TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 263, Number 1, January 1981

ON A RELATION BETWEEN $\widetilde{SL}_2$ CUSP FORMS AND CUSP FORMS ON TUBE DOMAINS ASSOCIATED TO ORTHOGONAL GROUPS

BY

S. RALLIS AND G. SCHIFFMANN

ABSTRACT. We use the decomposition of the discrete spectrum of the Weil representation of the dual reductive pair $(\widetilde{SL}_2, O(Q))$ to construct a generalized Shimura correspondence between automorphic forms on $O(Q)$ and $\widetilde{SL}_2$. We prove a generalized Zagier identity which gives the relation between Fourier coefficients of modular forms on $\widetilde{SL}_2$ and $O(Q)$. We give an explicit form of the lifting between $\widetilde{SL}_2$ and $O(n, 2)$ in terms of Dirichlet series associated to modular forms. For the special case $n = 3$, we construct certain Euler products associated to the lifting between $\widetilde{SL}_2$ and $\mathrm{Sp}_2 \cong O(3, 2)$ (locally).

Introduction. Let Q_1 be a nondegenerate quadratic form on $\mathbf{R}^{k-2}$ having signature $(k-3, 1)$. Let $\mathfrak{L}$ be a lattice in $\mathbf{R}^{k-2}$ so that $Q_1(\mathfrak{L}, \mathfrak{L}) \subseteq 2 \cdot \mathbf{Z}$, i.e., Q_1 is even-integer-valued on $\mathfrak{L}$. (Also $\mathfrak{L}$ must satisfy certain other technical assumptions relative to Q_1.) Let f be a holomorphic modular cusp form of weight s for the Hecke congruence group $\Gamma_0(t\, N_{\mathfrak{L}})$, where $N_{\mathfrak{L}}$ is the exponent of $\mathfrak{L}$ and t is a certain positive integer. Suppose that

$$f(z) = \sum_{n \geq 1} a_f(n) e^{2\pi\sqrt{-1}\, nz}$$

($z \in H$, the upper half plane) is the corresponding Fourier development of f at ∞. Then let $\xi \in \mathfrak{L}$ so that $Q_1(\xi, \xi) < 0$. We may write $\xi = m\xi_0$ with ξ_0 a primitive element in $\mathfrak{L}$ and $m \in \mathbf{Z}$, the integers. Then define

$$a_{\#}(\xi) = \sum_{\{\nu | \nu | m\}} \chi(\nu)\nu^{s+k/2-3} a_f\left(\frac{m^2}{\nu^2}\left|\frac{1}{2}Q_1(\xi_0, \xi_0)\right|\right) \tag{I-1}$$

where χ is a Dirichlet character mod $N_{\mathfrak{L}}$. Then $a_{\#}$ is defined on the set of all $\mathfrak{L}$ lattice points in the self-dual cone $W = \{X \in \mathbf{R}^{k-2} | Q_1(X, X) < 0\}$. It is easy to see $a_{\#}$ is invariant by the arithmetic group $\Gamma^{\mathfrak{L}} = \{g \in O(Q_1) | g(\mathfrak{L}) = \mathfrak{L}\}$, i.e., $a_{\#}(g \cdot \xi) = a_{\#}(\xi)$ for $g \in \Gamma^{\mathfrak{L}}$ and $\xi \in \mathfrak{L} \cap W$.

Then we can define a Dirichlet series

$$\sum_{\mathfrak{L} \cap W/\Gamma^{\mathfrak{L}}} a_{\#}(\xi) \frac{1}{\varepsilon(\xi)} |Q_1(\xi, \xi)|^{-s} \tag{I-2}$$

where the summation is over $\Gamma^{\mathfrak{L}}$ equivalence classes in $\mathfrak{L} \cap W$ and $\varepsilon(\xi) = \mathrm{order}(\Gamma^{\mathfrak{L}, \xi})$ (where $\Gamma^{\mathfrak{L}, \xi}$ is the stabilizer in $\Gamma^{\mathfrak{L}}$ of ξ). This Dirichlet series has several

Received by the editors February 15, 1978 and, in revised form, June 4, 1979.
AMS (MOS) subject classifications (1970). Primary 10D20.

features. First, after being suitably normalized by Γ factors, (I-2) admits an analytic continuation to the whole complex plane. Second (with the same normalization) (I-2) satisfies a functional equation in $\mathfrak{s}$. Third, (I-2) admits (in a certain sense) a type of Euler product when the Dirichlet series

$$D(\mathfrak{s}, f) = \sum_{n \geq 1} a_f(n) n^{-\mathfrak{s}}$$

has the same properties.

The three properties of (I-2) above are the standard type of properties exhibited by Dirichlet series attached to automorphic forms. Thus we must look for a cusp form on some reductive group so that (I-2) is the Dirichlet series "associated" to this cusp form. We start by noting that there is a natural tube domain in the above considerations. Namely we let $\mathfrak{R}_+ = \mathbf{R}^{k-2} + \sqrt{-1}\, W$ and we consider the connected component of the full group of analytic automorphisms of $\mathfrak{R}_+$. We know that this group is isomorphic to the connected component of $O(Q)$, the orthogonal group of the quadratic form $Q_1 \oplus x \cdot y$, where $x \cdot y$ denotes a "hyperbolic plane". Following the well-known theory of automorphic forms in tube domains, we ask whether there is a cusp form F on $\mathfrak{R}_+$, automorphic relative to $O(Q)$ and a suitable arithmetic subgroup $\Gamma^L(Q)$ of $O(Q)$, so that (I-2) is the Mellin transform of F along the direction of the cone.

The answer to the above question is in the affirmative. The main idea behind such a result is to use the correspondence between automorphic forms on $\widetilde{SL}_2$ and automorphic forms on $O(Q)$ given by the Weil representation of $\widetilde{SL}_2 \times O(Q)$ in **[II]**, **[III]** and **[IV]**.

This paper is the sequel to **[II]**. It is part of our program to show that the so-called Shimura correspondence given in **[8]**, **[10]** and **[12]** can be interpreted in terms of the theory developed in **[II]**. We now work with an arbitrary quadratic form Q on $\mathbf{R}^k$ (no restriction on the signature except that sgn $Q = (a, b)$ satisfies $a \geq b \geq 1$).

In **[8]** and **[10]** a correspondence is set up between $\widetilde{SL}_2$ cusp forms and $O(Q)$ cusp forms (see also **[2]**, **[4]**). Essentially starting with a Schwartz function φ, which transforms according to a finite dimensional representation of a maximal compact subgroup of $\widetilde{SL}_2 \times O(Q)$, we form the θ series $T_\varphi^L(G, g)$ with $G \in \widetilde{SL}_2$ and $g \in O(Q)$ (where L is a lattice in $\mathbf{R}^k$ so that $\mathfrak{L}$ is a direct summand of L). Then if f is a cusp form in the $\widetilde{SL}_2$ variable we form the Petersson inner product $\langle T_\varphi^L(\ , g) | f(\ ) \rangle$ (note here we view f simultaneously as a function on H and $\widetilde{SL}_2$). Thus the map $f \rightsquigarrow \langle T_\varphi^L(\ , g) | f(\ ) \rangle$ gives a correspondence between $\widetilde{SL}_2$ cusp forms and $O(Q)$ automorphic forms. However in the different cases (**[4]** and **[8]**) it must be shown directly that $g \rightsquigarrow \langle T_\varphi^L(\ , g) | f(\ ) \rangle$ is a cusp form, etc. On the other hand if we start with φ_D, a function belonging to the discrete spectrum of the Weil representation of $\widetilde{SL}_2 \times O(Q)$, then we can also define $T_{\varphi_D}^L(G, g)$. In a similar fashion we consider $\langle T_{\varphi_D}^L(\ , g) | f(\ ) \rangle$. Then we have another correspondence between $\widetilde{SL}_2$ cusp forms and $O(Q)$ automorphic forms. However the cuspidal properties of

$$g \rightsquigarrow \langle T_{\varphi_D}^L(\ , g) | f(\ ) \rangle$$

are evident from the cuspidal properties of $T^L_{\varphi_D}(\ ,\ )$, which are set forth in §§3 and 5 of **[II]**. We recall from **[II]** that

$$T^L_{\varphi_D}\left(\begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix}, g\right) = \sum_{n < -1} \beta_n(g) e^{2\pi\sqrt{-1}\, n\bar{z}} \tag{I-3}$$

with $z = -y/x + \sqrt{-1}\,/x^2 \in H$, $^-$ denotes complex conjugation, and

$$\beta_n(g) = \sum_{\{\lambda \in L | Q(\lambda,\lambda) = n\}} \varphi_D(g^{-1}\lambda).$$

We know for the special case when Q has signature $(k-2, 2)$ that each β_n is a *cusp form* on $O(Q)$. Thus $\langle T^L_{\varphi_D}(\ , g) | f(\ ) \rangle$ is a cusp form in the $O(Q)$ variable. Moreover if f corresponds to an Eisenstein-Poincaré series G_n for general Q, then $\langle T^L_{\varphi_D}(\ , g) | G_n(\ ) \rangle = \beta_n(g)$.

