TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 262, Number 1, November 1980

NONINVARIANCE OF AN APPROXIMATION PROPERTY FOR CLOSED SUBSETS OF RIEMANN SURFACES

BY

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ABSTRACT. A closed subset E of an open Riemann surface M is said to have the approximation property $\mathcal{Q}$ if each continuous function on E which is analytic at all interior points of E can be approximaed uniformly on E by functions which are everywhere analytic on M. It is known that $\mathcal{Q}$ is a topological invariant (i.e., preserved by homeomorphisms of the pair (M, E)) when M is of finite genus but not in general, not even for C^∞ quasi-conformal automorphisms of M. The principal result of this paper is that $\mathcal{Q}$ is not invariant even under a real-analytic isotopy of quasi-conformal automorphisms (of a certain M). M is constructed as the two-sheeted unbranched cover of the plane minus a certain discrete subset of the real axis, and the isotopy is induced by $(x + iy, t) \mapsto x + ity$, for $t > 0$; E can be taken to be that portion of M which lies over a horizontal strip.

Let M be an open Riemann surface and E be a closed subset of M. Denote by $A(E)$ the collection of continuous functions on E which are analytic on the interior of E. Say that E has property $\mathcal{Q}$ in M if each element of $A(E)$ can be approximated uniformly on E by functions which are analytic everywhere on M. By definition $\mathcal{Q}$ is a property of the pair (M, E) and is a conformal invariant; that is, if one pair is related to another by an analytic homeomorphism, the both pairs have $\mathcal{Q}$ or neither one does. It is natural to ask whether $\mathcal{Q}$ is invariant for other types of equivalence. The famous theorem of Bishop and Mergelyan implies that $\mathcal{Q}$ is a topological invariant in case E is compact, and the theorem of Arakelyan shows that $\mathcal{Q}$ is a toplogical invariant when M is planar. By "topological invariant" I mean an invariant for the equivalence defined by homeomorphism of pairs. It is known [S] that $\mathcal{Q}$ is a topological invariant when M is of finite genus but is not a topological invariant in general. In fact, $\mathcal{Q}$ is not an invariant even for the finer partition induced by this relation: call (M, E) equivalent to (M', E') if and only if M is conformally equivalent to M' and there is a quasi-conformal homeomorphism of M onto M' which carries E onto E' [S]. When I showed him the known example illustrating this phenonmenon, Dennis Sullivan asked me whether the example could be improved to one in which $\mathcal{Q}$ fails to be preserved by an isotopy. The intent of this article is to provide an affirmative answer by demonstrating that a real-analytic isotopy need not preserve $\mathcal{Q}$, even though it is "affine", being

Received by the editors November 8, 1979.

1980 *Mathematics Subject Classification*. Primary 30E10, 30F99.

[1]Partially supported by NSF Grant MCS77-01848. I am grateful to the Hebrew University, Jerusalem, and the Mathematical Institute, Oxford, for their gracious hospitality during my visit in the spring of 1978, when much of this work was done.

0002-9947/80/0000-0508/$04.50

definable in local coordinates by $(x + iy, t) \to x + ity$ for $0 < t < \infty$, and all the homeomorphisms of M throughout the homotopy are quasi-conformal equivalences. The precise statement is given below as the theorem. Let us write "$E \in \mathcal{Q}$" in place of "E has property $\mathcal{Q}$ in M".

THEOREM. *There is a connected open Riemann surface M, a homotopy U: $M \times \mathbf{R}^+ \to M$, and two closed connected subsets E^+ and E^- of M such that the following hold, where $U_t(p)$ is written in place of $U(p, t)$.*

(a) *M can be given as a two-sheeted unbranched cover of a plane region by Z: $M \to \mathbf{C} - \mathfrak{D}$, where $\mathfrak{D}$ is a discrete subset of the real axis, and the homotopy U is induced by the affine plane homotopy u: $\mathbf{C} \times \mathbf{R}^+ \to \mathbf{C}$ defined by $u(x + iy, t) = x + ity$; that is, $Z(U(p, t)) = u(Z(p), t)$ for all $p \in M$ and $t \in \mathbf{R}^+$.*

(b) *U is real-analytic on $M \times \mathbf{R}^+$.*

(c) *Each U_t is a real-analytic quasi-conformal automorphism of M.*

(d) *U_1 is the identity of M.*

(e) *As $t \to 1$ the distortion $|\partial U_t/\partial \bar{Z}|\ |\partial U_t/\partial Z|^{-1} \to 0$ uniformly on M.*

(f) $\{t\colon U_t(E^+) \in \mathcal{Q}\} = (0, 1]$.

(g) $\{t\colon U_t(E^-) \in \mathcal{Q}\} = (0, 1)$.

Because of the relation between U and u one may call U or U_t "affine". For t near 1 the maps U_t are as close to conformal and as close to the identity as may be desired; yet the slight movement of E^+ to $U_{1+\varepsilon}(E^+)$ destroys $\mathcal{Q}$ and the slight movement of E^- to $U_{1-\varepsilon}(E^-)$ creates $\mathcal{Q}$.

The proof of the theorem will be along the following lines. A locally finite collection of closed intervals J will be selected in $\mathbf{R}$ according to certain technical requirements. M will be formed as follows: two copies of $\mathbf{C}$ minus all the J's will be joined in the standard way along corresponding slits J, leaving out the set $\mathfrak{D}$ consisting of all the endpoints of the J's. The natural projection Z: $M \to \mathbf{C} - \mathfrak{D}$ exhibits M as a two-sheeted unbranched covering of a plane region. The homotopy defined on $\mathbf{C} \times \mathbf{R}^+$ by $u(x + iy, t) = x + ity$ transfers via Z to a homotopy defined on each component of $Z^{-1}(V) \times \mathbf{R}^+$, where V is any vertical strip in $\mathbf{C} - \mathfrak{D}$ whose projection on the y-axis is $\mathbf{R}$, $\mathbf{R}^+$, or $\mathbf{R}^-$. These homotopies agree whenever there is an overlap; so they define a homotopy U on M. Assertions (a)–(e) will follow immediately.

Define S_λ to be the strip $\{x + iy\colon |y| \leq \lambda\}$ and S_λ^+ (resp., S_λ^-) to be the intersection of S_λ with the closed right (resp., left) half-plane. Define $E_\lambda^\pm = Z^{-1}(S_\lambda^\pm)$. The intervals J will be arranged in $\mathbf{R}$ in such a fashion that for a certain $\lambda_0 > 0$ none of the sets S_λ^+ for $\lambda > \lambda_0$ (resp., S_λ^- for $\lambda \geq \lambda_0$) supports a nontrivial bounded analytic function which vanishes on $S_\lambda^+ \cap \mathfrak{D}$ (resp., $S_\lambda^- \cap \mathfrak{D}$). This will imply that $\mathcal{Q}$ fails for the corresponding E_λ^+ (resp., E_λ^-). Because $U_t(E_\lambda^\pm) = E_{t\lambda}^\pm$, this will mean that one-half of (f) and (g), namely with "$\subseteq$" in place of "$=$", will be true for $E^+ = E_{\lambda_0}^+$ and $E^- = E_{\lambda_0}^-$.

The proof of the other half of (f) and (g) is technically more difficult and will require the bulk of the work. These ideas will be involved. Fix one E_λ^+, $\lambda \leq \lambda_0$, or one E_λ^-, $\lambda < \lambda_0$, and call it E. Because of the spatial arrangement of $\mathfrak{D}$ there will exist a sequence of meromorphic functions H_n on $\mathbf{C}$ each of which has a zero or a pole at each point of $\mathfrak{D}$ and nowhere else and each of which is small on a large bounded set X_n, is large on a co-compact subset Y_n of $S_\lambda^{\pm}$, and is nearly 1 on a vertical strip σ_n which separates X_n from Y_n. Multiplying H_n by an exponential function and extracting a square root, we obtain an analytic function π_n on M which separates these four sets: $Z^{-1}(X_n)$, the two components of $Z^{-1}(\sigma_n)$, and $Z^{-1}(Y_n)$. $E - \bigcup Z^{-1}(\sigma_n)$ consists of separated pieces of finite genus; so if $f \in A(E)$, there is an analytic Φ on M such that $g = f - \Phi$ is very small on $E - \bigcup Z^{-1}(\sigma_n)$. g can be approximated by $\Sigma\, g_n$, where g_n is C^1 on $E^\circ \cup Z^{-1}(X_n)$, is supported in $Z^{-1}(\sigma_n)$, and has small $\bar{Z}$-derivative. Because π_n separates the four sets indicated above, g_n can be written $g_n = \tilde{g}_n \circ \pi_n$, where $\tilde{g}_n$ is (essentially) a smooth function of compact support in $\mathbf{C}$ having small $\bar{z}$-derivative. Because of this property of $\tilde{g}_n$ and of the nature of $\pi_n(Z^{-1}(X_n \cup \sigma_n \cup Y_n))$, $\tilde{g}_n$ can be approximated reasonably well by a rational function k_n. Thus, $k_n \circ \pi_n$ approximates g_n on $E \cup Z^{-1}(X_n)$. The poles of $k_n \circ \pi_n$ in M can be removed by a multiplier without essentially altering the goodness of the approximation to g_n. The resulting analytic functions ψ_n on M can be summed to an analytic function Ψ, because $Z^{-1}(X_n)\uparrow M$, and thus $\Phi + \Psi$ will approximate f.

