TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 262, Number 2, December 1980

ALGEBRAS OF AUTOMORPHIC FORMS WITH FEW GENERATORS

BY

PHILIP WAGREICH[1]

ABSTRACT. Those finitely-generated Fuchsian groups G for which the graded algebra of automorphic forms A is generated by 2 or 3 elements are classified. In these cases the structure of A is described.

Suppose G is a finitely-generated Fuchsian group of the first kind and A_G is the (graded) ring of automorphic forms relative to G. The purpose of this paper is to give a self-contained proof of the classification of those groups G for which the ring A_G is generated (as an algebra over $\mathbf{C}$) by $\leqslant 3$ elements. The first results in this direction were announced by Dolgachev **[D1]** who studied the case where G is a triangle group G_{n_1,n_2,n_3}. That is, G is the subgroup of orientation-preserving maps of the group generated by reflections in the sides of a hyperbolic triangle with angles π/n_1, π/n_2 and π/n_3. He showed that exactly 14 triangles have the property that A_G is generalized by 3 elements. The remaining groups for which H_+/G is compact were studied independently by Dolgachev **[D2]** and myself. Milnor **[M]** has studied rings of automorphic forms with fractional weight. This paper will describe the classification for arbitrary G. The idea of the proof is elementary and makes heavy use of the fact A_G is a graded ring (Definition (1.1)) i.e., $A_G = \bigoplus_{k \geqslant 0} A_k$, where A_k is the vector space of k-forms. Suppose A_G is generated as a $\mathbf{C}$-algebra by 3 elements x_0, x_1, x_2. Elementary arguments (1.5) show that the generators can be chosen to be homogeneous. Let $x_i \in A_{q_i}$. We let $S = \mathbf{C}[X_0, X_1, X_2]$ and define a grading on S by letting degree $X_i = q_i$. Define φ: $S \to A_G$ by $\varphi(X_i) = x_i$ and let $I = \text{kernel } \varphi$. Then we have

PROPOSITION (2.7). *I is a principal ideal.*

Now the kernel of a homomorphism of graded algebras is a homogeneous ideal, i.e., $I = \bigoplus_{n=0}^{\infty} I_n$ where $I_n = I \cap A_n$. One can easily show that I is generated by a homogeneous element. Suppose that the element is $f \in S$. Then

$$f(t^{q_0}Z_0, t^{q_1}Z_1, t^{q_2}Z_2) = t^d f(Z_0, Z_1, Z_2)$$

where d is an integer called the (weighted) degree of f. Such a polynomial is called a weighted homogeneous polynomial, i.e., homogeneous with respect to the grading of S defined above. We let $R = S/I$. The dimension of R_i can be computed easily

Received by the editors November 4, 1977.

1980 *Mathematics Subject Classification*. Primary 10D15, 30F35; Secondary 14J17.

Key words and phrases. Automorphic form, Fuchsian group, graded ring, Poincaré power series.

[1]Research partially supported by grants from the National Science Foundation.

© 1980 American Mathematical Society
0002-9947/80/0000-0552/$06.75

in terms of q_0, q_1, q_2 and d. On the other hand, $\dim A_i$ can be computed using the Riemann-Roch theorem.

By equating $\dim R_i$ and $\dim A_i$ and observing some elementary properties of rings of the form S/I we can eliminate all but a list of 42 types of groups. Direct calculation then gives the structure of A_G for that list of groups.

This paper is organized as follows. The first section is devoted to a review of properties of graded rings and the Poincaré power series of a graded ring. In §2 we calculate the Poincaré power series of A_G and as a consequence develop relations between invariants of G (the signature of G) and q_0, q_1, q_2 and d. Then in §3 we find a list of groups G so that all groups with A_G generated by $\leqslant 3$ elements are on the list. The fourth section is devoted to proving that all of the groups on the list are, in fact, generated by 2 or 3 elements.

This paper is essentially self-contained. In a subsequent paper **[W2]**, we will use the theory of singularities of complex surfaces to give more general results on the structure of algebras of automorphic forms.

I would like to thank Irwin Kra and Igor Dolgachev for stimulating correspondence.

1. Graded algebras. Suppose K is a field and R is a K-algebra.

DEFINITION (1.1). A *grading* on R is a collection of K-vector subspaces R_i of R so that

(i) $R = \bigoplus_{i=-\infty}^{\infty} R_i$.

(ii) $R_i R_j \subset R_{i+j}$.

The K-algebra R together with a grading is called a *graded K-algebra*. We say $x \in R$ is *homogeneous of degree i* if $x \in R_i$.

EXAMPLE (1.2). $R = K[X_0, \ldots, X_n]$. Fix integers $q_0, \ldots, q_n$. Define $R_i = K$-subspace generated by $X_0^{i_0} \cdots X_n^{i_n}$ so that $i_0 q_0 + \cdots + i_n q_n = i$. Note that

$$f \in R_i \Leftrightarrow f(t^{q_0}X_0, \ldots, t^{q_n}X_n) = t^i f(X_0, \ldots, X_n),$$

t an indeterminate. If $q_0 = q_1 = \cdots = q_n = 1$ we get the usual grading on R.

DEFINITION (1.3). A homomorphism $\varphi\colon S \to R$ of graded algebras is said to be *homogeneous of degree d* if $\varphi(S_i) \subset R_{i+d}$, for all i.

DEFINITION (1.4). We say R is a graded *K-algebra of finite type* if there are homogeneous elements $x_0, \ldots, x_n \in R$ so that the homomorphism

$$\varphi\colon K[X_0, \ldots, X_n] \to R$$

defined by $\varphi(X_i) = x_i$ is onto.

Henceforth we shall assume R is *positively graded* (i.e., $R_i = \{0\}$ for $i < 0$) and $R_0 = K$.

(1.5) If we are as before and degree $x_i = q_i$ and we grade $S = K[X_0, \ldots, X_n]$ as in Example (1.2) then φ is a graded homomorphism of degree 0. It follows that $I = \operatorname{kernel} \varphi$ is a homogeneous ideal, i.e., $I = \bigoplus_i I_i$. Thus I is generated by homogeneous elements. Let $\mathfrak{m} = \bigoplus_{i>0} R_i$. Then $\mathfrak{m}$ is a maximal ideal of R. The *embedding dimension* of R is defined to be K-dimension of the vector space $\mathfrak{m}/\mathfrak{m}^2$. The algebra R is of finite type if and only if $\dim_K \mathfrak{m}/\mathfrak{m}^2 < \infty$ and in that case

$x_0, \ldots, x_n$ generate R as a K-algebra if and only if their residues $\bar{x}_0, \ldots, \bar{x}_n \in \mathfrak{m}/\mathfrak{m}^2$ generate $\mathfrak{m}/\mathfrak{m}^2$ as a K vector space. Note that $\mathfrak{m}/\mathfrak{m}^2$ is a graded vector space and we can choose n *homogeneous* generators for $\mathfrak{m}/\mathfrak{m}^2$. These lift to n homogeneous algebra generators of R. The above discussion shows that the embedding dimension of R is the minimal n so that the algebraic variety Spec(R) can be embedded in affine n-space over K.

DEFINITION (1.6). Suppose R is a graded K-algebra, M is a graded R-module and $a_i = \dim_K M_i < \infty$ for all i. Let $\mathcal{P}_M(t) = \sum_{i=0}^{\infty} a_i t^i$, the *Poincaré power series* of M.

We now sketch the calculation of $\mathcal{P}_R(t)$ in the case we will be studying.

PROPOSITION (1.7). *Suppose $S = K[X_0, \ldots, X_n]$ is graded as in Example* (1.2), *$f \in S_d$, $I = (f)$ and $R = S/I$. Then*

(i) $\mathcal{P}_S(t) = 1/\prod_{i=0}^{n}(1 - t^{q_i})$,

(ii) $\mathcal{P}_R(t) = (1 - t^d)/\prod_{i=0}^{n}(1 - t^{q_i})$.

PROOF.

LEMMA. *Suppose $0 \to M' \xrightarrow{\varphi'} M \xrightarrow{\varphi} M'' \to 0$ is an exact sequence of graded S modules, φ is homogeneous of degree* 0, *φ' is homogeneous of degree α. Then*

$$t^{\alpha}\mathcal{P}_{M'}(t) - \mathcal{P}_M(t) + \mathcal{P}_{M''}(t) = 0.$$

The proof of the lemma is immediate. To prove (i) one can proceed by induction by considering the exact sequence

$$0 \to K[X_0, \ldots, X_i] \xrightarrow{\varphi'} K[X_0, \ldots, X_i] \xrightarrow{\varphi} K[X_0, \ldots, X_{i-1}] \to 0$$

where $\varphi'(g) = X_i g$ and φ is defined by $\varphi(X_j) = X_j, j \neq i, \varphi(X_i) = 0$.

To prove (ii) we use the exact sequence

$$0 \to S \xrightarrow{\varphi'} S \to R \to 0$$

where $\varphi'(g) = fg$. □

REMARK. One can show as above that for any graded K-algebra of finite type $\mathcal{P}_R(t)$ is a rational function **[A-M]**.

In §2 we will be interested in the principal part of $\mathcal{P}_R(t)$.

