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LINEAR SPACES WITH AN H^*-ALGEBRA-VALUED INNER PRODUCT

BY

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ABSTRACT. The paper deals with a particular class of VH-spaces of Loynes [5] whose inner product assumes its values in a trace-algebra associated with an H^*-algebra. It is shown that these spaces admit a structure of a "nonassociative module", and this structure could be used to characterize such spaces. Also we characterize other related spaces.

1. Introduction. The class of Hilbert modules constitutes an important collection of examples of LVH-spaces of Loynes [5]. It has been studied by Goldstine and Horwitz, the author, Giellis and Smith.

The present paper deals with a more general class of LVH-spaces. The inner product of these spaces assumes its value in an H^*-algebra but we do not postulate existence of a module structure. There is a rather interesting theory associated with these spaces that allows for a new way of looking at the results of Loynes [5], [6]. We would like to bring this theory to the attention of the mathematical community.

We shall use the definitions and the notation of [5] and the previous papers of the author.

2. General remarks and examples. Let A be a proper H^*-algebra with the norm denoted by $|\ |$. Then its trace-class τA [16] is strongly admissible in the sense of Loynes [5, p. 167] (hence every VH-space over τA is an LVH-space [5, p. 168]). This fact can be easily verified: the property (3) in [5] on page 167 follows from the fact that the norm $\tau(\ )$ is additive on positive members of τA and the property (6) (the same page) follows from the fact that τA is monotone complete in the sense of Wright [18].

Note also that the scalar product $(\ ,\ )$ of A can be expressed in terms of the trace tr of τA, $(x, y) = \operatorname{tr}(y^*x) = \operatorname{tr}(xy^*)$.

Let S be a space and let $\mathfrak{A}$ be a σ-ring of subsets of S. In particular $\mathfrak{A}$ could be the class of Borel subsets of a locally compact Hausdorff space S. Let η be a positive τA-valued measure on $\mathfrak{A}$. Consider the space $K = L^2(S, \eta)$ of all $\mathfrak{A}$-measurable complex-valued functions $x(s)$ on S such that $|x(s)|^2$ is η-summable. Then K is an LVH-space over τA with the inner product $[\ ,\ ]$ defined by $[x, y] = \int x(s)\bar{y}(s)\, d\eta(s)$. It is not a Hilbert module.

Another example of an LVH-space could be constructed by taking an arbitrary closed linear subspace K of a Hilbert module H. Note that for this example it is

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possible to introduce a certain structure on K which resembles the structure of a module. Let pr be the projection of H on to K. Define the map $\cdot$ of $K \times A$ into K by setting $f \cdot a = \mathrm{pr}(fa)$ ($f \in K$, $a \in A$). Then this operation $\cdot$ satisfies all the axioms of a module except for the fact that the equality $(f \cdot a) \cdot b = f \cdot (a \cdot b)$ does not hold.

3. Semimodule structure on a VH-space over τA. It turns out that *every* VH-space over τA has a similar operation, and this fact can be used to characterize these spaces. Also it gives a new insight into the theory.

THEOREM 1. *Let H be a VH-space over τA with the τA-valued inner product denoted by* [,] *and let $f \in H$, $a \in A$. Then there exists a unique member fa of H such that* $\mathrm{tr}[fa, g] = \mathrm{tr}(a[f, g]) = \mathrm{tr}([f, g]a)$ *for all $g \in H$. The map $f, a \to fa$ has the following properties* ($f, g \in H$, λ *is a complex number and* $a, b \in A$):

(1) $(f + g)a = fa + ga$;

(2) $f(a + b) = fa + fb$;

(3) $(\lambda f)a = \lambda(fa) = f(\lambda a)$;

(4) $\mathrm{tr}[f, ga^*] = \mathrm{tr}[fa, g] = \mathrm{tr}([f, g]a)$;

(5) *the operators Ta: $f \to fa$ are continuous for all $a \in A$.*

(*Note that $(fa)b \neq f(ab)$, in general.*)

PROOF. According to the definition [5, pp. 167–168], the inner product [,] is the map $H \times H \to \tau A$ with the following properties:

(i) $[x, x] \geqslant 0$, and $[x, x] = 0$ if and only if $x = 0$;

(ii) $[x, y]^* = [y, x]$;

(iii) $[\lambda x + \eta y, z] = \lambda[x, z] + \eta[y, z]$ (here $x, y, z \in H$ and λ, η are complex numbers).

