

ISOTOPY TYPES OF KNOTS OF CODIMENSION TWO*

BY

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ABSTRACT. In this paper the classification of n -dimensional knots in S^{n+2} , bounding r -connected manifolds, where $3r > n + 1 > 6$, in terms of stable homotopy theory is suggested.

The problem of isotopy classification is the fundamental one of knot theory. It was solved by A. Haefliger and J. Levine for knots of codimension ≥ 3 . The first step in reduction of a classification of knots of codimension two to a homotopy problem was made by R. Lashof and J. Shaneson [6]. They showed that in the class of n -dimensional knots with group Z for $n \geq 5$ there are at most two different knots having homotopy equivalent exteriors. In 1970 J. Levine [10] gave an algebraic classification of $(2q - 1)$ -dimensional knots in S^{2q+1} bounding $(q - 1)$ -connected manifolds. It turned out that the only invariant determining the isotopy type of such knots is the Seifert matrix considered to within S -equivalence. Later Trotter [16] and C. Kearton [5] obtained a classification of knots, studied by J. Levine, in terms of Blanchfield pairing. C. Kearton has partially analyzed the more difficult problem of classification of $2q$ -dimensional knots in S^{2q+2} bounding $(q - 1)$ -connected manifolds.

In the present paper the classification of a wider class of higher-dimensional knots is obtained. It is the class of n -dimensional knots in S^{n+2} bounding r -connected manifolds, where $3r \geq n + 1 \geq 6$. The main result of this paper is the construction of a one-to-one correspondence between the set of isotopy types of such knots and some set described in purely homotopic terms.

The plan of the paper is as follows:

In §1 submanifolds of the sphere of codimension one are considered. Such a submanifold is assigned some pairing in the sense of the Spanier-Whitehead theory. This pairing describes the homotopy linking of the submanifold with its copy translated in the direction of the positive normal field. This pairing induces the usual Seifert pairing on middle dimensional homology groups and is called the homotopy Seifert pairing. The main result of §1 is the construction of a one-to-one correspondence between the set of isotopy types of such submanifolds and the set of homotopy pairings.

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§2 deals with a situation in which nonisotopic Seifert manifolds are bounded by isotopic knots. I call this situation contiguity. In this section there is found a necessary and sufficient condition for the homotopy Seifert pairings to correspond to contiguous submanifolds.

In §3 it is proved that two submanifolds of a sphere bound the same knot if and only if they may be connected by a sequence of contiguities. This result together with the previous theorems bring us to the classification of knots.

From these general results an algebraic classification of some classes of knots may be obtained by standard homotopy methods. This is illustrated in §3, where the case of simple odd-dimensional knots is considered and algebraic classification in terms of Seifert matrices is obtained. This classification is essentially the same as J. Levine's [10]; the only difference is that we obtain the equivalence relation between Seifert matrices in another form. The results on algebraic classification of other classes of knots will be stated in a separate paper.

The terminology is given in the smooth category although the results of the paper are true in the piecewise linear category too.

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1. Classification of imbeddings of Seifert manifolds. An n -dimensional knot is a pair $K = (S^{n+2}, k^n)$, where S^{n+2} is an oriented sphere and k^n is its oriented n -dimensional submanifold which is a homotopy sphere. A Seifert manifold of a knot K is any compact connected oriented $(n+1)$ -dimensional submanifold $V \subset S^{n+2}$ with $\partial V = k$. In this section it is proved that the isotopy type of V is determined by the homotopy type of V and some homotopy pairing. Here also necessary and sufficient conditions are determined under which the finite complex with the given homotopy pairing may be realized by the $(n+1)$ -dimensional submanifold $V \subset S^{n+2}$ for which ∂V is the homotopy sphere.

1.1. Let V be a connected oriented $(n+1)$ -dimensional submanifold of the oriented sphere S^{n+2} . Suppose that the boundary ∂V is nonempty. Let Y be the closure of the complement of a tubular neighborhood of V in S^{n+2} . Let $u: V \wedge Y \rightarrow S^{n+1}$ be the canonical pairing (see, for example, [15, Chapter 3]) which is the Spanier-Whitehead duality [13]. Denote by $i_+: V \rightarrow Y$ the mapping which is given by small translation along the field of positive normals to V . By homotopy Seifert pairing of V we shall call the composition

$$\theta: V \wedge V \xrightarrow{1 \wedge i_+} V \wedge Y \xrightarrow{u} S^{n+1}.$$

It is clear that θ is determined by imbedding $V \subset S^{n+2}$ uniquely up to homotopy.

If $n = 2q - 1$, then homotopy pairing θ defines homology pairing $H_q(V) \otimes H_q(V) \rightarrow \mathbb{Z}$ by the formula $z_1 \otimes z_2 \mapsto \theta_*(z_1 \wedge z_2) \in H_{2q}(S^{2q}) = \mathbb{Z}$. This homology pairing coincides with Seifert pairing [9], [12].

1.2. **THEOREM.** *Let V_1 and V_2 be two compact oriented $(n+1)$ -dimensional submanifolds of sphere S^{n+2} with ∂V_1 and ∂V_2 homotopy spheres. Let $\theta_i: V_i \wedge V_i \rightarrow S^{n+1}$, $i = 1, 2$, be corresponding homotopy Seifert pairings. Suppose that the manifolds V_1 and V_2 are r -connected, where $3r \geq n + 1$, $n \geq 5$. If there is a homotopy*

equivalence $f: V_1 \rightarrow V_2$ for which $\theta_2 \circ (f \wedge f)$ is homotopic to θ_1 , then there exists isotopy of sphere S^{n+2} which transfers V_1 on V_2 with preservation of orientations.

1.3. THEOREM. Let K be a finite r -connected k -dimensional complex and $\theta: K \wedge K \rightarrow S^{n+1}$ be a continuous map. If $2n > 3k$, $2k \leq n + r$, $n \geq 4$, $r \geq 1$, then there exists an $(n + 1)$ -dimensional oriented submanifold $V \subset S^{n+2}$ with $\pi_1(\partial V) = 1$ and homotopy equivalence $g: V \rightarrow K$ such that $\theta \circ (g \wedge g): V \wedge V \rightarrow S^{n+1}$ is homotopic to the homotopy Seifert pairing of V .

The proofs of Theorems 1.2 and 1.3 will be given at the end of this section.

When we shall apply Theorem 1.3 it will be important to know under which conditions the homotopy pairing $\theta: K \wedge K \rightarrow S^{n+1}$ is realized by submanifold V^{n+1} for which ∂V is the homotopy sphere. The following theorem answers this question.

1.4. THEOREM. Let V be a connected $(n + 1)$ -dimensional oriented submanifold of sphere S^{n+2} with $\partial V \neq \emptyset$, and let $\theta: V \wedge V \rightarrow S^{n+1}$ be its homotopy Seifert pairing. The boundary ∂V is a homology sphere if and only if the following pairing

$$\theta + (-1)^{n+1} \theta': V \wedge V \rightarrow S^{n+1}$$

is the Spanier-Whitehead duality. Here θ' is the composition of the map $T: V \wedge V \rightarrow V \wedge V$, which transposes the coordinates, and θ ; the signs plus and minus are understood as operations in the cohomotopy group $\pi^{n+1}(V \wedge V)$.

REMARK. This cohomotopy group exists since $H^i(V \wedge V) = 0$ for $i > 2n$.

PROOF OF THEOREM 1.4. Let Y be a complement of an open tubular neighborhood of V in S^{n+2} and let $i_+: V \rightarrow Y$ and $i_-: V \rightarrow Y$ be given by translations in the directions of positive and negative normals to V , respectively.

Let us first prove that ∂V is the homology sphere if and only if the homomorphisms

$$i_{+*} - i_{-*}: H_k V \rightarrow H_k Y$$

are isomorphisms for $k > 0$. Consider the isomorphism

$$\Psi: H_k Y \rightarrow H_k(V, \partial V), \quad 0 < k < n + 1,$$

which is the composition

$$H_k Y \xrightarrow{\partial^{-1}} H_{k+1}(S^{n+2}, Y) \xrightarrow{\approx} H_{k+1}(N, \partial N) \xrightarrow{\approx} H_k(V, \partial V),$$

where the first homomorphism is reverse to boundary homomorphism, the second is an isomorphism of excision (here N is the tubular neighborhood of V in S^{n+2}) and the third is an isomorphism which exists by virtue of $(N, \partial N) = (V, \partial V) \times (I, \partial I)$.

The composition

$$H_k V \xrightarrow{i_{+*} - i_{-*}} H_k Y \xrightarrow{\Psi} H_k(V, \partial V)$$

coincides with the homomorphism induced by inclusion $V \rightarrow (V, \partial V)$. Since Ψ is an isomorphism, then $i_{+*} - i_{-*}$ is an isomorphism if and only if the inclusion

$V \rightarrow (V, \partial V)$ induces an isomorphism of k -dimensional homologies for $0 < k < n + 1$. But this is equivalent to ∂V being a homology sphere.

Consider the following diagram:

$$\begin{array}{ccccc}
 & & & & 1 \wedge i_+ \\
 & & & & \rightarrow \\
 T \nearrow & & V \wedge V & & V \wedge Y \\
 & & \downarrow i_- \wedge 1 & & \downarrow u \\
 & & Y \wedge V & \xrightarrow{v} & S^{n+1} \\
 & & \downarrow T & & \downarrow (-1)^n \\
 1 \wedge i_- \searrow & & V \wedge Y & \xrightarrow{u} & S^{n+1}
 \end{array}$$

Here u and v are canonical pairings which are Spanier-Whitehead dualities and $(-1)^n$ denotes the map of degree $(-1)^n$. This diagram is homotopically commutative. Therefore $\theta' = \theta \circ T = u \circ (1 \wedge i_+) \circ T \sim (-1)^n u \circ (1 \wedge i_-)$. If $s \in H^{n+1}S^{n+1}$ is a generator and $z \in H_k V$, then

$$\begin{aligned}
 [\theta + (-1)^{n+1}\theta']^*s/z &= \theta^*s/z + (-1)^{n+1}\theta'^*s/z \\
 &= (1 \wedge i_+)^* \circ u^*s/z - (1 \wedge i_-)^* \circ u^*s/z \\
 &= u^*s/i_{+*}z - u^*s/i_{-*}z = u^*s/(i_{+*} - i_{-*})z.
 \end{aligned}$$

Thus $[\theta + (-1)^{n+1}\theta']^*s/z = u^*s/(i_{+*} - i_{-*})z$. Since u is a Spanier-Whitehead duality, then it follows from this formula that the map $H_k V \rightarrow H^{n+1-k}V$, $0 < k < n + 1$ given by the formula

$$z \mapsto [\theta + (-1)^{n+1}\theta']^*s/z$$

is an isomorphism if and only if $i_{+*} - i_{-*}$ is an isomorphism.

This proves Theorem 1.4.