PROPOSITION (1.8). *If $p(t) = \prod_{i=0}^{n}(1 - t^{d_i})/\prod_{j=0}^{n+2}(1 - t^{q_j})$, then the principal part of p at $t = 1$ is $a/(1 - t)^2 + b/(1 - t)$ where*

$$a = \frac{\prod_{i=0}^{n} d_i}{\prod_{j=0}^{n+2} q_j} \quad \text{and} \quad b = a\left(\sum_{j=0}^{n+2} \frac{q_j - 1}{2} - \sum_{i=0}^{n} \frac{d_i - 1}{2}\right).$$

PROOF.

$$p(t) = \frac{1}{(1 - t)^2} \frac{\prod_{i=0}^{n}(1 + \cdots + t^{d_i - 1})}{\prod_{j=0}^{n+2}(1 + \cdots + t^{q_j - 1})}.$$

Now let $z = 1 - t$. Then $1 + \cdots + t^{m-1} = m - m(m-1)z/2 + f(z)$ where f consists of terms of degree ≥ 2 in z. Thus

$$p(t) = \frac{1}{z^2} \cdot \prod_{i=0}^{n} \left(d_i - \frac{d_i(d_i - 1)}{2} z \right) \cdot \prod_{j=0}^{n+2} \frac{1}{q_j} \left(1 + \frac{q_j - 1}{2} z \right) + g(z)$$

where $g(z)$ is holomorphic at $z = 0$. This gives us the desired result.

PROPOSITION (1.9). *Let $f(t) = (1 - t^d)/(1 - t^a)(1 - t^b)(1 - t^c)$ where a, b, c, d are positive integers. Suppose $\xi^e = 1$ and $\xi \neq 1$.*

(a) *If $e|a$, $e \nmid b, c, d$ then the principal part of f at $t = \xi$ is*

$$(-\xi(1 - \xi^d)/a(1 - \xi^b)(1 - \xi^c)) \cdot (1/(1 - t)).$$

(b) *If $e|a$, $e|b$, $e|d$, $e \nmid c$ then the principal part of f at $t = \xi$ is*

$$(-d\xi/ab(1 - \xi^c)) \cdot (1/(1 - t)).$$

PROOF. Let $z = t - \xi$ and substitute for t in the given function. The principal part is easily calculated.

DEFINITION (1.10). The *dimension* of a K-algebra R is the transcendence degree of the quotient field of R over K.

REMARK (1.11). (a) If R is finitely generated over K then $\dim R = \text{Krull} \dim R$.

(b) If R is a finitely-generated graded algebra then dimension R = order of pole of $\mathcal{P}_R(t)$ at $t = 1$ **[A-M]**.

DEFINITION (1.12). If $S = K[X_0, \ldots, X_n]$ is graded by letting degree $X_i = q_i > 0$, we define $c(q_0, \ldots, q_n)_i = \dim_K S_i$ which by Proposition (1.7) equals the coefficient of t^i in the power series $1/\prod_{i=0}^n (1 - t^{q_i})$.

When calculating this integer by hand it is frequently helpful to use the recursion formula

$$c(q_0, \ldots, q_n)_i = c(q_0, \ldots, q_{n-1})_i + c(q_0, \ldots, q_n)_{i-q_n} \qquad (1.10.1)$$

for all i. Also

$$c(1, \ldots, 1)_i = \binom{i+n}{n}.$$

The calculations in the later parts of this paper were done by using a computer, evaluating $c(q_0, \ldots, q_n)_i$ by multiplication of the power series $1/(1 - t^{q_i}) = \sum_{k=0}^{\infty} t^{kq_i}$.

2. Automorphic forms.

(2.1) The Lie group PSL(2, **R**) acts on the upper half plane H_+ by $\gamma \cdot z = (az + b)/(cz + d)$, where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is a representative for γ and $z \in H_+$. Henceforth we shall assume that G is a subgroup of PSL(2, **R**), that G acts properly discontinuously on H_+ and that $X = H_+/G$ is a compact Riemann surface with a finite number of punctures, i.e., there is a compact Riemann surface $\overline{X}$ and an open immersion of $X = H_+/G$ into $\overline{X}$ so that $\overline{X} - X$ consists of a finite number of points. Let σ be the number of punctures.

Let $p_1, \ldots, p_r \in X$ be the points where $H_+ \to X$ is branched and let e_i be the ramification index over p_i. We number the p_i so that $e_1 \leq e_2 \leq \cdots \leq e_r$. We let $e_{r+1} = \cdots = e_{r+\sigma} = \infty$. If $g = \text{genus}\, \overline{X}$ then $\{g; \sigma; e_1, \ldots, e_r\}$ is called the *signature* of G. We will also use the notation $\{g; e_1, \ldots, e_{r+\sigma}\}$ for the signature of G. By **[K**, p. 78] a set of integers as just mentioned arises from a group G as before if and only if

$$\sigma + \sum_{i=1}^{r}\left(1 - \frac{1}{e_i}\right) + 2g - 2 > 0. \tag{2.1.1}$$

Alternately $\sum_{i=1}^{r+\sigma}(1 - 1/e_i) + 2g - 2 > 0$.

(2.2) A meromorphic function $f(z)$ on H_+ is said to be an *unrestricted automorphic form of weight k for G* if

$$f((az + b)/(cz + d)) = (cz + d)^{2k}f(z) \tag{2.2.1}$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$. Equivalently

$$f(\gamma \cdot z) = (d\gamma/dz)^{-k}f(z). \tag{2.2.2}$$

For any parabolic element of G we can change coordinates so that its fixed point is at ∞. Let $A(z) = z + 1$ be the generator of the stabilizer of ∞. Then f has a Fourier expansion $f(z) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n z}$. We say f is *entire* if f is holomorphic in H_+ and $a_n = 0$ for $n < 0$ (for each parabolic subgroup of G). We say f is a *cusp form* if f is entire and $a_n = 0$ for $n \leqslant 0$ (see **[G**, II.7] for details).

Define A_k = the **C**-vector space of *entire* automorphic forms of weight k, $k > 0$.

$$C_k = \text{the subspace of cusp forms,} \qquad k \geqslant 1.$$

Define $A_G = \bigoplus_{k=0}^{\infty} A_k$, the algebra of automorphic forms,

$$\mathcal{C}_G = \bigoplus_{k=1}^{\infty} C_k, \quad \text{the ideal of cusp forms.}$$

Then A_G is a graded algebra and $\mathcal{C}_G$ is a homogeneous ideal. If we define $C_0 = \mathbf{C}$ and $C_G = \bigoplus_{k=0}^{\infty} C_k$ we call C_G the algebra of cusp forms.

Henceforth, when there can be no confusion we will denote A_G by A.

REMARK. In **[W2]** we prove that A_G is a finitely-generated **C**-algebra. It follows that $\mathcal{C}_G$ is a finitely-generated ideal. On the other hand C_G is not a finitely-generated **C**-algebra (see (4.1)).

(2.3) The dimension of the vector space A_k can be calculated using the Riemann-Roch theorem. For example, one can use the proof in **[G**, Chapter II, Theorem 1] slightly modified. If G has signature $\{g; \sigma; e_1, \ldots, e_r\}$ then

$$\dim A_k = \begin{cases} (2k-1)(g-1) + \sigma k + \sum_{i=1}^{r}\left[k\left(1 - \frac{1}{e_i}\right)\right] & \text{if } k > 1 \text{ or } \sigma > 0, \\ g & \text{if } k = 1 \text{ and } \sigma = 0, \\ 1 & \text{if } k = 0. \end{cases} \tag{2.3.1}$$

The symbol $[x]$ denotes the largest integer $\leqslant x$.

(2.4) An elementary calculation using (2.3) shows that the Poincaré power series (1.6) of A is

$$\mathcal{P}_A = \frac{2g - 2 + \sigma}{(1-t)^2} + \frac{3 - 3g - \sigma}{(1-t)} + \sum_{i=1}^{r} P_{e_i}(t) + \delta_{\sigma+1,1}t + g$$

where $P_{e_i}(t) = \Sigma_{k=0}^{\infty}[k(1 - 1/e_i)]t^k$ and δ denotes the Kronecker delta.

We shall be interested in the partial fraction decomposition of P_e. This is calculated in the following proposition.

PROPOSITION (2.5). (i) *If a and b are natural numbers and* $a_k = [k(a/b)]$ *then*

$$\sum_{k=0}^{\infty} a_k t^k = \frac{\Sigma_{k=0}^{b-1} a_k t^k}{(1 - t^b)} + \frac{a(\Sigma_{k=0}^{b-1} t^k)t^b}{(1 - t^b)^2}. \tag{2.5.1}$$

(ii) *The principal part of* $P_e(t)$ *at* $t = 1$ *is*

$$\frac{e-1}{e}\,\frac{1}{(1-t)^2} - \frac{3(e-1)}{2e}\,\frac{1}{(1-t)}. \tag{2.5.2}$$

(iii) *Suppose* $\xi^e = 1$ *and* $\xi \neq 1$. *Then* $P_e(t)$ *has a pole of order* 1 *at* ξ *with residue* $\xi^2/e(1 - \xi)$.

PROOF. (i) Observe that $a_{k+nb} = a_k + na$ and hence

$$\begin{aligned}\sum_{k=0}^{\infty} a_k t^k &= \sum_{k=0}^{b-1}\sum_{n=0}^{\infty} a_{k+nb}t^{k+nb} = \sum_{k=0}^{b-1}\sum_{n=0}^{\infty}(a_k + na)t^{k+nb}\\ &= \sum_{k=0}^{b-1}\left(a_k t^k \sum_{n=0}^{\infty} t^{nb} + a\sum_{n=0}^{\infty} nt^{k+nb}\right)\\ &= \sum_{k=0}^{b-1}\left(a_k t^k\left(\frac{1}{1-t^b}\right) + at^k\sum_{n=0}^{\infty} nt^{nb}\right)\\ &= \frac{\Sigma_{k=0}^{b-1} a_k t^k}{(1-t^b)} + \frac{a(\Sigma_{k=0}^{b-1} t^k)}{(1-t^b)^2}\cdot t^b.\end{aligned}$$

(ii) If we let $b = e$ and $a = e - 1$ in part (i), then $a_0 = a_1 = 0$, $a_i = i - 1$ for $i = 2, \dots, e - 1$. Let $z = 1 - t$ and substitute for t in (2.5.1). We get

$$P_e(t) = \frac{(e-2)(e-1)/2}{ez} + \frac{(e-1)\Sigma_{k=0}^{e-1}(1-z)^k(1-z)^e}{e^2z^2}$$

+ terms on nonnegative degree in z.