Note that H is a Hilbert space with respect to the scalar product $(x, y) = \mathrm{tr}[x, y]$.

Now we prove the existence of fa. Let us consider first the case when a is positive ($\mathrm{tr}(x^*ax) \geqslant 0$ for all $x \in A$). In this case the map e: $g \to \mathrm{tr}(a[f, g]) = e(g)$ is linear and bounded (because of the Schwarz inequality):

$$|\mathrm{tr}\, a[f, g]|^2 \leqslant \mathrm{tr}(a[f, f]) \cdot \mathrm{tr}(a[g, g]) \leqslant \|f\|^2 \|g\|^2 |a|^2.$$

Existence of fa and property (5) now follows from Riesz's theorem [4, 10G].

If a is arbitrary, then the first part of the theorem follows from the fact that a is a linear combination of positive members of A (if $a^* = a$ then $a = a^+ - a^-$; if a is arbitrary then $a = a_1 + ia_2$, where $a_1^* = a_1$, $a_2^* = a_2$).

The rest of the theorem is easy to verify (e.g.

$$\mathrm{tr}[f, ga^*] = \overline{\mathrm{tr}}\,[ga^*, f] = \overline{\mathrm{tr}}\,(a^*[g, f]) = \mathrm{tr}([g, f]^* a)$$
$$= \mathrm{tr}(a[f, g]) = \mathrm{tr}[fa, g]).$$

4. Relation to the theory of Loynes. Let H be a fixed VH-space over τA. Then H is an LVH-space (since τA is strongly admissible) as well as a Hilbert space with respect to the scalar product $(f, g) = \mathrm{tr}[f, g]$.

PROPOSITION 1. *A closed linear subspace M of H is accessible in the sense of Loynes* [5, *p.* 173] *if and only if it is closed under the map $f \to fa$ for each $a \in A$ (M is a "submodule" of H).*

PROOF. If M is accessible then $H = M \oplus M^P$, where

$$M^P = \{f \in H | [f, g] = 0 \text{ for all } g \in M\}$$

coincides with the orthogonal complement

$$M^\perp = \{f \in H | (f, g) = \operatorname{tr}[f, g] = 0 \text{ for all } g \in M\}.$$

Then it is easy to see that both M and M^P are "submodules" of H (if $f \in M^P$ then $\operatorname{tr}[fa, g] = \operatorname{tr}(a[f, g]) = 0$ for all $g \in M$ and each $a \in A$, i.e. M^P is closed under the maps $f \to fa$).

If M is closed under the map $f \to fa$ and $g \in M^\perp$, then $\operatorname{tr}(a[g, f]) = \operatorname{tr}[ga, f] = 0$ for each $a \in A$ which simply means that $[g, f]$ is orthogonal to each member a^* of A, i.e. $[g, f] = 0$. Thus $M^P = M^\perp$ and M is accessible.

As in a Hilbert module **[10]** we can define an A-linearity of an operator T on H by requiring $T(fa) = (Tf)a$ and $T(f + g) = Tf + Tg$ for all $f, g \in H$ and $a \in A$.

PROPOSITION 2. *Let T be a linear separator on H and let T^* be its adjoint* (*in the sense of a Hilbert space theory*). *Then $[Tf, g] = [f, T^*g]$ for all $f, g \in H$* (T^* *is the adjoint of T also in the sense of Loynes*) *if and only if $T(fa) = (Tf)a$ for all $f \in H$, $a \in A$* (T *is A-linear*). *A similar theorem is also valid if T and T^* are defined only on a dense subset of H.*

PROOF. If "$[Tf, g] = [f, T^*g]$" then T is A-homogeneous since

$$\begin{aligned}\operatorname{tr}[T(fa), g] &= \operatorname{tr}[fa, T^*g] = \operatorname{tr}(a[f, T^*g]) \\ &= \operatorname{tr}(a[Tf, g]) = \operatorname{tr}[(Tf)a, g]\end{aligned}$$

for all $f, g \in H$ and each $a \in A$. If T is A-linear then the equality

$$\begin{aligned}\text{“}\operatorname{tr}(a[f, T^*g]) &= \operatorname{tr}[(Tf)a, g] = \operatorname{tr}[T(fa), g] \\ &= \operatorname{tr}[fa, T^*g] = \operatorname{tr}(a[f, T^*g])\text{”}\end{aligned}$$

implies that $[Tf, g] - [f, T^*g]$ is orthogonal to each $a^* \in A$ (which means that $[Tf, g] = [f, T^*g]$) for all $f, g \in H$.

