

REGULARITY OF CERTAIN SMALL SUBHARMONIC FUNCTIONS

BY

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ABSTRACT. Suppose that u is subharmonic in the plane and that $\lim_{r \rightarrow \infty} B(r)/(\log r)^2 = \sigma < \infty$. It is known that, given $\varepsilon > 0$, there are arbitrarily large values of r such that $A(r) > B(r) - (\sigma + \varepsilon)\pi^2$. The following result is proved. Let u be subharmonic and let σ be any positive number. Then either $A(r) > B(r) - \pi^2\sigma$ for certain arbitrarily large values of r or, if this is false, then

$$\lim_{r \rightarrow \infty} (B(r) - \sigma(\log r)^2)/\log r$$

exists and is either $+\infty$ or finite.

1. Introduction. Let $u(z)$ be subharmonic in the plane and define

$$B(r) = B(r, u) = \max_{|z|=r} u(z),$$

$$A(r) = A(r, u) = \inf_{|z|=r} u(z).$$

In [3] the following result is proved.

THEOREM A. *Let $p > 1$ be given and suppose that $u(z)$ is subharmonic in the plane and satisfies*

$$\lim_{r \rightarrow \infty} \frac{B(r)}{(\log r)^p} = \sigma < \infty.$$

Then, given $\varepsilon > 0$,

$$A(r) > B(r) - (\sigma + \varepsilon)\operatorname{Re}\{(\log r)^p - (\log r + i\pi)^p\} \quad (1.1)$$

for r outside an exceptional set E for which

$$\lim_{r \rightarrow \infty} \frac{(p-1)}{(\log r)^{p-1}} \int_{E \cap [1, r]} \frac{(\log t)^{p-2}}{t} dt < \frac{\sigma}{\sigma + \varepsilon}.$$

Theorem A is related to certain results of P. D. Barry. (See Theorem 4 and the remarks in §7.4 of [1].) With Kjellberg's version of the $\cos \pi\lambda$ Theorem [5], [6] in view we might expect that functions extremal for Theorem A would have some kind of regular asymptotic behaviour. In this direction we shall prove

THEOREM 1. *Suppose that $u(z)$ is subharmonic in the plane and that σ is any positive number. Then either*

$$A(r) > B(r) - \pi^2\sigma \quad (1.2)$$

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for certain arbitrarily large values of r or, if this is not the case, then

$$\alpha = \lim_{r \rightarrow \infty} \frac{B(r) - \sigma(\log r)^2}{\log r} \quad (1.3)$$

exists and is either $+\infty$ or finite.

This result corresponds to $p = 2$ in Theorem A. It seems likely that, for the case of general p , the limit of (1.3) could be replaced by

$$\lim_{r \rightarrow \infty} \frac{B(r) - \sigma(\log r)^p}{(\log r)^{p-1}} > -\infty.$$

EXAMPLE. Any value of α admitted by Theorem 1 can in fact occur. Write $z = re^{i\theta}$, where $-\pi < \theta \leq \pi$. In the case $\alpha = +\infty$ we may let $u(z) = r \cos \theta$. If $\alpha = 0$ we set

$$\begin{aligned} u_0(z) &= 0, & r &\leq 1, \\ u_0(z) &= \max(\sigma(\log r)^2 - \sigma\theta^2, 0), & r &> 1. \end{aligned}$$

Evidently $u_0(z)$ is continuous and subharmonic except possibly on the segment $r > 1$ of the negative real axis, since $\sigma(\log r)^2 - \sigma\theta^2 = \operatorname{Re} \sigma(\log r + i\theta)^2$ is harmonic except for negative real z . If we write θ_1, θ_2 for the values of $\arg z$ which satisfy respectively

$$-2\pi < \theta_1 < 0, \quad 0 < \theta_2 < 2\pi,$$

and set

$$u_1(z) = \sigma(\log r)^2 - \sigma\theta_1^2, \quad u_2(z) = \sigma(\log r)^2 - \sigma\theta_2^2,$$

then u_1 and u_2 are harmonic near the negative real axis and so

$$u_0(z) = \max(u_1(z), u_2(z))$$

is subharmonic there and so everywhere in the plane. Also, for any real α , we define c by $2\sigma \log c = \alpha$, and set $u(z) = u_0(cz) - \sigma(\log c)^2$. Then

$$\begin{aligned} B(r, u) &= \sigma(\log r)^2 + \alpha \log r, & r &> 1/c, \\ A(r, u) &= B(r, u) - \pi^2\sigma, & r &> e^\pi/c, \end{aligned}$$

so that (1.2) fails and

$$\frac{B(r, u) - \sigma(\log r)^2}{\log r} = \alpha, \quad r > 1/c.$$

From the Riesz representation theorem for subharmonic functions it follows that there is a unique nonnegative measure μ defined on all bounded, Borel measurable subsets of the plane such that, if R is a given positive number,

$$u(z) = h_R(z) + \int_{|\xi| < R} \log \left| 1 - \frac{z}{\xi} \right| d\mu(\xi) \quad (1.4)$$

for $|z| < R$, where $h_R(z)$ is harmonic in $|z| < R$. In the proof it is assumed that u is harmonic at 0 but this may be achieved without loss of generality to our results by replacing u in a disc about 0 by the Poisson integral of its boundary values on the

disc. Throughout the paper we shall assume that $u(0) = 0$, as we may do without affecting the generality of our results. We define $\mu^*(r) = \mu(\{z: |z| < r\})$ for $r > 0$.