The first question is the relation between the two types of correspondences defined above. We show in Theorem 1-1 that given φ a Schwartz function, as above, there exists an $\widetilde{SL}_2 \times O(Q)$ intertwining projection operator P_s^+ onto the discrete spectrum (of the Weil representation of $\widetilde{SL}_2 \times O(Q)$) so that the difference $T^L_\varphi - T^L_{P_s^+(\varphi)}$ is orthogonal to all holomorphic $\widetilde{SL}_2$ cusp forms of weight $> \frac{1}{2}k + 1$ (relative to the Petersson inner product). Thus the two correspondences given above are essentially the same.

On the other hand the functions β_n have a highly transcendental nature. Specifically in [II, §7] a formula for the Fourier coefficients of β_n is given which involves an infinite sum of Bessel functions and certain types of trigonometric sums (such a sum is reminiscent of the Fourier coefficients of Eisenstein-Poincaré series). However we can express $T^L_{\varphi_D}$ in another way (this idea originated in **[14]** for the special case Q having type $(2, 2)$ and is extended to $(k, 2)$ for $k \geqslant 4$ in **[9]**). Namely we can write (valid for any $G \in \widetilde{SL}_2$ and certain g described below)

$$T^L_{\varphi_D}(G, g) = \sum_{n < -1} \beta_n^*(g) G_n(G) \tag{I-4}$$

where

$$\beta_n^*(g) = \sum_{\lambda \in X_n} \varphi_D(g^{-1}\lambda)$$

with X_n the subset of all lattice points λ in $\{X \in \mathbf{R}^k | Q(X, X) = n\} \cap L$ satisfying $Q(\lambda, v) = 0$ (where v is a given nonzero isotropic vector in L). Although β_n^* is not an automorphic form relative to $O(Q)$, it is invariant under the discrete group $\Gamma^L \cap O(Q)^{[v]}$, where $O(Q)^{[v]}$ is the parabolic subgroup of $O(Q)$ stabilizing the line determined by the vector v. The important point is that the validity of (I-4) depends on the "cuspidal" behavior of $\pi_Q(g^{-1})\varphi_D$ in the Weil representation. That is, if $\pi_Q(g^{-1})\varphi_D$ is a cusp form in the Weil representation (which means that $\pi_Q(g^{-1})\varphi_D$ satisfies the first and second Cusp Vanishing Theorems in **[II]** relative to the parabolic $O(Q)^{[v]}$) then the formula in (I-4) holds. Thus in §2 we have set up a rather elaborate technical machinery to deduce (I-4). We have adopted this point of view in order to prove a more general formula than in **[14]** and **[9]** and to show the dependence of the formula on the cuspidal properties of the Weil representation.

Using (I-4) we deduce that (when Q has signature $(k-2, 2)$)

$$\langle T^L_{\varphi_D}(\ , g)|f(\ )\rangle = \sum_{n < -1} a_f(|n|)\beta^*_n(g)|n|^{1-\mathfrak{s}}.$$

Thus we can compute the Fourier coefficient of $\langle T^L_{\varphi_D}(\ , g)|f(\ )\rangle$ and obtain (I-1) (modulo certain constants and a Gauss sum, see *Theorem* 5-1). Then we consider the Dirichlet series (I-2) associated with $\langle T^L_{\varphi_D}(\ , g)|f(\ )\rangle$. At this point we must however make the correspondence between modular forms and Dirichlet series more precise. Indeed in §3 we define a "Dirichlet series" associated to automorphic cusp forms on the tube domain $\mathfrak{R}_+$. Following the ideas in **[6]** we prove the analytic continuation and functional equation for such Dirichlet series. We note here that such results are of a fairly standard nature, but we could not find any reference in the literature for the particular case of the tube domain $\mathfrak{R}_+$; so we have included proofs of these statements in §3.

On the other hand we can determine the "Dirichlet series" of $\sum a_f(|n|)\beta^*_n(g)|n|^{1-\mathfrak{s}}$ as a type of Mellin transform

$$\sum_{n<-1} a_f(|n|)|n|^{1-\mathfrak{s}}\int_{O(Q_1)/\Gamma^{\mathfrak{L}}\times \mathbf{R}^\times_+} \beta^*_n(mA(r))r^{-\mathfrak{s}}\, d^*(r)\, d\lambda(m) \tag{I-5}$$

where $d\lambda$ is an $O(Q)$ invariant measure on $O(Q)/\Gamma^{\mathfrak{L}}$ and $A(r)$ is the group of positive dilations in $\mathfrak{R}_+$ with $d^*(r) = dr/r$ as invariant measure (on $A(r)$). Then in §4 we compute each inner integral as a function of $\mathfrak{s}$. As a result (*Theorem* 4-1) we deduce that (I-5) is given by the product of two Dirichlet series (aside from normalizing Γ functions and elementary functions like $a^{-\mathfrak{s}}$),

$$L(\chi, 2\mathfrak{s}+1-r)\cdot\left\{\sum_{n<-1} a_f(|n|)M(Q_1, \mathfrak{L}, n)|n|^{-\mathfrak{s}}\right\} \tag{I-6}$$

where $r = s + 2 - k/2$, where $L(\chi,\ )$ is the usual L function with Dirichlet character χ, and where $M(Q_1, \mathfrak{L},\ )$ are the Siegel mass numbers. The second term of (I-6) is the Rankin convolution of the $\widetilde{SL}_2$ Dirichlet series of f and the Siegel zeta function

$$\zeta_-(Q_1, \mathfrak{L}, \mathfrak{s}) = \sum_{n<-1} M(Q_1, \mathfrak{L}, n)|n|^{-\mathfrak{s}}.$$

This then allows us to make certain statements about the Euler product properties of (I-6).

From purely algebraic considerations (using Hadamard products of power series, see **[11]**) we know that if k is even and if both the Dirichlet series of f and $\zeta_-(Q_1, \mathfrak{L}, \mathfrak{s})$ admit the usual Euler product of the GL_2 theory, then the second term in (I-6) can be expressed as an Euler product with numerator of degree 2 and denominator of degree 4 for almost all primes p. On the other hand if k is odd then f and $\zeta_-(Q_1, \mathfrak{L}, \mathfrak{s})$ do not have the usual GL_2 type Euler product. However in **[10]** a modified theory of Euler products is set forth for $\widetilde{SL}_2$ automorphic forms of half integral weight. In particular we know that for d' a "fundamental discriminant", the series (having a Hecke eigenfunction property) $\sum_{m\geq 1} a_f(d'm^2)m^{-\mathfrak{s}}$ has an Euler product with numerator of degree 1 and denominator of degree 2 for almost all primes p (see Theorem 6-2). In the special case $k = 5$ we know from the theory of binary quadratic forms that

$$\sum M(Q_1, \mathfrak{L}, d'n^2)|n|^{-\mathfrak{s}}$$

has a similar type Euler product (see (6-7)). Then we deduce that the Rankin convolution of the latter two Dirichlet series has an Euler product with numerator of degree 3 and denominator of degree 4 for almost all primes p. We then deduce (for $k = 5$ and a special choice of L) that there is an Euler product formula for the partial Dirichlet series

$$\sum_{\substack{\xi \in \mathfrak{L} \cap W/\Gamma^{\mathfrak{L}} \\ Q_1(\xi,\xi) = d'f^2}} a_{\#}(\xi) \frac{1}{\varepsilon(\xi)} |Q_1(\xi,\xi)|^{-\mathfrak{s}} \tag{I-7}$$

with d', a "fundamental discriminant" (Theorem 6-2 and Remark 6-4). Essentially we view d' as determining an imaginary quadratic extension $\mathbf{Q}(\sqrt{d'})$ of $\mathbf{Q}$, the rationals. It is consistent to expect (by the Andrianov theory in [**1**]; see also [**5**]) that such a partial Dirichlet series admits an Euler product. We will make explicit the relation of these Euler products to the Andrianov theory in a future paper.

We have given above an outline of the main results of this paper. We here thank Shimura for his valuable advice and insight, especially in discussion of the types of Euler products expected in the above theory. We also thank M. Vergne for very carefully reading an earlier version of this work and pointing out certain errors.