COROLLARY. *A projection onto any accessible subspace of H is A-linear.*

PROPOSITION 3. *Assume that H is a Hilbert module* **[10]** *i.e. $f(ab) = (fa)b$ for all a, $b \in A$ and each $f \in H$. Then an A-linear operator T: $H \to H$ is bounded if and only if it is bounded in the sense of Loynes* **[5**, *p.* 169] *and* $\|T\| = \operatorname{glb}\{k > 0\colon [Tf, Tf] \leqslant k^2[f, f]$ *for all* $f \in H\} = \|T\|_L$. *If T is selfadjoint then*

$$\operatorname{glb}\{(Tf, f)\colon \|f\| \leqslant 1\} = \operatorname{lub}\{k\colon k[f, f] \leqslant [Tf, f]\}$$

and

$$\operatorname{lub}\{(Tf, f)\colon \|f\| \leqslant 1\} = \operatorname{glb}\{K\colon [Tf, f] \leqslant K[f, f]\}.$$

($\|T\|_L$ *denotes the norm of T in the sense of Loynes* **[5**, *p.* 169].)

PROOF. Assume that T is bounded. Then

$$\operatorname{tr}(a[Tf, Tf]a^*) = \operatorname{tr}[T(fa), T(fa)] \leqslant \|T\|^2\|fa\|^2 = \|T\| \operatorname{tr}(a[f, f]a^*)$$

for all $f \in H$, $a \in A$, which simply means that $[Tf, Tf] \leqslant \|T\|^2[f, f]$ for all $f \in H$, i.e. T is bounded in the sense of Loynes and $\|T\|_L \leqslant \|T\|$. The fact that $\|T\| \leqslant \|T\|_L$ is even easier to verify (we use the fact that the trace tr preserves the order and we do not need the property "$f(ab) = (fa)b$"). The rest of the theorem is established in a similar fashion.

5. Some applications. An important consequence of the prior theory is the fact that the Spectral Theorem (Theorem 7 on page 174 of **[5]**) can be easily derived from the classical case, the way it was done for a Hilbert module (note that the proof of Theorem 7 in **[10]** does not depend on the property "$(fa)b = f(ab)$"). Also we have a rather simple proof of Stone's theorem for arbitrary locally compact groups (we can use the proof of Theorem 3 of **[11]** with almost no change). Note that Loynes in **[5]**, **[6]** succeeded in proving Stone's theorem only for the case when the group is either the group Z of integers (Theorem D in **[6]**) or the group R of real numbers (Theorem E in **[6]**). We can use Stone's theorem to derive spectral representation theorems for stationary processes. One can define an H-valued stationary process (the way it was done in **[13]**) as a map ξ: $G \to H$ of a locally compact commutative group G (into H) which has the property that $[\xi(t), \xi(s)] = [\xi(t + r), \xi(s + r)]$ for all $r, s, t \in G$. Then we have the following representation theorem.

THEOREM 2. *For each H-valued stationary process ξ there exists an H-valued countably additive orthogonally scattered measure ν defined on Borel subsets of $\hat{G}$ such that*

$$\xi(t) = \int_G \overline{(t, \alpha)}\, d\nu(\alpha) \quad \textit{for all } t.$$

The measure ν has the property that $[\nu(\Delta_1), \nu(\Delta_2)] = 0$ for any two disjoint Borel sets Δ_1, Δ_2 (we may refer to the measure ν as a "generalized orthogonally scattered measure" since this property is a generalization of property (iii) *on page* 64 *of* **[8]** (*Definition* 1.2)).