The principal part of the above is

$$\frac{(e-1)(e-2)}{2e} + \frac{(e-1)e}{e^2z^2} + \frac{e-1}{e^2z^2}\left[-ez - \sum_{k=1}^{e-1} kz\right] = \frac{(e-1)}{ez^2} - \frac{3(e-1)}{2ez}.$$

(iii) $P_e(t)$ has a pole of order 1 at $t = \xi$, hence the residue $= \lim_{t\to\xi}(t - \xi)P_e(t)$ which can be evaluated using (2.5.1) and l'Hôpital's rule. □

REMARK. Since $\mathcal{P}_A$ has a pole of order 2 at $t = 1$, A is an algebra of dimension 2 (cf. [A-M]). In particular this tells us that if A is generated by n elements, then the ideal of relations has at least $n - 2$ generators.

THEOREM (2.6). *Suppose G is a group with signature* $\{g; \sigma; e_1, \dots, e_r\}$. *If the algebra of entire automorphic forms* A_G *is generated by* 3 *elements* f_0, f_1, f_2 *of weights* q_0, q_1, q_2 *respectively, then the ideal of relations is a principal ideal generated by a* *homogeneous element of degree d and*

$$2(g-1)+\sigma+\sum_{i=1}^{r}\frac{e_i-1}{e_i}=\frac{d}{q_0q_1q_2}, \tag{2.6.1}$$

$$3(1-g)-\sigma-\frac{3}{2}\sum_{i=1}^{r}\frac{e_i-1}{e_i}=-\frac{d(d-q_0-q_1-q_2+2)}{2q_0q_1q_2}, \tag{2.6.2}$$

$$(1-g)-\frac{1}{2}\sum_{i=1}^{r}\frac{e_i-1}{e_i}=-\frac{d(d-q_0-q_1-q_2)}{2q_0q_1q_2}. \tag{2.6.3}$$

Moreover

$$\text{g.c.d.}(q_0, q_1, q_2)=1. \tag{2.6.4}$$

PROOF. By (2.4) and Proposition (2.5) the principal part of $\mathcal{P}_A(t)$ at $t=1$ is

$$\left(2g-2+\sigma+\sum_{i=1}^{r}\frac{e_i-1}{e_i}\right)\frac{1}{(1-t)^2} + \left(3-3g-\sigma-\frac{3}{2}\sum_{i=1}^{r}\frac{e_i-1}{e_i}\right)\frac{1}{(1-t)}.$$

On the other hand we define φ: $\mathbf{C}[X_0, X_1, X_2]\to A$ by $\varphi(X_i)=f_i$, and we grade $S=\mathbf{C}[X_0, X_1, X_2]$ by letting degree $X_i=q_i$. We see that A is isomorphic to $R=S/I$ as a graded algebra, where $I=$ kernel φ.

LEMMA (2.7). *I is a principal ideal generated by a weighted homogeneous polynomial F of some degree d.*

We shall prove the lemma below. Returning to the proof of the theorem we apply Proposition (1.8) to find the principal part of $\mathcal{P}_R(t)$. Equating the coefficient of $1/(1-t)^2$ in $\mathcal{P}_R(t)$ and $\mathcal{P}_A(t)$ gives (2.6.1).

Similarly equating the coefficient of $1/(1-t)$ gives (2.6.2). The third equation follows from the first two. Finally g.c.d.$(q_0, q_1, q_2)=1$ since $\mathcal{P}_A(t)=\mathcal{P}_R(t)$ has a pole of order at most 1 at each nontrivial root of unity. □

REMARK. The case that A is generated by 2 elements is a special case of the above. If f_0, f_1 are generators of A define φ: $S\to A$ by $\varphi(X_i)=f_i$, $i=0, 1$, $\varphi(X_2)=0$. Grade S by letting degree $X_i=q_i$, $i=0, 1$, and degree $X_2=q_2$, arbitrary. Then I is the principal ideal generated by X_2, so that $d=q_2$.

PROOF OF LEMMA (2.7). If φ: $\mathbf{C}[X_0, X_1, X_2]\to A_G$ is a surjective graded homomorphism, then $I=$ kernel φ is a principal ideal generated by a homogeneous element.

PROOF. By (2.4) and Proposition (2.5), $\mathcal{P}_A(t)$ has a pole of order $\leqslant 2$ at $t=1$. By (2.1.1) the order of the pole is precisely 2. Hence by (1.11)(b), A has dimension 2. Now A is isomorphic to S/I hence I is a height 1 prime ideal. But S is a unique factorization domain, hence every height 1 prime ideal is principal [**Z**, Chapter V, §14]. If F generates I let $F=F_d+F_{d+1}+\cdots$ where $F_i\in S_i$. Now I is a homogeneous ideal so $F_d\in I$. Thus $F|F_d$ and therefore $F=F_d$. □

PROPOSITION (2.8). *Suppose G has signature* $\{g; \sigma; e_1, \ldots, e_r\}$ *and* A_G *is generated by* $f_i \in A_{q_i}$, $i = 0, 1, 2$. *For each i let* $\xi_i = \exp(2\pi\sqrt{-1}/e_i)$ *and* ν_i = *the number of j so that* $e_i | e_j$. *Then*

$$\mathcal{P}_{A_G}(t) = (1 - t^d)/(1 - t^{q_0})(1 - t^{q_1})(1 - t^{q_2}) \tag{2.8.1}$$

if and only if

(a) (2.6.1), (2.6.2) *and* (2.6.4) *hold.*

(b) *For each i*

$$\frac{\nu_i \xi_i^2}{e_i(1 - \xi_i)} = \begin{cases} \dfrac{-\xi_i(1 - \xi_i^d)}{q_{i_0}(1 - \xi_i^{q_{i_1}})(1 - \xi_i^{q_{i_2}})} & \text{if } e_i | q_{i_0} \text{ and } e_i \nmid q_{i_1}, q_{i_2}, \\ \dfrac{-d\xi_i}{q_{i_0} q_{i_1}(1 - \xi_i^{q_{i_2}})} & \text{if } e_i | q_{i_0}, e_i | q_{i_1}, e_i \nmid q_{i_2}. \end{cases} \tag{2.8.2}$$

(c) *For all i and j so that* $i \neq j$, $(q_i, q_j) | d$.

(d)

$$\begin{aligned} d &= q_0 + q_1 + q_2 + 1 && \text{if } \sigma = 0, \\ d &= q_0 + q_1 + q_2 && \text{if } \sigma > 0 \text{ and } g > 0, \\ d &\leq q_0 + q_1 + q_2 - 1 && \text{if } \sigma > 0 \text{ and } g = 0. \end{aligned} \tag{2.8.3}$$

PROOF. Let $f(t) = (1 - t^d)/(1 - t^{q_0})(1 - t^{q_1})(1 - t^{q_2})$. Decompose $f(t)$ and $\mathcal{P} = \mathcal{P}_{A_G}(t)$ as partial fractions over the complex numbers. The poles of $\mathcal{P}$ are at e_ith roots of unity for some e_i or at $t = 1$. Equation (2.6.4) is equivalent to f having a pole of order at most 1 at roots of unity. Equations (2.6.1) and (2.6.2) are equivalent to the coefficients of $1/(1 - t)$ and $1/(1 - t)^2$ being the same for f and $\mathcal{P}$. By Propositions (2.5)(iii) and (1.9), equations (2.8.2) are equivalent to the equality of the principal parts of f and $\mathcal{P}$ at ξ, where $\xi \neq 1$ is a root of unity. Now the polynomial part of $\mathcal{P}_A(t)$ is $\delta_{\sigma+1,1}t + g$. Thus $\sigma = 0$ if and only if $\delta_{\sigma+1,1} = 1$ if and only if $d = q_0 + q_1 + q_2 + 1$. If $\sigma > 0$ and $g > 0$ then the polynomial part of $\mathcal{P}$ is a nonzero constant, i.e., $\mathcal{P}$ is holomorphic and nonzero at ∞. But f is holomorphic and nonzero at ∞ if and only if $d = q_0 + q_1 + q_2$. Finally $\sigma > 0$ and $g = 0$ is equivalent to $\mathcal{P}$ having a zero at ∞ and f has a zero at ∞ if and only if $d \leq q_0 + q_1 + q_2 - 1$. When all other conditions are satisfied the constant term of $\mathcal{P}$ and f must be equal since $f(0) = \mathcal{P}(0) = 1$. □

3. Groups for which A_G may have few generators. Henceforth we assume G is a finitely-generated Fuchsian group of the first kind with signature $\{g; e_1, \ldots, e_r, \ldots, e_{r+\sigma}\}$ where $e_i < \infty$ for $1 \leq i \leq r$ and $e_i = \infty$ for $i > r$. We arrange the e_i so that $e_1 \leq e_2 \leq \cdots \leq e_r$.