0. Notation and terminology. We denote by $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{C}$ the ring of integers, the rational numbers, the real numbers and the complex numbers respectively. We denote by $\mathbf{Z}_+$, $\mathbf{Q}_+$ and $\mathbf{R}_+$ the set of positive elements in the respective rings.

1. We let $\mathbf{R}^k$ be a k dimensional space. Let Q be a nondegenerate quadratic form on $\mathbf{R}^k$ with an orthogonal basis $\{e_1, \ldots, e_k\}$ of $\mathbf{R}^k$ where $e_i \perp e_j$ (relative to Q) if $i \neq j$. Moreover

$$Q(e_i, e_i) = \begin{cases} 1 & \text{if } i \leqslant a, \\ -1 & \text{if } i > a, \end{cases}$$

where the signature of Q is (a, b) with $a \geqslant b \geqslant 1$. Also let

$$v_{a+j} = (e_j - e_{a+j})/\sqrt{2}\,, \qquad \tilde{v}_{a+j} = (e_j + e_{a+j})/\sqrt{2}\,, \qquad 1 \leqslant j \leqslant b.$$

Then we have a Q splitting of $\mathbf{R}^k$ as

$$\langle v_a, \tilde{v}_a \rangle \perp \langle v_{a+1}, \tilde{v}_{a+1} \rangle \perp \cdots \perp \langle v_k, \tilde{v}_k \rangle \perp \langle e_{b+1}, \ldots, e_a \rangle$$

where $\langle v_i, \tilde{v}_i \rangle$ is a hyperbolic plane.

2. Let $F_i = \langle v_a, \ldots, v_{a+i-1} \rangle$, $F_i^* = \langle \tilde{v}_a, \ldots, \tilde{v}_{a+i-1} \rangle$ and $L_i = \langle e_i, \ldots, e_a \rangle$. We denote Q restricted to L_i by Q_i. Let $O(Q)$ be the orthogonal group of Q.

Let $P_{F_i} = \{ g \in O(Q) |\, g(F_i) = F_i \}$. Then we know that every maximal parabolic subgroup of $O(Q)$ is $O(Q)$ conjugate to P_{F_i} for some i. Let $A_j(r)$ $(j = a, \ldots, k)$ be the torus subgroup of $O(Q)$ given by $v_j \to rv_j$, $\tilde{v}_j \to r^{-1}v_j$ and identity on $\langle v_j, \tilde{v}_j \rangle^{\perp}$. Moreover let $\{ N_1(W) |\, W \in L_1 \}$ be the unipotent subgroup of $O(Q)$ given by the following operations.

$$N_1(W)(v_a) = v_a, \qquad N_1(W)(\tilde{v}_a) = \tilde{v}_a - Q(W, W)v_a/2 + W,$$
$$N_1(W)(Y) = Y - Q(Y, W)v_a$$

with $Y \in L_1$. Then we know that $O(Q_1) \cdot A_a(r) \cdot N_1$ determines the *Langlands* decomposition of P_{F_1}.

3. Let $\widetilde{SL}_2$ be the twofold cover of $SL_2(\mathbf{R})$ determined by the Kubota cocycle relation as given in **[I]**. Let $\tilde{G}_2(Q) = \widetilde{SL}_2 \times O(Q)$ be the product group. Then let $\pi_{\mathfrak{M}}$ be the so-called "Weil representation" of $\tilde{G}_2(Q)$ (see **[II]**) in $L^2(\mathbf{R}^k)$. Let $\mathbf{F}_Q$ be the space of C^∞ vectors of the representation $\pi_{\mathfrak{M}}$. Let

$$\tilde{K} = \left\{ k(\theta, \varepsilon) = \left(\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \varepsilon \right) \middle| -\pi < \theta \leqslant \pi, \varepsilon = \pm 1 \right\}$$

be a maximal compact subgroup of $\widetilde{SL}_2$. Let

$$A = \left\{ a(r) = \left(\begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix}, 1 \right) \middle| r > 0 \right\}$$

and

$$N = \left\{ n(x) = \left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, 1 \right) \middle| x \in \mathbf{R} \right\}.$$

Let $\mathfrak{a}$, $\mathfrak{n}$, and $\mathfrak{k}$ be the infinitesimal generators of A, N, and $\tilde{K}$ respectively. Let

$$N_+ = \mathfrak{k} + \sqrt{-1}\,(\mathfrak{n} + \mathrm{Ad}(w_0)\mathfrak{n}), \qquad N_- = \mathfrak{k} - \sqrt{-1}\,(\mathfrak{n} + \mathrm{Ad}(w_0)\mathfrak{n})$$

with

$$w_0 = \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, -1 \right) \in \widetilde{SL}_2$$

and Ad the adjoint representation of $\widetilde{SL}_2$. Let $\omega_{SL_2} = -\mathfrak{k}^2 + \mathfrak{a}^2 + (\mathfrak{n} + \mathrm{Ad}(w_0)\mathfrak{n})^2$ be the Casimir element of $\widetilde{SL}_2$.

Then we set $\mathbf{F}_Q(\lambda) = \{\varphi \in \mathbf{F}_Q | \pi_{\mathfrak{M}}(\omega_{SL_2})\varphi = \lambda\varphi\}$, $\mathbf{F}_Q^{\pm}(\lambda) = \{\varphi \in \mathbf{F}_Q(\lambda) | \varphi$ vanishes on Ω_- (Ω_+ resp.)$\}$ with

$$\begin{cases} \Omega_+ = \{X | Q(X, X) > 0\}, \\ \Omega_- = \{X | Q(X, X) < 0\}. \end{cases}$$

Then the important properties of the spaces $\mathbf{F}_Q^{\pm}(\lambda)$ are summarized in **[III]**. We recall that $\mathbf{F}_Q^+(\lambda)$ ($\mathbf{F}_Q^-(\lambda)$ resp.) is nonzero if and only if $\lambda = s^2 - 2s$ with $s > 1$ and $s \equiv k/2 \bmod 1$ ($\lambda = s^2 + 2s$ with $s < -1$ and $s \equiv k/2 \bmod 1$ and $b > 1$ resp.).

We let $K = O(a) \times O(b)$ be a maximal compact subgroup of $O(Q)$. The irreducible representations of K are parametrized by $[s_1]_a \otimes [s_2]_b$ where s_1, s_2 are nonnegative integers ($s_2 = 0$ or 1 if $b = 1$) corresponding to spherical harmonic representations of degree s_1 and s_2 of $O(a)$ and $O(b)$ respectively. Let $E_{\mathfrak{M}}(s^2 - 2s, m, s_1, s_2) = \{\varphi \in \mathbf{F}_Q^+(s^2 - 2s) | \varphi$ belongs to $\tilde{K} \times K$ isotypic component in $\mathbf{F}_Q^+(s^2 - 2s)$ transforming according to the character $k(\theta, \varepsilon) \rightsquigarrow (\mathrm{sgn}\,\varepsilon)^{2m} e^{\sqrt{-1}\,\theta m}$ of $\tilde{K}$ and according to $[s_1]_a \otimes [s_2]_b$ on $K\}$. Again the important properties of $E_{\mathfrak{M}}(s^2 - 2s, m, s_1, s_2)$ are summarized in **[III]**. We recall that $E_{\mathfrak{M}}(s^2 - 2s, m, s_1, s_2)$ is a nonzero space if and only if $s_1 - s_2 = s - \frac{1}{2}(a - b) + 2l$, l a nonnegative integer, and $m = s + 2j$, j a nonnegative integer.

Similarly, $E_{\mathfrak{M}}(s^2 - 2s, m, s_1, s_2) \subseteq \mathbf{F}_Q^-(s^2 - 2s)$ is a nonzero space if and only if $s_2 - s_1 = s - \frac{1}{2}(b - a) + 2l'$, l' a nonnegative integer, and $m = -(s + 2j')$, j' a nonnegative integer.

4. Let L be a lattice in $\mathbf{Q} \otimes_{\mathbf{Z}} \{e_1, \ldots, e_k\}$ which is Q integral, i.e., $Q(\xi_1, \xi_2) \in \mathbf{Z}$ for all $\xi_1, \xi_2 \in L$. Let $L_*(Q) = \{\eta \in Q \otimes_{\mathbf{Z}} \{e_1, \ldots, e_k\} | Q(\eta, L) \subseteq \mathbf{Z}\}$, the Q integral dual of L. Let n_L be the exponent of L, i.e., the smallest positive integer n_L so that $n_L \cdot \eta \in L$ for all $\eta \in L_*(Q)$. Let $\Gamma^L(Q) = \{\gamma \in O(Q) | \gamma(L) = L\}$. We know that $\Gamma^L(Q)$ is an arithmetic subgroup of $O(Q)$. If $\eta \in L_*(Q)$, we set $\Gamma^L(Q)_\eta = \{\gamma \in \Gamma^L(Q) | \gamma \cdot \eta \equiv \eta \bmod L\}$. Then if $\{\xi_i\}$ is a $\mathbf{Z}$-basis of L, let $D_{Q(L)} = \det\{Q(\xi_i, \xi_j)\}$, the *discriminant* of L relative to Q.