THEOREM 2. *Suppose that $u(z)$ is subharmonic in the plane and that σ is a positive number such that*

$$A(r) \leq B(r) - \pi^2 \sigma$$

for all large r . Suppose further that the limit α of (1.3) is finite. Let

$$u_1(z) = \int_{|\xi| < \infty} \log \left| 1 + \frac{z}{\xi} \right| d\mu(\xi) = \int_0^\infty \log \left| 1 + \frac{z}{t} \right| d\mu^*(t)$$

and define $B_1(r) = \max_{|z|=r} u_1(z)$. Then

$$\lim_{r \rightarrow \infty} \frac{B_1(r) - \sigma(\log r)^2}{\log r} = \alpha.$$

THEOREM 3. *Under the conditions of Theorem 2*

$$\lim_{r \rightarrow \infty} (\mu^*(r) - 2\sigma \log r) = \alpha.$$

The first part of the paper is devoted to showing that, under the conditions expressed in the second alternative of Theorem 1, $\mu^*(r) = O(\log r)$ when

$$\lim (B(r) - \sigma(\log r)^2)/\log r < \infty.$$

This is rather more drawn out than might be expected due to certain tiresome modifications to u that seem to be necessary in the subsequent parts of the proof. In §6 and §7 the growth properties of u and u_1 are considered and the theorems are proved more or less together.

2. Decomposition of u . In [2] Barry has put into subharmonic form results derived by Kjellberg [5, pp. 190–192] in the case $u(z) = \log |f(z)|$, where f is an entire function. Some of these are as follows.

With $\mu^*(t) = \mu(|z| < t)$ define

$$u_1(z, R) = \int_{|\xi| < R} \log \left| 1 - \frac{z}{\xi} \right| d\mu(\xi), \quad (2.1)$$

$$u_2(z, R) = \int_{|\xi| < R} \log \left| 1 + \frac{z}{\xi} \right| d\mu(\xi) = \int_0^R \log \left| 1 + \frac{z}{t} \right| d\mu^*(t), \quad (2.2)$$

$$u_3(z, R) = u(z) - u_1(z, R). \quad (2.3)$$

Then, with $B_j(r, R) = \max_{|z|=r} u_j(z, R)$, $A_j(r, R) = \inf_{|z|=r} u_j(z, R)$, $j = 1, 2, 3$,

$$A_2(r, R) \leq A_1(r, R) \leq B_1(r, R) \leq B_2(r, R); \quad (2.4)$$

and

$$-\frac{4r}{R} B(2R) < A_3(r, R) < B_3(r, R) < \frac{4r}{R} B(2R). \quad (2.5)$$

for $0 \leq r \leq \frac{1}{2}R$.

We note finally the subharmonic analogue of Jensen's Theorem [4, p. 473]: for $r > 0$

$$u(0) + \int_0^r \log \frac{r}{t} d\mu^*(t) = \frac{1}{2\pi} \int_0^{2\pi} u(re^{i\theta}) d\theta \leq B(r). \quad (2.6)$$

Concerning $u_2(z, R)$ we have the following lemma.

LEMMA 1. *Let R_1, R_2 and R be positive numbers satisfying $R_1 < R_2 < R$. Then*

$$\begin{aligned} & \int_{R_1}^{R_2} \frac{A_2(t, R) - B_2(t, R)}{t} dt \\ &= \mu^*(R) \int_0^\pi \text{Arg} \left(1 - \frac{R_2}{R} e^{i\theta} \right) d\theta - \int_0^R \frac{\mu^*(t)}{t} \log \left| \frac{t + R_2}{t - R_2} \right| dt \\ & \quad - \mu^*(R) \int_0^\pi \text{Arg} \left(1 - \frac{R_1}{R} e^{i\theta} \right) d\theta + \int_0^R \frac{\mu^*(t)}{t} \log \left| \frac{t + R_1}{t - R_1} \right| dt. \end{aligned}$$

Also

$$\begin{aligned} I(R_1, R_2, R) &= \int_{R_1}^{R_2} \frac{ds}{s} \int_0^s \frac{A_2(t, R) - B_2(t, R)}{t} dt \\ &= \mu^*(R) \int_{R_1}^{R_2} \frac{ds}{s} \int_0^\pi \text{Arg} \left(1 - \frac{s}{R} e^{i\theta} \right) d\theta \\ & \quad - \int_0^R \frac{\mu^*(t)}{t} dt \int_{R_1}^{R_2} \frac{1}{s} \log \left| \frac{s + t}{s - t} \right| ds. \end{aligned}$$

The first part is contained in the proof of the Lemma in [3] and the second part follows immediately from the first on integration.