PROOF. Let $H\xi$ be the closed subspace of H generated by the vectors of the form $\xi(t)$, $t = G$, i.e. $H\xi$ is the closure of the set $\{\sum_{k=1}^n \lambda_k \xi(t_k)$: $t_1, \ldots, t_n \in G$ and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are complex numbers$\}$. Then introduce the operation $f \to fa$ on $H\xi$ ($a \in A$), and for each $t \in G$ define the operator $U_t(\sum \lambda_k \xi(t_k)) = \sum \lambda_k \xi(t_k + t)$. Then it is easy to verify that $[U_t f, U_t g] = [f, g]$ and $(U_t f, U_t g) = (f, g)$, i.e. U_t is unitary in both the standard and the generalized senses. Hence we can apply Stone's theorem, established in the same fashion as Theorem 3 in **[11]**, to conclude that there exists a generalized spectral measure P: $\Delta \to P_\Delta$ on $\hat{G}$ such that each P_Δ is an A-linear projection and $U_t = \int_{\hat{G}} \overline{(t, \alpha)}\, dP_\alpha$. The spectral measure P has the property that $[P_\Delta(H\xi), P_{\Delta'}(H\xi)] = 0$ if Δ and Δ' are disjoint Borel sets. Then we define the "generalized orthogonally scattered measure" ν on the Borel subsets of $\hat{G}$ by setting $\nu(\Delta) = P_\Delta \xi(1)$, where 1 is the identity of G. Then it is easy to verify that

$$\xi(t) = \int_{\hat{G}} (t, \alpha)\, d\nu(\alpha) \quad \text{for each } t \in G.$$

6. Characterization of VH**-spaces over** τA. Let H again be a VH-space with a τA-valued inner product.

PROPOSITION 4. *Let* $\{e_\alpha\}$ *be an approximate identity for* A. *Then*

$$\lim_\alpha (fe_\alpha, g) = (f, g) \quad \text{for all } f, g \in H.$$

PROOF. Let $a, b \in A$ be such that $[f, g] = ab$. Then Proposition 4 follows from the following facts:

$$\begin{aligned} |(fe_\alpha, g) - (f, g)| &= |\mathrm{tr}(e_\alpha[f, g]) - \mathrm{tr}[f, g]| \\ &= |\mathrm{tr}(e_\alpha ab - ab)| \leqslant |e_\alpha a - a| \cdot |b|, \end{aligned}$$

and the last expression tends to zero.

PROPOSITION 5. *If* $f \in H$ *and* $a \in A$ *then* $|(fa, f)| \leqslant \|La\| \, \|f\|^2$, *where* La *is the operator* $x \to ax$ *acting on* A.

PROOF. The proposition follows from Lemma 5 of **[16]**:

$$|(fa, f)| = |\mathrm{tr}(a[f, f])| \leqslant \tau(a[f, f]) \leqslant \|La\| \tau[f, f] = \|La\| \cdot \|f\|^2.$$

Using the last two propositions it is now easy to state an equivalent set of axioms for our class of VH-spaces.

THEOREM 3 (A CHARACTERIZATION OF VH-SPACES OVER τA). *Let* H *be a Hilbert space which has a structure of nonassociative module. In other words, we are assuming that there is a map* $f, a \to fa$ *of* $H \times A$ *into* H *with following properties* ($f, g \in H$, λ *is a complex number and* $a, b \in A$):

(1) $(f + g)a = fa + ga$;

(2) $f(a + b) = fa + fb$;

(3) $\lambda(fa) = (\lambda f)a = f(\lambda a)$.

(*We do not assume* "$f(ab) = (fa)b$".) *Assume further that*

(4) $(fa, g) = (f, ga^*)$,

(5) $(fa^*, f) \geqslant 0$,

(6) $|(fa, f)| \leqslant \|La\| \cdot \|f\|^2$ (*here* La: $x \to ax$, $a, x \in A$),

(7) $\lim_\alpha (fe_\alpha, g) = (f, g)$ *for each approximate identity* $\{e_\alpha\}$ *of* A.

Then there exists a τA*-valued inner product* $[f, g]$ *on* H *such that* H *is a* VH*-space* (*and also an* LVH*-space*) *with respect to* $[\ ,\ ]$, *and* $(f, g) = \mathrm{tr}[f, g]$, $\mathrm{tr}(fa, g) = \mathrm{tr}(a[f, g])$ *for all* $f, g \in H$ *and each* $a \in A$.