A lattice L is called *Type* II when $Q(\xi_1, \xi_2)$ is an even integer for all $\xi_1, \xi_2 \in L$. Moreover L is said to by *Type* II* if L is Type II and $n_L Q(\xi, \xi)$ is even for all $\xi \in L_*(Q)$.

5. The quadratic residue symbol $(-)$ is given as in **[10]**. Moreover

$$\varepsilon_d = \begin{cases} 1 & \text{if } d \equiv 1 \bmod 4, \\ \sqrt{-1} & \text{if } d \equiv 3 \bmod 4. \end{cases}$$

6. Let

$$\Gamma_0(w) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z}) | c \equiv 0 \bmod w \right\},$$

$$\Gamma^0(w) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z}) | c \equiv 0 \bmod w, a \equiv d \equiv 1 \bmod w \right\}.$$

Let

$$\Gamma_0^0(w_1, w_2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z}) | c \equiv \bmod w_1, b \equiv 0 \bmod w_2 \right\}$$

with w_2 dividing w_1.

Let ψ be the covering map of $\widetilde{SL}_2$ to $SL_2(\mathbf{R})$ given by $\psi(g, \varepsilon) = g$. Then we set $\Delta_w = \psi^{-1}(\Gamma_0(w))$, $\tilde{\Delta}_{2w} = \psi^{-1}(\Gamma_0^0(2w, 2))$, $U_w = \psi^{-1}(\Gamma^0(w))$ and $\tilde{U}_{2w} = \psi^{-1}(\Gamma_0^0(2w, 2)) \cap U_{2w}$.

Let s_Q^L be the multiplier on the group $\Gamma_0^0(2n_L, 2)$ ($\Gamma_0(n_L)$ when L is Type II*, resp.) given by

$$s_Q^L(\gamma) = \left(\overline{\varepsilon_{d_\gamma}}\right)^k \left(\frac{2c_\gamma}{d_\gamma}\right)^k \left(\frac{D_{Q(L)}}{d_\gamma}\right)$$

for

$$\gamma = \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix} \quad \text{with } c_\gamma \neq 0.$$

Let $s_Q^{L,\eta}$ be the multiplier on the group $\Gamma_0^0(2n_L, 2) \cap \Gamma^0(2n_L)$ ($\Gamma^0(n_L)$ when L is Type II* resp.) given by

$$s_Q^{L,\eta}(\gamma) = \exp\left(\pi\sqrt{-1}\ \delta_\gamma \beta_\gamma Q(\eta, \eta)\right) s_Q^L(\gamma).$$

REMARK. We note that s_Q^L given above is a multiplier on the group $\Gamma_0(2n_L)$ when L is Type II.

7. If Γ_1 is any arithmetic subgroup of $SL_2(\mathbf{Z})$, σ any multiplier on Γ_1 of weight d, then $[\Gamma_1, d, \sigma]$ is the space of all f so that

(i) f is holomorphic on H, the upper half-plane,

(ii) $f(h \cdot z) = (cz + d)^d \sigma(h) f(z)$ for all $h = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1$,

(iii) at each parabolic cusp $G^{-1}(\infty)$ of Γ_1 on $\mathbf{Q} \cup \{\infty\}$ (with $G \in GL_2(\mathbf{Z})$), f has a Fourier expansion of the form

$$(\tilde{c}z + \tilde{d})^{-d} f(z) = \sum_{n+\kappa \geqslant 0} c_n \exp\left(2\pi\sqrt{-1}\left(\frac{n+\kappa}{N_{\Gamma_1}}\right)G(z)\right)$$

where N_{Γ_1} is the smallest positive integer ν so that

$$G^{-1}\begin{pmatrix} 1 & N_{\Gamma_1} \\ 0 & 1 \end{pmatrix} G \in \Gamma_1$$

and κ is the ramification of σ at $G^{-1}(\infty)$ (see **[II]** for precise definition).

We say $f \in [\Gamma_1, d, \sigma]$ is a *cusp form* if $n + \kappa \geqslant 0$ above is replaced by $n + \kappa > 0$ at each parabolic cusp of Γ_1. Let $[\Gamma_1, d, \sigma]_0$ be the linear space of such cusp forms.

Let $[\Gamma_1, d, \sigma]^*$ be the space of all f which satisfy (i), (ii), and (iii) above when H is replaced by $\overline{H}$, the lower half plane in (i) and $n + \kappa \geqslant 0$ in (iii) is replaced by $n + \kappa \leqslant 0$.

We note that via the map $\varphi(z) \rightsquigarrow f(z) = \overline{\varphi(\bar{z})}$, the space $[\Gamma_1, d, \sigma]^*$ maps bijectively onto $[\Gamma_1, d, \bar{\sigma}]$.

8. Let $\varphi \in E_{\mathfrak{M}}(s^2 - 2s, s, s_1, s_2)$ with $s > k/2$. Then define for $\eta \in L_*(Q)$, $G \in \widetilde{SL}_2$, $g \in O(Q)$,

$$T^L_{\varphi,\eta}(G, g) = \sum_{\xi \in L} \pi_{\mathfrak{M}}((G, g))^{-1}(\varphi)(\xi + \eta).$$

The properties of $T^L_{\varphi,\eta}$ are given in Theorem 2 of **[IV]** (for the case $\eta = 0$) and Theorem 2.5 of **[II]**. We know that for $(\Gamma, \gamma) \in \tilde{U}_{2N_L} \times \Gamma^L(Q)$ ($U_{N_L} \times \Gamma^L(Q)$ if L is a Type II lattice, respectively) that

$$T^L_{\varphi,\eta}(G\Gamma, g\gamma) = c^L_Q(\Gamma) \exp\left(-\pi\sqrt{-1}\, d_\Gamma b_\Gamma Q(\eta, \eta)\right) T^L_{\varphi,\eta}(G, g)$$

where

$$\Gamma = \left(\begin{pmatrix} a_\Gamma & b_\Gamma \\ c_\Gamma & d_\Gamma \end{pmatrix}, \varepsilon\right).$$

We note the relationship $s^L_Q(\psi(\Gamma)) = \overline{c^L_Q(\Gamma)}$ (ψ, the covering homomorphism of $\widetilde{SL}_2$ to SL_2), where

$$\Gamma = \left(\begin{pmatrix} a_\Gamma & b_\Gamma \\ c_\Gamma & d_\Gamma \end{pmatrix}, 1\right)$$

satisfies the condition $c_\Gamma \neq 0$.

If Z is an element of the enveloping algebra of $\tilde{G}_2(Q) = \widetilde{SL}_2 \times O(Q)$, then we have that

$$T^L_{\pi_{\mathfrak{M}}(Z)\cdot\varphi,\eta}(G, g) = Z * T^L_{\varphi,\eta}(G, g)$$

where $Z *$ denotes differentiation on the left. Hence $\omega_{\mathrm{SL}_2} * T^L_{\varphi,\eta} = (s^2 - 2s) \cdot T^L_{\varphi,\eta}$.

Then we let

$$\tilde{T}^{L}_{\varphi,\eta}(z, g) = \left[\operatorname{Im}(z)\right]^{-s/2} T^{L}_{\varphi,\eta}\left(\left(\begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix}, 1\right), g\right)$$

with $z = -y/x + \sqrt{-1}\,/x^2 \in H$, the upper half-plane. Then we have that $\check{T}^{L}_{\varphi,\eta} \in \left[\Gamma^0(2n_L, 2) \cap \Gamma^0(2n_L), s, s_Q^{L,\eta}\right]_0$ $\left(\in \left[\Gamma^0(n_L), s, s_Q^{L,\eta}\right]_0 \text{ if } L \text{ is Type II* resp.}\right)$. Then (again for $\varphi \in E_{\mathfrak{M}}(s^2 - 2s, s, s_1, s_2)$ with $s > k/2$) there is a decomposition formula:

$$\check{T}^{L}_{\varphi,\eta}(z, g) = \sum_{r \in X_\eta^+} r^{s-1} e^{\pi\sqrt{-1}\, rz} T^{L,r}_{\varphi,\eta}(g)$$

where $T^{L,r}_{\varphi,\eta}$ satisfies the functional equation $T^{L,r}_{\varphi,\eta}(g\gamma) = T^{L,r}_{\varphi,\eta}(g)$ for $\gamma \in \Gamma^L(Q)_\eta$ (here $X_\eta^+ = \{Q(\xi + \eta) | \xi + \eta \in \Omega_+ \text{ with } \xi \in L\}$).