THEOREM (3.1). *If the algebra* A_G *is generated by* 2 *elements then it is a polynomial ring in two variables. Those signatures and the degrees* q_0, q_1 *of the generators are given below.*

Signature	(q_0, q_1)
$\{0; \infty, \infty, \infty\}$	(1, 1)
$\{0; 2, \infty, \infty\}$	(1, 2)
$\{0; 2, 3, \infty\}$	(2, 3)

If the algebra A_G is generated by 3 *elements, then G is on Table* 1 *which follows. For these signatures we determine the degrees q_0, q_1, q_2 of the generators and the degree d of the relation. The converse of this theorem will be proven in* §4.

TABLE 1

Signature	$(d;q_0,q_1,q_2)$	Signature	$(d;q_0,q_1,q_2)$
{3}, non hyperelliptic	(4;1,1,1)	{0,2,2,2,2,3}	(8;2,2,3)
{2}	(6;1,1,3)	{0;2,2,2,2,2}	(10;2,2,5)
{2;2}	(5;1,1,2)	{0;2,3,3,3}	(9;2,3,3)
{1;2,2,2}	(6;1,2,2)	{0;2,2,3,4}	(10,2,3,4)
{1,2,2}	(8;1,2,4)	{0;2,2,3,3}	(12;2,3,6)
{1;2,3}	(7;1,2,3)	{0;2,2,2,5}	(12;2,4,5)
{1;2}	(12;1,4,6)	{0;2,2,2,4}	(14;2,4,7)
{1;3}	(10;1,3,5)	{0;2,2,2,3}	(18;2,6,9)
{1;4}	(9;1,3,4)	{0;4,4,4}	(12;3,4,4)
{1;∞,∞,∞}	(3;1,1,1)	{0,3,4,5}	(13;3,4,5)
{1;∞,∞}	(4;1,1,2)	{0;3,4,4}	(16;3,4,8)
{1;∞}	(6;1,2,3)	{0;3,3,4}	(24;3,8,12)
{0;∞,∞,∞,∞}	(2;1,1,1)	{0;3,3,5}	(18;3,5,9)
{0;2,∞,∞,∞}	(3;1,1,2)	{0;3,3,6}	(15;3,5,6)
{0;2,2,∞,∞}	(4;1,2,2)	{0;2,5,6}	(16;4,5,6)
{0;3,∞,∞}	(4;1,2,3)	{0;2,5,5}	(20;4,5,10)
{0;2,2,2,∞}	(6;2,2,3)	{0;2,4,7}	(18;4,6,7)
{0;3,3,∞}	(6;2,3,3)	{0;2,4,6}	(22;4,6,11)
{0;2,4,∞}	(6;2,3,4)	{0;2,4,5}	(30;4,10,15)
		{0;2,3,9}	(24;6,8,9)
		{0;2,3,8}	(30;6,8,15)
		{0;2,3,7}	(42;6,14,21)

PROOF. For each possible signature we shall determine the degrees q_0, q_1, q_2 of the generators. Then d is determined by (2.6.1). For clarity we shall divide the proof into sections.

(3.2) Suppose A is generated by forms f_0, f_1, f_2 and $q_i = \text{degree} f_i$. We let $S = \mathbf{C}[X_0, X_1, X_2]$ and define $\varphi: S \to A$ by $\varphi(X_i) = f_i$. We grade S by letting degree $X_i = q_i$. Then φ is onto and A is isomorphic to $R = S/I$, where $I = \text{kernel } \varphi$. Now by (2.3)

$$\dim A_1 = \begin{cases} (g-1) + \sigma & \text{if } \sigma > 0, \\ g & \text{if } \sigma = 0, \end{cases}$$

and $\dim A_1 \leqslant 3$ since A is generated by 3 elements. Thus

$$(g-1) + \sigma \leqslant 3 \text{ if } \sigma > 0, \quad \text{and} \quad g \leqslant 3 \text{ if } \sigma = 0. \tag{3.2.1}$$

In either case $g \leqslant 3$.

(3.3) Suppose $g = 3$. Then by (3.2.1) either $\sigma = 0$ or $\sigma = 1$. In either case $\dim A_1 = 3$ implies there are at least 3 generators of weight 1. Hence $q_0 = q_1 = q_2 = 1$. Applying (2.3.1) we get $\dim A_2 = 6 + 2\sigma + r$. But $\dim S_2 = c(1, 1, 1)_2 = 6$ (see Definition (1.10)), hence, $\sigma = r = 0$. Thus $\{3\}$ is the only possible signature with $g = 3$. If X is hyperelliptic, then the one-forms do not generate A_G (this follows from [S, p. 293]).

(3.4) Now suppose $g = 2$ or $g = 1$. Then Table 2 gives $\dim A_K$ for all signatures which do not appear on Table 1. A generator of A_G is indicated by the letters a, b, c, d. We stop counting generators after we have found that there must be four generators. As an example, consider the signature $\{1; 2, e_2\}$, $e_2 > 3$. Clearly there is a generator $a \in A_1$. Now $\dim A_2 = 2$, so there must be an element $b \in A_2$ so that a^2 and b are linearly independent. Next there must be an element $c \in A_3$ so that a^3, ab and c are independent. Finally, a, b and c generate a subspace of dimension $\leqslant c(1, 2, 3)_4 = 4$ in A_4, so there must be a fourth generator d.

TABLE 2. $\dim A_k$ for signatures with embedding dimension > 3, $g = 1$ or 2

	k	1	2	3	4	5
$\{2; e_1, \ldots, e_r\}$	$r \geq 2$	a, b 2	c, d $3+r$			
$\{2; e_1\}$	$e_1 > 2$	a, b 2	c 4	d 7		
$\{2; \sigma; e_1, \ldots, e_r\}$	$\sigma \geq 3$	a, b, c, d ≤ 4				
$\{2; \sigma; e_1, \ldots, e_r\}$	$\sigma = 2$	a, b, c 3	d ≥ 7			
$\{2; \sigma; e_1, \ldots, e_r\}$	$\sigma = 1$	a, b 2	c, d ≥ 5			
$\{1; e_1, \ldots, e_r\}$	$r \geq 3$	a 1	b, c, d r			
$\{1; e_1, e_2, e_3\}$	$e_3 > 2$	a 1	b, c 3	d ≥ 4		
$\{1; e_1, e_2\}$	$e_1 > 2$	a 1	b 2	c, d 4		
$\{1; 2; e_2\}$	$e_2 > 3$	a 1	b 2	c 3	d 5	
$\{1; e_1\}$	$e_1 > 4$	a 1	1	b 2	c 3	d 4
$\{1; \sigma; e_1, \ldots, e_r\}$	$\sigma > 3$	a, b, c, d 4				
$\{1; 3; e_1, \ldots, e_r\}$	$r > 0$	a, b, c 3	d ≥ 7			
$\{1; 2; e_1, \ldots, e_r\}$	$r > 0$	a, b 2	c, d ≥ 5			
$\{1; e_1; \infty\}$		a 1	b, c 3	d ≥ 4		
$\{0; \sigma; e_1, \ldots, e_r\}$	$\sigma > 4$	a, b, c, d $\sigma - 1$				
$\{0; 4; e_1, \ldots, e_r\}$		a, b, c 3	$r+5$	7		

(3.5) Suppose $g = 0$. Then Table 3 lists all the groups with signatures not on Table 1. This table suffices to rule out all the cases except $\sigma = 2$, $r = 1$, $e_1 > 4$. In that case $q_0 = 1$, $q_1 = 2$ and $q_2 = 3$. By (2.6.1), $d/6 = (e_1 - 1)/e_1$ which implies $e_1 = 3$ or 6. But $e_1 = 6$ does not satisfy (2.6.3). Thus $e_1 = 3$.

$\{0; 2, 2, \infty\}$ is not realizable by (2.1.1).

4. Classification. In this section we prove the converse of Theorem (3.1); for every group G on Table 1, A_G is generated by 3 or 2 elements. In order to do this, we first reinterpret the ring A_G as a ring of functions on X.

(4.1) Let ω^* be an arbitrary meromorphic differential form on $\bar{X}$ and K the divisor of ω^*. Then ω^* pulls back to a meromorphic form ω on H_+. If f is an automorphic form of weight k, $g = f/\omega^k$ is a G-invariant meromorphic function on H_+ and hence induces a meromorphic function on $\bar{X}$. By [G, p. 24] f is holomorphic if and only if

$$(g) + k \cdot K + \sum_{i=1}^{r} \left[k\left(1 - \frac{1}{e_i}\right) \right] p_i + \sum_{r+1}^{r+\sigma} kp_i > 0$$

where $p_1, \ldots, p_r$ are the elliptic points and $p_{r+1}, \ldots, p_{r+\sigma}$ are the cusp points. Here ( ) denotes the divisor of a function or differential. Thus for any k we have an isomorphism

$$\varphi_k \colon A_k \overset{\approx}{\to} L\left(k \cdot K + \sum_{i=1}^{r} \left[k\left(1 - \frac{1}{e_i}\right) \right] p_i + \sum_{i=1}^{\sigma} kp_{r+i} \right)$$

TABLE 3. dim A_k for signatures with embedding dimension > 3, $g = 0$, $\sigma \leqslant 2$