The reader acquainted with [S] will recognize a similarity in spirit between the above scheme for showing $E \in \mathcal{Q}$ and the method used in §11 of [S]. However, the method of [S] is technically simpler in several respects. For example, the projection π of the surface in [S] served to separate all the components of all the $\pi^{-1}(\sigma_n)$ at once, and it was not necessary to allow singularities to arise in intermediate steps for later removal. The punctures in the surface of [S] allowed the existence of enough globally analytic functions but prevented the existence of an isotopy. The surface M of this article is punctured in order to allow construction of the separating functions π_n; since the punctures lie over the real axis, they do not interfere with the isotopy. I thank Ted Gamelin for asking me whether a related result utilized an iteration of some sort. This remark prompted me to look for an approximation of $g = f - \Phi$ by means of separate approximations of "pieces" of g; this in turn opened up the possiblity of individual separating functions π_n for each "piece". And I thank Dennis Sullivan for raising the question which this paper answers.

The rest of the paper is devoted to a proof of the theorem. Most of the technical aspects are gathered into manageable or convenient aggregates and termed lemmas or corollaries.

LEMMA 1 (a). *If* $|z| \leqslant \frac{1}{2}$, *then* $|\log(1 + z) - z| \leqslant |z|/2$.
(b) *If* $|w| \leqslant 1$, *then* $|e^w - 1| \leqslant (e - 1)|w| \leqslant 2|w|$.
(c) *If* $\Sigma|a_n| \leqslant \frac{1}{2}$, *then* $|\Pi(1 + a_n) - 1| \leqslant 3\,\Sigma|a_n|$.

PROOF. (a) and (b) are well known and follow readily from easy manipulations of Taylor series. Assume $\Sigma|a_n| \leqslant \frac{1}{2}$. From (a) it follows that $|\log \Pi(1 + a_n) - \Sigma\, a_n| \leqslant \frac{1}{2}\,\Sigma|a_n|$; so $|\log \Pi(1 + a_n)| \leqslant \frac{3}{2}\,\Sigma|a_n| \leqslant \frac{3}{4}$. Now an application of (b) yields (c).

LEMMA 2. *For* $0 < \lambda < \infty$ *and complex* z_0 *and* z *define*

$$\tau(z; z_0, \lambda) = (a^z - a^{z_0})/(a^z + a^{\bar{z}_0}),$$

where $a = \exp(\pi/2\lambda)$. *Suppose* $0 < \lambda < \pi(2 \log 2)^{-1}$ *and* $|x - x_0| > 1$, *where* $x = \operatorname{Re} z$ *and* $x_0 = \operatorname{Re} z_0$. *Let* $\omega = \operatorname{signum}(x - x_0)$. *Then*

$$|1 - \omega\tau(z, z_0, \lambda)^{\pm 1}| \leqslant 4a^{-|x-x_0|} = 4 \min(a^x \cdot a^{-x_0}, a^{x_0} \cdot a^{-x}).$$

PROOF. In case $x \geqslant x_0 + 1$ compute

$$|1 - \tau^{+1}| = \left|\frac{a^{z_0} + a^{\bar{z}_0}}{a^z + a^{\bar{z}_0}}\right| = a^{x_0 - x}\left|\frac{1 + a^{\bar{z}_0 - z_0}}{1 + a^{\bar{z}_0 - z}}\right| \leqslant a^{x_0 - x}\frac{2}{1 - a^{x_0 - x}} \leqslant 4a^{x_0 - x},$$

because $|a^{\bar{z}_0 - z_0}| = 1$ and $a^{x_0 - x} \leqslant a^{-1} < \frac{1}{2}$. The computations for τ^{-1} and for the case $x \leqslant x_0 - 1$ are very similar.

LEMMA 3. *If* x_n *is a sequence of distinct real numbers such that* $|x_n| \to \infty$ *and* $0 < \lambda < \infty$, *the following are equivalent.*

(a) *There exists a nonzero bounded analytic function on* S_λ *which vanishes on* $X = \{x_n : n \geqslant 1\}$.

(b) *On* S_λ^+ *there is a nonzero bounded analytic function which vanishes on* $X \cap S_\lambda^+$, *and the analogous statement holds for* S_λ^-.

(c) $\sum a^{-|x_n|} < \infty$, *where* $a = \exp(\pi/2\lambda)$.

(d) $\prod_n \operatorname{sgn}(x_n)(a^{x_n} - a^z)/(a^{x_n} + a^z)$ *converges normally on the plane to a function which is analytic on* $S_{2\lambda}^\circ$ (*the interior of* $S_{2\lambda}$), *is bounded by* 1 *on* S_λ, *and vanishes in* $S_{2\lambda}$ *precisely on* X.

PROOF. The conformal map $w = \varphi(z) = \exp(\pi z/2\lambda)$ carries S_λ to the closed right half-plane minus 0 and takes X to a positive sequence $\{w_n\}$ which clusters only at 0 and/or ∞. The Blaschke condition for a real sequence w_n tending to 0 (resp. ∞) in the right half-plane is easily seen to be $\sum w_n < \infty$ (resp., $\sum w_n^{-1} < \infty$). Because a convergent Blaschke product on the open half-plane is analytic on the closed half-plane minus the cluster set of its zeros, (a) is equivalent to (c). Trivially, (a) implies (b). The above mapping $w = \varphi(z)$ (resp., $w = 1/\varphi(z)$) sends S_λ^- (resp., S_λ^+) onto the half-disc $\{w : |w| \leqslant 1, \operatorname{Re} w > 0\} - \{0\}$, which contains the disc $\Delta = \{w : |w - \frac{1}{2}| < \frac{1}{2}\}$. So (b) implies the Blaschke condition for the real sequence $w_n = \varphi(x_n) \to 0$ (resp., $w_n = 1/\varphi(x_n) \to 0$) in Δ, which implies (c). Recall that a sequence, series, or product of meromorphic functions f_n is said to converge normally on a region R if for each compact subset K of R all the functions f_n are analytic on K for $n \geqslant N(K)$ and the sequence, series, or product of the f_n for $n \geqslant N(K)$ converges uniformly on K. Lemma 2 and simple observations about $\tau(z; x_n, \lambda)$ reveal that (c) implies (d). Finally, (d) trivially implies (a).

LEMMA 4. *There exist sequences* s_n, t_n, K_n, *and* N_n *of positive integers such that*

(1) $1 \leqslant s_1 < t_1 < s_2 < t_2 < \cdots$ *and* $1 \leqslant K_1 < N_1 < K_2 < N_2 < \cdots$,

(2) $2^{-t_n/n} \sum_1^n 2^{s_j} N_j \to 0$ *as* $n \to \infty$; *in particular*, $t_n - s_n \to \infty$,

(3) $16^{t_n} \sum_{n+1}^\infty 4^{-s_j}(K_j + 4^{-s_j/j} N_j) \to 0$ *as* $n \to \infty$; *in particular*, $s_{n+1} - t_n \to \infty$ *and* $\sum_1^\infty 4^{-s_n}(K_n + 4^{-s_n/n} N_n) < \infty$, *and*

(4) $\sum 4^{-s_n} N_n = \infty = \sum 4^{-s_n} 4^{\varepsilon s_n} K_n$ *for every* $\varepsilon > 0$.