1.5. For the proofs of Theorems 1.2 and 1.3 some facts of the thickening theory [17] are needed.

Let K^k be a finite connected CW-complex of dimension k with base point $*$. Let M^m be a compact manifold with base point $*$ $\in \partial M$ and fixed orientation of the tangent space to M in the point $*$. Suppose that $m \geq k + 3$ and the inclusion $i: \partial M \rightarrow M$ induces an isomorphism $\pi_1(\partial M) \rightarrow \pi_1(M)$. The simple homotopy equivalence $\varphi: (K, *) \rightarrow (M, *)$ is called the m -dimensional thickening of K . Two m -dimensional thickenings $\varphi_1: (K, *) \rightarrow (M_1, *)$ and $\varphi_2: (K, *) \rightarrow (M_2, *)$ are called equivalent if there is a diffeomorphism $h: M_1 \rightarrow M_2$ preserving the base points and the orientation in them, and such that

$$h \circ \varphi_1 \sim \varphi_2: (K, *) \rightarrow (M_2, *).$$

1.6. THEOREM. Let M^m be a manifold, K^k be a finite CW-complex, $f: K \rightarrow M$ be some $(2k - m + 1)$ -connected map and $k \leq m - 3$. Then there exist a compact submanifold N^m of M^m with $\pi_1(\partial N) = \pi_1(N)$ and a simple homotopy equivalence $g: K \rightarrow N$ such that $g \sim f: K \rightarrow M$ and this homotopy preserves base points.

1.7. THEOREM. Let $f_1: K \rightarrow N_1^m$ and $f_2: K \rightarrow N_2^m$ be two simple homotopy equivalences where N_1 and N_2 are submanifolds of manifold M^m with $\pi_1(\partial N_1) = \pi_1(N_1)$, $\pi_1(\partial N_2) = \pi_1(N_2)$ and $k = \dim K \leq m - 3$, $m \geq 6$. If f_1 and f_2 are homotopic in $M \text{ rel}^*$ and the compositions

$$K \xrightarrow{f_\nu} N_\nu \xrightarrow{\subset} M, \quad \nu = 1, 2,$$

are $(2k - m + 2)$ -connected, then there exists an isotopy $h_t: M \rightarrow M$ such that $h_0 = \text{id}$, $h_t(*) = 1$, $h_1(N_1) = N_2$ and the diagram

$$\begin{array}{ccc} & K & \\ f_1 \swarrow & & \searrow f_2 \\ N_1 & \xrightarrow{h_1} & N_2 \end{array}$$

is homotopy commutative.

Theorem 1.6 is the first part of Wall's Embedding theorem [17, p. 76]. Theorem 1.7 makes precise the second part of the Wall theorem with regard to the results of [3], [11].

The simple homotopy equivalence $g: K \rightarrow N$ which existence is affirmed in Theorem 1.6 is called the thickening induced by the map $f: K \rightarrow M$. Theorem 1.7 means uniqueness of induced thickening.

For the proofs of Theorems 1.2 and 1.3 we also need to use the next statement.

1.8. PROPOSITION. Let N^{n+2} be a compact submanifold of sphere S^{n+2} having the homotopy type of a finite k -dimensional r -connected CW-complex, where $r \geq 1$, $n \geq 4$, $n > k$. Let $Y = \text{cl}(S^{n+2} - N)$. The inclusions $i: \partial N \rightarrow N$ and $j: \partial N \rightarrow Y$ determine the map $\psi: \partial N \rightarrow N \times Y$, $\psi(x) = (i(x), j(x))$. The map $\psi_*: [L, \partial N] \rightarrow [L, N] \times [L, Y]$ induced by ψ is surjective if $\dim L \leq n + r - k + 1$ and bijective if $\dim L \leq n + r - k$.

PROOF. It is enough to show that the spaces N , Y , ∂N are simply connected and the map ψ induces isomorphism of homology groups in dimensions $\leq n + r - k + 1$. N is simply connected by hypotheses. Since $n + 2 \geq 6$, then N has a handle decomposition with handles of index $\leq k$. The dual decomposition begins on $\partial N \times [0, 1]$ and has the handles of index $> n + 2 - k \geq 3$. Therefore $\pi_1(\partial N) = \pi_1(N) = 1$. It follows now from Van Kampen's theorem that $\pi_1(Y) = 1$.

Notice that Y is $(n - k)$ -connected since $H_s Y = H^{n+1-s} N = 0$, if $s \leq n - k$. Therefore any class $z' \in H_s(N \times Y)$ for $s \leq (n - k + 1) + (r + 1)$ is uniquely presented as $z' = z_1 \times 1 + 1 \times z_2$, where $z_1 \in H_s N$ and $z_2 \in H_s Y$. If $z \in H_s(\partial N)$, then it is clear that $\psi_* z = i_* z \times 1 + 1 \times j_* z$. It follows from the Mayer-Vietoris sequence of the pair $\{N, Y\}$ that we have isomorphism

$$\tilde{H}_s(\partial N) \xrightarrow{i_* \otimes j_*} \tilde{H}_s N \oplus \tilde{H}_s Y.$$

Consequently $\psi_*: H_s(\partial N) \rightarrow H_s(N \times Y)$ is an isomorphism for $s \leq n + r - k + 1$.

The proposition is proved.

1.9. PROOF OF THEOREM 1.2. Denote $N_\nu = V_\nu \times [0, 1]$, $\nu = 1, 2$. Identify V_ν with $V_\nu \times 0 \subset N_\nu$ and denote by i_ν the corresponding inclusion $V_\nu \rightarrow N_\nu$, $\nu = 1, 2$. The imbedding of V_ν in S^{n+2} may be extended to imbedding $N_\nu \rightarrow S^{n+2}$ such that for each point $x \in V_\nu$ the curve $\{(x, t); t \in [0, 1]\}$ goes out from $x = (x, 0)$ along the direction of the negative normal to V_ν , $\nu = 1, 2$.

Manifolds V_ν are simply connected and for $s > n - r$ we have $H^s V_\nu = H_{n+1-s}(V_\nu, \partial V_\nu) = \tilde{H}_{n+1-s} V_\nu = 0$, $\nu = 1, 2$. Here we have used that ∂V_ν is a homotopy sphere. Thus there exists a finite CW-complex K of dimension $\leq n - r$ and homotopy equivalence $\varphi_1: K \rightarrow V_1$. Denote the composition $\varphi_2 = f \circ \varphi_1$ by $\varphi_2: K \rightarrow V_2$.

The maps $i_1 \circ \varphi_1: K \rightarrow N_1$ and $i_2 \circ \varphi_2: K \rightarrow N_2$ are $(n + 2)$ -dimensional thickenings of complex K . According to Theorem 1.7 (here all conditions of this theorem are satisfied) there exists isotopy h_t of sphere S^{n+2} such that $h_0 = \text{id}$, $h_1(N_1) = N_2$ and $h_1 \circ i_1 \circ \varphi_1$ is homotopic to $i_2 \circ \varphi_2$ in N_2 .

I want to prove that manifolds V_2 and $h_1(V_1)$ are isotopic on ∂N_2 . Since V_2 is the thickening induced by the map

$$\alpha_2: K \xrightarrow{\varphi_2} V_2 \xrightarrow{\subset} \partial N_2,$$

and $h_1(V_1)$ is the thickening induced by the map

$$\alpha_1: K \xrightarrow{\varphi_1} V_1 \xrightarrow{h_1} h_1(V_1) \xrightarrow{\subset} \partial N_2,$$

then by virtue of Theorem 1.7 it is enough to show that each of the maps α_1 and α_2 are $2(n - r) - (n + 1) + 2$ -connected and $\alpha_1 \sim \alpha_2: K \rightarrow \partial N_2$. The connectivity conditions for α_1 and α_2 are obviously satisfied since $3r > n + 1$.

Let us show that $\alpha_1 \sim \alpha_2$. By virtue of Proposition 1.8 it is enough to prove that

$$(1) i \circ \alpha_1 \sim i \circ \alpha_2 \text{ in } N_2,$$

$$(2) j \circ \alpha_1 \sim j \circ \alpha_2 \text{ in } Y,$$

where $i: \partial N_2 \rightarrow N_2$ and $j: \partial N_2 \rightarrow Y$ are inclusions and $Y = S^{n+2} - \text{int } N_2$. The relation (1) can be easily verified:

$$i \circ \alpha_2 = i_2 \circ \varphi_2 \sim h_1 \circ i_1 \circ \varphi_1 = i \circ \alpha_1.$$

Let us verify the relation (2). Let N be obtained from N_2 by removing some collar of the boundary ∂N_2 . By $r: N_2 \rightarrow \partial N$ we denote a natural retraction and by $u: N \wedge Y \rightarrow S^{n+1}$ a canonical pairing which is a Spanier-Whitehead duality. It is clear that the composition $V_2 \wedge V_2 \xrightarrow{i \wedge j} N_2 \wedge Y \xrightarrow{r \wedge 1} N \wedge Y \xrightarrow{u} S^{n+1}$ is homotopic to homotopy Seifert pairing θ_2 of manifold V_2 . Analogously the composition

$$V_1 \wedge V_1 \xrightarrow{h_1 \wedge h_1} h_1(V_1) \wedge h_1(V_1) \xrightarrow{i \wedge j} N_2 \wedge Y \xrightarrow{r \wedge 1} N \wedge Y \xrightarrow{u} S^{n+1}$$

is homotopic to θ_1 . Therefore

$$\theta_2 \circ (\varphi_2 \wedge \varphi_2) \sim u \circ (r \circ i \circ \varphi_2 \wedge j \circ \varphi_2),$$

$$\theta_1 \circ (\varphi_1 \wedge \varphi_1) \sim u \circ (r \circ i \circ h_1 \circ \varphi_1 \wedge j \circ h_1 \circ \varphi_1).$$

On the other hand, since $\varphi_2 = f \circ \varphi_1$, then

$$\theta_2 \circ (\varphi_2 \wedge \varphi_2) = \theta \circ (f \wedge f) \circ (\varphi_1 \wedge \varphi_1) \sim \theta_1 \circ (\varphi_1 \wedge \varphi_1).$$

Thus

$$u \circ (r \circ i \circ \varphi_2 \wedge j \circ \varphi_2) \sim u \circ (r \circ i \circ h_1 \circ \varphi_1 \wedge j \circ h_1 \circ \varphi_1).$$

We have

$$\begin{aligned} i \circ \varphi_2 &= i \circ \alpha_1 \sim i \circ \alpha_2 = i \circ h_1 \circ \varphi_1, \\ j \circ \varphi_2 &= j \circ \alpha_2, \quad j \circ h_1 \circ \varphi_1 = j \circ \alpha_1. \end{aligned}$$

Consequently

$$u \circ (\beta \wedge j \circ \alpha_2) \sim u(\beta \wedge j \circ \alpha_1), \quad \text{where } \beta = r \circ i \circ \varphi_2 \sim r \circ i \circ h_1 \circ \varphi_1.$$

It follows from the last relation that $j \circ \alpha_2 \sim j \circ \alpha_1$. In fact, since u is a Spanier-Whitehead duality and β is homotopy equivalent, then the map $\{K, Y\} \rightarrow \{K \wedge K, S^{n+1}\}$, which transfers the class of S -map $\gamma: K \rightarrow Y$ into the class of $u \circ (\beta \wedge \gamma)$, is isomorphism. It means that $j \circ \alpha_2$ is stably homotopic to $j \circ \alpha_1$. On the other hand $[K, Y] = \{K, Y\}$ since Y is r -connected and $\dim K \leq n - r$.