3. Preliminaries. To prove the theorems we assume that for all large r

$$A(r) \leq B(r) - \pi^2 \sigma \quad (3.1)$$

and that

$$\alpha = \lim_{r \rightarrow \infty} \frac{B(r) - \sigma(\log r)^2}{\log r} < +\infty \quad (3.2)$$

and aim to prove the existence of the finite limit (1.3). The first step is to show that (3.1) and (3.2) together imply

$$\mu^*(r) = O(\log r) \quad \text{as } r \rightarrow \infty \quad (3.3)$$

and in order to do this we assume that

$$\overline{\lim}_{r \rightarrow \infty} \frac{\mu^*(r)}{\log r} = +\infty \quad (3.4)$$

and deduce a contradiction. We have

LEMMA 2. Suppose that (3.1), (3.2) and (3.4) hold. Then there exists a subharmonic function $U(z)$ which satisfies the following conditions:

$$(i) \quad \lim_{r \rightarrow \infty} \frac{B(r, U) - \sigma(\log r)^2}{\log r} < -A < -\frac{4\sigma}{\pi^2}(\pi^2 + 90); \quad (3.5)$$

$$(ii) \quad A(r, U) = B(r, U) - \pi^2\sigma \quad (3.6)$$

holds for $e^\pi \leq r \leq r_0$ and for $2r_0 \leq r < \infty$, for some $r_0 > e^\pi$;

$$(iii) \quad \overline{\lim}_{r \rightarrow \infty} \frac{\mu^*(r, U)}{\log r} = \infty. \quad (3.7)$$

Set $v_1(z) = \max\{u(z), B(|z|) - \pi^2\sigma\}$. Since $B(|z|)$ is subharmonic [7, §3.20] so also is $v_1(z)$. Also, since v_1 and u have the same maximum on each circle about the origin, (3.2) holds for v_1 ; and it follows from (2.6) that (3.4) is equivalent to

$$\overline{\lim}_{r \rightarrow \infty} \frac{1}{(\log r)^2} \int_0^{2\pi} u(re^{i\theta}) d\theta = \infty,$$

so that (3.4) persists likewise for v_1 . Now set $v_2(z) = v_1(cz)$, where c is a positive constant. Then $B(r, v_2) = B(cr, v_1)$ so that

$$\lim_{r \rightarrow \infty} \frac{B(r, v_2) - \sigma(\log r)^2}{\log r} = \alpha + 2\sigma \log c,$$

and we choose c so that condition (i) obtains. Moreover

$$A(r, v_2) = B(r, v_2) - \pi^2\sigma$$

for all large r , say $r \geq R_0$, and (3.4) holds for v_2 .

Since (3.4) holds we conclude that, given any positive number M , there is some number $r_0 > R_0 + 2$ such that

$$B(2r_0, v_2) - B(r_0, v_2) > M \log r_0.$$

For if there were no such r_0 we should have, for all large positive integers n ,

$$\begin{aligned} B(2^n, v_2) &= B(2^n, v_2) - B(2^{n-1}, v_2) + B(2^{n-1}, v_2) \\ &\leq M(n-1)\log 2 + B(2^{n-1}, v_2) - B(2^{n-2}, v_2) + B(2^{n-2}, v_2) \\ &\leq \dots \leq \frac{1}{2}n^2M \log 2 + O(1) = O(\log 2^n)^2. \end{aligned}$$

Thus $B(r, v_2) = O(\log r)^2$ and so $B(r, u) = O(\log r)^2$. This together with (2.6) contradicts (3.4).

Now $w(z)$ defined by

$$w(z) = 0, \quad |z| \leq 1,$$

$$w(z) = \max\{\sigma(\log|z|)^2 - \sigma(\operatorname{Arg} z)^2, 0\}, \quad |z| > 1,$$

is subharmonic in the plane (as was shown in the example following Theorem 1).

We define a new function

$$U(z) = \begin{cases} w(z) + D, & |z| \leq r_0, \\ h(z), & r_0 \leq |z| \leq 2r_0, \\ v_2(z), & |z| \geq 2r_0, \end{cases} \quad (3.8)$$

where $D = B(r_0, v_2) - \sigma(\log r_0)^2 + \pi^2\sigma$ and $h(z)$ is the harmonic function in $r_0 < |z| < 2r_0$ taking boundary values $w(z) + D$ on $|z| = r_0$, $v_2(z)$ on $|z| = 2r_0$. Clearly $U(z)$ satisfies conditions (i) and (ii) of Lemma 2. Moreover, once we have shown that $U(z)$ is subharmonic, it is evident that (3.7) holds.

To show that U is subharmonic it is enough to show that h dominates both v_2 and $w + D$ for $r_0 \leq |z| \leq 2r_0$. For $|z| = r_0$,

$$h(z) = w(z) + D \geq \sigma(\log r_0)^2 - \pi^2\sigma + D = B(r_0, v_2) \geq v_2(z),$$

while for $|z| = 2r_0$,

$$\begin{aligned} h(z) &= v_2(z) \geq A(2r_0, v_2) = B(2r_0, v_2) - \pi^2\sigma > B(r_0, v_2) + M \log r_0 - \pi^2\sigma \\ &= \sigma(\log r_0)^2 + M \log r_0 + D - 2\pi^2\sigma > \sigma(\log 2r_0)^2 + D \geq W(z) + D \end{aligned}$$

provided M is large enough. This completes the proof.