PROOF. Define three sequences of integers by the recursion: $r_1 = 1$, $s_n = nr_n$, $t_n = n(3s_n + n)$, $r_{n+1} = 2t_n + n$. It is easy to see that r_n and s_n are strictly increasing with n and that $s_{n+1} > nr_{n+1} > t_n > s_n$. Put $N_n = 4^{s_n}$ and $K_n = N_n 4^{-s_n/n} = N_n 4^{-r_n} = 4^{s_n - r_n} = 4^{(n-1)r_n}$. (1) and (4) are immediate.

Each of the sums Σ_1^n in (2) and Σ_{n+1}^∞ in (3) is readily seen to be a sum of distinct powers of 2. Therefore, each sum is at most twice its largest term. For the sum Σ_1^n in (2) this estimate is $2 \cdot 2^{s_n} N_n = 2 \cdot 2^{3s_n} = 2 \cdot 2^{t_n/n} \cdot 2^{-n}$, and (2) follows. For the sum Σ_{n+1}^∞ in (3) the estimate is $2 \cdot 4^{-s_{n+1}} \cdot 2K_{n+1} = 4 \cdot 4^{nr_{n+1} - s_{n+1}} = 4 \cdot 4^{-r_{n+1}} = 4 \cdot 4^{-2t_n} \cdot 4^{-n}$, and (3) follows.

Let us turn now to the specification of M, U, E^+, and E^-. Put $\lambda_0 = \pi(4 \log 2)^{-1}$; any value could serve for λ_0, but this one is convenient because $\exp(\pi/2\lambda_0) = 4$. Select s_n, t_n, K_n, and N_n according to Lemma 4. For each $n > 0$ select K_n disjoint closed subintervals of the real interval $(s_n, s_n + 1)$ and N_n disjoint closed subintervals of $(-s_n - 1, -s_n)$; refer to each of these tiny intervals as J. Consider the intervals J as slits in the plane and join two copies of the plane slit by all the J's in the standard fashion by joining the upper edge of each J in each plane with the lower edge of the same J in the other plane. Call the resulting surface $\overline{M}$ and let $\bar{\pi}$: $\overline{M} \to \mathbf{C}$ be the natural projection of $\overline{M}$ onto $\mathbf{C}$. $\overline{M}$ is exhibited as a branched two-sheeted cover of $\mathbf{C}$. Let $B \subseteq \overline{M}$ be the set of branch points of this covering; that is $\beta \in B$ if and only if $d\bar{\pi}(\beta) = 0$ if and only if $\bar{\pi}(\beta)$ is an endpoint of one of the intervals J. Now define $M = \overline{M} - B$, $Z = \bar{\pi}|_M$, and $\mathfrak{D} = \bar{\pi}(B) =$ the set of endpoints of the J's. Z: $M \to \mathbf{C} - \mathfrak{D}$ realizes M as a two-sheeted unbranched cover of the region $\mathbf{C} - \mathfrak{D}$. Denote by $E_\lambda^\pm$ the sets $Z^{-1}(S_\lambda^\pm)$ in M and put $E^+ = E_{\lambda_0}^+$ and $E^- = E_{\lambda_0}^-$.

As previously indicated, U will be induced by the plane isotopy $u(x + iy, t) = x + ity$, $t > 0$, in this manner. M is covered by open sets T for which $Z|_T$ is a homeomorphism and $Z(T) = I \times I'$, where I is an open interval of $\mathbf{R}$ and $I' = \mathbf{R}$ or $\mathbf{R}^+$ or $\mathbf{R}^-$. I' can equal $\mathbf{R}$ precisely when I is disjoint from $\mathfrak{D}$. For each such T define a homotopy U_T: $T \times \mathbf{R}^+ \to T$ by $U_T(p, t) = (Z|_T)^{-1} u(Z(p), t)$. It is clear that if $T_1 \cap T_2 \neq \varnothing$, then for all $p \in T_1 \cap T_2$ and all $t > 0$, $U_{T_1}(p, t) = U_{T_2}(p, t) = U_{T_1 \cap T_2}(p, t)$. Therefore, we may define $U(p, t) = U_T(p, t)$ for any T which contains p and for all $t > 0$. Because Z is a local coordinate at every point of M and is an analytic homeomorphism on each T, it is immediate from the definition of U and elementary properties of u that (a)–(e) hold.

Because $u_t(S_\lambda^\pm) = S_{t\lambda}^\pm$ it is immediate that $U_t(E_\lambda^\pm) = E_{t\lambda}^\pm$. So (f) and (g) are equivalent to the following statements:

(f′) $E_\lambda^+ \in \mathcal{Q}$ if and only if $\lambda \leqslant \lambda_0$.

(g′) $E_\lambda^- \in \mathcal{Q}$ if and only if $\lambda < \lambda_0$.

The "only if" parts of (f′) and (g′) are proved as follows. Fix $\lambda \geqslant \lambda_0$ and define $a = a(\lambda) = \exp(\pi/2\lambda)$. $a(\lambda_0) = 4$, and $a(\lambda) = 4^{1-\varepsilon}$ for some $\varepsilon > 0$ when $\lambda > \lambda_0$. For $0 \leqslant s < x < s + 1$ and $a \leqslant 4$ we find $a^{-x} > 4^{-1}a^{-s}$ and $4^{-x} > 4^{-1}4^{-s}$. From Lemma 3 and from part (4) of Lemma 4 it then follows that if φ is a bounded analytic function on S_λ^- for $\lambda \geqslant \lambda_0$ (or on S_λ^+ for $\lambda > \lambda_0$) which vanishes at every point of $\mathfrak{D} \cap S_\lambda^-$ (or $\mathfrak{D} \cap S_\lambda^+$), then φ vanishes identically. The following lemma then yields that the corresponding sets $E_\lambda^\pm = Z^{-1}(S_\lambda^\pm)$ in M do not belong to $\mathcal{Q}$. Special cases of this lemma were used in **[S]**.

LEMMA 5. *Suppose π_1: $M_1 \to R$ is a two-sheeted branched cover of a connected open subset R of* $\mathbf{C}$ *with branch set $B_1 \subseteq M_1$. Let $B_0 \subseteq B_1$; put $M_2 = M_1 - B_0$ and $\pi_2 = \pi_1|_{M_2}$. If S is a subregion of R such that $\bar{S} \neq R$ and* 0 *is the only bounded analytic function on S which vanishes at all points of $\pi_1(B_1) \cap S$, then $\pi_2^{-1}(\bar{S})$, which is $\pi_1^{-1}(\bar{S}) - B_0$, does not have property $\mathcal{Q}$ in M_2.*

PROOF. For any function g on $\pi_2^{-1}(S)$, define Δg on $S - \pi_1(B_1)$ by $\Delta g(z) = (g(p_1) - g(p_2))^2$, where $\{p_1, p_2\} = \pi_2^{-1}(z)$. (See **[RS]**, where this idea is used.) Δg is analytic on $S - \pi_1(B_1)$ whenever g is analytic on $\pi_2^{-1}(S)$. If furthermore g is bounded on $\pi_2^{-1}(S)$, Riemann's theorem on removable singularities implies that g extends to a bounded analytic function on $\pi_1^{-1}(S) \subseteq M_1$ and that Δg extends similarly to S. The extended Δg vanishes at every point $z_1 \in \pi_1(B_1) \cap S$, because as z tends to z_1 the two points p_1 and p_2 coalesce to the single point of B_1 lying over z_1. By hypothesis Δg must therefore vanish identically on S whenever g is a bounded analytic function on $\pi_2^{-1}(S)$.

Now choose $z_0 \in R - (\bar{S} \cup \pi_1(B_1))$ and let $\{p_1, p_2\} = \pi_2^{-1}(z_0)$. Select a meromorphic function f on M_2 which has a pole at p_1 as its only singularity **[BS]**. Then f is analytic on $\pi_2^{-1}(\bar{S})$ and yet f cannot be approximated uniformly on $\pi_2^{-1}(\bar{S})$ by an analytic function F on M_2. For if F were analytic on M_2 and $|f - F| < 1$ on $\pi_2^{-1}(\bar{S})$, the foregoing paragraph shows that $\Delta(f - F) \equiv 0$ on S. By uniqueness of analytic functions $\Delta(f - F) \equiv 0$ on $R - (\pi_1(B_1) \cup \{z_0\})$. However, this is contradicted by the fact that $f - F$ is bounded near p_2 and unbounded near p_1. So such an F does not exist, and $\pi_2^{-1}(\bar{S})$ does not have property $\mathcal{Q}$ in M_2.