PROOF. For any fixed $f \in H$ consider the linear map

$$1_f\colon La \to 1_f(La) = (fa, f).$$

It is bounded, since $|(fa, f)| \leqslant \|La\| \, \|f\|^2$, and is defined on a dense subset of $C(A)$ **[12**, p. 101]. Hence it could be extended to entire $C(A)$. It follows from Theorem 1 of **[12]** that there exists a member $[f, f]$ of τA such that $(fa, f) = \mathrm{tr}(a[f, f])$ for all $a \in A$. Next we define

$$\begin{aligned} [f, g] = \tfrac{1}{4}\{&[f + g, f + g] - [f - g, f - g] \\ &+ i[f + ig, f + ig] - i[f - ig, f - ig]\} \end{aligned}$$

and verify that $\operatorname{tr}(a[f, g]) = (fa, g)$. Then one can easily see that $[\ ,\ ]$ is a generalized inner product in the sense of Loynes [5, p. 168] and the property (7) can be used to show that $\operatorname{tr}[f, g] = (f, g)$ for all $f, g \in H$. More specifically, take any approximate identity $\{e_\alpha\}$ of A and verify

$$\operatorname{tr}[f, g] = \lim_\alpha \operatorname{tr}(e_\alpha[f, g]) = \lim_\alpha (fe_\alpha, g) = (f, g).$$

To prove that $[\ ,\ ]$ is positive definite we use property 3 (5): $\operatorname{tr}(a^*[f, f]a) = \operatorname{tr}(aa^*[f, f]) = (faa^*, f) \geqslant 0$. To show that $[f, g]^* = [g, f]$ we consider the following equality, $a \in A$:

$$\operatorname{tr}(a[f, g]) = (fa, g) = (f, ga^*) = \overline{(ga^*, f)} = \overline{\operatorname{tr}(a^*[g, f])}$$
$$= \operatorname{tr}([g, f]^*a) = \operatorname{tr}(a[g, f]^*).$$

7. Characterization of other spaces. As other applications of the previous theory we shall present a simple characterization of a Hilbert module (Theorem 4) as well as a characterization of the space $L^2(S, \eta)$ where η is some Borel τA-valued measure defined on a compact subset S of the real line (Theorem 5).

THEOREM 4. *Let H be a VH-space over τA. If the operation $f, a \to fa$ has the property that $[fa, g] = [f, g]a$ for all f, g in H and $a \in A$, then H is a Hilbert module* [10] *with respect to the τA-valued inner product $[f, g]' = [g, f]$.*

PROOF. It is sufficient to prove that $(fa)b = f(ab)$ for all $a, b \in A$ and each $f \in H$. But this is a consequence of the following identity ($g \in H$):

$$\operatorname{tr}[f(ab), g] = \operatorname{tr}([f, g]ab) = \operatorname{tr}([fa, g]b) = \operatorname{tr}[(fa)b, g].$$

THEOREM 5. *Let H be a VH-space over τA which has a bounded selfadjoint operator T with a simple spectrum* [7, p. 149] *(there exists $q \in H$ such that the set $\{p(T)q: p(T) = \lambda_0 I + \sum_{k=1}^n \lambda_k T^k$ is a polynomial in $T\}$ is dense in H). Then there exists a compact subset S of the real line and a τA-valued regular Borel measure η on S such that H is isometrically isomorphic to $L^2(S, \eta)$ and T corresponds to the multiplication of members of $L^2(S, \eta)$ with some continuous real-valued function $h(s)$ on S.*

PROOF. Let B be the Banach algebra generated by the operator T and the identity I. Then B is a commutative C^*-algebra, hence it is isomorphic to the space $C(S)$ of continuous complex-valued functions on the space S of maximal ideals of B. It is well known that S is homeomorphic to the spectrum of T, hence it could be identified with a compact subset of the real line.

Let $f \leftrightarrow T_f$ be the Gelfand correspondence between $C(S)$ and B; let $h \in C(S)$ be such that $T = T_h$. The h is continuous and real valued.

Now we can apply the Daniell theory discussed in [15] to the positive linear map $J: f \to [T_f(q), q] = J(f)$ to introduce the Borel τA-valued measure η on S. Then it is easy to verify that H is isomorphic to $L^2(S, \eta)$ and T corresponds to the multiplication with h.

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