Thus α_1 is homotopic to α_2 on ∂N_2 and so the isotopy of ∂N_2 exists which translates V_2 on $h_1(V_1)$ with preservation of orientations. This isotopy may be extended to the isotopy of sphere S^{n+2} .

The theorem is proved.

1.10. PROOF OF THEOREM 1.3. Let S^{n+1} be the equator of sphere S^{n+2} . Since $r \geq 2k - n$, then by virtue of Theorem 1.6 an $(n + 1)$ -dimensional trivial thickening of complex K exists, i.e. submanifold $N_0^{n+1} \subset S^{n+1}$ and homotopy equivalence $\varphi: K \rightarrow N_0$. Fix some orientation on N_0 . Let us identify manifold N_0 with $N_0 \times 0 \subset N_0 \times [0, 1] = N$. Imbed manifold N in S^{n+2} so that this imbedding will be the extension of the imbedding $N_0 \subset S^{n+1}$ and so that for each point $x \in N_0$ the curve $\{(x, t); t \in [0, 1]\}$ goes out from x along the direction of negative normal to N_0 in S^{n+2} .

Let Y be a closure of complement to N in sphere S^{n+2} and let Y' be obtained from Y by removing some small collar of boundary ∂Y . By $v: N \wedge Y' \rightarrow S^{n+1}$ denote the canonical Spanier-Whitehead duality and by $r: Y \rightarrow Y'$ the natural retraction. The map $u: N \wedge Y \xrightarrow{1 \wedge r} N \wedge Y' \xrightarrow{v} S^{n+1}$ is also a Spanier-Whitehead duality. By virtue of Spanier-Whitehead theory there exists a unique element $\{h\} \in \{K, Y\}$ such that the composition

$$K \wedge K \xrightarrow{\alpha \wedge h} N \wedge Y \xrightarrow{u} S^{n+1}$$

is stably homotopic to θ , where $\alpha: K \rightarrow N$ is the composition of homotopy equivalence $\varphi: K \rightarrow N_0$ and imbedding $N_0 \rightarrow N$. Note that Y is $(n - k)$ -connected and since $3k \leq 2n$, then $\{K, Y\} = [K, Y]$. Besides $\{K \wedge K, S^{n+1}\} = [K \wedge K, S^{n+1}]$ since $k < n$. Thus we can regard that $h: K \rightarrow Y$ is the mapping and $\theta \sim u \circ (\alpha \wedge h)$.

By virtue of Proposition 1.8 there is the map $e: K \rightarrow \partial N$ with $\psi_*([e]) = ([\alpha], [h])$, where $\psi: [K, \partial N] \rightarrow [K, N] \times [K, Y]$. We want to apply Theorem 1.6 to the mapping $e: K \rightarrow \partial N$. The manifold ∂N is $\min\{r, n - k\}$ -connected and since $2k - (n + 1) + 1 \leq \min\{r, n - k\}$, then in this case Theorem 1.6 is applicable. Let $f: K \rightarrow V^{n+1}$ be the thickening induced by the mapping $e: K \rightarrow \partial N$. Here V^{n+1} is the submanifold of ∂N with $\pi_1(\partial V) = \pi_1(V) = 1$ and f is homotopy equivalent.

Orient V so that N_0 and V define the same orientation of ∂N .

Let $\beta: V \rightarrow \partial N$ be an inclusion. Homotopy Seifert pairing of V is homotopic to the composition

$$V \wedge V \xrightarrow{\beta \wedge \beta} \partial N \wedge \partial N \xrightarrow{i \wedge j} N \wedge Y \xrightarrow{u} S^{n+1},$$

where $i: \partial N \rightarrow N, j: \partial N \rightarrow Y$ are inclusions. According to the construction

$$\beta \circ f \sim e, \quad i \circ e \sim \alpha, \quad j \circ e \sim h, \quad u \circ (\alpha \wedge h) \sim \theta.$$

Thus the following diagram

$$\begin{array}{ccccccc} V \wedge V & \xrightarrow{\beta \wedge \beta} & \partial N \wedge \partial N & \xrightarrow{i \wedge j} & N \wedge Y & \xrightarrow{u} & S^{n+2} \\ & \searrow f \wedge f & & \nearrow e \wedge e & \nearrow \alpha \wedge h & & \nearrow \theta \\ & & K \wedge K & & & & \end{array}$$

is homotopy commutative. Since the composition of the horizontal line of this diagram is homotopic to the homotopy Seifert pairing of V , then the latter is homotopic to $\theta \circ (g \wedge g)$, where g is the homotopy equivalent inverse to f .

The theorem is proved.

1.11. By $M_{r,n}$ we denote the set of isotopy classes of r -connected $(n+1)$ -dimensional oriented submanifolds of sphere S^{n+2} bounding the homotopy spheres.

Consider also the set of pairs (K, θ) , where K is a finite r -connected complex and and $\theta: K \wedge K \rightarrow S^{n+1}$ is a homotopy pairing for which $\theta + (-1)^{n+1}\theta'$ is the Spanier-Whitehead duality. We shall call two such pairs (K_i, θ_i) , $i = 1, 2$, equivalent if there is homotopy equivalence $f: K_1 \rightarrow K_2$ such that θ_1 is homotopic to $\theta_2 \circ (f \wedge f)$. We denote the set of equivalence classes by $Z_{r,n}$.

The next statement follows from Theorems 1.2, 1.3 and 1.4.

1.12. COROLLARY. *If $3r \geq n+1 \geq 6$, then the map $M_{r,n} \rightarrow Z_{r,n}$ which assigns to the submanifold of codimension one of sphere S^{n+2} its homotopy Seifert pairing is bijective.*

Actually, by virtue of Theorem 1.2 this map is injective and according to Theorem 1.4 its values belong to $Z_{r,n}$. Surjectivity of this map follows from Theorem 1.3 (it is necessary to note that if (K, θ) is some element of $Z_{r,n}$, then duality $K \wedge K \rightarrow S^{n+1}$ exists and thus K has a homotopy type $(n-r)$ -connected complex) and Theorem 1.4.

2. Contiguity. In this section the simplest situation is considered when nonisotopic $(n+1)$ -dimensional submanifolds of sphere S^{n+2} bound isotopic knots, and relations are found between their homotopy Seifert pairings.

2.1. PROPOSITION. *Let N be a connected $(n+2)$ -dimensional compact submanifold of sphere S^{n+2} and let D^{n+1} be a small disk lying on ∂N . Denote $V = \partial N - \text{int } D^{n+1}$, $Y = S^{n+2} - \text{int } N$. Let $i: V \rightarrow N$ and $j: V \rightarrow Y$ be inclusions. Then there exists an S -map $f: N \vee Y \rightarrow V$ which is stable homotopy equivalent and such that*

$$i \circ f \circ k \stackrel{S}{\sim} 1_N, \quad i \circ f \circ l \stackrel{S}{\sim} 0, \quad j \circ f \circ k \stackrel{S}{\sim} 0, \quad j \circ f \circ l \stackrel{S}{\sim} 1_Y,$$

where $k: N \rightarrow N \vee Y$ and $l: Y \rightarrow N \vee Y$ are inclusions; $\text{sign } \stackrel{S}{\sim}$ means "stably homotopic".

PROOF. Let manifold N_1 be obtained from N by removing some small collar of the boundary of N . Analogously, Y_1 is obtained from Y by removing the small collar of ∂Y . By $r: N \rightarrow N_1$ and $s: Y \rightarrow Y_1$, we denote the natural retractions. Since N_1 and Y_1 are disjoint subsets of sphere S^{n+2} , then the canonical pairings

$$v: N_1 \wedge Y_1 \rightarrow S^{n+1}, \quad u: Y_1 \wedge N_1 \rightarrow S^{n+1}$$

are defined. Since N_1 and Y_1 are complementary to each other, then u and v are Spanier-Whitehead dualities.

Let ω be a simple smooth arc in sphere S^{n+2} which does not intersect $V \cup \text{int } N_1 \cup \text{int } Y_1$ and such that its origin lies on ∂N_1 and its end lies on ∂Y_1 . Suppose that ω transversally intersects ∂N_1 and ∂Y_1 . Obviously, the space $Z = N_1 \cup Y_1 \cup \omega$ is a deformation retract of complement $S^{n+2} - V$ and therefore the canonical pairing

$$w: V \wedge Z \rightarrow S^{n+1}$$

is a Spanier-Whitehead duality. If $i_1: N_1 \rightarrow Z$ and $i_2: Y_1 \rightarrow Z$ are inclusions, then

$$w \circ (1_V \wedge i_1) \stackrel{S}{\sim} u \circ (s \circ j \wedge 1_{N_1}), \quad (1)$$

$$w \circ (1_V \wedge i_2) \stackrel{S}{\sim} v \circ (r \circ i \wedge 1_{Y_1}). \quad (2)$$

Let the map $M: Z \wedge Z \rightarrow S^{n+1}$ be given by the matrix $\begin{pmatrix} 0 & 0 \\ u & v \end{pmatrix}$, i.e.

$$\begin{aligned} M \circ (i_1 \wedge i_1) &= 0, & M \circ (i_1 \wedge i_2) &= v, \\ M \circ (i_2 \wedge i_1) &= u, & M \circ (i_2 \wedge i_2) &= 0. \end{aligned} \quad (3)$$

It is easy to verify that M is a Spanier-Whitehead duality. Since M and w are the dualities, then by virtue of Spanier-Whitehead theory there exists an S -map $h: Z \rightarrow V$ which is stable homotopy equivalent such that

$$w \circ (h \wedge 1_Z) \stackrel{S}{\sim} M. \quad (4)$$

Let us prove that

$$r \circ i \circ h \circ i_1 \stackrel{S}{\sim} 1_{N_1}, \quad (5)$$

$$r \circ i \circ h \circ i_2 \stackrel{S}{\sim} 0, \quad (6)$$

$$s \circ j \circ h \circ i_1 \stackrel{S}{\sim} 0, \quad (7)$$

$$s \circ j \circ h \circ i_2 \stackrel{S}{\sim} 1_{Y_1}. \quad (8)$$

We have

$$\begin{aligned} v \circ (r \circ i \circ h \circ i_1 \wedge 1_{Y_1}) \\ = v \circ (r \circ i \wedge 1_{Y_1}) \circ (h \circ i_1 \wedge 1_{Y_1}) \stackrel{S}{\sim} w \circ (1_v \wedge i_2) \circ (h \circ i_1 \wedge 1_{Y_1}) \\ = w \circ (h \wedge 1_Z) \circ (i_1 \wedge i_2) \stackrel{S}{\sim} M \circ (i_1 \wedge i_2) = v. \end{aligned}$$

Here homotopies (2), (4) and (3) were used. Thus $v \circ (r \circ i \circ h \circ i_1 \wedge 1_{Y_1}) \stackrel{S}{\sim} v$. It follows now that $r \circ i \circ h \circ i_1 \stackrel{S}{\sim} 1_{N_1}$ since v is a Spanier-Whitehead duality. Thus (5) is proved.