4. Modification of U . Let U be the function of Lemma 2. From (3.7) it follows that given any positive number K there are arbitrarily large values of r such that $B(r, U) > K(\log r)^2$. Given one such value, say r_1 , take $2R$ to be the smallest number greater than r_1 such that

$$\frac{\sigma(\log 2R)^2 - B(2R, U)}{\log 2R} = A. \quad (4.1)$$

From (3.5) such an R exists; and clearly (if K is large enough) $r_1 < R$ so that

$$\frac{\sigma(\log R)^2 - B(R, U)}{\log R} < A.$$

That is

$$B(R, U) > \sigma(\log R)^2 - A \log R. \quad (4.2)$$

Also

$$B(R, U) < B(2R, U) = \sigma(\log 2R)^2 - A \log 2R. \quad (4.3)$$

We modify $U(z)$ in $|z| > R$ so as to obtain a subharmonic function which is not too large when $|z|$ is large. Let $h(z)$ be the harmonic function in $R < |z| < 2R$ which takes boundary values $U(z)$ on $|z| = R$ and $B(2R, U)$ on $|z| = 2R$. This function clearly dominates U on $R \leq |z| \leq 2R$ so that

$$U_1(z) = \begin{cases} U(z), & |z| \leq R, \\ h(z), & R \leq |z| \leq 2R, \end{cases}$$

is subharmonic in $|z| < 2R$. Further, $h(z) \geq h_1(z)$, where $h_1(z)$ is the harmonic function in $R < |z| < 2R$ taking boundary values $V_1 = \sigma(\log R)^2 - A \log R - \pi^2\sigma$ on $|z| = R$ and $V_2 = B(2R, U)$ on $|z| = 2R$. (This follows from (3.6) and (4.2).) $h_1(z)$ may be written explicitly as

$$h_1(z) = \frac{V_2 - V_1}{\log 2} \log \frac{|z|}{R} + V_1. \quad (4.4)$$

We define

$$U_2(z) = 2\sigma(\log r)^2 + B(2R, U) - 2\sigma(\log 2R)^2$$

and show that, if R is large, $U_2(z) \leq h(z)$ for $R < |z| \leq 2R$ so that

$$W_R(z) = \begin{cases} U_1(z), & |z| \leq 2R, \\ U_2(z), & z \geq 2R, \end{cases} \quad (4.5)$$

is subharmonic. For $R < |z| < 2R$ we obtain, after some simplification,

$$\begin{aligned} h(z) - U_2(z) &\geq h_1(z) - U_2(z) \\ &= \left(\frac{B(2R, U) + A \log R + \pi^2 \sigma - \sigma(\log R)^2}{\log 2} - 2\sigma \log 2|z|R \right) \log \frac{|z|}{2R} \\ &\geq \left(\frac{\sigma(\log 2R)^2 - A \log 2 + \pi^2 \sigma - \sigma(\log R)^2}{\log 2} - 2\sigma \log 2R^2 \right) \log \frac{|z|}{2R} \\ &> 0 \end{aligned}$$

if R is large enough.

We note that $W_R(z) = U_1(z) = U(z)$ for $|z| \leq R$ and that

$$B(R^2, W_R) = W_R(R^2) < 10\sigma(\log R)^2. \quad (4.6)$$

5. Behaviour of $\mu^*(r)$. Throughout this section the functions A , B , μ and μ^* will be understood to refer to the function W_R . Set $\rho = R^{3/2}$. Given $t \leq \frac{1}{2}R$ we have from (2.5)

$$\begin{aligned} A(t) &= \int_{|\xi| < \rho} \log \left| 1 - \frac{z_1}{\xi} \right| d\mu(\xi) + O\left(\frac{t}{\rho} B(2\rho)\right), \\ B(t) &= \int_{|\xi| < \rho} \log \left| 1 - \frac{z_2}{\xi} \right| d\mu(\xi) + O\left(\frac{t}{\rho} B(2\rho)\right), \end{aligned}$$

where $|z_1| = |z_2| = t$, and thus

$$\begin{aligned} A(t) - B(t) &= \int_{|\xi| < R} \log \left| 1 - \frac{z_1}{\xi} \right| d\mu(\xi) - \int_{|\xi| < R} \log \left| 1 - \frac{z_2}{\xi} \right| d\mu(\xi) \\ &\quad + \int_{R < |\xi| < \rho} \log \left| \frac{\xi - z_1}{\xi - z_2} \right| d\mu(\xi) + O\left(\frac{t}{\rho} B(2\rho)\right) \\ &\geq A_1(t, R) - B_1(t, R) - \int_R^\rho \log \left| \frac{s+t}{s-t} \right| d\mu^*(s) + O\left(\frac{t}{\rho} B(2\rho)\right) \\ &\geq A_2(t, R) - B_2(t, R) - \mu^*(\rho) \log \left(\frac{R+t}{R-t} \right) + O\left(\frac{t}{\rho} B(2\rho)\right). \end{aligned}$$