		1	2	3	4	5
$\{0;e_1,\ldots,e_r,\infty,\infty\}$	$r \geq 3$	a 1	b,c,d 1+r			
$\{0;e_1,e_2,\infty,\infty\}$	$e_2 > 2$	a 1	b,c 3	d ≥ 4		
$\{0;e_1,\infty,\infty\}$	$e_1 \geq 4$	a 1	b 2	c 3		
$\{0;e_1,\ldots,e_r,\infty\}$	$r \geq 4$	0	a,b,c r-1	d ≥ 2		
$\{0;e_1,e_2,e_3,\infty\}$	$e_3 > 2$	0	a,b 2	c,d ≥ 2		
$\{0;e_1,e_2,\infty\}$	$e_1 \geq 3$, $e_2 \geq 4$	0	a 1	b,c 2	d ≥ 2	
$\{0;2,e_2,\infty\}$	$e_2 \geq 5$	0	a 1	b 1	c 2	d 2
$\{0;e_1,\ldots,e_r\}$	$r \geq 6$	0	a,b,c r-3	d ≥ 1		
$\{0;e_1,\ldots,e_5\}$	$e_4 > 2$	0	a,b 2	c,d ≥ 2		
$\{0;2,2,2,2,e_5\}$	$e_5 \geq 4$	0	a,b 2	c 1	d 4	
$\{0;e_1,e_2,e_3,e_4\}$	$e_1 \geq 3$	0	a 1	b,c,d 3		
$\{0;2,e_2,e_3,e_4\}$	$e_2 \geq 3$, $e_4 > 3$	0	a 1	b,c 2	d ≥ 2	

TABLE 3 (continued)

		1	2	3	4	5	6	7	8	9	10
$\{0;2,2,e_3,e_4\}$	$e_3 \geq 4$	0	a 1	b 1	c,d 3						
$\{0,2,2,3,e_4\}$	$e_4 \geq 5$	0	a 1	b 1	c 2	d 2					
$\{0;2,2,2,e_4\}$	$e_4 \geq 6$	0	a 1	0	b 2	c 1	d 3				
$\{0;e_1,e_2,e_3\}$	$e_1 \geq 4,\ e_3 > 4$	0	0	a 1	b,c 2	d ≥ 1					
$\{0;3,e_2,e_3\}$	$e_2 \geq 5$	0	0	a 1	b 1	c,d 2					
$\{0;3,4,e_3\}$	$e_3 \geq 6$	0	0	a 1	b 1	c 1	d 2				
$\{0;3,3,e_3\}$	$e_3 \geq 7$	0	0	a 1	0	b 1	c 2	d 1			
$\{0;2,e_2,e_3\}$	$e_2 \geq 6$	0	0	0	a 1	b 1	c,d 2				
$\{0;2,5,e_3\}$	$e_3 \geq 7$	0	0	0	a 1	b 1	c 1	d 1			
$\{0;2,4,e_3\}$	$e_3 \geq 8$	0	0	0	a 1	0	b 1	c 1	d 2		
$\{0;2,3,e_3\}$	$e_3 \geq 10$	0	0	0	0	0	a 1	0	b 1	c 1	d 1

defined by $\varphi_k(f) = f/\omega^k$. Let

$$L_k = L\left(k \cdot K + \sum_{i=1}^{r}\left[k\left(1 - \frac{1}{e_i}\right)\right]p_i + \sum_{i=1}^{\sigma} kp_{r+i}\right).$$

Then we have an isomorphism of **C**-algebras

$$\varphi\colon A_G \xrightarrow{\approx} \bigoplus_{k \geqslant 0} L_k. \tag{4.1.1}$$

Let L_G denote $\bigoplus_{k \geqslant 0} L_k$.

REMARK. φ_k induces an isomorphism of the space of cusp forms of weight k with a similar vector space of functions

$$\varphi_k\colon C_k \xrightarrow{\approx} L\left(k \cdot K + \sum_{i=1}^{r}\left[k\left(1 - \frac{1}{e_i}\right)\right]p_i + \sum_{i=1}^{\sigma} (k-1)p_{r+i}\right).$$

One can use this to see that the algebra of cusp forms is not finitely generated, as follows. Linear combinations of products of cusp forms of degree $< k$ are all contained in

$$\overline{C}_k = L\left(k \cdot K + \sum_{i=1}^{r}\left[k\left(1 - \frac{1}{e_i}\right)\right]p_i + \sum_{i=1}^{\sigma} (k-2)p_{r+i}\right).$$

If $g \geqslant 1$, it follows immediately from Riemann-Roch that $\dim C_k/\overline{C}_k \geqslant \sigma$, and hence that there are at least σ generators of *every* degree. If $g = 0$ then we have $A_k \supset C_k \supset \overline{C}_k$ and again by Riemann-Roch $\dim C_k/\overline{C}_k > 0$, provided $\dim A_k \geqslant \sigma$. Now A is an algebra of dimension 2 (see Remark preceding Theorem (2.6)) hence there are an infinite number of k so that $\dim A_k \geqslant \sigma$. Thus C_k requires an infinite number of generators.

(4.2) Let $\mathbf{C}(X)$ denote the field of meromorphic functions on X. Note that the inclusions $L_k \to \mathbf{C}(X)$ do not induce an inclusion $L_G \to \mathbf{C}(X)$. (Since, for example, $L_k \subset L_{2k}$.) Let t be an indeterminate and define

$$L_k \to \mathbf{C}(X)[t], \qquad f \mapsto f \cdot t^k.$$

This extends to an injective map $i_G: L_G \to \mathbf{C}(X)[t]$.

Henceforth, we identify L_G with its image in $\mathbf{C}(X)[t]$, i.e., an element $(0, \ldots, 0, f, 0, \ldots)$ is identified with $f \cdot t^k$ if f lies in the kth component. Thus the exponent of t serves as a reminder of which L_k f comes from.

EXAMPLE (4.3). Suppose G is a group with signature $\{1; 4\}$. Then $L_k \approx L([k\frac{3}{4}]p)$ where $p \in X$ is the branch point. Let f be the Weierstrass $\wp$ function with pole (of order 2) at p and $g = f'$. Then letting $\langle\ \rangle$ denote "vector space generated by", we have

$$L(np) = \begin{cases} \langle 1 \rangle, & n = 0, 1, \\ \langle 1, f, g, f^2, f^2 g, \ldots, f^{n/2} \rangle, & n \text{ even}, \geqslant 2, \\ \langle 1, f, g, f^2, f^2 g, f^3, \ldots, f^{n-3/2} g \rangle, & n \text{ odd}, \geqslant 3. \end{cases}$$

Now

$$L_k = \begin{cases} L(3np), & k = 4n, \\ L(3np), & k = 4n + 1, \\ L((3n + 1)p), & k = 4n + 2, \\ L((3n + 2)p), & k = 4n + 3. \end{cases}$$

$L_G = \langle 1 \rangle \oplus \langle 1 \rangle t \oplus \langle 1 \rangle t^2 \oplus \langle 1, f \rangle t^3 \oplus \langle 1, f, g \rangle t^4 \oplus \langle 1, f, g \rangle t^5 \ldots$. It can easily be verified that L_G is generated by $z_0 = 1 \cdot t$, $z_1 = f \cdot t^3$ and $z_2 = g \cdot t^4$. The map

$$\Phi: \mathbf{C}[Z_0, Z_1, Z_2] \to L_G$$

defined by $\Phi(Z_i) = z_i$ is surjective. By Lemma (2.7) the kernel of Φ is a principal ideal generated by a homogeneous polynomial f of degree d. Now by Proposition (2.8)(d), $d = q_0 + q_1 + q_2 + 1 = 9$ and we see that

$$f(Z_0, Z_1, Z_2) = Z_0 Z_2^2 - 4Z_1^3 + g_2 Z_0^6 Z_1 + g_3 Z_0^9$$

where g_2 and g_3 are constants depending on G so that $g_2^3 - 27g_3^2 \neq 0$. □

For the groups on Table 1 we have natural candidates for generators of A_G. To show that these elements actually generate we use the following.

PROPOSITION (4.4). *Suppose A is a graded algebra over* $\mathbf{C}$,

$$\mathcal{P}_A(t) = \frac{1 - t^d}{(1 - t^{q_0})(1 - t^{q_1})(1 - t^{q_2})}, \qquad z_i \in A_{q_i} \textit{ for each } i = 0, 1, 2,$$

and the z_i satisfy a relation f of degree d in A. Suppose, moreover, that

(1) *f is irreducible,*

(2) *the z_i generate a field $\mathbf{C}(z_0, z_1, z_2)$ of transcendence degree* 2.

Then the canonical map

$$\Phi: \mathbf{C}[Z_0, Z_1, Z_2]/(f) \to A$$

is an isomorphism.

PROOF. Let $R = \mathbf{C}[Z_0, Z_1, Z_2]/(f)$ and $I = \text{kernel } \Phi$. Now f irreducible implies R is an integral domain. Thus $I \neq (0)$ implies [A-M, §11] $\dim R/I < \dim R = 2$. But transcendence degree $\mathbf{C}(z_0, z_1, z_2) = 2$ implies $\dim R/I = 2$ [A-M, 11.25]. Thus $I = (0)$ and hence Φ is one-one. Finally, $\mathcal{P}_R(t) = \mathcal{P}_A(t)$ implies Φ is onto. □

REMARK (4.5). Suppose A is an integral domain, A, $\mathcal{P}_A$ and z_i are as in Proposition (4.4) and the z_i satisfy a relation f of degree d.

(1) If the z_i satisfy no relation of degree $< d$ then f is irreducible.

(2) If for some $i \neq j$, $z_i^{q_j}/z_j^{q_i}$ is not a constant then the z_i generate a field $\mathbf{C}(z_0, z_1, z_2)$ of transcendence degree 2.

PROOF. (1) If f were reducible then $f = gh$ where g and h are homogeneous of degree $< d$. But A is an integral domain and hence either g or h is a relation in A. This contradicts the hypothesis.