The proof of the "if" parts of (f′) and (g′), namely, that $E_\lambda^+ \in \mathcal{Q}$ for $0 < \lambda \leq \lambda_0$ and that $E_\lambda^- \in \mathcal{Q}$ for $0 < \lambda < \lambda_0$ will require the construction of certain auxiliary functions on $\mathbf{C}$ and on M. Henceforth let b be a variable ranging over $\mathfrak{D} = \bar{\pi}(B)$ = the set of endpoints of the intervals J, and let s_n, t_n, K_n, and N_n be as in Lemma 4. Define functions A_n, B_n, C_n, and D_n as follows, where for $b < 0$ we let $k = k(b)$ be the unique integer such that $s_k < -b < s_k + 1$, $\lambda_b = k\lambda_0(1 + k)^{-1}$, and $a_b = \exp(\pi/2\lambda_b) = 4 \cdot 4^{1/k}$.

$$A_n(z) = \prod_{b<-t_n} \tau(z; b, \lambda_b)^{-1} = \prod \frac{a_b^z + a_b^b}{a_b^z - a_b^b},$$

$$B_n(z) = \prod_{-t_n<b<t_n} \tau(z; b, n\lambda_0) = \prod \frac{4^{z/n} - 4^{b/n}}{4^{z/n} + 4^{b/n}},$$

$$C_n(z) = \prod_{b>t_n} \left(-\tau(z; b, \lambda_0)^{-1}\right) = \prod \frac{4^b + 4^z}{4^b - 4^z},$$

$$D_n(z) = A_n(z)B_n(z)C_n(z).$$

LEMMA 6. (a) *The product for D_n converges normally on the plane to a meromorphic function all of whose zeros and poles are simple.*

(b) $\{z\colon D_n(z) = 0$ *or* ∞ *and* $|\mathrm{Im}\, z| < 4\lambda_0/3\} = \mathfrak{D}$.

(c) *If* $D_n(z) = \infty$ *and* $|\operatorname{Re} z| < t_n$, *then either* $z \in \mathfrak{D}$ *or else* $|\operatorname{Im} z| \geqslant 2n\lambda_0$.
For every $\delta > 0$ *and* $t > 0$ *the following hold for all large enough* n.

(d) $|D_n(z) - 1| < \delta$ *whenever* $t_n - t \leqslant |\operatorname{Re} z| \leqslant t_n + t$.

(e) $|D_n(z)| < 1 + \delta$ *on* $S_{n\lambda_0} \cap \{z: |\operatorname{Re} z| \leqslant t_n + t\}$.

(f) $|D_n(z)| > 1 - \delta$ *on* $(S_{\lambda_0} \cap \{z|\operatorname{Re} z \geqslant t_n - t\}) \cup (S_{\lambda_0 - \delta} \cap \{z: \operatorname{Re} z \leqslant -t_n + t\})$.

PROOF. Let ε be very small, let $x = \operatorname{Re} z$, and let $-s_k - 1 < b < -s_k \leqslant x - 1$. By Lemma 2

$$|1 - \tau(z; b, \lambda_b)^{-1}| \leqslant 4 \cdot 4^{(k+1)x/k} \cdot 4^{-(k+1)s_k/k} \leqslant 4 \cdot 4^{2x} \cdot 4^{-s_k} \cdot 4^{-s_k/k}.$$

From Lemma 1 and part (3) of Lemma 4 it follows that A_n is normally convergent on $\mathbf{C}$ and that $|A_n(z) - 1| < \varepsilon$ for $-t_n - t \leqslant \operatorname{Re} z$, for all large n. In a similar manner we find that C_n is normally convergent and $|C_n(z) - 1| < \varepsilon$ for $\operatorname{Re} z \leqslant t_n + t$, for large n. B_n is convergent, being a finite product, and Lemma 1, Lemma 2, and part (2) of Lemma 4 give $|B_n(z) - 1| < \varepsilon$ for $\operatorname{Re} z > t_n - t$, for large n. Because there are an even number of b's in $(-t_n, t_n)$, $B_n(z) = \prod\tau = \prod(-\tau)$, and the same argument as above shows that $|B_n(z) - 1| < \varepsilon$ for $|\operatorname{Re} z| > t_n - t$, for large n. Each $\tau(z; b, \lambda)$ is periodic with period $4\lambda i$, and z_0 is a zero of τ if and only if $z_0 + 2\lambda i$ is a pole. The smallest λ involved in the product for D_n is $\lambda = k\lambda_0(1 + k)^{-1}$ for $k = n + 1$; so $\lambda \geqslant 2\lambda_0/3$ and $2\lambda \geqslant 4\lambda_0/3$. (a), (b), and (c) are clear, and if ε is small enough (d) follows from the estimates above. Because $|A_n|$ and $|C_n|$ are each bounded by $1 + \varepsilon$ in $|\operatorname{Re} z| \leqslant t_n + t$ and $|B_n| \leqslant 1$ in $S_{n\lambda_0}$, we have (e) for small ε and large n. Finally, because $|B_n(z) - 1| < \varepsilon$ for $|\operatorname{Re} z| > t_n - t$, $|C_n| \geqslant 1$ in S_{λ_0}, and $|A_n| \geqslant 1$ in $S_{(n+1)\lambda_0/(n+2)}$, we have (f) for small ε and large n.

LEMMA 7. *If* $z_0 \in \mathbf{C}$, $\delta > 0$, *and* V *is an open neighborhood of an arc which connects* z_0 *to* ∞, *then there exists an entire function* h *having a simple zero at* z_0 *with no other zeros such that* $|h - 1|_{\mathbf{C}-V} \leqslant \delta$.

PROOF. $|e^\alpha - 1| \leqslant 2|\alpha|$ for $|\alpha| \leqslant 1$, by Lemma 1(b); so $|\alpha - \beta| < 1$ implies $|e^\alpha - e^\beta| \leqslant 2|\alpha - \beta|\,|e^\beta|$. Choose a branch of $\log(z - z_0)^{-1} = -\log(z - z_0)$ in the complement of the given arc γ which joins z_0 to ∞. In V choose a connected simply connected neighborhood V_1 of γ which is the interior of a locally polygonal set. Then $V_1 \cup \{\infty\}$ is connected and locally connected; so Arakelyan's Theorem **[A1]**, **[A2]** can be applied to the function $\log(z - z_0)^{-1}$ on $\mathbf{C} - V_1$ with $\varepsilon = \min(1, \delta/2)$ to yield an entire function g so that $|g(z) - \log(z - z_0)^{-1}| < \varepsilon$ for $z \in \mathbf{C} - V_1 \supseteq \mathbf{C} - V$. Put $\alpha = g(z)$ and $\beta = \log(z - z_0)^{-1}$ in the opening sentence of this proof, and we obtain $|\exp(g(z)) - (z - z_0)^{-1}| \leqslant 2\varepsilon|z - z_0|^{-1} \leqslant \delta/|z - z_0|$ for $z \in \mathbf{C} - V$. Thus, $|(z - z_0)\exp(g(z)) - 1| \leqslant \delta$ for $z \in \mathbf{C} - V$. The function $h(z) = (z - z_0)\exp(g(z))$ has the desired behavior.

COROLLARY 8. *There is a sequence* H_n *of meromorphic functions on* $\mathbf{C}$ *which have these properties. Let* $t > 0$ *and* $\delta > 0$ *be arbitrary.*

(1) $H_n(\bar{z}) = \overline{H_n(z)}$.

(2) *The zeros of* H_n *are simple and comprise the set* $\{b: |b| < t_n\}$.

(3) *The poles of* H_n *are simple and comprise the set* $\{b: |b| > t_n\}$.

(4) $\sup\{|H_n(z) - 1|: t_n - t \leq |\text{Re } z| \leq t_n + t\} \to 0$ *as* $n \to \infty$.

(5) $\sup\{|H_n(z)| : |\text{Re } z| \leq t_n + t$ *and* $|\text{Im } z| \leq n\lambda_0\} \to 1$ *as* $n \to \infty$.

(6) $\inf\{|H_n(z)|: \text{Re } z \geq t_n - t$ *and* $|\text{Im } z| \leq \lambda_0\} \to 1$ *as* $n \to \infty$.