Hence, for $1 < R_1 < R_2 = \frac{1}{2}R$,

$$\begin{aligned} \Delta(R_1, R_2) &= \int_{R_1}^{R_2} \frac{ds}{s} \int_0^s \frac{A(t) - B(t)}{t} dt \\ &\geq \int_{R_1}^{R_2} \frac{ds}{s} \int_0^s \frac{A_2(t, R) - B_2(t, R)}{t} dt \\ &\quad - \mu^*(\rho) \int_{R_1}^{R_2} \frac{ds}{s} \int_0^s \frac{1}{t} \log \left(\frac{R+t}{R-t} \right) dt + O\left(\frac{R_2}{\rho} B(2\rho)\right) \\ &= T_1 - T_2 + o(1), \end{aligned} \quad (5.1)$$

from (4.6), and we estimate T_1 and T_2 in turn. From Lemma 1

$$T_1 = \mu^*(R) \int_{R_1}^{R_2} \frac{1}{s} ds \int_0^\pi \operatorname{Arg} \left(1 - \frac{s}{R} e^{i\theta} \right) d\theta \\ - \int_0^R \frac{\mu^*(t)}{t} dt \int_{R_1}^{R_2} \frac{1}{s} \log \left| \frac{t+s}{t-s} \right| ds.$$

Also

$$\operatorname{Arg} \left(1 - \frac{s}{R} e^{i\theta} \right) \geq -\operatorname{Arc} \sin \frac{s}{R} \geq -\frac{\pi s}{2R}$$

so

$$T_1 \geq -\frac{1}{4}\pi^2\mu^*(R) - I(R_2) + I(R_1), \quad (5.2)$$

where

$$I(r) = \int_0^R \frac{\mu^*(t)}{t} dt \int_0^r \frac{1}{s} \log \left| \frac{t+s}{t-s} \right| ds. \quad (5.3)$$

We have

$$I(R_2) \leq \int_0^R \frac{\mu^*(t)}{t} dt \int_0^\infty \frac{1}{s} \log \left| \frac{t+s}{t-s} \right| ds \\ = \int_0^R \frac{\mu^*(t)}{t} dt \int_0^\infty \frac{1}{s} \log \left| \frac{s+1}{s-1} \right| ds \\ = \frac{1}{2}\pi^2 \int_0^R \frac{\mu^*(t)}{t} dt \leq \frac{1}{2}\pi^2 B(R) + O(1),$$

from (2.6), and combining this with (5.2) we obtain

$$T_1 \geq -\frac{1}{4}\pi^2\mu^*(R) - \frac{1}{2}\pi^2 B(R) + I(R_1) + O(1). \quad (5.4)$$

A straightforward estimate yields

$$T_2 = \mu^*(\rho) \int_{R_1/R}^{R_2/R} \frac{ds}{s} \int_0^s \frac{1}{t} \log \left(\frac{1+t}{1-t} \right) dt \leq 2\mu^*(\rho) \quad (5.5)$$

and combining (5.1), (5.4) and (5.5) we obtain

$$\Delta(R_1, R_2) \geq -\left(2 + \frac{1}{4}\pi^2\right)\mu^*(\rho) - \frac{1}{2}\pi^2 B(R) + I(R_1) + O(1) \\ \geq -\frac{(4 + \frac{1}{2}\pi^2)}{\log R} \int_{R^{3/2}}^{R^2} \frac{\mu^*(t)}{t} dt - \frac{1}{2}\pi^2 B(R) + I(R_1) + O(1) \\ \geq -\frac{10}{\log R} B(R^2) - \frac{1}{2}\pi^2 B(R) + I(R_1) + O(1) \\ \geq -110\sigma \log R - \frac{1}{2}\pi^2 B(R) + I(R_1), \quad (5.6)$$

from (4.6). On the other hand $W(z) = U(z)$ for $|z| \leq R$ and thus from (3.6)

$$\Delta(R_1, R_2) \leq \int_{R_1}^{R_2} \left(\int_{e^\pi}^s \frac{-\pi^2\sigma}{t} dt + \int_{r_0}^{2r_0} \frac{\pi^2\sigma}{t} dt \right) \frac{ds}{s} \\ \leq -\frac{1}{2}\pi^2\sigma(\log R_2)^2 + \frac{1}{2}\pi^2\sigma(\log R_1)^2 + 2\pi^3\sigma \log R_2. \quad (5.7)$$

Since $2\pi^3 < 70$, (5.6) and (5.7) together yield

$$\frac{1}{2}\pi^2(B(R) - \sigma(\log R_2)^2) + 180\sigma \log R \geq I(R_1) - \frac{1}{2}\pi^2\sigma(\log R_1)^2. \quad (5.8)$$

Since $R_2 = \frac{1}{2}R$ the left-hand side of (5.8) is, from (4.3), no larger than

$$\begin{aligned} & \frac{1}{2}\pi^2(\sigma(\log 2R)^2 - A \log 2R - \sigma(\log \tfrac{1}{2}R)^2) + 180\sigma \log R \\ &= \log R(2\pi^2\sigma \log 2 - \tfrac{1}{2}A\pi^2 + 180\sigma) - \tfrac{1}{2}\pi^2A \log 2 < 0 \end{aligned}$$

when R is large, from (3.5). Thus (5.8) reduces to

$$\frac{1}{2}\pi^2\sigma(\log R_1)^2 \geq I(R_1). \quad (5.9)$$

Now $W = U_1 = U$ for $|z| \leq R$ and U satisfies (3.7). Thus, by taking R sufficiently large, we can find R_1 such that $1 < R_1 < R_1^{3/2} < \frac{1}{2}R$ and $\mu^*(R_1^{1/2}) > 5\sigma \log R_1$. For such an R_1