(2) If $\mathbf{C}(z_0, z_1, z_2)$ has transcendence degree $\leqslant 1$, then z_0 and z_1 are algebraically dependent. Then there is an irreducible polynomial $f(Z_0, Z_1)$ satisfied by z_0 and z_1. But f is weighted homogeneous of the form $Z_0^\alpha - \varepsilon Z_1^\beta$ where $\varepsilon \in \mathbf{C}$, $\alpha = q_1/(q_0, q_1)$ and $\beta = q_0/(q_0, q_1)$ by [O-W, Lemma 3.6]. Thus $z_0^{q_1}/z_1^{q_0}$ is a power of ε, contradicting the hypothesis. □

THEOREM (4.6). *For every group on Table* 1 *in* §3, A_G *is generated by* $\leqslant 3$ *elements. The degrees of the generators and the relation are given in Table* 1.

PROOF. For each group one finds natural candidates, $z_i \in L_{q_i} \cdot t^{q_i}$, $i = 0, 1, 2$, for the generators of L_G using Table 4. Then we show that these actually generate using Proposition (4.4) and Remark (4.5). One first verifies that for each group G

$$\mathcal{P}_{A(t)} = (1 - t^d)/(1 - t^{q_0})(1 - t^{q_1})(1 - t^{q_2})$$

where q_0, q_1, q_2 and d are given in Table 1. This verification can be carried out using Proposition (2.8) or (2.4) and Proposition (2.5) and we shall not include these calculations here. The fact that the z_i satisfy a relation f of degree d is also easily seen. Hence all that remains is to show that Proposition (4.4)(1) or Remark (4.5)(1) and Proposition (4.4)(2) or Remark (4.5)(2) are satisfied. We first prove a technical lemma which is used in the proof.

LEMMA (4.6.1). *Suppose* $f_0, \ldots, f_r$ *are analytic functions, none of which is identically zero. Suppose that for each i there is a point p so that* $\nu_p(f_i) < \nu_p(f_j)$ *for all* $i < j$, *where* $\nu_p(f)$ *is the order of the pole of f at p. Then* $f_0, \ldots, f_r$ *are linearly independent.*

PROOF. Proof by induction. The assertion is true for $r = 0$. Now suppose it is true for $r - 1$ functions. If $f_0, \ldots, f_r$ are as before and $a_0 f_1 + \cdots + a_{r-1} f_{r-1} + a_r f_r = 0$ then $a_0 f_0 + \cdots + a_{r-1} f_{r-1} = -a_r f_r$. If $a_r \neq 0$, then by hypothesis there is a point p so that the right-hand side has a pole at p of order, say $n = \nu_p(f_r)$, while the left-hand side has a pole of lesser order. This is impossible, so $a_r = 0$. Now by the inductive hypothesis $f_0, \ldots, f_{r-1}$ are linearly independent so $a_0 = a_1 = \cdots = a_{r-1} = 0$. Thus $f_0, \ldots, f_r$ are linearly independent. □

We return to the proof of the theorem, examining each signature in turn.

TABLE 4. dim A_k for signatures on Table 1

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
{3}	3	6	10	14																				
{2}	2	3	5	7	9	11																		
{2;2}	2	4	6	9	11																			
{1;2,2,2}	1	3	3	6	6	8																		
{1;2,2}	1	2	2	4	4	6	6	9																
{1;2,3}	1	2	3	4	5	7	7																	
{1;2}	1	1	1	2	2	3	3	4	4	5	5	6												
{1;3}	1	1	2	2	3	4	4	5	6	6														
{1;4}	1	1	2	3	3	4	5	6	6															
{1;∞,∞,∞}	3	6	9																					
{1;∞,∞}	2	4	6	8																				
{1;∞}	1	2	3	4	5	6																		
{0;∞,∞,∞,∞}	3	5																						
{0;2,∞,∞,∞}	2	4	5																					
{0;∞,∞,∞}	2	3	4	5																				
{0;2,2,∞,∞}	1	3	3	5																				
{0;3,∞,∞}	1	2	3	3																				
{0;2,∞,∞}	1	2	2	3	3	4	4																	
{0;2,2,2,∞}	0	2	1	3	2	4																		
{0;3,3,∞}	0	1	2	1	2	3																		
{0;2,4,∞}	0	1	1	2	1	2																		
{0;2,3,∞}	0	1	1	1	1	2	1	2																
{0;2,2,2,2,3}	0	2	1	3	2	5	3	6																
{0;2,2,2,2,2}	0	2	0	3	1	4	2	5	3	6														

TABLE 4 (continued)

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
{0;2,3,3,3}	0	1	2	1	2	4	2	4	5															
{0;2,2,3,4}	0	1	1	2	1	3	2	4	3	4														
{0;2,2,3,3}	0	1	1	1	1	3	1	3	3	3	3	5												
{0;2,2,2,5}	0	1	0	2	1	2	1	3	2	4	2	4												
{0;2,2,2,4}	0	1	0	2	0	2	1	3	1	3	2	4	2	4										
{0;2,2,2,3}	0	1	0	1	0	2	0	2	1	2	1	3	1	3	2	3	2	4						
{0;4,4,4}	0	0	1	2	0	1	2	3	1	2	3	4												
{0;3,4,5}	0	0	1	1	1	1	1	2	2	2	2	3	2											
{0;3,4,4}	0	0	1	1	0	1	1	2	1	1	2	3	1	2	3	3								
{0;3,3,6}	0	0	1	0	1	2	0	1	2	1	2	3	1	2	3									
{0;3,3,5}	0	0	1	0	1	1	0	1	2	1	1	2	1	2	3	1	2	3						
{0;3,3,4}	0	0	1	0	0	1	0	1	1	0	1	2	0	1	2	1	1	2	1	2	2	1	2	3

TABLE 4 (continued)

		1	2	3	4	5	6	7	8	9	10	11	12
	k	13	14	15	16	17	18	19	20	21	22	23	24
		25	26	27	28	29	30	31	32	33	34	35	36 etc.
{0;2,5,6}		0	0	0	1	1	1	0	1	1	2	1	2
		1	2	2	2								
{0;2,5,5}		0	0	0	1	1	0	0	1	1	2	0	1
		1	2	2	1	1	2	2	3				
{0;2,4,7}		0	0	0	1	0	1	1	1	0	1	1	2
		1	2	1	2	1	2						
{0;2,4,6}		0	0	0	1	0	1	0	1	0	1	1	2
		0	1	1	2	1	2	1	2	1	2		
{0;2,4,5}		0	0	0	1	0	0	0	1	0	1	0	1
		0	1	1	1	0	1	1	2	0	1	1	2
		1	1	1	2	1	2						
{0;2,3,9}		0	0	0	0	0	1	0	1	1	0	0	1
		0	1	1	1	1	2	0	1	1	1	1	2
{0:2,3,8}		0	0	0	0	0	1	0	1	0	0	0	1
		0	1	1	1	0	1	0	1	1	1	1	2
		0	1	1	1	1	2						
{0;2,3,7}		0	0	0	0	0	1	0	0	0	0	0	1
		0	1	0	0	0	1	0	1	1	0	0	1
		0	1	1	1	0	1	0	1	1	1	1	1
		0	1	1	1	1	2						

$\{3\}$: By Noether's theorem **[S-D]** A_G is generated by 1-forms if and only if X is nonhyperelliptic. In fact, we can see this rather explicitly in the case of genus 3. If X is nonhyperelliptic then one can choose a basis ω_1, ω_2, ω_3 for the space of 1-forms and then

$$h(x) = (\omega_1(x): \omega_2(x): \omega_3(x))$$

defines the canonical embedding of X in P^2. Now one can see that A_G is isomorphic to the homogeneous coordinate ring of $h(X)$. Hence A_G is generated by the ω_i, and the ω_i satisfy a (nonsingular) relation of degree 4. This relation is precisely the equation of $h(X)$ in P^2. Conversely, if X is hyperelliptic we can write X as the Riemann surface of the plane curve

$$y^2 = \prod_{i=1}^{7} (x - \varepsilon_i).$$

By [S, §10.10] the 1-forms are generated by dx/y, xdx/y, x^2dx/y and the 2-forms by $dx^2/y^2, \ldots, x^4dx^2/y^2$, dx^2/y. Now the last 2-form is not in the subspace generated by products of 1-forms, hence A is not generated by 1-forms. One can show that A is generated by the three 1-forms and dx^2/y.

$\{2\}$: We may write X as the Riemann surface of the curve

$$y^2 = (x - \varepsilon_1)(x - \varepsilon_2)(x - \varepsilon_3)(x - \varepsilon_4)(x - \varepsilon_5) \tag{4.6.1}$$

where the ε_i are distinct nonzero complex numbers.

$$A_1 = \langle dx/y, xdx/y \rangle,$$
$$A_2 = \langle dx^2/y^2, xdx^2/y^2, x^2dx^2/y^2 \rangle,$$
$$A_3 = \langle dx^3/y^3, xdx^3/y^3, x^2dx^3/y^3, x^3dx^3/y^3, dx^3/y^2 \rangle,$$
$$L_1 = \langle 1, x \rangle, \qquad L_2 = \langle 1, x, x^2 \rangle, \qquad L_3 = \langle 1, x, x^2, x^3, y \rangle.$$

Let $z_0 = 1 \cdot t$, $z_1 = x \cdot t$, $z_2 = y \cdot t^3$. Then the z_i generate a field of transcendence degree 2 and satisfy an irreducible polynomial relation

$$(z_2)^2 - z_0 \prod_{i=1}^{5} (z_1 - \varepsilon_i z_0) = 0. \tag{4.6.2}$$

Thus by Proposition (4.4) the z_i generate L_G.