9. We let Q have signature $(a, 1)$ (of *Lorentz* type). Then let L be a Q integral lattice in $\mathbf{R}^k$ $(k = a + 1)$. Let $r \in X^- = \{Q(\xi, \xi) \in \Omega_- | \xi \in L\}$. The group $\Gamma^L(Q)$ operates on the set $L \cap \{X \in \mathbf{R}^k | Q(X, X) = r\}$ and we know that there exist finitely many orbits ("finiteness of class number"). Then if $X_1, \ldots, X_t$ form a set of representatives of $\Gamma^L(Q)$ orbits we know that $\Gamma^L(Q)^{X_i}$ is a finite group. We let $\varepsilon_{X_i} = \operatorname{order}(\Gamma^L(Q)^{X_i})$. Then the *Siegel mass number* is

$$M(Q, L, r) = \sum_{i=1}^{t} \frac{1}{\varepsilon_{X_i}}.$$

We note that this number is independent of the choice of representatives $X_1, \ldots, X_t$ above.

10. If L is a Type II lattice then for each $(\Omega, \gamma) \in \psi^{-1}(SL_2(\mathbf{Z})) \times \Gamma^L(Q)$ we know that there is a unique $d_L \times d_L$ unitary matrix $\{\sigma^L_{\eta_i\eta_j}(\ ,\ )\}$ so that

$$T^{L}_{\varphi,\eta_i}(G\Omega, g\gamma) = \sum_{j=1}^{j=d_L} \sigma^L_{\eta_i\eta_j}(\Omega, \gamma) T^{L}_{\varphi,\eta_j}(G, g)$$

where $\{\eta_1, \ldots, \eta_{d_L}\}$ is a set determining the distinct coset representatives of $L_*(Q)/L$ (see Theorem 2.1 of **[II]**).

11. A function h on $\mathbf{R}^m$ satisfies the *Poisson Summation formula* relative to a lattice $L \subseteq \mathbf{R}^k$ if

(A) h is continuous and integrable (L') on $\mathbf{R}^k$,

(B) the series $F(X) = \sum_{\xi \in L} h(X + \xi)$ is absolutely convergent and defines a continuous function on $\mathbf{R}^m$.

(C) the series $\sum_{\xi^* \in L^*} \hat{h}(\xi^*)$ is absolutely convergent with $L^* = \{\xi^* \in \mathbf{R}^k | [\xi^*, L] \subseteq \mathbf{Z}\}$, where $[\ ,\ ]$ is the bilinear form on $\mathbf{R}^k$ given by $[\vec{X}, \vec{Y}] = \sum_{i=1}^{k} x_i y_i$ (with $\vec{X} = \sum x_i e_i$, $\vec{Y} = \sum y_i e_i$). Here ^ represents Fourier transform given by

$$\hat{h}(W) = \int_{\mathbf{R}^k} h(Z) e^{2\pi\sqrt{-1}\,[W,Z]}\, dZ.$$

Then if h satisfies (A), (B), and (C), we have the Poisson formula

$$\sum_{\xi \in L} h(X + \xi) = \sum_{\mu \in L^*} \hat{h}(\mu) e^{2\pi\sqrt{-1}\,[X,\mu]}.$$

1. Shimura correspondence. Our problem is to show the relationship of **[II]** to the results of Shimura reported in **[10]**. The connection will be to view $T^L_{\varphi,\eta}$ given in §0 as a certain "kernel operator" transforming automorphic forms on $\widetilde{SL}_2$ to automorphic forms on $O(Q)$. We make this precise in the next several paragraphs.

For this we must extend the definition of $T^L_{\varphi,\eta}$. We assume that $\varphi \in E_{\mathfrak{M}}(s^2 - 2s, s, s_1, s_2)$ ($E_{\mathfrak{M}}(s^2 + 2s, s, s_1, s_2)$, resp.) with $|s| > \frac{1}{2}k$. Moreover $\eta \in L_*(Q)$, the Q dual lattice to L. We let $J_L(\eta) =$ the group generated by η in $L_*(Q)/L$. We let

$$\Gamma^L(Q)^*_\eta = \{\gamma \in \Gamma^L(Q) | \gamma\eta \equiv a(\gamma)\eta \bmod L\}$$

(with $a(\gamma) \in \mathbf{Z}$). Then $\Gamma^L(Q)^*_\eta \supseteq \Gamma^L(Q)_\eta$ (defined in §0), and we have a representation of $\Gamma^L(Q)^*_\eta$ on $J_L(\eta)$ given by $\gamma\eta \equiv a(\gamma)\eta \bmod L$ (where the kernel of the representation is clearly $\Gamma^L(Q)_\eta$).

LEMMA 1-1. *Let χ be a multiplicative Dirichlet character on $\mathbf{Z}/O(J_L(\eta)) \cdot \mathbf{Z}$ (with $O(J_L(\eta)) =$ order of $J_L(\eta)$). Let $\Omega \in \tilde{\Delta}_{N_L}$ and $\gamma \in \Gamma^L(Q)^*_\eta$. Then if φ satisfies the hypotheses in the above paragraph, we have*

$$\sum_{r \in \mathbf{Z}/O(J_L(\eta))\mathbf{Z}} \chi(r) T^L_{\varphi,r\eta}(G\Omega, g\gamma) = c^L_Q(\Omega)(\chi(\delta a(\gamma)))^{-1} \cdot \sum_{u \in \mathbf{Z}/O(J_L(\eta))\mathbf{Z}} e^{-\pi\sqrt{-1}\,\alpha\beta u^2 a(\gamma^{-1})^2 Q(\eta,\eta)} \chi(u) T^L_{\varphi,u\eta}(G, g). \quad (1\text{-}1)$$

PROOF. The proof follows directly from subsection 8 of §0 and the multiplicative properties of χ. Q.E.D.

Then we let

$$\Delta^\eta_{N_L} \times \Gamma^L(Q)'_\eta = \Big\{(\Omega, \gamma) \in \tilde{\Delta}_{N_L} \times \Gamma^L(Q)^*_\eta \Big| \exp\big(-\pi\sqrt{-1}\,\alpha\beta v^2 a(\gamma^{-1})^2 Q(\eta,\eta)\big) = 1 \text{ for all } v = 1, \dots, O(J_L(\eta)) - 1 \text{ and } \gamma \in \Gamma^L(Q)^*_\eta\Big\}.$$

It follows from Lemma 1-1 that if $(\Omega', \gamma') \in \Delta^\eta_{N_L} \times \Gamma^L(Q)'_\eta$ and if

$$T^L_{\varphi,\eta,\chi}(G, g) = \sum_{r \in \mathbf{Z}/O(J_L(\eta))\mathbf{Z}} \chi(r) T^L_{\varphi,r\eta}(G, g),$$

then

$$T^L_{\varphi,\eta,\chi}(G\Omega, g\gamma) = c^L_Q(\Omega)\big[\chi(\delta a(\gamma))\big]^{-1} T^L_{\varphi,\eta,\chi}(G, g). \quad (1\text{-}2)$$

We note that $\Delta^\eta_{N_L} \times \Gamma^L(Q)'_\eta$ is the maximal subgroup of $\Delta_{N_L} \times \Gamma^L(Q)^*_\eta$ for which (1-2) is valid.

REMARK 1-1. If L is a Type II lattice then Lemma 1-1 is valid for $\Omega \in \Delta_{N_L}$ provided we have the additional hypothesis that $O(J_L(\eta))$ is even. It is then understood in the definition of $\Delta^\eta_{N_L} \times \Gamma^L(Q)'_\eta$ above that $\tilde{\Delta}_{N_L}$ is replaced by Δ_{N_L}.