For the proof of (8) we have

$$\begin{aligned} u \circ (s \circ j \circ h \circ i_2 \wedge 1_{N_1}) \\ = u \circ (s \circ j \wedge 1_{N_1}) \circ (h \circ i_2 \wedge 1_{N_1}) \stackrel{S}{\sim} w \circ (1_v \wedge i_1) \circ (h \circ i_2 \wedge 1_{N_1}) \\ = w \circ (h \wedge 1_Z) \circ (i_2 \wedge i_1) \stackrel{S}{\sim} M \circ (i_2 \wedge i_1) = u. \end{aligned}$$

From this it follows that $s \circ j \circ h \circ i_2 \stackrel{S}{\sim} 1_{Y_1}$ as above. The relations (6) and (7) may be proved analogously.

Let the map $g: Z \rightarrow N_1 \vee Y_1$ be as follows. g identically maps N_1 and Y_1 and maps the arc ω in the base point of $N_1 \vee Y_1$. Consider the composition

$$f: N \vee Y \xrightarrow{r \vee s} N_1 \vee Y_1 \xrightarrow{h} Z \rightarrow V$$

of the map $r \vee s$, homotopy equivalence, which is inverse to g , and the map h . It follows from relations (5), (6), (7), (8) that f satisfies all required conditions.

The proposition is proved.

2.2. COROLLARY. *Let N be a connected $(n+2)$ -dimensional compact submanifold of sphere S^{n+2} and let D^{n+1} be a small disk on ∂N . Let $V = \partial N - \text{int } D^{n+1}$, $Y = S^{n+2} - \text{int } N$. Suppose that the manifold V is oriented so that the field of positive normals to V is directed to the exterior of N . Then there exists an S -map $f: N \vee Y \rightarrow V$ which is stable homotopy equivalent and such that the map*

$$\theta \circ (f \wedge f): (N \vee Y) \wedge (N \vee Y) \rightarrow S^{n+1}$$

is given by the matrix $\begin{pmatrix} 0 & u \\ 0 & 0 \end{pmatrix}$ (this means that

$$\begin{aligned} \theta \circ (f \wedge f) \circ (k \wedge k) &\sim 0, & \theta \circ (f \wedge f) \circ (k \wedge l) &\sim u, \\ \theta \circ (f \wedge f) \circ (l \wedge k) &\sim 0, & \theta \circ (f \wedge f) \circ (l \wedge l) &\sim 0, \end{aligned}$$

where $k: N \rightarrow N \vee Y$ and $l: Y \rightarrow N \vee Y$ are inclusions), where $\theta: V \wedge V \rightarrow S^{n+1}$ is the homotopy Siefert pairing of V and u is a Spanier-Whitehead duality.

PROOF. It follows from the definition that the homotopy pairing θ is homotopic to the composition

$$V \wedge V \xrightarrow{i \wedge j} N \wedge Y \xrightarrow{v \circ (r \wedge s)} S^{n+1},$$

where the notations are the same as in the proof of Proposition 2.1. Let $f: N \vee Y \rightarrow V$ be the stable homotopy equivalence constructed in Proposition 2.1.

Then

$$\theta \circ (f \wedge f) \sim v \circ (r \wedge s) \circ (i \wedge j) \circ (f \wedge f) = v \circ (r \wedge s) \circ (i \circ f \wedge j \circ f).$$

Therefore

$$\begin{aligned} \theta \circ (f \wedge f) \circ (k \wedge k) &\sim v \circ (r \wedge s) \circ (i \circ f \circ k \wedge j \circ f \circ k) \\ &\sim v \circ (r \wedge s) \circ (1_N \wedge 0) = 0, \end{aligned}$$

and

$$\begin{aligned} \theta \circ (f \wedge f) \circ (k \wedge l) &\sim v \circ (r \wedge s) \circ (i \circ f \circ k \wedge j \circ f \circ l) \\ &\sim v \circ (r \wedge s) \circ (1_N \wedge 1_Y) = v \circ (r \wedge s) = u \end{aligned}$$

is a Spanier-Whitehead duality. It may be proved analogously that

$$\theta \circ (f \wedge f) \circ (l \wedge k) \sim 0, \quad \theta \circ (f \wedge f) \circ (l \wedge l) \sim 0.$$

Thus Corollary 2.2 is proved.

This corollary describes some homotopy Seifert pairings which may have a Seifert manifold of trivial knot.

2.3. DEFINITION. We shall call two homotopy pairings $\theta_i: K_i \wedge K_i \rightarrow S^{n+1}$, $i = 1, 2$ contiguous if there exist complexes L_1 and L_2 , pairings $\alpha: K_1 \wedge K_2 \rightarrow S^{n+1}$ and $u: L_1 \wedge L_2 \rightarrow S^{n+1}$ from which the latter is a Spanier-Whitehead duality, and an S -map $f: (K_1 \vee K_2) \rightarrow (L_1 \vee L_2)$ which is a stable homotopy equivalence and such that $\eta \circ (f \wedge f)$ is stably homotopic to ξ , where the homotopy pairings

$$\begin{aligned} \eta: (L_1 \vee L_2) \wedge (L_1 \vee L_2) &\rightarrow S^{n+1} \quad \text{and} \\ \xi: (K_1 \vee K_2) \wedge (K_1 \vee K_2) &\rightarrow S^{n+1} \end{aligned}$$

are given by the following matrices

$$\begin{pmatrix} 0 & u \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \theta_1 & \alpha \\ (-1)^n \alpha' & (-1)^n \theta_2' \end{pmatrix},$$

respectively. (The latter means that $\eta \circ (l_i \wedge l_j) = a_{ij}$, $\xi \circ (k_i \wedge k_j) = b_{ij}$, where $i = 1, 2$, $l_i: L_i \rightarrow L_1 \vee L_2$ and $k_i: K_i \rightarrow K_1 \vee K_2$ are inclusions, and a_{ij} and b_{ij} are the elements of corresponding matrices.) Here prime denotes "transposition", i.e. the composition with the map which changes the position of factors.

The following two theorems are main results of this section. They state that the contiguity relation of homotopy Seifert pairings corresponds to some simple geometric situation (geometric contiguity) for Seifert manifolds.

2.4. THEOREM. Let N^{n+2} be a compact submanifold of the sphere S^{n+2} . Suppose that $\partial N = V_1^{n+1} \cup U \cup V_2^{n+1}$, where $U = \Sigma^n \times [0, 1]$, $V_1 \cap U = \partial V_1 = \Sigma^n \times 0$, $V_2 \cap U = \partial V_2 = \Sigma^n \times 1$, $V_1 \cap V_2 = \emptyset$ and Σ^n is a homotopy sphere (Figure 1). Let us orient V_1 and V_2 so that they define different orientations of ∂N . Then the corresponding homotopy Seifert pairings $\theta_i: V_i \wedge V_i \rightarrow S^{n+1}$, $i = 1, 2$, are contiguous.

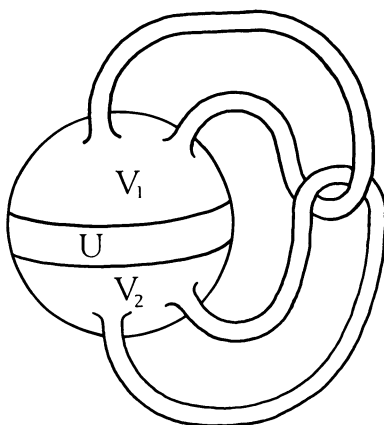


FIGURE 1

PROOF. Let us notice that if θ is a homotopy Seifert pairing of some oriented submanifold $V^{n+1} \subset S^{n+2}$, then $(-1)^n \theta'$ is the homotopy Seifert pairing of the same submanifold with opposite orientation. It follows from the diagram in the proof of Theorem 1.4.

For the proof of Theorem 2.4 consider the small disk D^{n+1} in $\text{int } U = \Sigma^n \times (0, 1)$. Let $V = \partial N - \text{int } D^{n+1}$. We have conditions as in Corollary 2.2 and therefore there is an S -map $h: N \vee Y \rightarrow V$ which is stable homotopy equivalent and such that the map $\theta \circ (h \wedge h)$ is given by the matrix $\begin{pmatrix} \theta & u \\ 0 & 0 \end{pmatrix}$, where θ is a homotopy Seifert pairing of manifold V , $u: N \wedge Y \rightarrow S^{n+1}$ is a Spanier-Whitehead duality and $Y = S^{n+2} - \text{int } N$.

Let ω be a simple arc in $U - D^{n+1}$ such that its ends are situated on ∂V_1 and ∂V_2 . Let $H = V_1 \cup V_2 \cup \omega$. The inclusion $g: H \rightarrow V$ is homotopy equivalent. Denote by $i_1: V_1 \rightarrow H$ and $i_2: V_2 \rightarrow H$ the inclusions. Then

$$\theta_1 = \theta \circ (g \circ i_1 \wedge g \circ i_1), \quad (-1)^n \theta'_2 = \theta \circ (g \circ i_2 \wedge g \circ i_2).$$

The distinction here is caused by the fact that the orientation of V_2 is opposite to the orientation of V (see the remark at the beginning of the proof). If by α we denote $\theta \circ (g \circ i_1 \wedge g \circ i_2)$, then

$$\theta \circ (g \circ i_2 \wedge g \circ i_1) \sim (-1)^n \alpha'.$$

It follows from these relations that if the space H is identified with $V_1 \vee V_2$ and if by f we denote the composition of the map g and homotopy equivalence $V \rightarrow N \vee Y$ inverse to h , then all conditions of Definition 2.3 will be satisfied.

The theorem is proved.

The following theorem states that under some connectivity assumptions the contiguity of homotopy Seifert pairings implies "a geometrical contiguity", i.e. the situation which has been considered in Theorem 2.4.

2.5. THEOREM. *Let V_1 and V_2 be r -connected $(n+1)$ -dimensional compact submanifolds of S^{n+2} and $\theta_i: V_i \wedge V_i \rightarrow S^{n+1}$, $i = 1, 2$, are corresponding homotopy Seifert pairings. Suppose that $3r > n+1 > 6$. If ∂V_i are homotopy spheres and*

pairings θ_1 and θ_2 are contiguous, then there exist isotopies g_t and h_t of S^{n+2} and a compact submanifold $N^{n+2} \subset S^{n+2}$ such that

$$\partial N = g_1(V_1) \cup U \cup h_1(V_2),$$

where

$$U = \sum^n \times [0, 1], \quad g_1(V_1) \cap U = \sum^n \times 0 = \partial g_1(V_1),$$

$$h_1(V_2) \cap U = \partial h_1(V_2) = \sum^n \times 1, \quad g_1(V_1) \cap h_1(V_2) = \emptyset$$

and the orientations of $g_1(V_1)$ and $h_1(V_2)$ define opposite orientations of ∂N .