(7) $\inf\{|H_n(z)|: \text{Re } z \leq -t_n + t$ *and* $|\text{Im } z| \leq \lambda_0 - \delta\} \to 1$ *as* $n \to \infty$.

PROOF. Start with the meromorphic functions D_n of Lemma 6; note that $D_n(\bar{z}) = \overline{D_n(z)}$. Enumerate the zeros and poles of D_n in $\{z: \text{Im } z > 0\}$ as $z_1, z_2, \ldots$; then $\bar{z}_1, \bar{z}_2, \ldots$ are the zeros and poles in the lower half-plane. For each j let V_j be a small neighborhood of the line segment $L_j = \{z = x + iy: x = \text{Re } z_j$ and $y \geq \text{Im } z_j\}$ which joins z_j to ∞. Because L_j is disjoint from the set $T_n = S_{\lambda_0} \cup \{z: |\text{Re } z| \leq t_n \text{ and } |\text{Im } z| \leq n\lambda_0\} \cup \{z: |\text{Re } z| \in \bigcup [s_k, s_k + 1]\}$, we may assume that $V_j \cap T_n = \varnothing$, as well. By Lemma 7 there is an entire function h_j which is zero only at z_j and which satisfies $|h_j - 1| \leq \delta_j = 2^{-n-j}$ outside V_j. Put $k_j(z) = \overline{h_j(\bar{z})}$. Let $F_n = \prod(h_j k_j)^{\pm 1}$, where the exponent is chosen to be $+1$ in case $D_n(z_j) = \infty = D_n(\bar{z}_j)$ and is chosen to be -1 in case $D_n(z_j) = D_n(\bar{z}_j) = 0$. Because $\sum 2\delta_j = 2 \cdot 2^{-n} < \infty$ and each compact set meets only finitely many of the V_j or their conjugates, the product for F_n converges normally on the plane to a meromorphic function which by Lemma 1 satisfies $|F_n - 1| \leq 62^{-n}$ on T_n, for $n \geq 2$. From Lemma 6 it is clear that $H_n = F_n D_n$ has all the required properties.

Define $R(x_0; t, \lambda)$ to be the rectangle $\{z = x + iy: |x - x_0| \leq t \text{ and } |y| \leq \lambda\}$. For $n > 0$ define $t_{-n} = -t_n$, $G_n(z) = 2^{z-t_n}H_n(z)$, and $G_{-n}(z) = 2^{t_n - z}H_n(z)$, where H_n is as in Lemma 8. Fix $\lambda_+ \leq \lambda_0$ and $\lambda_- < \lambda_0$, define $\lambda_n = \lambda_+$ for $n > 0$ and $\lambda_n = \lambda_-$ for $n < 0$, and put $S(n) = S^+_{\lambda_n}$ for $n > 0$ and $S(n) = S^-_{\lambda_n}$ for $n < 0$. Define these sets:

$$\sigma_n = \{z \in S(n): 16^{-1} \leq |G_n(z)| \leq 16\},$$

$$X_n = \{z: |\text{Re } z| \leq s_{|n|} + 1 \text{ and } |\text{Im } z| \leq |n|\lambda_0\} \cup (S(n) \cap \{z: \text{Re } z/t_n \leq 1\}) - \sigma_n,$$

$$Y_n = (S(n) \cap \{z: \text{Re } z/t_n \geq 1\}) - \sigma_n.$$

LEMMA 9. *The following statements hold for* $|n|$ *sufficiently large.*

(1) G_n *is meromorphic on* $\mathbf{C}$ *with a simple zero or pole at each* $b \in \mathfrak{D}$ *and no other zeros nor poles*; $G_n(\bar{z}) = \overline{G_n(z)}$.

(2) G_n *is an analytic homeomorphism on a neighborhood of* $R(t_n; 5, \lambda_n)$.

(3) $R(t_n; 3, \lambda_n) \subseteq \sigma_n \subseteq R(t_n; 5, \lambda_n)$.

(4) $|G_n| < 16^{-1}$ *on* X_n, $|G_n| > 16$ *on* Y_n, $16^{-1} \leq |G_n| \leq 16$ *on* σ_n, *and* G_n *is an analytic homeomorphism on* σ_n.

(5) *There is a smooth function* Θ_n: $[16^{-1}, 16] \to (0, \pi)$ *such that* $G_n(\sigma_n) = \{w: 16^{-1} \leq |w| \leq 16$ *and* $|\arg w| \leq \Theta_n(|w|)\}$.

PROOF. For definiteness let us consider the case $n < 0$; the case $n > 0$ is treated in a very similar manner. (1) is clear because the same thing is true for H_n. By Corollary 8 $H_{-n}(z + t_n) \to 1$ uniformly on $R(0; 7, 3\lambda_0)$ as $n \to -\infty$. Because 2^{-z} is an analytic homeomorphism on $S_{3\lambda_0}$ and $G_n(z + t_n) = 2^{-z}H_{-n}(z + t_n) \to 2^{-z}$ uniformly on $R(0; 7, 3\lambda_0)$, $G_n(z + t_n)$ is an analytic homeomorphism on $R(0; 6, 2\lambda_0)$ for large $n < 0$, and (2) follows.

From parts (7) and (5) of Corollary 8 we obtain $\frac{1}{2} < |H_{-n}(z)|$ for $\operatorname{Re} z \leqslant t_n + 5$ and $|H_{-n}(z)| < 2$ for $\operatorname{Re} z \geqslant t_n - 5$ for large $n < 0$. If $\operatorname{Re} z < t_n - 5$, we have $|2^{t_n - z}| > 32$; so $|G_n(z)| > 16$. If $\operatorname{Re} z > t_n + 5$, we have $|2^{t_n - z}| < 32^{-1}$; so $|G_n(z)| < 16^{-1}$. Thus, $\sigma_n \subseteq R(t_n; 5, \lambda_n)$. Because $\frac{1}{2} < |H_{-n}(z)| < 2$ in $R(t_n; 5, \lambda_n)$, which contains $R(t_n; 3, \lambda_n)$, and $8^{-1} \leqslant |2^{t_n - z}| \leqslant 8$ on $R(t_n; 3, \lambda_n)$, we see that $16^{-1} \leqslant |G_n(z)| \leqslant 16$ on $R(t_n; 3, \lambda_n)$. Thus, $R(t_n; 3, \lambda_n) \subseteq \sigma_n$ and (3) is proved.

Because $t_{|n|} - s_{|n|} \gg 0$ for large $|n|$, we have from Corollary 8 that $|H_{-n}(z)| < 2$ for $z \in X_n$ for large $n < 0$. By (3) and the definition of X_n, $\operatorname{Re} z \geqslant t_n + 1$ for $z \in X_n$; so $|2^{t_n - z}| \leqslant \frac{1}{2}$ and $|G_n(z)| < 2 \cdot \frac{1}{2} = 1$ for $z \in X_n$. Because of (3) and the definition of σ_n, this means that $|G_n| < 16^{-1}$ on X_n. In a similar manner we obtain $|G_n| > 16$ on Y_n. By (2) and (3), G_n is an analytic homeomorphism on σ_n, and (4) is proved.

Because G_n is a diffeomorphism and does not vanish on a neighborhood of $R(t_n; 5, \lambda_n)$, the functions $r = |G_n(z)|$ and $\theta = \arg(G_n(z))$, where $-\pi < \theta < \pi$, constitute a global differentiable coordinate pair on a neighborhood of $R(t_n; 5, \lambda_n)$. We can therefore parametrize $G_n(x - i\lambda_n)$ as $G_n(x - i\lambda_n) = r \exp[i\Theta_n(r)]$ for a smooth function Θ_n. Because $\arg 2^{t_n - z} > 0$ for $\operatorname{Im} z = -\lambda_n$ and H_{-n} is nearly 1, Θ_n takes values in $(0, \pi)$ for large $n < 0$. The set $\gamma = \{r = r_0\} \cap R(t_n; 5, \lambda_n)$ consists of regular arcs which have no endpoints in $R(t_n; 5, \lambda_n)^\circ$. If we knew that γ consisted of just one arc which meets the boundary of $R(t_n; 5, \lambda_n)$ in just two points, we could complete the argument as follows. For $r_0 \in [16^{-1}, 16]$ γ does not meet $\{x - t_n = \pm 5\}$, for on the latter set $|G_n|$ is approximately 32 or 32^{-1}, which is not close to 16 or 16^{-1}. So γ connects $\operatorname{Im} z = -\lambda_n$ to $\operatorname{Im} z = +\lambda_n$. Because $G_n(\bar{z}) = \overline{G_n(z)}$ and $G_n > 0$ on $[t_n - 5, t_n + 5]$, γ is parametrized by a single symmetric interval $-\theta_0 \leqslant \theta \leqslant \theta_0$. Evidently (r_0, θ_0) corresponds to a point of $\{\operatorname{Im} z = \lambda_n\}$ or to a point of $\{\operatorname{Im} z = -\lambda_n\}$. As we have previously observed, $\{\operatorname{Im} z = -\lambda_n\}$ corresponds to positive θ. Since $G(x - i\lambda_n) = r \exp(i\Theta_n(r))$, this shows that (5) holds.