$\{2; 2\}$: The curve X is given by equation (4.6.1). Let p be the branch point on X. Suppose $p = \infty$. Then $L_1 = L(K) = \langle 1, x \rangle$, $L_2 = L(2K + p) = L(5\infty) = \langle 1, x, x^2, y \rangle$. Let $z_0 = 1 \cdot t$, $z_1 = x \cdot t$, $z_2 = y \cdot t^2$. Then the z_i satisfy the irreducible relation

$$z_0 z_2^2 - \prod_{i=1}^{5} (z_1 - \varepsilon_i z_0) = 0. \tag{4.6.3}$$

Hence by Proposition (4.4) the z_i generate L_G.

Now suppose $p \neq \infty$. Then $L_1 = \langle 1, x \rangle$, $L_2 = \langle 1, x, x^2, f \rangle$ where f has a pole of order 1 at p. By subtracting a suitable linear combination of x and x^2 from f we may replace f by a function with a pole of odd order at infinity (since the canonical divisor $K = 2\infty$). By the Riemann-Roch theorem $l(\infty + p) = 1 + l(\infty - p) = 1$, hence $L(\infty + p)$ consists only of constants. Therefore f must have a pole of order 3 at ∞. Now $L_3 \supseteq \langle 1, x, x^2, x^3, f, xf \rangle$ and by using Lemma (4.6.1) we can show that those six functions are linearly independent. The dimension of L_3 is 6, hence the inclusion above is an equality. Similarly $L_4 = \langle 1, x, x^2, x^3, x^4, f, xf, x^2f, f^2 \rangle$ and these functions are linearly independent. We can now exhibit candidates for generators of L, namely let $z_0 = 1 \cdot t$, $z_1 = x \cdot t$, $z_2 = f \cdot t^2$. One can easily verify that there are 12 monomials of (weighted) degree 5 in z_0, z_1, z_2 and that dim $L_5 = 11$. Thus there must be a relation of degree 5 among the z_i. Moreover one can easily see from the preceding remarks that there is no relation of degree less than 5. Hence by Remark (4.5) and Proposition (4.4) the z_i generate L and satisfy a relation of degree 5.

$\{1; 2, 2, 2\}$: In the case $g = 1$, the canonical divisor $K = 0$. Thus $L_1 = L(0) = \langle 1 \rangle$, $L_2 = L(p_1 + p_2 + p_3) = \langle 1, f, g \rangle$ where we choose f and g so that the divisor of poles of f, $(f)_\infty = p_1 + p_2$ and $(g)_\infty = p_2 + p_3$. Let $z_0 = 1 \cdot t$, $z_1 = f \cdot t^2$, $z_2 = g \cdot t^2$. There is a relation of degree 6 among the z_i since dim $A_6 = 9$, while there are 10 monomials of degree 6 in the z_i. Now by Remark (4.5)(1) it is sufficient to show that there is no relation of degree < 6. Now $L_2 = L_3$ so there is no relation of degree 3. Next we see that

$$L_4 = L(2p_1 + 2p_2 + 2p_3) \supseteq \langle 1, f, g, f^2, fg, g^2 \rangle \tag{4.6.4}$$

and dim $L_4 = 6$. By considering the orders of the poles of the above functions at the p_i we can see that they are linearly independent. Thus we have equality in (4.6.4) and there can be no relation of degree 4. Finally $L_4 = L_5$ so there is no relation of degree 5.

$\{1; 2, 2\}$: In this case $L_1 = L(0) = \langle 1 \rangle$, $L_2 = L(p_1 + p_2) = \langle 1, f \rangle$, $L_3 = L_2$, $L_5 = L_4 = L(2p_1 + 2p_2) = \langle 1, f, g \rangle$ where we can choose f and g so that $(f)_\infty = p_1 + p_2$ and $(g)_\infty = 2p_1$. Then one can easily verify that $L_6 = L_7 = \langle 1, f, f^2, f^3, g, fg \rangle$ since these functions are linearly independent. There is a relation of degree 8 among the z_i since dim $A_8 = 8$ and there are 9 monomials of degree 9 in the z_i. Now by the above remarks if we let $z_0 = 1 \cdot t$, $z_1 = f \cdot t^2$, $z_2 = g \cdot t^4$ then there are no relations of degree less than 8, hence by Proposition (4.4) and Remark (4.5) the z_i generate L_G.

$\{1; 2, 3\}$: In this case $L_1 = \langle 1 \rangle$, $L_2 = \langle 1, f \rangle$, $L_3 = \langle 1, f, g \rangle$ where $(f)_\infty = p_0 + p_1$ and $(g)_\infty = 2p_1$. Then one can easily verify that a basis for L_i, $i = 4, 5, 6$, is given as follows.

$$L_4 = \langle 1, f, g, f^2 \rangle, \qquad L_5 = \langle 1, f, g, f^2, fg \rangle,$$
$$L_6 = \langle 1, f, g, f^2, fg, g^2, f^3 \rangle.$$

Letting $z_0 = 1 \cdot t$, $z_1 = f \cdot t^2$, $z_2 = g \cdot t^3$ we see that there is a relation of degree 7 and no relation of degree < 7. Hence again by Proposition (4.4) and Remark (4.5) the z_i generate L_G.

$\{1; n\}$: Let p be the branch point, $\wp$ the Weierstrass function with pole at p. Let $f = \wp$ and $g = \wp$ and $z_0 = 1 \cdot t$. If $n = 2$, let $z_1 = f \cdot t^4 \in L_4 \cdot t^4$, $z_2 = g \cdot t^6 \in L_6 \cdot t^6$. If $n = 3$, let $z_1 = f \cdot t^3 \in L_3 \cdot t^3$ and $z_2 = g \cdot t^5 \in L_5 \cdot t^5$. If $n = 4$, let $z_1 = f \cdot t^3 \in L_3 \cdot t^3$ and $z_2 = g \cdot t^4 \in L_4 \cdot t^4$. If $n = \infty$, let $z_1 = f \cdot t^2$ and $z_2 = g \cdot t^3$. By

Proposition (4.4) it is sufficient to show that in each case the z_i satisfy an irreducible relation in $L_d \cdot t^d$.

$$\{1; 2\}: z_2^2 - 4z_1^3 + g_2 z_0^8 z_1 + g_3 z_0^{12} = 0,$$
$$\{1; 3\}: z_2^2 - 4z_0 z_1^3 + g_2 z_0^7 z_1 + g_3 z_0^{10} = 0,$$
$$\{1; 4\}: z_0 z_2^2 - 4z_1^3 + g_2 z_0^6 z_1 + g_3 z_0^9 = 0,$$
$$\{1; \infty\}: z_2^2 - 4z_1^3 + g_2 z_0^4 z_1 + g_3 z_0^6 = 0,$$

where g_2, g_3 are complex numbers so that $g_2^3 - 27g_3^2 \neq 0$. These can be seen to be irreducible by considering them as polynomials in z_2.

$\{1; \sigma\}$, $\sigma = 2$ or 3: These can be explicitly constructed as above using the fact that on an elliptic curve X any divisor of degree n is linearly equivalent to a divisor of the form np for some $p \in X$ **[W1]**.

All of the groups on Table 1 with $\sigma > 0$ are seen to have L_G generated by ≤ 3 elements by elementary application of Proposition (4.4), Remark (4.5), (A.1) and (A.2). For example in the case of signature $\{0; 2, 2, 2, \infty\}$ L has two generators z_0, $z_1 \in L_2$ and one generator $z_2 \in L_3$. Now z_0 and z_1 generate the quotient field of L_G, hence Remark (4.5)(2) is satisfied. As for Remark (4.5)(1), there are clearly no relations in L_1, L_2 or L_3. Applying (A.2) with $i = r = 2$ we see there are no relations in L_4 and applying (A.1) with $i = 3$ and $j = 2$ we see there are no relations in L_5.

Finally, we consider the groups with $g = \sigma = 0$. We number the branch points $p_1, \ldots, p_r \in X = \hat{\mathbf{C}}$, the Riemann sphere. We may assume $p_1 = 0$, $p_2 = 1$ and $p_3 = \infty \in \hat{\mathbf{C}}$ and the canonical divisor is $K = -2 \cdot p_1$.

$\{0; 2, 2, 2, 2, 3\}$: In this case $L_1 = \{0\}$, $L_2 = L(-3 \cdot 0 + 1 + \infty + p_4 + p_5)$. The latter is generated by

$$z_0 = x^3/(x-1)(x-p_4)(x-p_5)$$

and

$$z_1 = x^4/(x-1)(x-p_4)(x-p_5).$$

Rearrange the subscripts so that $e_1 = 3$. Then $L_3 = L(-4 \cdot 0 + 1 + \infty + p_4 + p_5)$ is generated by $z_2 = x^4/(x-1)(x-p_4)(x-p_5)$. Clearly Remark (4.5)(2) is satisfied. To verify Remark (4.5)(1) we can easily see using (A.1) and (A.2) that the only possible relation of degree $< d = 8$ is in degree 6. The monomials in L_6 are z_2^2, z_0^3, $z_0^2 z_1$, $z_0 z_1^2$ and z_1^3. At the point 0 these functions have a zero of order 8, 9, 10, 11 and 12 respectively. Hence by (4.6.1) they must be linearly independent. Thus Remark (4.5)(1) is satisfied and we have the desired result.

$\{0; 2, 2, 2, 2, 2\}$: Let $L_2 = \langle z_0, z_1 \rangle$ and $L_5 = \langle z_2 \rangle$. Several applications of (A.1) and (A.2) show that Remark (4.5)(1) and (2) are satisfied.

