bar{x}_1)(a' + b'\bar{x}_1) = (aa' + bb'\bar{x}_1^2) + (ab' + a'b)\bar{x}_1$.

Claim 5.

$$\beta_*: H_r(X^1 \wedge MSO; Z_2) \to H_{r-1}(X^1 \wedge MSO; Z_2)$$

operates by $\beta_*(a + bx_1) = b$.

N.B. This is the homomorphism induced by β: $X^1 \wedge MSO \to S(X^1 \wedge MSO)$ and not the Z_2 homology Bockstein.

This is clear from the definitions.

Claim 6. $\beta u_{2n} \in \pi_{2n-1}(X^1 \wedge MSO) \subset \pi_{2n-1}(MO)$ is indecomposable.

PROOF. We show that $h(\beta u_{2n})$ is indecomposable.

$$h(\beta u_{2n}) = \beta_*(h(u_{2n})) = \beta_*(\bar{x}_{2n} + f(\bar{x}_1, \ldots, \bar{x}_{2n-2})).$$

If we write $\bar{x}_{2n} + f(\bar{x}_1, \ldots, \bar{x}_{2n-2}) = a + \bar{x}_1 b$,

$$a = \bar{x}_{2n} + \varepsilon \bar{x}_1 \bar{x}_{2n-1} + g(\bar{x}_1, \ldots, \bar{x}_{2n-2}),$$
$$b = \delta \bar{x}_{2n-1} + h(\bar{x}_1, \ldots, \bar{x}_{2n-2}).$$

Then $\varepsilon = \delta$. Since $da = 0$, $\varepsilon = 1$. Thus $\beta_*(h(u_{2n})) = \bar{x}_{2n-1} + h(\bar{x}_1, \ldots, \bar{x}_{2n-2})$ which is indecomposable. Now let $u_{2n-1} = \beta u_{2n}$. Since β factors through $S^1 \wedge MSO$, $\beta u_{2n-1} = 0$ and (c) follows. This completes the proof of Theorem 4.3.

5. In this section we will exploit Corollary 4.4 to calculate MSO_*.

$$W_* \cong \pi_*(X^1 \wedge MSO) \cong MSO_*(*; Z_2).$$

LEMMA 5.1. $H(MSO_*(*; Z_2), \beta) \cong Z_2[(u_{2n})^2]$.

PROOF. Since $MSO_*(*; Z_2)$ is the tensor product of differential algebras, it is sufficient, by Künneth theorem, to calculate the homology of each. The homology of $Z_2[u_{2n}, u_{2n-1}]$ is $Z_2[(u_{2n})^2]$ and the homology of $Z_2[(u_{2^r})^2]$ is $Z_2[(u_{2^r})^2]$, so the result follows.

LEMMA 5.2. *Suppose $E_*(X)$ has no torsion prime to p and ρ: $E_*(X) \to E_*(X; Z_p)$ induces an onto map $E_*(X) \to H(E_*(X; Z_p), \beta)$. Then $E_*(X) \otimes Z_p \cong \ker \beta$ and* $\mathrm{TORS}\, E_*(X) \cong \mathrm{im}\, \beta$. *All torsion is simple. Thus $E_*(X) \cong (Z)^a \oplus (Z_p)^b$ where* $b = \mathrm{rank\ im}\, \beta$ *and* $a = \mathrm{rank\ ker}\, \beta - \mathrm{rank\ im}\, \beta = \mathrm{rank}\, H(E_*(X; Z_p), \beta)$.

PROOF. Consider the exact sequence

$$\cdots E_*(X) \xrightarrow{\rho} E_*(X; Z_p) \xrightarrow{\delta} E_*(X) \xrightarrow{p} E_*(X) \xrightarrow{\rho} \cdots.$$

We first show that $\rho|\mathrm{im}\, \delta$ is a monomorphism. Suppose $\rho\delta(u) = 0$. Since $\beta = \rho\delta$, $u = \rho(v) = \beta(w)$. Thus $\delta(u) = 0$. Now $E_*(X) \otimes Z_p \cong \mathrm{im}\, \rho = \ker \delta = \ker \beta$ since ρ: $\ker \delta \to \ker \beta$ is an isomorphism. We now show that all torsion is simple. Let $\alpha \in \mathrm{TORS}\, E_*(X)$ with $p^r\alpha = 0$. Then $p^{r-1}\alpha = \delta(\gamma)$. If $r > 1$, $0 = \rho(p^{r-1}\alpha) = \rho\delta\gamma$. Thus $\delta\gamma = 0$ so $p^{r-1}\alpha = 0$. Thus it follows that $p\alpha = 0$. Finally $\mathrm{TORS}\, E_*(X) = \ker p = \mathrm{im}\, \delta \cong \mathrm{im}\, \rho\delta = \mathrm{im}\, \beta$.