Finally, to see that $\gamma = \{r = r_0\} \cap R(t_n; 5, \lambda_n)$ consists of a single arc, consider the following elementary calculation for an analytic nonvanishing f:

$$\begin{aligned} 2|f|\frac{\partial |f|}{\partial x} &= \frac{\partial |f|^2}{\partial x} = \frac{\partial (ff^-)}{\partial x} = \frac{\partial f}{\partial x}\bar{f} + f\frac{\partial \bar{f}}{\partial x} \\ &= \bar{f}\frac{\partial f}{\partial x} + f\left(\frac{\partial f}{\partial x}\right)^- = \bar{f}f' + f(f')^- = 2\operatorname{Re}\bar{f}f'; \end{aligned}$$

so $\partial|f|/\partial x = |f|^{-1}\operatorname{Re}(\bar{f}f')$. Apply this formula to G_n, taking into account the fact that H_{-n} is very close to 1 on a neighborhood of $R(t_n; 5, \lambda_n)$ and hence H'_{-n} is very close to 0. The result is

$$\begin{aligned} \frac{\partial |G_n|}{\partial x} &= |G_n|^{-1}\operatorname{Re}\left(\overline{G}_n G_n'\right) \approx |2^{t_n - z}|^{-1}\operatorname{Re}\left((2^{t_n - z})^-(2^{t_n - z})'\right) \\ &= \frac{\partial}{\partial x}|2^{t_n - z}| = (\log 2)|2^{t_n - z}| \leqslant -(\log 2)32^{-1} \end{aligned}$$

on $R(t_n; 5, \lambda_n)$. So for large negative n, $\partial r/\partial x = \partial|G_n|/\partial x < 0$ on $R(t_n; 5, \lambda_n)$. This means that each horizontal line $\{\operatorname{Im} z = \text{constant}\}$ meets γ in at most one point;

and γ consists of at most one arc, since it has no endpoints in $R(t_n;\, 5, \lambda_n)^\circ$.

Define for large $|n|$ the following subsets of $\mathbf{C}$. [See Figure 1.]

$$V_n^+ = \left\{ z\colon 4^{-1} < |z| < 4 \text{ and } \tfrac{1}{2}\Theta_n(|z|) < \arg z < \pi - \tfrac{1}{2}\Theta_n(|z|) \right\},$$

$$V_n^- = \left\{ z\colon \bar{z} \in V_n^+ \right\},$$

$$W = \left\{ z\colon 4^{-1} \leqslant |z| \leqslant 4 \text{ and } |\arg z| \leqslant \tfrac{1}{2}\Theta_n(|z|) \right\},$$

$$W' = \left\{ z\colon -z \in W \right\}.$$

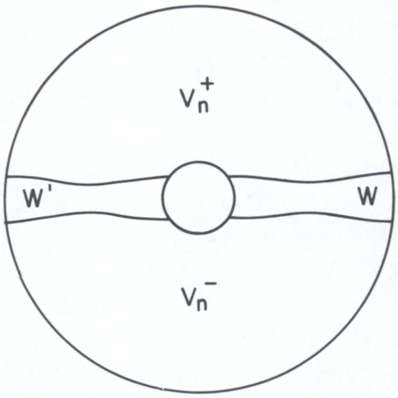

FIGURE 1

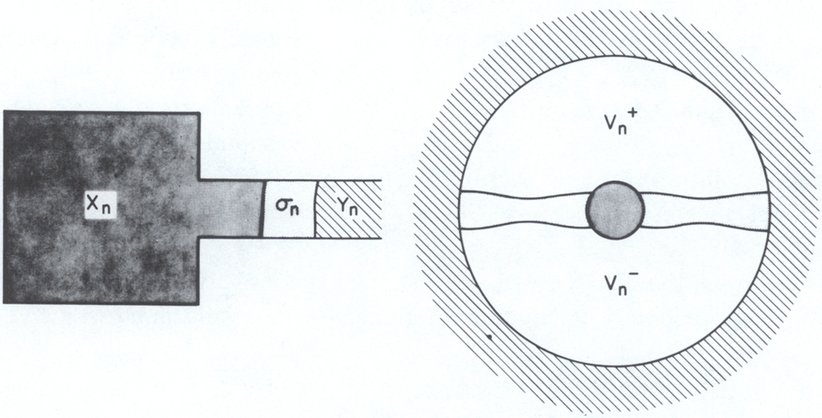

FIGURE 2

LEMMA 10. *For large $|n|$ there is an analytic function π_n on M which enjoys these properties* [*see Figure* 2].

(1) $\pi_n^2 = G_n \circ Z$;

(2) π_n *maps* $Z^{-1}(X_n \cup \sigma_n \cup Y_n)$ *into* $\mathbf{C} - (V_n^+ \cup V_n^-)$;

(3) $\pi_n(Z^{-1}(X_n)) \subseteq \{z\colon |z| < 4^{-1}\}$, $\pi_n(Z^{-1}(Y_n)) \subseteq \{z\colon |z| > 4\}$, *and π_n is an analytic homeomorphism of $Z^{-1}(\sigma_n)$ onto $W \cup W'$.*

PROOF. Recall the surface $\overline{M}$ and its projection $\bar{\pi}$ onto $\mathbf{C}$, which has branch set $B \subseteq \overline{M}$. G_n has a simple zero or pole at each point b of $\mathfrak{D} = \bar{\pi}(B)$, and no other zeros nor poles. From this fact and the definition of $\overline{M}$ and $\bar{\pi}$ it follows that the function $G_n \circ \bar{\pi}$ has a single-valued square root, call it $\bar{\pi}_n$, on $\overline{M}$. Indeed, $\overline{M}$ can be thought of as the classical Riemann surface constructed from the multiple-valued function $\sqrt{G_n}$ on the plane. Restricting $\bar{\pi}_n$ to M we have $\sqrt{G_n \circ Z}$ as a single-valued analytic function, call it π_n, on M having no zeros nor poles. Properties (2) and (3) are immediate from Lemma 9. Note that $\pi_n(p^+) = -\pi_n(p^-)$ if $\{p^+, p^-\} = Z^{-1}(z)$ for $z \in \sigma_n$.

LEMMA 11. *Let $|n|$ be large. For every $\delta > 0$ and every C^1 function g on $Z^{-1}(X_n \cup \sigma_n \cup Y_n)$ such that $g = 0$ on the closure of $Z^{-1}(X_n \cup Y_n)$ and $|\partial g/\partial \bar{Z}| < \delta$ on $Z^{-1}(\sigma_n)$ there exists a meromorphic function ψ on M such that $|\psi - g| < 17\delta$ on $Z^{-1}(X_n \cup \sigma_n \cup Y_n)$. The poles of ψ lie outside $Z^{-1}(X_n \cup \sigma_n \cup Y_n)$.*

PROOF. Let $|n|$ be so large that Lemma 10 holds. By Lemma 10 and the hypothesis on g we can find a C^1 function $\tilde{g}$ on $\mathbf{C}$ such that $\tilde{g} = 0$ on $\{z: |z| < 4^{-1} + \varepsilon$ or $|z| > 4 - \varepsilon\}$ for some $\varepsilon > 0$ and $\tilde{g} \circ \pi_n = g$ on $Z^{-1}(X_n \cup \sigma_n \cup Y_n)$. Calculate

$$\frac{\partial g}{\partial \bar{Z}} = \frac{\partial(\tilde{g} \circ \pi_n)}{\partial \bar{Z}} = \frac{\partial(\tilde{g} \circ \pi_n \circ Z^{-1})}{\partial \bar{z}} \circ Z = \frac{\partial \tilde{g}}{\partial \bar{z}} \circ \pi_n \circ Z^{-1} \circ Z = \frac{\partial \tilde{g}}{\partial \bar{z}} \circ \pi_n.$$

So $|\partial \tilde{g}/\partial \bar{z}| < \delta$ on $W \cup W'$. Now we apply a method of Mergelyan [**M**, §3, Chapter I] to approximate $\tilde{g}$ by a rational function k. Then $k \circ \pi_n$ will approximate $\tilde{g} \circ \pi_n = g$. Let $\Gamma^{\pm}$ be curves oriented positively (counterclockwise) inside $V_n^{\pm}$ which are within ε of the boundary of $V_n^{\pm}$. See Figure 3.