EXAMPLE. We let $\mathbf{R}^3 = \{(x_1, x_2, x_3) | x_i \in \mathbf{R}\}$ and $Q_N(X, X) = 2[x_2^2 - 4x_1x_3]/N$. Then we have the example of Niwa in **[8]**. In particular let

$$L = \{(4Nm_1, Nm_2, Nm_3/4) | m_1, m_2, m_3 \in \mathbf{Z}\}.$$

Then L is a Type II lattice relative to Q_N, and we have

$$L_*(Q_N) = \left\{\left(\frac{1}{2}m_1', \frac{1}{4}m_2', \frac{1}{32}m_3'\right) \middle| m_i' \in \mathbf{Z}\right\}.$$

Then $N_L = 8N$. If we choose $\eta = (1, 0, 0)$ as in **[8]**, then $J_L(\eta)$ is a cyclic group of order $4N$. Moreover since $Q_N(\eta, \eta) = 0$, we see that $\Delta^{\eta}_{N_L} = \Delta_{N_L}$ in this case (recall L is Type II and the convention stated in Remark 1-1).

REMARK 1-2. We deduce from (1-2) and arguments similar to those in Lemma 3.3 of **[II]** that the map

$$x \underset{s^L_{Q,\chi}}{\rightsquigarrow} s^L_Q(x)(\chi(d)) \quad \text{for } x = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \psi(\Delta^{\eta}_{N_l})$$

is a projective representation of weight $\frac{1}{2}k$ for $\psi(\Delta^{\eta}_{N_L})$. Moreover we note that in the z variable $\check{T}^L_{\varphi,\eta,\chi}(z, g) \in [\psi(\Delta^{\eta}_{N_L}), |s|, s^L_{Q,\chi}]_0$ by arguments similar to those in Proposition 3.4 of **[II]**.

Then for any arithmetic subgroup Γ_1 in $SL_2(\mathbf{Z})$ we denote by $\mathfrak{D}_{\Gamma_1}$ the fundamental domain of Γ_1 in H.

We say that the pair (Γ_1, β), with Γ_1 an arithmetic subgroup of $SL_2(\mathbf{Z})$ and β a projective representation of Γ_1, has *the* (L, η) ((L, η, χ) *resp.*) *property* if

(i) $\Gamma_1 \subseteq \psi(\tilde{U}_{N_L})$ ($\Gamma_1 \subseteq \psi(\Delta^{\eta}_{N_L})$ resp.)

(ii) $\beta = s_Q^{L,\eta}$ restricted to Γ_1 ($\beta = s^L_{Q,\chi}$ restricted to Γ_1).

Then we let $f \in [\Gamma_1, |s|, \beta]$ where (Γ_1, β) *has the* (L, η, χ) *property*. We define

$$F_f(g|\varphi, L, \eta, \chi, \Gamma_1) = \int_{\mathfrak{D}_{\Gamma_1}} \check{T}^L_{\varphi,\eta,\chi}(z, g)\overline{f(z)}\,(\mathrm{Im}\, z)^{|s|-2}\, dx\, dy. \tag{1-3}$$

We note that (1-3) represents the *inner product* of $f(\ )$ and $\check{T}^L_{\varphi,\eta,\chi}(\ , g)$ in the *Petersson metric* on $\mathfrak{D}_{\Gamma_1}$ (note that $\check{T}^L_{\varphi,\eta,\chi}$ is a cusp form but f is only an integral form).

We note immediately from Lemma 1-1 and (1-2) that

$$F_f(g\gamma|\varphi, L, \eta, \chi, \Gamma_1) = (\chi(a(\gamma)))^{-1}F_f(g|\varphi, L, \eta, \chi, \Gamma_1) \tag{1-4}$$

for *all* $\gamma \in \Gamma^L(Q)'_{\eta}$.

We call the correspondence given in (1-3) the *abstract Shimura correspondence*. We shall see shortly that our notation is well chosen in that the correspondence given in **[8]** will be a special case of our theory.

Our problem is to make a thorough study of the map (1-3). For this we examine what happens if f is an *Eisenstein-Poincaré series* for Γ_1.

We recall that for an arbitrary arithmetic subgroup $\Gamma \subseteq SL_2(\mathbf{Z})$, an Eisenstein-Poincaré series given by

$$G_d(\tau, \beta, \Gamma, m) = \sum_{\gamma \in \Gamma \cap P \backslash \Gamma} \exp\left(2\pi\sqrt{-1}\,\frac{(m+\kappa)}{N_\Gamma}\,\gamma(\tau)\right)\overline{\beta(\gamma)}\left(\frac{1}{c_\gamma \tau + d_\gamma}\right)^d \tag{1-5}$$

defines an element of $[\Gamma, d, \beta]_0$, where $m \geqslant 1$,

$$\gamma = \begin{bmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{bmatrix},$$

and $d > 2$. Here

$$e^{2\pi\sqrt{-1}\,\kappa} = \beta\left(\begin{pmatrix} 1 & N_\Gamma \\ 0 & 1 \end{pmatrix}\right)$$

and

$$P = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \in SL_2(\mathbf{R}) | a \in \mathbf{R}^*, b \in \mathbf{R} \right\}.$$

Moreover we know that $\{G_d(\tau, \beta, \Gamma, m) | m \geqslant 1\}$ forms a spanning set of the finite-dimensional vector space $[\Gamma, d, \beta]_0$.

We now interpret the functions in (1-5) in terms of the group $\widetilde{SL}_2$ and the representation $\pi_{\mathfrak{M}}$. Again we assume that

$$\varphi \in E_{\mathfrak{M}}(s^2 - 2s, s, s_1, s_2) \quad \left(E_{\mathfrak{M}}(s^2 + 2s, s, s_1, s_2) \text{ resp.}\right),$$

where $|s| > \frac{1}{2}k$.

Let Γ be an arithmetic subgroup of $SL_2(\mathbf{Z})$ and ρ_Γ some unitary character on $\psi^{-1}(\Gamma)$ *with the compatibility condition* $\pi_{\mathfrak{M}}(M)^{-1}(f)(X) = \rho_\Gamma(M)^{-1} f(X)$ for *all* $M \in \psi^{-1}(\Gamma) \cap \tilde{P}$ and for each $f \in S(\mathbf{R}^k)$. Then we define

$$U^L_{\varphi,X}(G, \Gamma, \rho_\Gamma) = \sum_{M \in \psi^{-1}/\psi^{-1}(\Gamma) \cap \tilde{P}} \pi_{\mathfrak{M}}(GM)^{-1}(\varphi)(X)\rho_\Gamma(M)^{-1} \tag{1-6}$$

(where $\tilde{P} = \psi^{-1}(P)$ and

$$M = \left(\begin{pmatrix} a_M & b_M \\ c_M & d_M \end{pmatrix}, \varepsilon\right).$$

We show in the next lemma the relationship between $U^L_{\varphi,X}$ and the series defined in (1-5).

LEMMA 1-2. *Let* (Γ, β) *have the* (L, η) *or* (L, η, χ) *property. Let*

$$\varphi \in E_{\mathfrak{M}}(s^2 - 2s, s, s_1, s_2) \quad \left(E_{\mathfrak{M}}(s^2 + 2s, s, s_1, s_2) \textit{ resp.}\right)$$

be given by

$$\begin{cases} Q(X, X)^{s-1} e^{-\pi Q(X,X)} \|X_+\|^{-(s+k/2+s_1+s_2-2)} P_{s_1}(X_+) P_{s_2}(X_-) \textit{ on } \Omega_+, \\ \left(Q(X, X)^{|s|-1} e^{\pi Q(X,X)} (\|X_-\|)^{-(|s|+k/2+s_1+s_2-2)} \tilde{P}_{s_1}(X_+) \tilde{P}_{s_2}(X_-) \textit{ on } \Omega_- \textit{ resp.}\right), \end{cases} \tag{1-7}$$

where P_{s_1} *and* P_{s_2} ($\tilde{P}_{s_1}$ *and* $\tilde{P}_{s_2}$ *resp.*) *are harmonic polynomials of degree* s_1 *and* s_2 *in* $\mathbf{R}^a$ *and* $\mathbf{R}^b$. *Then let* $Z \in \Omega_+$ (Ω_- *resp.*) *so that*

$$Q(Z, Z) = 2(n + \kappa)/N_\Gamma > 0 \quad (2(n + \kappa)/N_\Gamma < 0 \textit{ resp.}).$$