$$\begin{aligned} I(R_1) &\geq \int_{R_1^{1/2}}^{R_1^{3/2}} \frac{\mu^*(t)}{t} dt \int_0^{R_1} \frac{1}{s} \log \left| \frac{t+s}{t-s} \right| ds \\ &\geq \mu^*(R_1^{1/2}) \int_{R_1^{1/2}}^{R_1^{3/2}} \frac{1}{t} dt \int_0^{R_1} \frac{1}{s} \log \left| \frac{t+s}{t-s} \right| ds \\ &> 5\sigma \log R_1 \int_{R_1^{-1/2}}^{R_1^{1/2}} \frac{1}{t} dt \int_0^{1/t} \frac{1}{s} \log \left| \frac{s+1}{s-1} \right| ds \\ &> 5\sigma \log R_1 \int_{R_1^{-1/2}}^1 \frac{1}{t} dt \int_0^1 \frac{1}{s} \log \left| \frac{s+1}{s-1} \right| ds \\ &= \frac{5\sigma}{8} \pi^2 (\log R_1)^2, \end{aligned}$$

which contradicts (5.9). The assumption which leads to this contradiction, namely (3.4), is thus mistaken and we deduce (3.3).

6. Bounds on the growth of $B(r)$. The remainder of the paper is concerned only with the functions u and u_1 occurring in the statement of the theorems. In this section we aim to show that (3.1) and (3.2) together imply

$$-\infty < \alpha = \lim_{r \rightarrow \infty} \frac{B(r) - \sigma(\log r)^2}{\log r} < \beta = \overline{\lim}_{r \rightarrow \infty} \frac{B(r) - \sigma(\log r)^2}{\log r} < +\infty. \quad (6.1)$$

As we have shown, (3.3) holds and we may thus introduce

$$u_1(z) = \lim_{R \rightarrow \infty} u_2(z, R) = \operatorname{Re} \left(z \int_0^\infty \frac{\mu^*(t)}{t(t+z)} dt \right).$$

We write $B_1(r) = \max_{|z|=r} u_1(z)$, $A_1(r) = \inf_{|z|=r} u_1(z)$ and take the limit as $R \rightarrow \infty$ in the first equation of Lemma 1 to obtain for all large $R_1 < R_2$

$$0 \geq \int_{R_1}^{R_2} \frac{A_1(t) - B_1(t) + \pi^2\sigma}{t} dt = \nu(R_1) - \nu(R_2), \quad (6.2)$$

where

$$\nu(r) = \int_0^\infty \frac{\mu^*(t)}{t} \log \left| \frac{t+r}{t-r} \right| dt - \pi^2\sigma \log r. \quad (6.3)$$

We conclude that $\nu(r)$ is nondecreasing for all large r . Integrating (6.3) over $(1, r)$ we obtain

$$\int_0^\infty \frac{\mu^*(t)}{t} dt \int_1^r \frac{1}{s} \log \left| \frac{t+s}{t-s} \right| ds - \frac{1}{2} \pi^2 \sigma(\log r)^2 = \int_1^r \frac{\nu(s)}{s} ds$$

so that

$$\int_0^\infty \frac{\mu^*(t)}{t} dt \int_0^r \frac{1}{s} \log \left| \frac{t+s}{t-s} \right| ds - \frac{1}{2} \pi^2 \sigma(\log r)^2 = \int_1^r \frac{\nu(s)}{s} ds + O(1). \quad (6.4)$$

Thus

$$\begin{aligned} B_1(r) &= r \int_0^\infty \frac{\mu^*(t)}{t(t+r)} dt \\ &= \sigma(\log r)^2 + \frac{2}{\pi^2} \int_1^r \frac{\nu(t)}{t} dt + J(r) + O(1), \end{aligned} \quad (6.5)$$

where

$$J(r) = \int_0^\infty \frac{\mu^*(t)}{t} \left(\frac{r}{r+t} - \frac{2}{\pi^2} \int_0^r \frac{1}{s} \log \left| \frac{s+t}{s-t} \right| ds \right) dt. \quad (6.6)$$

Two changes of variable in the integrals of (6.6) yield

$$J(r) = \int_0^\infty \frac{\mu^*(rt)}{t} h(t) dt, \quad (6.7)$$

where

$$h(t) = \frac{1}{1+t} - \frac{2}{\pi^2} \int_0^{1/t} \frac{1}{s} \log \left| \frac{1+s}{1-s} \right| ds.$$

We have

LEMMA 3. $h(t) \geq 0$ for $t \geq 1$ and $h(1/t) = -h(t)$ for $t \geq 1$.