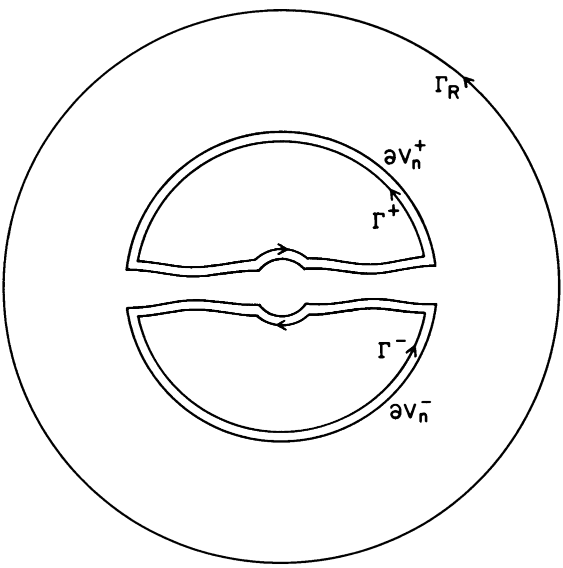

FIGURE 3

Let Γ_R be the circle of radius R, centered at 0. For $z_0 \in \mathbf{C} - (V_n^+ \cup V_n^-)$ we can write the generalized Cauchy formula as follows (see [**M**, pp. 304–305], [**G**, p. 26], [**S**, §7]), for $R > |z_0|$.

$$\tilde{g}(z_0) = \frac{1}{2\pi i}\left(\int_{\Gamma_R} - \int_{\Gamma^+ \cup \Gamma^-}\right)\tilde{g}(z)(z - z_0)^{-1}\, dz - \frac{1}{\pi}\int\int_{\Sigma_R} \frac{\partial \tilde{g}}{\partial \bar{z}}(z - z_0)^{-1}\, dx\, dy,$$

where Σ_R is the set of points inside Γ_R and outside both Γ^+ and Γ^-. Because $\tilde{g}$ vanishes on Γ_R, the line integral reduces to

$$I(z_0) = -(2\pi i)^{-1} \int_{\Gamma^+ \cup \Gamma^-} \tilde{g}(z)(z - z_0)^{-1}\, dz.$$

Because $\tilde{g}$ vanishes on most of Σ_R, the integral over Σ_R reduces to an integral over T_n = the two components of $[\{4^{-1} + \varepsilon \leqslant |z| \leqslant 4 - \varepsilon\} - (\Gamma^+ \cup \Gamma^-)]$ which contain ± 1. See Figure 4, in which T_n is shaded.

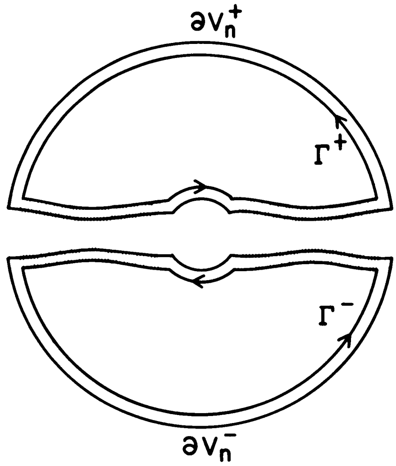

FIGURE 4

Because $\tilde{g}$ is C^1 and $|\partial\tilde{g}/\partial\bar{z}| \leqslant \delta$ on $\mathbf{C} - (V_n^+ \cup V_n^-)$, we can take ε so small that $|\partial\tilde{g}/\partial\bar{z}| < 2\delta$ on T_n. Then

$$\left|\pi^{-1}\int\int_{\Sigma_R}\right| = \pi^{-1}\left|\int\int_{T_n}\right| \leqslant 2\delta\pi^{-1}\int\int_{T_n}|z - z_0|^{-1}\, dx\, dy$$

$$\leqslant 2\delta\pi^{-1}\int\int_{|z|<4}|z - z_0|^{-1}\, dx\, dy \leqslant 2\delta\pi^{-1}\sqrt{4\pi\cdot\pi\cdot 4^2} = 16\delta.$$

The last inequality is Lemma 3.1.1 of **[B]**. Now for $z_0 \in \mathbf{C} - (V_n^+ \cup V_n^-)$ and $z \in \Gamma^+ \cup \Gamma^-$, the distance $|z - z_0|$ is bounded away from 0; thus, $I(z_0)$ can be uniformly approximated for $z_0 \in \mathbf{C} - (V_n^+ \cup V_n^-)$ by a finite Riemann sum for this integral, which is manifestly a rational function $k(z_0)$ having its poles in $V_n^+ \cup V_n^-$. Choosing such a k for which $|I(z_0) - k(z_0)| < \delta$ for all $z_0 \in \mathbf{C} - (V_n^+ \cup V_n^-)$, we have $|\tilde{g} - k| \leqslant 17\delta$ on $\mathbf{C} - (V_n^+ \cup V_n^-)$ and so $|k \circ \pi_n - \tilde{g} \circ \pi_n| < 17\delta$ on $\pi_n^{-1}(\mathbf{C} - (V_n^+ \cup V_n^-)) \supseteq Z^{-1}(X_n \cup \sigma_n \cup Y_n)$. Thus, $\psi = k \circ \pi_n$ does what is required. The poles of ψ are not in $Z^{-1}(X_n \cup \sigma_n \cup Y_n)$, because ψ is bounded there.

LEMMA 12. *Let S be a closed subset of* $\mathbf{C}$ *which is star-shaped with respect to* 0, *let z_n be a sequence of points of $\mathbf{C} - S$ tending to ∞, and let k_n be a sequence of nonnegative integers.f For each $\varepsilon > 0$ there is an entire function φ such that $|1 - \varphi| < \varepsilon$ on S and for each n, φ has a zero of order at least k_n at z_n.*

PROOF. Given $\varepsilon > 0$, let $\varepsilon_n > 0$ be chosen so that $\Sigma\, k_n\varepsilon_n < \min(\frac{1}{2}, \varepsilon/3)$. Put $\gamma_n = \{rz_n : r \geqslant 1\}$; γ_n is an arc joining z_n to ∞ in $\mathbf{C} - S$, because S is star-shaped.

Each compact set meets at most finitely many γ_n. Choose a neighborhood V_n of each γ_n so that $V_n \subseteq \mathbf{C} - S$ and so that each compact meets only finitely many V_n. Using Lemma 7 choose an entire function h_n so that $h_n(z_n) = 0$ and $|1 - h_n| < \varepsilon_n$ on $\mathbf{C} - V_n$. By Lemma 1 the product $\prod h_n^{k_n}$ converges normally on the plane to a function φ having the desired properties.

COROLLARY 13. *In Lemma* 11 *we can require* ψ *to be analytic on* M *if we relax the approximation to* $|\psi - g| \leq 18\delta$.