In the proof of this theorem we shall use the following lemma.

2.6. LEMMA. *Let $2r + 2 \geq k > r \geq 1$ and X be a finite complex with $H_i(X; G) = 0$ for $i \leq r$ or $i > k$ and for any abelian group G . Then there exists a finite r -connected k -dimensional complex Y such that SX is homotopy equivalent to SY .*

This is the standard result (compare [14, Chapter 8, Exercise D1]).

2.7. PROOF OF THEOREM 2.5. Since the pairings $\theta_i: V_i \wedge V_i \rightarrow S^{n+1}$ are contiguous, then, according to the definition, there exist complexes L_1 and L_2 , pairings $\alpha: V_1 \wedge V_2 \rightarrow S^{n+1}$ and $v: L_1 \wedge L_2 \rightarrow S^{n+1}$ (from which the latter is a Spanier-Whitehead duality), and an S -map $f: (V_1 \vee V_2) \rightarrow (L_1 \vee L_2)$ which is stable homotopy equivalent and such that $\eta \circ (f \wedge f)$ is stably homotopic to ξ , where the homotopy pairings

$$\eta: (L_1 \vee L_2) \wedge (L_1 \vee L_2) \rightarrow S^{n+1}, \quad \text{and}$$

$$\xi: (V_1 \vee V_2) \wedge (V_1 \vee V_2) \rightarrow S^{n+1},$$

are given by matrices

$$\begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \theta_1 & \alpha \\ (-1)^n \alpha' & (-1)^n \theta_2' \end{pmatrix}$$

respectively.

The constructions of the desirable submanifolds N and isotopies g_t and h_t are performed in several steps.

Step 1. This is the construction of the $(n + 2)$ -dimensional compact submanifold $N \subset S^{n+2}$, which satisfies the following condition. If V^{n+1} is obtained from ∂N by removing the interior of some small disk $D^{n+1} \subset \partial N$, then there exists a stable homotopy equivalence $d: L_1 \vee L_2 \rightarrow V$ for which $\theta \circ (d \wedge d)$ is homotopic to η , where θ is the homotopy Seifert pairing of V .

For this, first note that by virtue of Lemma 2.6 and properties of V_1 and V_2 , we may suppose that L_1 and L_2 are r -connected complexes of dimension $\leq n - r$. Let $\tau: L_1 \rightarrow N^{n+2}$ be the trivial thickening of L_1 . Let $V = \partial N - \text{int } D^{n+1}$ and $Y = S^{n+2} - \text{int } N$. Consider stable homotopy equivalence $e: N \vee Y \rightarrow V$ which existence is established in Corollary 2.2. There exists an S -map $\sigma: L_2 \rightarrow Y$ for which $u \circ (\tau \wedge \sigma) \stackrel{S}{\sim} v$, where $u: N \wedge Y \rightarrow S^{n+1}$ is the pairing considered in Corollary 2.2. Since u and v are Spanier-Whitehead dualities, then σ is stable homotopy

equivalent. By virtue of the suspension theorem and our connectivity assumptions we may suppose that σ is the usual map and $u \circ (\tau \wedge \sigma) \sim v$. If we take d as the composition

$$L_1 \vee L_2 \xrightarrow{\tau \vee \sigma} N \vee Y \xrightarrow{e} V,$$

then

$$\theta \circ (d \wedge d) = \theta \circ (e \wedge e) \circ ((\tau \vee \sigma) \wedge (\tau \vee \sigma)) = \eta.$$

Step 2. This is the construction of isotopy l_t of S^{n+2} such that $V_1 \cap l_1(V_2) = \emptyset$ and the composition of the map $1 \wedge (l_1|_{V_2}): V_1 \wedge V_2 \rightarrow V_1 \wedge l_1(V_2)$ and the canonical pairing $V_1 \wedge l_1(V_2) \rightarrow S^{n+1}$ is homotopic to $\alpha: V_1 \wedge V_2 \rightarrow S^{n+1}$.

This may be done in the following way. If Z is the complement of a tubular neighborhood of V_1 in S^{n+2} and $w: V_1 \wedge Z \rightarrow S^{n+1}$ is the canonical pairing which is a Spanier-Whitehead duality, then there is a map $\psi: V_2 \rightarrow Z$ for which $w \circ (1 \wedge \psi) \sim \alpha$. (Here we do not mention the usual arguments about connectivity conditions and application of the suspension theorem.) Now we may construct the thickening $V_2 \rightarrow N_2$ induced by ψ . By the same arguments we have seen used in the proof of Theorem 1.3 we can construct the map $V_2 \rightarrow \partial N_2$ such that a thickening $\varphi: V_2 \rightarrow W_2$, induced by this map, has the following property: $\delta \circ (\varphi \wedge \varphi) \sim \theta_2$, where $\delta: W_2 \wedge W_2 \rightarrow S^{n+1}$ is the homotopy Seifert pairing of W_2 . Also, we shall have that the composition of the map $1 \wedge \varphi: V_1 \wedge V_2 \rightarrow V_1 \wedge W_2$ and the canonical pairing $V_1 \wedge W_2 \rightarrow S^{n+1}$ is homotopic to α . Now Theorem 1.2 is applicable and we obtain the desirable isotopy l_t with $l_1(V_2) = W_2$. See Figure 2.

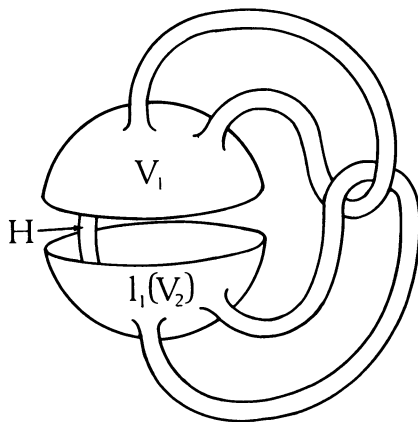


FIGURE 2

Step 3. Let H be an $(n+1)$ -dimensional disk in S^{n+2} with the following properties:

(1) $H \cap V_1 = H \cap \partial V_1 = \partial H \cap V_1 = D_1$ and $H \cap l_1(V_2) = H \cap \partial l_1(V_2) = \partial H \cap l_1(V_2) = D_2$ are two n -dimensional disks in ∂H .

(2) Orientations of D_1 and D_2 , defined by that of ∂V_1 and $\partial l_1(V_2)$ respectively, determine the opposite orientations of ∂H (Figure 2).

Let the manifold W be obtained from $V_1 \cup H \cup l_1(V_2)$ by smoothing corners. Choose the orientation of W according to that of V_1 . Then there is a homotopy equivalence $q: V_1 \vee V_2 \rightarrow W$ such that for the homotopy Seifert pairing $\bar{\theta}: W \wedge W \rightarrow S^{n+1}$ of W we shall have that $\bar{\theta} \circ (q \wedge q)$ is given by the matrix

$$\begin{pmatrix} \theta_1 & \alpha \\ (-1)^n \alpha' & (-1)^n \theta_2' \end{pmatrix},$$

i.e. the pairing $\bar{\theta} \circ (q \wedge q)$ coincides with ξ . Let F be the composition

$$W \rightarrow V_1 \vee V_2 \xrightarrow{f} L_1 \vee L_2 \xrightarrow{d} V,$$

where the first map is homotopy inverse to q . (Here we may regard f as usual but not as an S -map by virtue of our connectivity assumptions.) Then F is homotopy equivalent and $\theta \circ (F \wedge F) \sim \bar{\theta}$. Applying Theorem 1.2 we get an isotopy $m_i: S^{n+2} \rightarrow S^{n+2}$ such that $m_1(W) = V$.

Now we may put

$$g_i = m_i, \quad h_i = m_i \circ l_i.$$

Then $g_1(V_1)$ and $h_1(V_2)$ are disjoint submanifolds of V . Let $U = \partial N - \text{int } g_1(V_1) - \text{int } h_1(V_2)$. Then it is easy to see that U is the h -cobordism between $\partial g_1(V_1)$ and $\partial h_1(V_2)$ and the h -cobordism theorem implies the desired result.

2.8. COROLLARY. *Under the assumption of Theorem 2.5 the oriented knots $(S^{n+2}, \partial V_i)$, $i = 1, 2$ are equivalent.*

The equivalence of these knots is realized by isotopy of sphere S^{n+2} which translates $\Sigma^n \times 0$ on $\Sigma^n \times 1$ along U .

The next statement will establish the reflexivity of the contiguity relation.

2.9. PROPOSITION. *If K is a finite polyhedron and $\theta: K \wedge K \rightarrow S^{n+1}$ is the homotopy pairing for which $\theta + (-1)^{n+1}\theta'$ is a Spanier-Whitehead duality, then θ is contiguous to itself.*

PROOF. By virtue of Spanier-Whitehead theory there are S -maps $f, g: K \rightarrow K$ for which

$$(\theta + (-1)^{n+1}\theta') \circ (f \wedge 1) \stackrel{S}{\sim} \theta, \quad (\theta + (-1)^{n+1}\theta') \circ (g \wedge 1) \stackrel{S}{\sim} (-1)^n \theta'.$$

Since $(\theta + (-1)^{n+1}\theta') \circ ((f - g) \wedge 1) \stackrel{S}{\sim} (\theta + (-1)^{n+1}\theta')$, then $f - g \stackrel{S}{\sim} 1_K$. Here 1_K is the identity mapping of K . Let $f \vee g: K \vee K \rightarrow K$ be the S -map determined by f and g ; let $e: K \vee K \rightarrow K$ be a map which identifies the summands of the bouquet; and let $i_1, i_2: K \rightarrow K \vee K$ be a natural inclusion. Determine an S -map $h: K \vee K \rightarrow K \vee K$ by the formula $h = i_1 \circ (f \vee g) + i_2 \circ e$. Let us prove that h is a stable homotopy equivalence. For this it is enough that h induces the surjective map of integral homology. If $z_1, z_2 \in H_r K$, then

$$h_*(i_{1*}z_1 + i_{2*}z_2) = i_{1*}(f_*z_1 + g_*z_2) + i_{2*}(z_1 + z_2).$$

Any class $a \in H_r(K \vee K)$ may be uniquely presented in the form $a = i_{1*}a_1 + i_{2*}a_2$, where $a_1, a_2 \in H_r K$. Denote $z_1 = a_1 - g_*a_2$, $z_2 = f_*a_2 - a_1$. Then it is not

difficult to verify that $h_*(i_{1*}z_1 + i_{2*}z_2) = i_{1*}a_1 + i_{2*}a_2 = a$. Therefore h_* is an epimorphism and h is a stable homotopy equivalence.