$\{0; 2, 3, 3, 3\}$: Let $e_1 = 2$. Then $L_2 = L(-3 \cdot 0 + 1 + \infty + p_4)$ and $L_3 = L(-5 \cdot 0 + 2 \cdot 1 + 2\infty + 2p_4)$. Let

$$z_0 = x^3/(x-1)(x-p_4) \in L_2, \quad z_1 = x^5/(x-1)^2(x-p_4)^2$$

and

$$z_2 = x^6/\,(x-1)^2(x-p_4)^2.$$

Then $L_2 = \langle z_0 \rangle$ and $L_3 = \langle z_1, z_2 \rangle$. Clearly Remark (4.5)(2) is satisfied. Now by (A.1) the only possible nontrivial relations of degree $< d = 9$ are in L_6 or L_8. The monomials in L_6 are $z_0^3, z_1^2, z_1z_2, z_2^2$. At the point 0 these functions have a zero of order 9, 10, 11 and 12 respectively. Hence they are linearly independent. A similar calculation in L_8 shows that Remark (4.5)(1) is satisfied.

$\{0; 2, 2, 3, 4\}$: Let $e_1 = 2, e_2 = 2, e_3 = 3, e_4 = 4$ and define

$$z_0 = x^3/\,(x-1)(x-p_4) \in L_2, \quad z_1 = x^5/\,(x-1)(x-p_4)^2 \in L_3$$

and

$$z_2 = x^7/\,(x-1)^2(x-p_4)^3.$$

Then $L_2 = \langle z_0 \rangle$, $L_3 = \langle z_1 \rangle$, $L_4 = \langle z_0^2, z_2 \rangle$ and $L_5 = \langle z_0 z_1 \rangle$. Now L_6 contains the monomials z_0^3, z_1^2 and $z_0 z_2$. At the point ∞ these have poles of order 3, 4 and 3 respectively. Thus any relation is of the form $\alpha z_0^3 + \beta z_0 z_2 = 0$. But any such relation factors, and hence is a consequence of a relation of lower degree. This is impossible. There are no relations of degree 7 or 9 by Proposition (A.1). Finally, the monomials of degree 8 are $z_0^4, z_0^2 z_2, z_0 z_1^2$ and z_2^2. At p_4 these have poles of order 4, 4, 5 and 6 respectively. It follows easily that there is no relation of degree 8.

$\{0; 2, 2, 3, 3\}$: Let $L_2 = \langle z_0 \rangle$, $L_3 = \langle z_1 \rangle$. Then z_0^3 and z_1^2 are linearly independent since z_0^3 has a pole of order 3 at ∞ and z_1^2 has a pole of order 4. Using Table 4 we see there is a $z_2 \in L_6$ so that $L_6 = \langle z_0^3, z_1^2, z_2 \rangle$. Then the fact that L_G is an integral domain and repeated application of (A.1) shows there is no relation among the z_i of degree < 12. Thus Remark (4.5) and Proposition (4.4) imply that the z_i generate L_G.

$\{0; 2, 2, 2, 5\}$: Let $e_4 = 5$. Then $L_2 = \langle z_0 = x^3/(x-1)(x-p_4) \rangle$, $L_4 = \langle z_0^2, z_1 = x^6/(x-1)^2(x-p_4)^3 \rangle$, $L_5 = \langle z_2 = x^8/(x-1)^2(x-p_4)^4 \rangle$. Applying Propositions (A.1) and (A.2) we see that the only possible relation of degree $< d = 12$ is in degree 10. Now the monomials of degree 10 are $z_0^5, z_0^3 z_1, z_0 z_1^2$ and z_2^2. At p_4 these have poles of order 5, 6, 7 and 8 respectively. Thus there can be no relation among them. Thus Remark (4.5)(1) is satisfied. Clearly (4.5)(2) holds, hence we get the desired result by Proposition (4.4).

$\{0; 2, 2, 2, 4\}$ and $\{0; 2, 2, 2, 3\}$: These two cases follow easily from Propositions (A.1), (A.2) and (4.4).

Consider the groups with $g = \sigma = 0$ and $r = 3$. In this case there is, up to conjugacy, a unique group for each signature. We can obtain our results by using Remark (4.5) as before. Alternately one can prove that the relation of degree d must be irreducible and apply Proposition (4.4) directly. A third proof can be developed using the theory of Seifert bundles. The crucial fact in that proof is that there is a *unique* singularity with Seifert invariants $\{0; b; (\alpha_1, \beta_1), (\alpha_2, \beta_2), (\alpha_3, \beta_3)\}$ **[O-W]**. □

Appendix.

PROPOSITION (A.1). *Suppose k is a field, L is a graded k-algebra, L is an integral domain, $z_n \in L_{q_n}$ for $n = 0, 1, 2$ and*

$$\mathcal{P}_{L^{(t)}} = (1 - t^d)/\left(\prod_n 1 - t^{q_n}\right).$$

Suppose we are given natural numbers i and j and d so that $i + j < d$, $\dim L_i = 1$, $\dim L_j = L_{i+j}$, *and there are no relations of degree j or i (among the z_n). Then there are no relations of degree $j + i$.*

PROOF. Let $S = K[Z_0, Z_1, Z_2]$, define f: $S \to L$ by $f(Z_n) = z_n$ and let $I =$ kernel f. If we grade S by letting degree $Z_n = q_n$, for $n = 0, 1, 2$ then f is a graded homomorphism of degree 0. Note that $\mathcal{P}_S(t) = 1/(\prod_n 1 - t^{q_n})$. Let $X \in S_i$ be any nonzero element and $x = f(X)$. Then $x \neq 0$ since there are no relations of degree i. Then we have a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \to & I_j & \to & S_j & \xrightarrow{f_j} & L_j \\ & & \downarrow & & \downarrow\varphi & & \downarrow\Psi \\ 0 & \to & I_{j+i} & \to & S_{j+i} & \xrightarrow{f_{j+i}} & L_{j+i} \end{array}$$

where $\varphi(z) = XZ$ and $\Psi(z) = xz$. Note that no relations of degree j means $I_j = \{0\}$. Moreover $\dim S_j =$ coefficient of t_j in $\mathcal{P}_S =$ coefficient of t^j in $\mathcal{P}_L = \dim L_j$. Hence f_j is an isomorphism. Now Ψ is one-one since L is a domain; thus $\dim L_j = \dim L_{j+i}$ implies Ψ is an isomorphism. Thus f_{j+i} is onto. But as above, $\dim L_{j+i} = \dim S_{j+i}$ and hence $I_{j+i} = \{0\}$. $\square$

PROPOSITION (A.2). *If L is a graded integral domain over an algebraically closed field k and $x, y \in L_i$ are linearly independent, then for all $r \geqslant 1$, $x^r, x^{r-1}y, \ldots, y^r$ are linearly independent in L_{ri}.*

PROOF. Suppose $a_0x^r + a_1x^{r-1}y + \cdots + a_ry^r = 0$. Let t be an indeterminate and let $f(t) = \sum_{i=0}^r a_it^i$. Then $f(t)$ factors into linear factors over k,

$$f(t) = \prod_{i=1}^{r} (\alpha_it - \beta_i).$$

But then $\prod_{i=1}^r(\alpha_iy - \beta_ix) = \sum_{i=0}^r a_ix^{r-i}y^i = 0$ and hence $\alpha_iy - \beta_ix = 0$ for some i. This implies $\alpha_i = \beta_i = 0$ and hence $a_i = 0$, for all i. $\square$

BIBLIOGRAPHY

[A-M] M. Atiyah and I. MacDonald, *Introduction to commutative algebra*, Addison-Wesley, Reading, Mass., 1969.

[C-R] P. Connor and F. Raymond, *Holomorphic Seifert fiberings*, Proc. Second Conf. Compact Transformation Groups, Lecture Notes in Math., vol. 299, Springer-Verlag, Berlin, 1972, pp. 1–75.

[C] J. Carrell, *Holomorphically injective complex toral actions*, Proc. Second Conf. Compact Transformation Groups, Lecture Notes in Math., vol. 299, Springer-Verlag, Berlin, 1972, pp. 205–236.

[D1] I. Dolgachev, *Automorphic forms and quasi-homogeneous singularities*, Functional Anal. Appl. **9** (1975), 149–151.

[D2] ______, *Automorphic forms and quasi-homogeneous singularities* (preprint).

[G] R. Gunning, *Lectures on modular forms*, Notes by Armand Brumer, Princeton Univ. Press, Princeton, N. J., 1962.

[K] I. Kra, *Automorphic forms and Kleinian groups*, Math. Lecture Notes Series, Benjamin, Reading, Mass., 1972.

[M] J. Milnor, *On the 3-dimensional Brieskorn manifolds, $M(p, q, r)$*, Knots, Groups and 3-Manifolds (Papers dedicated to the memory of R. H. Fox), Ann. of Math. Studies, no. 84, Princeton Univ. Press, Princeton, N. J., 1975, pp. 175–225.

[O-W] P. Orlik and P. Wagreich, *Algebraic surfaces with k^*-action*, Acta Math. **138** (1977), 43–81.

[S-D] B. Saint-Donat, *On Petri's analysis of the linear system of quadrics through a canonical curve*, Math. Ann. **206** (1973), 157–175.

[S] G. Springer, *Introduction to Riemann surfaces*, Addison-Wesley, Reading, Mass., 1957.

[W1] P. Wagreich, *Elliptic singularities of surfaces*, Amer. J. Math. **92** (1970), 419–454.

[W2] ______, *Automorphic forms and singularities with $\mathbf{C}^*$-action*, Illinois J. Math. (to appear).

[Z] O. Zariski and P. Samuel, *Commutative algebra*. I, Van Nostrand, Princeton, N. J., 1959.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS AT CHICAGO CIRCLE, CHICAGO, ILLINOIS 60680