For $t > 1$

$$\begin{aligned} (t+1) \frac{2}{\pi^2} \int_0^{1/t} \frac{1}{s} \log \left| \frac{1+s}{1-s} \right| ds &= (t+1) \frac{2}{\pi^2} \int_0^{1/t} \left(2 + \frac{2s^2}{3} + \frac{2s^4}{5} + \dots \right) ds \\ &= (t+1) \frac{4}{\pi^2} \left(\frac{1}{t} + \frac{1}{3^2 t^3} + \frac{1}{5^2 t^5} + \dots \right) \\ &= \frac{4}{\pi^2} \left(1 + \frac{1}{t} + \frac{1}{3^2 t^2} + \frac{1}{3^2 t^3} + \dots \right) \\ &< \frac{8}{\pi^2} \left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right) = 1. \end{aligned}$$

Thus $h(t) > 0$ for $t > 1$ and certainly $h(1) = 0$. Further, for $t > 1$,

$$\begin{aligned} h\left(\frac{1}{t}\right) &= \frac{t}{1+t} - \frac{2}{\pi^2} \int_0^t \frac{1}{s} \log \left| \frac{s+1}{s-1} \right| ds \\ &= \frac{t}{1+t} - \frac{2}{\pi^2} \int_{1/t}^\infty \frac{1}{s} \log \left| \frac{s+1}{s-1} \right| ds \\ &= \frac{t}{1+t} - \frac{2}{\pi^2} \left(\frac{\pi^2}{2} - \int_0^{1/t} \frac{1}{s} \log \left| \frac{s+1}{s-1} \right| ds \right) = -h(t) \end{aligned}$$

and this proves Lemma 3.

Now, from (6.7) and Lemma 3,

$$\begin{aligned} J(r) &= \int_0^1 \frac{\mu^*(rt)}{t} h(t) dt + \int_1^\infty \frac{\mu^*(rt)}{t} h(t) dt \\ &= \int_1^\infty \frac{\mu^*(r/t)}{t} h(1/t) dt + \int_1^\infty \frac{\mu^*(rt)}{t} h(t) dt \\ &= \int_1^\infty \frac{\mu^*(rt) - \mu^*(r/t)}{t} h(t) dt \geq 0. \end{aligned} \quad (6.8)$$

Combining this with (6.5) we obtain

$$B_1(r) \geq \sigma(\log r)^2 + \frac{2}{\pi^2} \int_1^r \frac{\nu(t)}{t} dt + O(1).$$

Hence

$$\alpha_1 = \lim_{r \rightarrow \infty} \frac{B_1(r) - \sigma(\log r)^2}{\log r} \geq \lim_{r \rightarrow \infty} \left(\frac{2}{\pi^2 \log r} \int_1^r \frac{\nu(t)}{t} dt \right) > -\infty, \quad (6.9)$$

since $\nu(t)$ is nondecreasing. Moreover, since for any $k > 1$

$$\begin{aligned} B_1(r) &= r \int_0^\infty \frac{\mu^*(t)}{t(t+r)} dt \\ &\leq \int_0^{kr} \frac{\mu^*(t)}{t} dt + O\left(r \int_{kr}^\infty \frac{\log t}{t(t+r)} dt\right) \\ &\leq B(kr) + O\left(\int_k^\infty \frac{\log r + \log t}{t(t+1)} dt\right) = B(kr) + O(\log r), \end{aligned} \quad (6.10)$$

we deduce at once from (6.9) that

$$\alpha = \lim_{r \rightarrow \infty} \frac{B(r) - \sigma(\log r)^2}{\log r} > -\infty. \quad (6.11)$$

But $\alpha < \infty$ by hypothesis and from this together with (6.10) we deduce that $\alpha_1 < \infty$. Thus, from (6.9),

$$\lim_{r \rightarrow \infty} \nu(t) = A_0$$

exists and is finite. From (6.5), then,

$$\beta_1 = \overline{\lim}_{r \rightarrow \infty} \frac{B_1(r) - \sigma(\log r)^2}{\log r} = \frac{2}{\pi^2} A_0 + \overline{\lim}_{r \rightarrow \infty} \frac{J(r)}{\log r}. \quad (6.12)$$

But

$$\begin{aligned} J(r) &= \int_1^\infty \frac{1}{t} (\mu^*(rt) - \mu^*(r/t)) h(t) dt \leq \int_1^\infty \frac{\mu^*(rt)}{t(1+t)} dt \\ &= O\left(\int_1^\infty \frac{\log t + \log r}{t(1+t)} dt\right) = O(\log r), \end{aligned}$$

so, from (6.12), $\beta_1 < \infty$. Since $\beta < \beta_1$ we deduce finally (6.1).

7. Conclusion of the proofs of Theorems 1 and 2. We prove

LEMMA 4. $\mu^*(r) = 2\sigma \log r + O(1)$.

Since $\lim_{r \rightarrow \infty} \nu(r)$ exists and is finite we deduce from (6.2) that

$$\int_1^\infty \frac{A_1(t) - B_1(t) + \pi^2 \sigma}{t} dt > -\infty.$$

Hence

$$A_1(r) > B_1(r) - 2\pi^2 \sigma \quad (7.1)$$

outside a set E of finite logarithmic measure, and thus

$$\begin{aligned} \int_0^r \frac{\mu^*(t)}{t} dt &= \frac{1}{2\pi} \int_0^{2\pi} u_1(re^{i\theta}) d\theta = B_1(r) + O(1) \\ &= \int_0^\infty \frac{r\mu^*(t)}{t(t+r)} dt + O(1) \end{aligned} \quad (7.2)$$

as $r \rightarrow \infty$ outside E . Rearranging (7.2) yields

$$\int_1^\infty \frac{\mu^*(rt) - \mu^*(r/t)}{t(t+1)} dt = O(1).$$

Using (6.8) and the fact that $h(t) \leq 1/(1+t)$, we deduce that $J(r) = O(1)$, and so (6.5) yields