Let $\eta: (K \vee K) \wedge (K \vee K) \rightarrow S^{n+1}$ be a homotopy pairing given by the following matrix:

$$\begin{pmatrix} 0 & \theta + (-1)^{n+1}\theta' \\ 0 & 0 \end{pmatrix}.$$

We have

$$\begin{aligned} \eta \circ (h \wedge h) &= \eta \circ ((i_1 \circ (f \vee g) + i_2 \circ e) \wedge (i_1 \circ (f \vee g) + i_2 \circ e)) \\ &= \eta \circ (i_1 \circ (f \vee g) \wedge i_1 \circ (f \vee g)) + \eta \circ (i_1 \circ (f \vee g) \wedge i_2 \circ e) \\ &\quad + \eta \circ (i_2 \circ e \wedge i_1 \circ (f \vee g)) + \eta \circ (i_2 \circ e \wedge i_2 \circ e) \\ &= \eta \circ (i_1 \circ (f \vee g) \wedge i_2 \circ e) = (\theta + (-1)^{n+1}\theta') \circ ((f \vee g) \wedge e). \end{aligned}$$

From this it follows that

$$\begin{aligned} \eta \circ (h \wedge h) \circ (i_1 \wedge i_1) &= (\theta + (-1)^{n+1}\theta') \circ ((f \vee g) \wedge e) \circ (i_1 \wedge i_2) \\ &= (\theta + (-1)^{n+1}\theta') \circ (f \wedge 1) \stackrel{S}{\sim} \theta. \end{aligned}$$

Analogously one can get

$$\begin{aligned} \eta \circ (h \wedge h) \circ (i_1 \wedge i_2) &\stackrel{S}{\sim} \theta, \\ \eta \circ (h \wedge h) \circ (i_2 \wedge i_1) &\stackrel{S}{\sim} (-1)^n \theta', \\ \eta \circ (h \wedge h) \circ (i_2 \wedge i_2) &\stackrel{S}{\sim} (-1)^n \theta', \end{aligned}$$

i.e. the pairing $\eta \circ (h \wedge h)$ is given by the following matrix:

$$\begin{pmatrix} \theta & \theta \\ (-1)^n \theta' & (-1)^n \theta' \end{pmatrix}.$$

This proves the proposition (in the notations of Definition 2.3, $L_1 = L_2 = K$, $u = \theta + (-1)^{n+1}\theta'$, $\alpha = \theta$).

3. The classification of knots.

3.1. Let $K_{r,n}$ be the set of isotopy types of n -dimensional knots (S^{n+2}, k^n) such that $\pi_i(S^{n+2} - k^n) = \pi_i(S^1)$ for $i < r$. There is the natural map $M_{r,n} \rightarrow K_{r,n}$ which assigns to a submanifold from $M_{r,n}$ its boundary (for the definition of the set $M_{r,n}$ see 1.11). J. Levine proved that this map is surjective. In Corollary 1.12 the map $M_{r,n} \rightarrow Z_{r,n}$ was constructed and it was shown that it is bijective if $3r > n + 1 > 6$. Thus, if $3r > n + 1 > 6$, then we have the map $\Phi: Z_{r,n} \rightarrow K_{r,n}$ which is the composition of the map inverse to the bijection of Corollary 1.12 and the map $M_{r,n} \rightarrow K_{r,n}$ defined above.

The map Φ is surjective. For the description of the set $K_{r,n}$ we shall find the equivalence relation which arises on $Z_{r,n}$ by virtue of mapping Φ .

3.2. DEFINITION. We shall call two pairs (K, θ) and (L, η) , representing two elements of $Z_{r,n}$, R -equivalent if there exists a finite sequence of pairs $(K_1, \theta_1), \dots, (K_s, \theta_s)$, where for $1 < m < s$, K_m is an r -connected complex and $\theta_m: K_m \wedge K_m \rightarrow S^{n+1}$ is a pairing for which $\theta_m + (-1)^{n+1}\theta'_m$ is a Spanier-Whitehead

duality, such that $(K_1, \theta_1) = (K, \theta)$, $(K_s, \theta_s) = (L, \eta)$, and pairings θ_m and θ_{m+1} are contiguous for all $1 \leq m \leq s-1$.

In other words R -equivalence is the equivalence relation on $Z_{r,n}$ which is generated by the contiguity relations. Note that contiguity is reflexive and a symmetric relation; it is not transitive.

3.3. THEOREM. *Let V_1 and V_2 be oriented r -connected $(n+1)$ -dimensional compact submanifolds of sphere S^{n+2} and $\theta_i: V_i \wedge V_i \rightarrow S^{n+1}$, $i = 1, 2$, are the corresponding Seifert pairings. If ∂V_i are homotopy spheres, $3r \geq n+1 \geq 6$ and pairs (V_i, θ_i) are R -equivalent in $Z_{r,n}$, then the oriented knots $(S^{n+2}, \partial V_i)$, $i = 1, 2$, are equivalent.*

PROOF. Since the pairs (V_1, θ_1) and (V_2, θ_2) are R -equivalent in $Z_{r,n}$, then according to the definition the finite sequence of pairs $(W_1, \eta_1) = (V_1, \theta_1)$, $(W_2, \eta_2), \dots, (W_s, \eta_s) = (V_2, \theta_2)$ exists, where W_m is a finite r -connected cell complex and $\eta_m: W_m \wedge W_m \rightarrow S^{n+1}$ is the homotopy pairing, $m = \overline{1, s}$, and all conditions formulated in Definition 3.2 are fulfilled. By virtue of Theorem 1.3 and Theorem 1.4 we can assume that W_m is a compact oriented $(n+1)$ -dimensional submanifold of sphere S^{n+2} for which ∂W_m is a homotopy sphere and η_m is a homotopy Seifert pairing of W_m , $m = \overline{1, s}$. Since η_m and η_{m+1} are contiguous and $3r \geq n+1 \geq 6$, then by virtue of Corollary 2.8, the oriented knots $(S^{n+2}, \partial W_m)$ and $(S^{n+2}, \partial W_{m+1})$ are equivalent, $m = \overline{1, s-1}$. Therefore the knot $(S^{n+2}, \partial W_1) = (S^{n+2}, \partial V_1)$ is equivalent to the knot $(S^{n+2}, \partial W_s) = (S^{n+2}, \partial V_2)$.

The theorem is proved.

3.4. THEOREM. *Let V_1 and V_2 be oriented r -connected $(n+1)$ -dimensional compact submanifolds of sphere S^{n+2} and $\theta_i: V_i \wedge V_i \rightarrow S^{n+1}$, $i = 1, 2$, are the corresponding homotopy Seifert pairings where $r \geq 2$, $n \geq 4$. If ∂V_i are homotopy spheres and the oriented knots $(S^{n+2}, \partial V_i)$, $i = 1, 2$, are equivalent, then the pairs (V_i, θ_i) , $i = 1, 2$, are R -equivalent in $Z_{r,n}$.*

REMARK. The theorem is true without the assumptions that $r \geq 2$, $n \geq 4$. This more general statement may be deduced from the given theorem with the help of G. Bredon's suspension construction which permits us "to increase" r and n . In the present paper this more general statement will not be used.

The proof of Theorem 3.4 will be based on the following lemmas.

We shall suppose below that $n \geq 2r+1$ otherwise V_1 and V_2 must be contractible and the theorem becomes trivial.

3.5. LEMMA. *Under the assumptions of Theorem 3.4 there is an r -connected $(n+2)$ -dimensional compact submanifold V (with corners) of $S^{n+2} \times [1, 2]$ such that*

$$V \cap S^{n+2} \times \nu = V_\nu \times \nu, \quad \nu = 1, 2, \quad \text{and} \\ \partial V = V_1 \times 1 \cup V_2 \times 2 \cup X,$$

where X is the trace of isotopy translating ∂V_1 on ∂V_2 .

PROOF. The existence of V satisfying all required conditions besides r -connectedness was proved by J. Levine [10, p. 186]. In order to construct an r -connected V one may use the method of modification described in §§4 and 5 of [7]. This method

does not change ∂V and gives an r -connected manifold by virtue of analogy of Lemma 4 of [7] and the fact that $\pi_i(S^{n+2} \times [1, 2] - X) = \pi_i(S^1)$ for $i < r$.

Later on we shall identify V_ν with $V_\nu \times \nu \subset S^{n+2} \times [1, 2]$, where $\nu = 1, 2$.

3.6. LEMMA. *Let V be the same as in Lemma 3.5 and let Y be the closure of the complement of the tubular neighborhood of V in $S^{n+2} \times [1, 2]$. Denote $Y_\nu = Y \cap S^{n+2} \times \nu$, $\nu = 1, 2$. Fix some orientation of V and let $i_+, i_-: (V; V_1, V_2) \rightarrow (Y; Y_1, Y_2)$ be translations along the fields of positive and negative normals to V . If for some k the homomorphisms*

$$i_{+*}, i_{-*}: T_k(V, V_1) \rightarrow T_k(Y, Y_1)$$

are both monomorphisms, then $T_k(V, V_1) = 0 = T_k(Y, Y_1)$, where T_k denotes the torsion subgroup of k -dimensional homology. Besides, if $T_k(V, V_1) = 0$ and $H_k(V, V_1) \neq 0$, then there exists an indivisible element $\alpha \in H_k(V, V_1)$, $\alpha \neq 0$ for which $i_{+}(\alpha) = 0$ or $i_{-*}(\alpha) = 0$.*

PROOF. The infinite cyclic covering $p: \tilde{Z} \rightarrow S^{n+2} \times [1, 2] - X$ may be constructed from the infinite sequence of copies of the space Y by the same way as the infinite cyclic covering of a knot may be constructed by cutting a sphere along a Seifert manifold [9]. Let $\tilde{Z}_1 = p^{-1}((S^{n+2} \times 1) - \partial V_1)$. Since X is the trace of isotopy, it follows that $H_*(Z, Z_1) = 0$ and the arguments analogous to that used in [9, p. 541] prove that there is an isomorphism of Λ -modules

$$d: H_k(V, V_1) \otimes \Lambda \rightarrow H_k(Y, Y_1) \otimes \Lambda,$$

where

$$d(a \otimes 1) = i_{+*}(a) \otimes t - i_{-*}(a) \otimes 1, \quad a \in H_k(V, V_1).$$

Here $\Lambda = Z[t, t^{-1}]$ is the group ring of the infinite cyclic group.

Suppose that $i_{+*}, i_{-*}: T_k(V, V_1) \rightarrow T_k(Y, Y_1)$ are both monomorphisms. Let $b \in T_k(Y, Y_1)$, $b \neq 0$. Then there is a unique element $q \in H_k(V, V_1) \otimes \Lambda$ for which $d(q) = b \otimes 1$. Write q in the form

$$q = \sum_{i=m}^n z_i \otimes t^i,$$

where $m \leq n$, $z_i \in H_k(V, V_1)$. We may suppose that $z_n \neq 0$. Since d is an isomorphism, then z_i must belong to $T_k(V, V_1)$.