PROOF. Write $Z_n = Z^{-1}(X_n \cup \sigma_n \cup Y_n)$. Given g and δ satisfying the hypothesis of Lemma 11, let ψ_1 be meromorphic on M and satisfy $|\psi_1 - g|_{Z_n} \leq 17\delta$. Let P be the pole set of ψ_1 and enumerate $Z(P) = \{z_1, z_2, \dots\}$. Note that $P \cap Z_n = \varnothing$; so $Z(P) \cap (X_n \cup \sigma_n \cup Y_n) = \varnothing$. For each z_j let k_j be the larger of the orders of the poles of ψ_1 at the two points of $Z^{-1}(z_j)$. In Z_n g vanishes off a compact set; so $|g|_{Z_n} < \infty$. Thus $|\psi_1|_{Z_n} = K < \infty$. Apply Lemma 12 to the star-shaped $S = X_n \cup \sigma_n \cup Y_n$, the sequence $\{z_j\}$, the integers $\{k_j\}$, and $\varepsilon = \delta/K$ to find an entire function φ so that $|\varphi - 1|_S \leq \delta/K$ and φ has a zero of order at least k_j at each z_j. Put $\psi = (\varphi \circ Z)\psi_1$, which has no poles on M by construction.

$$|\psi - \psi_1|_{Z_n} = |(\varphi \circ Z)\psi_1 - \psi_1|_{Z_n} \leq |\psi_1|_{Z_n}|\varphi \circ Z - 1|_{Z_n} \leq K \cdot \delta/K = \delta.$$

Therefore, $|\psi - g|_{Z_n} \leq |\psi - \psi_1|_{Z_n} + |\psi_1 - g|_{Z_n} \leq \delta + 17\delta = 18\delta$.

Equipped with the foregoing technical tools we can now detail the proof that $E_\lambda^+ \in \mathcal{Q}$ for $\lambda \leq \lambda_0$ and $E_\lambda^- \in \mathcal{Q}$ for $\lambda < \lambda_0$. Fix one such E_λ^+ or E_λ^- and call it E. Let $f \in A(E)$ and $\varepsilon > 0$. By Corollary 13 there is a set of integers $\mathfrak{N} = \mathbf{Z} \cap [N, \infty)$, in case $E = E_\lambda^+$, or $\mathfrak{N} = \mathbf{Z} \cap (-\infty, -N]$, in case $E = E_\lambda^-$, such that for every $n \in \mathfrak{N}$ and every C^1 function g on $Z_n = Z^{-1}(X_n \cup \sigma_n \cup Y_n)$ which is supported in the relative (to Z_n) interior of $Z^{-1}(\sigma_n)$ there is an analytic function ψ on M such that $|g - \psi|_{Z_n} \leq 18|\partial g/\partial \overline{Z}|_{Z_n}$. (If $\delta = |\partial g/\partial \overline{Z}|_{Z_n} > 0$, this is Corollary 13; if $\delta = 0$, then g is analytic on the connected set Z_n°, and since it vanishes on $Z^{-1}(X_n)$, it vanishes identically and we can take $\psi = 0$.) For every $n \in \mathfrak{N}$ put $W_n = E \cap Z^{-1}\{z: |\operatorname{Re} z - t_n| < 1\}$. $E - \bigcup_{n \in \mathfrak{N}} W_n$ consists of a sequence of separated closed connected subsets, which we may number E_n, $n \in \mathfrak{N}$. Do this numbering so that W_n sits between E_n and $E_{n'}$ where $|n'| = |n| + 1$. Each E_n has a neighborhood of finite genus; indeed, the closure of E_n in $\overline{M}$ is compact. We may select these neighborhoods to be disjoint from each other. By Theorem 1.5 of [S] there is an analytic function Φ on M such that $|f - \Phi|_{E_n} \leq 2^{-|n|}\theta$, where $\theta = \varepsilon/40$. Put $g = f - \Phi$. Note that $|g|_{\cup E_n} \leq \theta/2$.

Select a C^1 function χ: $\mathbf{C} \to [0, 1]$ which depends only on $x = \operatorname{Re} z$ and which has these properties: $\chi \equiv 1$ on $[-1, 1]$, $\chi \equiv 0$ outside $(-2, 2)$, and $|\partial\chi/\partial x| < 2$. Then $|\partial\chi/\partial\bar{z}| \leq 1$. Put $\chi_n = \chi \circ (Z - t_n)$; then $|\partial\chi_n/\partial\overline{Z}| \leq 1$ on M, and we may assume that $\chi_m\chi_n \equiv 0$ for m and n in $\mathfrak{N}$, $m \neq n$, because of the large gaps between t_m and t_n for large $|n|$. Define $g_n = \chi_n g$ for $n \in \mathfrak{N}$. $|\Sigma g_n - g|_E \leq \theta/2$ because of the following. For each p there is an integer m such that $(\Sigma g_n)(p) = g_m(p)$, because at most one term in $\Sigma g_n(p)$ is nonzero. Thus, $|\Sigma g_n(p) - g(p)| = |g_m(p) - g(p)| = |\chi_m(p) - 1|\,|g(p)|$. If $\chi_m(p) - 1 \neq 0$, then $p \in \bigcup E_n$ and $|g(p)| \leq \theta/2$, while $|\chi_m(p) - 1| \leq 1$.

In E° we calculate

$$\frac{\partial g_n}{\partial \bar{Z}} = \frac{\partial(\chi_n g)}{\partial \bar{Z}} = \frac{\partial \chi_n}{\partial \bar{Z}} g + \chi_n \frac{\partial g}{\partial \bar{Z}} = \frac{\partial \chi_n}{\partial \bar{Z}} g,$$

because $g \in A(E)$. Now $((\partial\chi_n/\partial\bar{Z})g)(p) = 0$ unless $1 < |\operatorname{Re} Z(p) - t_n| < 2$, in which case $|(\partial\chi_n/\partial\bar{Z})(p)| \leq 1$ and $|g(p)| \leq \max\{2^{-|m|}\theta\colon m \in \mathfrak{N}, m = n-1, n, \text{ or } n+1\} \leq 2 \cdot 2^{-|n|}\theta$. Thus, the support of g_n in Z_n is contained in $R(t_n; 2, \lambda)$, which belongs to the relative (to Z_n) interior of σ_n, by Lemma 9, and $|\partial g_n/\partial\bar{Z}|_{Z_n} \leq 2 \cdot 2^{-|n|}\theta$.

Next we approximate g_n on Z_n by a C^1 function h_n on Z_n. Specifically, let s_n be the map of σ_n into $\mathbf{C}$ given by $s_n(z) = t_n + r_n(z - t_n)$, where $r_n < 1$ and r_n is close to 1, and define h_n by $h_n = g_n$ on $Z^{-1}(X_n \cup Y_n)$ and $h_n = (Z|_\Sigma)^{-1} \circ s_n \circ (Z|_\Sigma)$ for each component Σ of $Z^{-1}(\sigma_n)$. Because g_n vanishes on a neighborhood of $Z^{-1}(X_n \cup Y_n)$, the same will be true for h_n if r_n is close enough to 1. In this case h_n will be C^1 on Z_n and $|\partial h/\partial\bar{Z}|_{Z_n} = r_n|\partial g/\partial\bar{Z}|_{Z_n} \leq |\partial g/\partial\bar{Z}|_{Z_n} \leq 2 \cdot 2^{-|n|}\theta$. Because $Z^{-1}(\sigma_n)$ is compact we may take r_n so close to 1 that $|g_n - h_n|_{Z^{-1}(\sigma_n)} \leq 2^{-|n|}\theta$.

By Corollary 13 there is an analytic function ψ_n on M such that $|\psi_n - h_n|_{Z_n} \leq 18 \cdot 2 \cdot 2^{-|n|}\theta = 36 \cdot 2^{-|n|}\theta$. Because $\sum_{n\in\mathfrak{N}} 2^{-|n|} \leq 1$ and every compact set in M is contained in all but perhaps finitely many of the Z_n, the sum $\sum \psi_n$ converges normally on M to an analytic function Ψ. Let F be $\Phi + \Psi$, which is analytic on M, and estimate

$$\begin{aligned} |F - f| &= \left|\sum \psi_n + \Phi - f\right| \\ &= \left|\sum(\psi_n - h_n) + \sum(h_n - g_n) + \left(\sum g_n - g\right) + (g + \Phi - f)\right| \\ &\leq \sum|\psi_n - h_n| + \sum|h_n - g_n| + \left|\sum g_n - g\right| + |g + \Phi - f|. \end{aligned}$$

On E the first sum is at most $\sum_{n\in\mathfrak{N}} 36 \cdot 2^{-|n|}\theta \leq 36\theta$. The second sum has at most one nonzero term at any point of E; so it is dominated by $\max\{2^{-|n|}\theta\colon n \in \mathfrak{N}\} \leq \theta/2$. The third term $|\sum g_n - g|$ is at most $\theta/2$, as estimated earlier, and the last expression $|g + \Phi - f|$ is identically zero by definition of g. Therefore, we have $|F - f| \leq 37\theta = 37\varepsilon/40 < \varepsilon$ on E, and the proof is complete.

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