$$B_1(r) = \sigma(\log r)^2 + \frac{2}{\pi^2} \int_1^r \frac{\nu(t)}{t} dt + O(1) \quad (7.3)$$

as $r \rightarrow \infty$ outside E . Combining (7.2) and (7.3) we obtain

$$\int_0^r \frac{\mu^*(t)}{t} dt = \sigma(\log r)^2 + \frac{2}{\pi^2} \int_1^r \frac{\nu(t)}{t} dt + O(1)$$

as $r \rightarrow \infty$ outside E . Now, given r outside E , we may choose k satisfying $3 > k > 2$ such that kr is also outside E (this follows since E has finite logarithmic measure). Thus, for r outside E ,

$$\begin{aligned} \mu^*(r) \log k &\leq \int_r^{kr} \frac{\mu^*(t)}{t} dt \\ &= 2\sigma \log k \log r + \sigma(\log k)^2 + \frac{2}{\pi^2} \int_r^{kr} \frac{\nu(t)}{t} dt + O(1) \\ &\leq 2\sigma \log k \log r + \sigma(\log k)^2 + \frac{2}{\pi^2} \nu(kr) \log k + O(1) \end{aligned}$$

and so

$$\mu^*(r) \leq 2\sigma \log r + O(1) \quad (7.4)$$

for r outside E . Since E has finite logarithmic measure and $\mu^*(r)$ increases with r we easily deduce that (7.4) holds for all large r . Quite similarly we obtain $\mu^*(r) \geq 2\sigma \log r + O(1)$ and this proves the lemma.

From Lemma 4 and (6.8) it follows that $J(r) = O(1)$ as $r \rightarrow \infty$ and hence, from (6.5), $\alpha_1 = \beta_1$. From (7.1)

$$B(r) > A(r) > A_1(r) > B_1(r) - 2\pi^2 \sigma$$

for r outside E . But E has finite logarithmic measure and so, given any r , we may choose $k = k(r) > 1$ such that r/k is outside E and $k(r) \rightarrow 1$ as $r \rightarrow \infty$. Hence $B(r) > B(r/k) > B_1(r/k) + O(1)$ and finally

$$\begin{aligned}\alpha &= \lim_{r \rightarrow \infty} \frac{B(r) - \sigma(\log r)^2}{\log r} > \lim_{r \rightarrow \infty} \frac{B_1(r/k) - \sigma(\log r)^2}{\log r} \\ &= \alpha_1 - \lim_{r \rightarrow \infty} 2\sigma \log k = \alpha_1.\end{aligned}$$

But $\alpha \leq \beta \leq \beta_1 = \alpha_1$ so $\alpha = \beta = \alpha_1 = \beta_1$ and this completes the proof of Theorems 1 and 2. Let us note that, from (7.3), $\alpha = (2/\pi^2)A_0$.

8. Proof of Theorem 3. We write $\mu^*(r) = 2\sigma \log r + \varepsilon(r)$, where $\varepsilon(r)$ is bounded for $r \geq 1$. From (6.3) we obtain

$$\int_0^\infty \frac{\mu^*(t) - 2\sigma \log t}{t} \log \left| \frac{t+r}{t-r} \right| dt = \nu(r). \quad (8.1)$$

We introduce $\varepsilon_1(t) = \varepsilon(t)$ for $t \geq 1$, $\varepsilon_1(t) = \varepsilon(1)$ for $0 \leq t \leq 1$, and write

$$\nu_1(r) = \nu(r) + \int_0^1 (\varepsilon(1) - \varepsilon(t)) \log \left| \frac{t+r}{t-r} \right| dt/t.$$

Then $\nu_1(r) \rightarrow A_0$ as $r \rightarrow \infty$ and (8.1) may be rewritten as

$$\int_0^\infty \frac{\varepsilon_1(t)}{t} \log \left| \frac{t+r}{t-r} \right| dt = \nu_1(r).$$

With a change of variable we obtain

$$\int_{-\infty}^\infty \varepsilon_1(e^t) \log \left| \frac{e^{r-t} + 1}{e^{r-t} - 1} \right| dt = \nu_1(e^r).$$

Now $\varepsilon_1(e^t) = \mu^*(e^t) - 2\sigma t$ so

$$\lim_{\substack{x \rightarrow \infty \\ y > x \\ (y-x) \rightarrow 0}} (\varepsilon_1(e^y) - \varepsilon_1(e^x)) \geq 0;$$

that is, $\varepsilon_1(e^t)$ is slowly decreasing in the sense of [8, p. 209]. Moreover an application of contour integration yields

$$\int_{-\infty}^\infty e^{ixt} \log \left| \frac{e^t + 1}{e^t - 1} \right| dt = \frac{\pi}{|x|} \frac{1 - e^{-\pi|x|}}{1 + e^{-\pi|x|}}$$

for $x \neq 0$. We are thus able to apply Theorem 10a of [8, p. 211], to deduce that

$$\varepsilon_1(e^t) \rightarrow \frac{2A_0}{\pi^2} = \alpha$$

as $t \rightarrow \infty$ and this completes the proof of Theorem 3.

NOTE ADDED IN PROOF. The conjecture that follows the statement of Theorem 1 is false. Details of a correct theorem for $p \neq 2$ should appear in due course.

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