The two following cases are possible: (1) $n > 0$, (2) $n < 0$. If $n > 0$, then $d(q)$ contains as summand the element $i_{+*}(z_n) \otimes t^{n+1}$ and does not contain other elements of degree $n + 1$. Since Λ is a free abelian group, then it follows from equality $d(q) = b \otimes 1$ that $i_{+*}(z_n) = 0$. Since i_{+*} is a monomorphism, then $z_n = 0$ and we have obtained a contradiction. Analogously case (2) may be reduced to a contradiction by using that i_{-*} is a monomorphism. So $T_k(Y, Y_1) = 0$ and then $T_k(V, V_1) = 0$.

This proves the first statement of the lemma.

In order to prove the second statement let us notice that if $H_k(Y, Y_1) \neq 0$, then at least one of the homomorphisms

$$i_{+*}, i_{-*}: H_k(V, V_1) \rightarrow H_k(Y, Y_1)$$

must have nontrivial kernel. This may be deduced similarly to recently described arguments from the fact that d is an isomorphism. If $T_k(V, V_1) = 0$, then, as it has been proved above, the group $H_k(Y, Y_1)$ has no torsion. From this follows that if $a \in \ker i_{+*}$ and $a = pc$, where $p \in \mathbb{Z}$, $p \neq 0$ and c is indivisible, then $c \in \ker i_{+*}$. This implies the second statement of the lemma.

It is clear that in this lemma we may replace V_1 by V_2 and Y_1 by Y_2 .

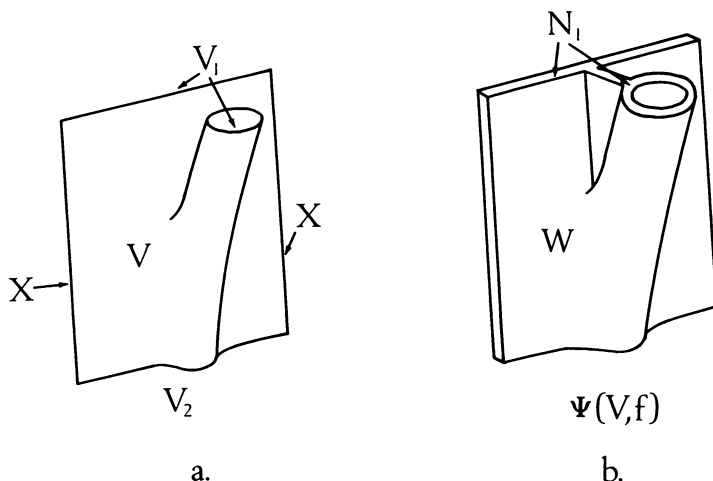


FIGURE 3

3.7. Let V^{n+2} be a manifold, the boundary of which is the union of three $(n+1)$ -manifolds V_1, V_2, X and has the corners along ∂V_1 and ∂V_2 (Figure 3a). Let $f: (D_+^k, \partial D_+^k) \times D^{n+2-k} \rightarrow (V, V_1)$ be a smooth imbedding, where D_+^k is the upper semisphere of S^k . Consider the manifold $\psi(V, f)$ which is obtained from the union $V \times [0, 1] \cup D^{k+1} \times D^{n+2-k}$ by identification of points $(x, y) \in D_+^k \times D^{n+2-k}$ with $(f(x, y), 1)$ (see Figure 3b) and then smoothing corners along $X \times 1$, $(f(D_+^k \times S^{n+1-k}), 1)$ and $(f(\partial D_+^k \times D^{n+2-k}), 1)$. Let

$$N_1 = V_1 \times [0, 1] \cup D_-^k \times D^{n+2-k},$$

where D_-^k is the low semisphere of S^k and the points $(x, y) \in \partial D_-^k \times D^{n+2-k}$ are identified with $(f(x, y), 1)$. Let

$$N_2 = V_2 \times [0, 1].$$

Then

$$\partial \psi(V, f) = V \cup N_1 \cup N_2 \cup W,$$

where V is identified with $V \times \theta$, and W is an $(n+2)$ -dimensional manifold with $\partial W = W_1 \cup W_2 \cup X$, where $W_\nu = W \cap N_\nu$, $\nu = 1, 2$.

Thus we have obtained $(\psi(V, f); V, N_1, N_2, W)$.

Of course, the manifold $\psi(V, f)$ is homeomorphic to $V \times [0, 1]$, but for us the given definition is more convenient.

3.8. LEMMA. Let V be the $(n+2)$ -dimensional r -connected submanifold of $S^{n+2} \times [1, 2]$, constructed in Lemma 3.5. Let $\alpha \in \pi_k(V, V_1)$ be such an element that $i_{+*}\alpha = 0 \in \pi_k(Y, Y_1)$, where Y, Y_1, i_+ are the same as in Lemma 3.6. Suppose that $k \leq n-1, 2k \leq n+r$. Then there exist an imbedding

$$f: (D_+^k, \partial D_+^k) \times D^{n+2-k} \rightarrow (V, V_1),$$

which realizes α , and an imbedding

$$g: \psi(V, f) \rightarrow S^{n+2} \times [1, 2],$$

extending the imbedding of V and such that

$$g(\psi(V, f)) \cap S^{n+2} \times \nu = N_\nu, \quad \nu = 1, 2.$$

PROOF. The possibility to realize α by imbedding $\varphi: (D_+^k, \partial D_+^k) \rightarrow (V, V_1)$ follows from Corollary 1.1 of Hudson's work [3]. Since $i_{+*}(\alpha) = 0$, then there exists a continuous map $h: D^{k+1} \rightarrow S^{n+2} \times [1, 2]$ extending φ and such that

$$h(D^{k+1}) \cap S^{n+2} \times 1 = h(D_-^k),$$

$$h(D^{k+1}) \cap S^{n+2} \times 2 = \emptyset,$$

$$h(D^{k+1}) \cap V = h(D_+^k).$$

By virtue of Theorem 1 of [3], applied to the manifold, which has been obtained from $S^{n+2} \times [1, 2]$ by cutting along V and smoothing corners, we may suppose that h is an imbedding. Now we may obtain the desired imbedding g by thickening V and $h(D^{k+1})$ in $S^{n+2} \times [1, 2]$.

This proves the lemma.

It is obvious that the analogy construction and statement are true if i_+ is replaced by i_- or V_1 by V_2 .

This lemma will enable us to prove Theorem 3.4 considering submanifold $g(W) \subset S^{n+2} \times [1, 2]$ instead of V , since $\partial V_1 = \partial g(W_1)$ and V_1 and $g(W_1)$ have contiguous homotopy Seifert pairings (as it follows from Theorem 2.4). The same is true for homotopy Seifert pairings of V_2 and $g(W_2)$. We shall say that $g(W)$ is obtained by killing α .

3.9. LEMMA. Let V, f and W be the same as in 3.7 and let $\alpha \in \pi_k(V, V_1)$ be the homotopy class realized by imbedding f . Let $\beta \in \pi_{n+1-k}(W, W_2)$ be the homotopy class realized by sphere $x_0 \times S^{n+1-k} \subset W$, where $x_0 \in D^{k+1}$. Then

- (a) $H_i(W, W_1) \approx H_i(V, V_1)$ for $i < k$;
- (b) $H_k(W, W_1) \approx H_k(V, V_1)/(h(\alpha))$, where h denotes the Hurewicz homomorphism and $(h(\alpha))$ is the subgroup generated by $h(\alpha)$;
- (c) $H_j(W, W_2) \approx H_j(V, V_2)$ for $j < n+1-k$;
- (d) $H_{n+1-k}(V, V_2) \approx H_{n+1-k}(W, W_2)/(h(\beta))$;
- (e) if $h(\alpha)$ has a finite order, then $h(\beta)$ is an element of infinite order;
- (f) if $h(\alpha)$ is an indivisible element, then $h(\beta) = 0$;
- (g) if V, V_1, V_2 are r -connected and $r < k < n+1-r$, then W, W_1, W_2 are also r -connected.

PROOF. It is easy to see that the pair (W, W_1) is homeomorphic to $(V - f(D_+^k \times \text{int } D^{n+2-k}), (V_1 - f(\partial D_+^k \times \text{int } D^{n+2-k}) \cup f(D_+^k \times \partial D^{n+2-k})))$. The inclusion of the last pair in $(V, V_1 \cup f(D_+^k \times D^{n+2-k}))$ is an excision map and induces the isomorphism of homology groups. Thus

$$H_*(W, W_1) \approx H_*(V, V_1 \cup f(D_+^k \times D^{n+2-k})).$$

Now (a) and (b) follow from consideration of the corresponding homology sequence.

Let us note that (W, W_2) is homeomorphic to $(V - f(D_+^k \times \text{int } D^{n+2-k}), V_2)$ and now the inclusion of the last pair to (V, V_2) gives the homomorphisms $H_j(W, W_2) \rightarrow H_j(V, V_2)$. From this one may easily deduce (c) and (d).

Let us prove (e). We shall identify (W, W_2) with $(V - f(D_+^k \times \text{int } D^{n+2-k}), V_2)$ by means of the evident homeomorphism. Then $h(\beta)$ will be identified with the homology class which is realized by $f(x_0 \times S^{n+1-k})$, where $x_0 \in D_+^k$. Suppose that $h(\beta)$ has finite order p , say. Then there is an $(n+2-k)$ -chain c in $V - f(D_+^k \times \text{int } D^{n+2-k})$ such that

$$\partial c = pz_0 + u,$$

where z_0 is the fundamental circle of $f(x_0 \times S^{n+1-k})$ and u is some chain in V_2 . Denote by y_0 the chain in $f(x_0 \times D^{n+2-k})$ for which $\partial y_0 = z_0$. Let $c_1 = py_0 - c$. Then c_1 is a circle module V_2 and the intersection number of $h(\alpha)$, and $\{c_1\}$ is equal to $\pm p \neq 0$. But it is impossible if $h(\alpha)$ has a finite order. This proves (e).

If $h(\alpha)$ is an indivisible element, then Poincaré duality implies that there is a homology class $u \in H_{n+2-k}(V)$ such that the intersection number $h(\alpha) \cdot u$ is equal to 1. We may realize u by a circle of the form $a + b$, where a is a chain in $f(D_+^k \times D^{n+2-k})$ and b is a chain in $V - f(D_+^k \times \text{int } D^{n+2-k})$. It is clear that a determines the homology class of the pair $(f(D_+^k \times D^{n+2-k}), f(D_+^k \times S^{n+1-k}))$ which is homologous to y_0 . Thus there is some $(n+2-k)$ -chain e in $f(D_+^k \times S^{n+1-k})$ such that $\partial e = z_0 - \partial a$. Consider $e + b$. It is a chain in $V - f(D_+^k \times \text{int } D^{n+2-k})$ and its boundary is equal to z_0 . Therefore z_0 is homologous to zero and so $h(\beta) = 0$. This proves (f).