We have

$$U^L_{\varphi,Z}\left(\left(\begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix}, 1\right), \Gamma, \rho_\Gamma\right) = \left|\frac{2(n + \kappa)}{N_\Gamma}\right|^{|s|-1}$$

$$\begin{cases} (\operatorname{Im} z)^{|s|/2} G_s(z, \beta, \Gamma, n)(\|Z_+\|)^{-(s+k/2+s_1+s_2-2)} P_{s_1}(Z_+) P_{s_2}(Z_-), \\ (\operatorname{Im} \bar{z})^{|s|/2} G_s(\bar{z}, \beta, \Gamma, n)(\|Z_-\|)^{-(|s|+k/2+s_1+s_2-2)} P_{s_1}(Z_+) P_{s_2}(Z_-), \end{cases} \tag{1-8}$$

where $z = -y/x + x^{-2}\sqrt{-1}$ *and where* ρ_Γ *is the homomorphism of* $\psi^{-1}(\Gamma)$ *given by*

$$\rho_\Gamma(\Omega) = \begin{cases} c_Q^L(\Omega)e^{-\pi\sqrt{-1}\,\delta\beta Q(\eta,\eta)} & \textit{if } (\Gamma, \beta) \textit{ satisfies } (L, \eta) \textit{ condition}, \\ c_Q^L(\Omega)\chi(\delta)^{-1} & \textit{if } (\Gamma, \beta) \textit{ satisfies } (L, \eta, \chi) \textit{ condition}, \end{cases} \quad (1\text{-}9)$$

where

$$\Omega = \left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \varepsilon\right).$$

PROOF. We let $\beta_\varphi(G) = \pi_{\mathfrak{M}}(G)(\varphi)(Z)$. Then by hypothesis on φ we have $\beta_\varphi(Gk(\theta, \varepsilon)) = e^{\sqrt{-1}\,s\theta}(\operatorname{sgn}\varepsilon)^k\beta_\varphi(G)$. Then it follows from Remark 3.1 of **[II]** that (with $z = x^{-2}\sqrt{-1} - y/x$)

$$\tilde{\beta}_\varphi\left(\left(\begin{bmatrix} x^{-1} & -y \\ 0 & x \end{bmatrix}, 1\right)\cdot\sqrt{-1}\right)$$
$$= a_\varphi Q(Z, Z)^{s-1}e^{\pi\sqrt{-1}\,zQ(Z,Z)}\|Z_+\|^{-(s+k/2+s_1+s_2-2)}P_{s_1}(Z_+)P_{s_2}(Z_-).$$

Hence again by Remark 3.1 of **[II]**, for any $(M, 1) \in \widetilde{SL}_2$ with $M = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$, we have

$$\tilde{\beta}_\varphi\left(\frac{dz - b}{-cz + a}\right)(-cz + a)^{-s}(\psi_2(M))^{2s}(\operatorname{Im} z)^{|s|/2} = \beta_\varphi\left(\left(\begin{bmatrix} x & y \\ 0 & x^{-1} \end{bmatrix}M, 1\right)^{-1}\right). \quad (1.10)$$

Thus we have from (1-10)

$$U^L_{\varphi,Z}\left(\left(\begin{bmatrix} x & y \\ 0 & x^{-1} \end{bmatrix}, 1\right), \Gamma, \rho_\Gamma\right) = (\operatorname{Im} z)^{s/2}\left|\frac{2(n + \kappa)}{N_\Gamma}\right|^{s-1}$$
$$\cdot\|Z_+\|^{-(s+k/2+s_1+s_2-2)}P_{s_1}(Z_+)P_{s_2}(Z_-)\sum_{M\in\Gamma/\Gamma\cap P}(-c_Mz + a_M)^{-s}$$
$$\cdot\exp\left(\pi\sqrt{-1}\,M^{-1}(z)\frac{2(n + \kappa)}{N_\Gamma}\right)$$
$$\cdot c_Q^L((M, 1))^{-1}[\psi_2(M)]^k\exp\left(\pi\sqrt{-1}\,b_Md_MQ(\eta, \eta)\right)$$

for

$$M = \begin{bmatrix} a_M & b_M \\ c_M & d_M \end{bmatrix} \in \Gamma.$$

We note that $\Gamma/\Gamma \cap P \cong \psi^{-1}(\Gamma)/\psi^{-1}(\Gamma) \cap \tilde{P}$. Moreover the inverse map $X \to X^{-1}$ carries $\Gamma/\Gamma \cap P$ onto $\Gamma \cap P \setminus \Gamma$. Finally we note that since (Γ, β) has the (L, η) property, then

$$\beta(M) = s_Q^{L,\eta}(M) = c_Q^L((M, 1))^{-1}[\psi_2(M)]^ke^{\pi\sqrt{-1}\,b_Md_MQ(\eta,\eta)}$$

by definition. Q E.D.

Then given this interpretation of $U^L_{\varphi_1,Z}(\ , \Gamma, \rho_\Gamma)$ as "Eisenstein-Poincaré" series (φ_1 as in Lemma 1-2), we can compute the inner product (on $\widetilde{SL}_2/\psi^{-1}(\Gamma)$) of $U^L_{\varphi_1,Z}(\ , \Gamma, \rho_\Gamma)$ with $T^L_{\varphi,\eta,\chi}$, where φ is a $\tilde{K} \times K$ finite function in

$$E_{\mathfrak{M}}(s^2 - 2s, s, s_1, s_2) \quad (E_{\mathfrak{M}}(s^2 + 2s, s, s_1, s_2) \text{ resp.})$$

when $s > \frac{1}{2}k$ ($s < -\frac{1}{2}k$ resp.). We let $d\sigma$ be an $\widetilde{SL}_2$ invariant measure on $\widetilde{SL}_2/\psi^{-1}(\Gamma)$.

LEMMA 1-3. *Let φ_1 be given by* (1-7) *in Lemma* 1-2. *Let* (Γ, β_Γ) *satisfy the* (L, η, χ) *property. Then*

$$\beta_\Gamma\begin{pmatrix} 1 & N_\Gamma \\ 0 & 1 \end{pmatrix} = 1.$$

Let $Z \in \Omega_+$ (Ω_- *resp.*) *so that* $Q(Z, Z) = 2n/N_\Gamma > 0$ ($Q(Z, Z) = 2n/N_\Gamma < 0$ *resp.*).

Then there exists a suitably normalized $\widetilde{SL}_2$ *invariant measure* $d\mu_0$ *on* $\widetilde{SL}_2/N \cdot \mathfrak{Z}$ *with* $\mathfrak{Z} = \operatorname{Center}(\widetilde{SL}_2)$ *so that*

$$\int_{\widetilde{SL}_2/\psi^{-1}(\Gamma)} U^L_{\varphi_1,Z}(G, \Gamma, \rho_\Gamma)\overline{T^L_{\varphi,\eta,\chi}(G, g)}\, d\sigma(G)$$
$$= \sum_{r \in \mathbf{Z}/O(J_L(\eta))\mathbf{Z}} \chi(r)\Bigg\{ \sum_{\{\xi \in L \mid Q(\xi + r\eta, \xi + r\eta) = 2n/N_\Gamma\}} \cdot \int_{\widetilde{SL}_2/\mathfrak{Z}\cdot N} \overline{\pi_{\mathfrak{M}}((G, g))^{-1}(\varphi)(\xi + r\eta)}$$
$$\cdot \pi_{\mathfrak{M}}(G)^{-1}(\varphi_1)(Z)\, d\mu_0(G)\Bigg\}, \tag{1-11}$$

where ρ_Γ *is given by* (1-9).

PROOF. By definition of $U^L_{\varphi,Z}(\ , \Gamma, \rho_\Gamma)$ we deduce (using essentially the Rankin convolution trick) that

$$\int_{\widetilde{SL}_2/\psi^{-1}(\Gamma)} U^L_{\varphi_1,Z}(G, \Gamma, \rho_\Gamma)\overline{T^L_{\varphi,\eta,\chi}(G, g)}\, d\sigma(G)$$
$$= \int_{SL_2/\psi^{-1}(\Gamma)} \Bigg\{ \sum_{M \in \psi^{-1}(\Gamma)/\psi^{-1}(\Gamma) \cap \tilde{P}} \pi_{\mathfrak{M}}(GM)^{-1}(\varphi_1)(Z) c^L_Q(M)^{-1}\chi(d_M)\Bigg\} \cdot \overline{T^L_{\varphi,\eta,\chi}(G, g)}\, d\sigma(G)$$
$$= \int_{\widetilde{SL}_2/\psi^{-1}(\Gamma)\cap\tilde{P}} \pi_{\mathfrak{M}}(G)^{-1}(\varphi_1)(Z)\overline{T^L_{\varphi,\eta,\chi}(G, g)}\, d\tilde{\sigma}(G), \tag{1-12}$$

where $d\tilde{\sigma}$ is an $\widetilde{SL}_2$ invariant measure on $\widetilde{SL}_2/\psi^{-1}(\Gamma) \cap \tilde{P}$. But we observe that $\psi^{-1}(\Gamma) \cap \tilde{P} = \mathfrak{Z} \cdot \{n(k \cdot N_\Gamma) | k \in \mathbf{Z}\}$. Thus

$$\widetilde{SL}_2/\psi^{-1}(\Gamma) \cap \tilde{P} = \widetilde{SL}_2/\mathfrak{Z} \cdot \{n(kN_\Gamma) | k \in \mathbf{Z}\}$$

and we have that

$$\int_{\widetilde{SL}_2/\psi^{-1}(\Gamma)\cap\tilde{P}} \pi_{\mathfrak{M}}(G)^{-1}(\varphi_1)(Z)\overline{T^L_{\varphi,\eta,\chi}(G, g)}\, d\tilde{\sigma}(G)$$
$$= \int_{\widetilde{SL}_2/\mathfrak{Z}\cdot N}\Bigg\{\int_0^{N_\Gamma} \overline{T^L_{\varphi,\eta,\chi}(Gn(x), g)} \exp\left(-2\pi\sqrt{-1}\,\frac{n}{N_\Gamma}x\right) dx\Bigg\}$$
$$\cdot \pi_{\mathfrak{M}}(G^{-1})(\varphi_1)(Z)\, d\mu_0(G). \tag{1-13}$$