Now we apply Lemma 5.2 with $p = 2$, and $E = MSO$, and $X = *$.

LEMMA 5.3. *$MSO_* \to Z_2[(u_{2n})^2]$ is onto.*

PROOF. $(RP^{2n})^2 \sim CP^{2n}$, as already observed, and u_{2n-1} is orientable. Thus any square in MO_* is cobordant to an orientable manifold.

LEMMA 5.4. *β: $W_* \to W_*$ is a derivation.*

PROOF. According to Araki and Toda [2], $E_*(\ ; Z_2)$ has a multiplication with β a derivation if η induces the 0 homomorphism in E_*. This is clear when $E = MSO$ since η represents a framed circle which is an oriented boundary. According to Komornicki [4], the multiplications in $E_*(\ ; Z_2)$ are in 1-1 correspondence with $E_2(*; Z_2)$ if one exists. In our case this is 0 by inspection of the spectral sequence. Hence we are done. Since all torsion in W_* is 2 torsion [7], we conclude

THEOREM 5.5. *Let $W_* = Z_2[x_{2n}, x_{2n-1}, (x_{2^r})^2, r \geq 0, n \neq 2^s]$ and $\beta(x_{2n}) = x_{2n-1}$, $\beta(x_{2^r})^2 = 0$, β a derivation. Then $MSO_* \otimes Z_2 \cong \ker \beta$ and $\mathrm{TORS}(MSO_*) \cong \mathrm{im}\, \beta$ since all torsion is of order 2. Thus $MSO_n = (Z)^a \oplus (Z_2)^b$ where $b = \mathrm{rank}(\mathrm{im}\, \beta)_n$, $a = \mathrm{rank}(\ker \beta)_n - \mathrm{rank}(\mathrm{im}\, \beta)_n = \mathrm{rank}\, Z[CP^{2n}; n \geq 1]$.*

6. It is reasonable to apply this method to other bordism theories. The case that is most analogous is MSU. We have $BSU \times CP^\infty \cong BU$ and $MSU \wedge X \cong MU$ where $S^2X \cong CP^\infty$. In this case it is easy to see that $E^3 = E^\infty$ (the d_2 differential is determined by $d_2(x_2) = \eta$, $x_2 \in H_2(CP^\infty)$). It seems that one could pursue this method, using characteristic number arguments to control filtration. Whether it would be much of an improvement over [3], [12], [8] is dubious.

Appendix. In [9], Taylor gives another algebraic discussion of MSO. He calculates $MSOZ_2$ via the Atiyah-Hirzebruch spectral sequence for $(MSOZ_2)_*(HZ)$ and relating this to the spectral sequence for $(MSOZ_4)_*(HZ)$ he concludes that MSO_* has no elements of order 4 and that all k-invariants for $MSOZ_{(2)}$ are trivial (where $Z_{(2)}$ is the rationals with odd denominators). It is not hard to use this information to calculate MSO_* (using Lemmas 5.2 and 5.4).

BIBLIOGRAPHY

1. J. F. Adams, *Stable homotopy and generalized homology*, Univ. of Chicago Press, Chicago, Ill., 1974.

2. S. Araki and H. Toda, *Multiplicative structures in* mod *q cohomology theories*. I, Osaka J. Math. **2** (1965), 71–115.

3. P. E. Conner and E. E. Floyd, *Torsion in SU-bordism*, Mem. Amer. Math. Soc. No. 60 (1966).

4. Wojeciech Komornicki, *Multiplications in two-cell complexes and related spectra*, Thesis, Univ. of Illinois, 1976.

5. Arunas Liulevicius, *Lecture notes on characteristic classes*, Aarhus Universitet, 1968.

6. J. P. May, *E_∞ ring spaces and E_∞ ring spectra*, Lecture Notes in Math., vol. 577, Springer-Verlag, Berlin, 1977.

7. J. W. Milnor, *On the cobordism ring* Ω^*, Notices Amer. Math. Soc. **5** (1958), 457.

8. R. E. Stong, *Notes on cobordism theory*, Princeton Univ. Press, Princeton, N. J., 1968.

9. L. R. Taylor, *2-local cobordism theories*, J. London Math. Soc. (2) **14** (1976), 303–308.

10. R. Thom, *Quelques properiétés globales des variétés différentiables*, Comment. Math. Helv. **28** (1954), 17–86.

11. C. T. C. Wall, *Determination of the cobordism ring*, Ann. of Math. (2) **72** (1960), 292–311.

12. ______, *Addendum to a paper of Conner and Floyd*, Proc. Cambridge Philos. Soc. **62** (1966), 171–175.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CHICAGO AT CHICAGO CIRCLE, CHICAGO, ILLINOIS 60680