Assertion (g) is standard and well known.

3.10. LEMMA. Suppose we have conditions as in Theorem 3.4. Let q be some integer, $2q \leq n$. Then there is an $(n+2)$ -dimensional compact submanifold W (with corners) of $S^{n+2} \times [1, 2]$, satisfying the following conditions:

- (a) $\partial W = W_1 \cup W_2 \cup X$, where $W_\nu = W \cap S^{n+2} \times \nu$, $\nu = 1, 2$, and X is the trace of isotopy, translating ∂V_1 and ∂V_2 ;
- (b) manifolds W, W_1, W_2 are r -connected;
- (c) from (a). It follows that $\partial W_\nu = \partial V_\nu$, $\nu = 1, 2$, and if we orient W_ν so that this equality will be true as for oriented manifolds, then homotopy Seifert pairings of W , and V_ν are R -equivalent in $Z_{r,n}$, $\nu = 1, 2$;
- (d) $H_i(W, W_\nu) = 0$ for $i \leq q$, $\nu = 1, 2$.

PROOF. We construct W by induction. First we have the manifold V constructed in Lemma 3.5. This manifold satisfies all required conditions for $q = r$.

Consider the group $\pi_{r+1}(V, V_1) = H_{r+1}(V, V_1)$. If this group is nonzero, then by virtue of Lemma 3.6 there is $\alpha \in H_{r+1}(V, V_1)$, $\alpha \neq 0$ for which $i_{+*}\alpha = 0$ or $i_{-*}\alpha = 0$. Lemma 3.8 enables us to construct manifold V' obtained from V by killing α . Then V' will satisfy the condition (a) (as it follows from the construction) and (b) (by virtue of 3.9(g)). The condition (c) will also be satisfied (it follows from Theorem 2.4, see remark after proof of Lemma 3.8). Besides

$$H_i(V', V'_1) = 0 \quad \text{for } i < r \quad (\text{see 3.9(a)});$$

$$H_j(V', V'_2) = 0 \quad \text{for } j < r \quad (\text{see 3.9(c)});$$

and the group $H_{r+1}(V', V'_1)$ will be a proper factor-group of $H_{r+1}(V, V_1)$ (by virtue of 3.9(b)).

Since $H_{r+1}(V, V_1)$ is a finitely generated abelian group, this procedure may be iterated a finite number of times after which i_{+*} and i_{-*} will both be monomorphisms. Then, by Lemma 3.6, we shall have V'' with $H_i(V'', V''_1) = 0$ for $i < r + 1$.

By the same way we may kill group $H_{r+1}(V'', V''_2)$ without changing groups $H_i(V'', V''_1)$ for $i \leq r + 1$ because $2(r + 1) \leq 2q < n$ (see Lemma 3.9(c)).

This shows that we can proceed by induction on q .

This completes the proof.

3.11. The previous lemma permits us to kill homology groups below the middle dimension. In order to kill middle dimensional homology groups we must consider two cases.

Case 1. n is even. Let $n = 2q$. Then Lemma 3.10 gives a manifold W and we want to kill groups $H_{q+1}(W, W_1) = \pi_{q+1}(W, W_1)$ and $H_{q+1}(W, W_2) = \pi_{q+1}(W, W_2)$. Since $\partial W = W_1 \cup W_2 \cup X$, where $X = \Sigma^n \times [1, 2]$, Poincaré duality and the universal coefficients theorem imply that these groups are isomorphic and free abelian. If these groups are nonzero, then by virtue of Lemma 3.6 there is an indivisible element $\alpha \in \pi_{q+1}(W, W_1)$ with $i_{+*}\alpha = 0$ or $i_{-*}\alpha = 0$. If W' is obtained by killing α , then $h(\beta) = 0$ (see Lemma 3.9(f)) and so $H_q(W', W'_2) = H_q(W, W_2) = 0$ by Lemma 3.9(d). Besides rank of $H_{q+1}(W', W'_1)$ is smaller than rank of $H_{q+1}(W, W_1)$ and so, by a finite number of steps we shall have W'' with $H_i(W'', W''_1) = 0$ for $i < q + 1$ and $H_j(W'', W''_2) = 0$ for $j < q$. But then Poincaré duality implies that

$$H_*(W'', W''_\nu) = 0, \quad \nu = 1, 2.$$

Case 2. n is odd, $n = 2q + 1$. Let W be the manifold constructed in Lemma 3.10. If group $\text{Tors } \pi_{q+1}(W, W_1)$ is nonzero, then by Lemma 3.6 this group contains $\alpha \neq 0$ with $i_{+*}\alpha = 0$ or $i_{-*}\alpha = 0$. Let W' be obtained from W by killing α . Then W' also satisfies the conditions of Lemma 3.10 and $\text{Tors } H_{q+1}(W', W'_1)$ is a proper factor-group of $\text{Tors } H_{q+1}(W, W_1)$. Thus after a finite number of such operations we obtain manifold W'' with $\text{Tors } H_{q+1}(W'', W''_1) = 0$. Now we may kill by the same way the group $\text{Tors } H_{q+1}(W'', W''_2)$. Under this, the group $H_{q+1}(W'', W''_1)$ can increase, but its torsion subgroup will remain zero as it follows from Lemma 3.9(e). So we shall obtain submanifold W''' satisfying all conditions of Lemma 3.10

and

$$\text{Tors } H_{q+1}(W''', W_1''') = \text{Tors } H_{q+1}(W''', W_2''') = 0.$$

After this, indivisible elements of groups $H_{q+1}(W''', W_1''')$ and $H_{q+1}(W''', W_2''')$ may be killed as in Case 1 (by virtue of Lemmas 3.6 and 3.9(f)). Now Poincaré duality implies that we have killed all relative homology groups.

Thus in both cases we have proved the following:

3.12. LEMMA. *Under assumptions of Theorem 3.4 there is an $(n+2)$ -dimensional compact submanifold W (with corners) of $S^{n+2} \times [1, 2]$ which has properties (a), (b), (c) of Lemma 3.10 and*

$$H_*(W, W_\nu) = 0 \quad \text{for } \nu = 1, 2.$$

3.13. PROOF OF THEOREM 3.4. Let W be a submanifold constructed as in Lemma 3.12. Then W is a relative h -cobordism and, by virtue of the h -cobordism theorem, the pair (W, X) is diffeomorphic to $(W_1, \partial W_1) \times [1, 2]$. So we obtain a concordance between W_1 and W_2 . By theorems of Rourke [11], there is an isotopy of $S^{n+2} \times [1, 2]$ carrying W on $W_1 \times [1, 2]$. Thus we obtain an isotopy of S^{n+2} carrying W_2 on W_1 with preservation of orientations, specified in Lemma 3.10(c). So the homotopy Seifert pairings of W_1 and W_2 are R -equivalent in $Z_{r,n}$. Since the homotopy Seifert pairing of V_ν is R -equivalent in $Z_{r,n}$ to that of W_ν , $\nu = 1, 2$ (by Lemma 3.10(c)), Theorem 3.4 follows.

Theorems 3.3 and 3.4 give the main result of this paper:

3.14. THEOREM. *Suppose that $r \geq 2$, $n \geq 4$. There is a mapping*

$$K_{r,n} \rightarrow Z_{r,n}/R$$

which assigns to a knot the R -equivalence class of the homotopy Seifert pairing of any r -connected Seifert manifold of this knot. If $3r \geq n+1 \geq 6$, then this mapping is bijective.

The first assertion of this theorem is true without assumptions that $r \geq 2$, $n \geq 4$ (see remark after the formulation of Theorem 3.4).

3.15. As an example we consider the class of simple odd-dimensional knots studied by J. Levine and deduce from Theorem 3.14 an algebraic classification of such knots in terms of Seifert matrices. The result will be essentially the same as Levine's [10].

For this class of knots $n = 2q - 1$, $r = q - 1$. Suppose that $q \geq 3$. Let (K, θ) be representative of some element from $Z_{q-1, 2q-1}$. This means that K is a finite $(q-1)$ -connected complex and $\theta: K \wedge K \rightarrow S^{2q}$ is a homotopy pairing for which $\theta + \theta'$ is a Spanier-Whitehead duality. It follows that $H^i(K; G)$ is isomorphic to $H_{2q-i}(K; G)$ and therefore is equal to zero, for $i > q$. Thus K has a homotopy type of bouquet of q -dimensional spheres. For each of these spheres choose a corresponding homology class $z_i \in H_q K$. Consider the integral square matrix A with elements $a_{ij} = \theta_*(z_i \wedge z_j) \in H_{2q}(S^{2q}) = \mathbb{Z}$. It follows from Hopf's theorem that two pairs (K_1, θ_1) and (K_2, θ_2) represent the same element of $Z_{q-1, 2q-1}$ if and only

if the corresponding matrices A_1 and A_2 are congruent (this means that the integral unimodular square matrix P exists such that $P'A_1P = A_2$). Since $\theta + \theta'$ is a Spanier-Whitehead duality, then $A + (-1)^q A'$ is a unimodular matrix, where the prime means transposition. Thus the set $Z_{q-1,2q-1}$ is in one-to-one correspondence with the set of congruence classes of integral square matrices A for which $\det(A + (-1)^q A') = \pm 1$.

We shall call two integral square matrices A_1 and A_2 , of sizes $r \times r$ and $s \times s$, respectively, contiguous if $r = s \bmod 2$ and an integral rectangular $(r \times s)$ -matrix B exists such that the matrix

$$\begin{pmatrix} A & B \\ (-1)^{q+1}B' & (-1)^{q+1}A_2' \end{pmatrix}$$

is congruent to matrix of the form $\begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix}$, where C is a square unimodular matrix of size $(r+s)/2 \times (r+s)/2$ and 0 means a zero square matrix of the same size (compare with Definition 2.3).

The equivalent definition may be obtained from the previous one replacing C by the identity matrix.

We shall call two integral square matrices A_1 and A_2 R -equivalent if there exists a sequence of integral square matrices $D_1, D_2, \dots, D_k$ such that D_i and D_{i+1} are contiguous for $i = \overline{1, k+1}$, matrix $D_i + (-1)^q D_i'$ is unimodular for $i = \overline{1, k}$ and $A_1 = D_1, A_2 = D_k$.

It is clear that the R -equivalence relation on $Z_{q-1,2q-1}$ in the sense of 3.2 coincides, by means of the above-mentioned one-to-one correspondence, with the R -equivalence relation just introduced on the set of square matrices.

Now it follows from Theorem 1.14 that the map which assigns to a knot from $K_{q-1,2q-1}$ the R -equivalence class of any Seifert matrix is a bijection of $K_{q-1,2q-1}$ on the set of R -equivalence classes of square matrices A with

$$\det(A + (-1)^q A') = \pm 1.$$

It is not difficult to show that the R -equivalence relation on the set of Seifert matrices coincides with the S -equivalence relation. It implicitly follows from the comparison of the results of the present paper with the work [10].

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