But in the inner integral of (1-13) we can switch the order of integration and summation (by using the estimates in (1-15) of **[II]**:

$$\int_0^{N_\Gamma} \overline{T^L_{\varphi,\eta,\chi}(Gn(x), g)} \exp\left(-2\pi\sqrt{-1}\,\frac{nx}{N_\Gamma}\right) dx$$
$$= \sum_{r\in(\mathbf{Z}/O(J_L(\eta))\mathbf{Z})} \chi(r) \sum_{\{\xi|\xi\in L\}} \overline{\pi_{\mathfrak{M}}((G,g))^{-1}(\varphi)(\xi + r\eta)}$$
$$\cdot \left\{ \int_0^{N_\Gamma} \exp\left(\pi\sqrt{-1}\,x\left[Q(\xi + r\eta, \xi + r\eta) - \frac{2n}{N_\Gamma}\right]\right) dx \right\}. \tag{1-14}$$

But then since (Γ, β) has the (L, η, χ) property, we deduce that $\Gamma \subset \psi(\Delta^{\eta}_{N_L})$, and then using the paragraphs following Lemma 1-1, we have $e^{\pi\sqrt{-1}\,r^2N_\Gamma Q(\eta,\eta)} = 1$. Thus

$$\int_0^{N_\Gamma} \overline{T^L_{\varphi,\eta,\chi}(Gn(x), g)} \exp\left(-\pi\sqrt{-1}\,x\frac{2n}{N_\Gamma}\right) dx$$
$$= \sum_{r\in\mathbf{Z}/O(J_L(n))\mathbf{Z}} \chi(r) \left\{ \sum_{\{\xi\in L|Q(\xi+r\eta,\xi+r\eta)=2n/N_\Gamma\}} \overline{\pi_{\mathfrak{M}}((G,g))^{-1}(\varphi)(\xi + r\eta)} \right\}. \tag{1-15}$$

But then, by adapting the arguments in the proof of Lemma 1.4 of **[II]**,

$$|f(r\cdot X)| \leqslant M' r^{s-k/2} Q(X,X)^{s-1} \exp\left(-\tfrac{1}{2}\pi r^2 Q(X,X)\right)\left(\frac{1}{\|X_+\|}\right)^{s+k/2-2}$$

for all $X \in \mathbf{R}^k$ (where $f = \varphi$ or φ_1) and M' some positive constant independent of X.

Thus we have the estimates

$$\int_{\widetilde{SL}_2/N\cdot 3} |\pi_{\mathfrak{M}}(G^{-1})(\varphi_1)(Z)|$$
$$\cdot \left\{ \sum_{\{\xi\in L|Q(\xi+r\eta,\xi+r\eta)=2n/N_\Gamma\}} |\pi_{\mathfrak{M}}((G,g))^{-1}(\varphi)(\xi + r\eta)| \right\} d\mu_0(G)$$
$$\leqslant M' \|g^{-1}\|^{s+k/2-2} \left\{ \int_0^{+\infty} |r^{-s}e^{-\beta/r^2}|^2 r\, dr \right\}$$
$$\cdot \left\{ \sum_{\{\xi\in L|Q(\xi+r\eta,\xi+r\eta)=2n/N_\Gamma\}} \left(\frac{1}{\|(\xi + r\eta)_+\|}\right)^{s+k/2-2} \right\} \tag{1-16}$$

(here we have used the fact $d\mu_0(G) \cong d\tilde{K} \otimes r dr$ in the Iwasawa deomposition of $\widetilde{SL}_2$). But we know from the arguments used in the proof of Theorem 1.5 of **[II]** that the series on the right-hand side of (1-16) is absolutely convergent. Moreover the integral on the right-hand side of (1-16) is clearly convergent. Thus we can switch the order of integration and summation in (1-13) and deduce the desired result. Q.E.D.

COROLLARY 1 TO LEMMA 1-3. *Let $\varphi = \varphi_1$ in Lemma 1-3. Then with the same hypotheses in effect as in Lemma 1-3,*

$$\int_{\widetilde{SL}_2/\psi^{-1}(\Gamma)} U^L_{\varphi,Z}(G, \Gamma, \rho_\Gamma)\, \overline{T^L_{\varphi,\eta,\chi}(G, g)}\, d\sigma(G)$$
$$= c_1 \cdot Q(Z, Z)^{s-1} \|Z_+\|^{-(k/2+s+s_1+s_2-2)} P_{s_1}(Z_+) P_{s_2}(Z_-) \sum_{r \in \mathbf{Z}/O(J_L(\eta))\mathbf{Z}} \chi(r)$$
$$\cdot \Bigg\{ \sum_{\{\xi \in L \mid Q(\xi + r\eta, \xi + r\eta) = 2n/N_\Gamma\}} \| g(\xi + r\eta)_+ \|^{-(k/2+s+s_1+s_2-2)}$$
$$\cdot \overline{P_{s_1}(g(\xi + r\eta)_+)}\ \overline{P_{s_2}(g(\xi + r\eta)_-)} \Bigg\}, \tag{1-17}$$

with c_1 a positive constant which depends only on the normalization of certain measures and the number s.

PROOF. If we let $d\mu_0(G) = d\tilde{K} \otimes r\,dr$, then, using (1-11),

$$c(s) = c_1 \left| \frac{2n}{N_\Gamma} \right|^{2s-2} \left\{ \int_0^\infty \exp\left(-\frac{1}{r^2} \frac{4\pi n}{N_\Gamma} \right) r^{1-2s}\, dr \right\},$$

where c_1 arises from the normalization of the measures above in (1-13). Q.E.D.

REMARK 1-3. Using the notation of (3-20) of **[II]** we see from Corollary 1 to Lemma 1-3 that (with c_1' a nonzero constant)

$$\int_{\widetilde{SL}_2/\psi^{-1}(\Gamma)} U^L_{\varphi,Z}(G, \Gamma, \rho_\Gamma)\, \overline{T^L_{\varphi,\eta,\chi}(G, g)}\, d\sigma(G)$$
$$= c_1' e^{2\pi n/N_\Gamma} \varphi(Z) \sum_{r \in \mathbf{Z}/O(J_L(\eta))\mathbf{Z}} \chi(r)\, T^{L,2n/N_\Gamma}_{\bar{\varphi},r\eta}(g)$$
$$= c_1' e^{2\pi n/N_\Gamma} \varphi(Z)\, T^{L,2n/N_\Gamma}_{\bar{\varphi},\eta,\chi}(g). \tag{1-18}$$

The basic idea in the work of Niwa **[8]** in explicating the Shimura correspondence is to compute the inner product of $U^L_{\varphi,Z}(\ ,\Gamma,\rho_\Gamma)$ (with φ as in Lemma 1-3) with $T^L_{f,\eta,\chi}$ in $L^2(\widetilde{SL}_2/\psi^{-1}(\Gamma))$ *when f is a Schwartz function* on $\mathbf{R}^k$ which transforms according to a unitary character of $\tilde{K}$. In Lemma 1-3 we computed such an inner product when $f \in E_{\mathfrak{M}}(s^2 - 2s, s, s_1, s_2)$. Our next problem is to give a precise relationship between Lemma 1-3 and the main theorem of **[8]**.

Let P_s^+ (P_s^- resp.) be the $\tilde{G}_2(Q)$ intertwining projection map of $\mathbf{F}_Q$ into $\mathbf{F}_Q^+(s^2 - 2s)$ ($\mathbf{F}_Q^-(s^2 + 2s)$ resp.). In particular P_s^+ is obtained by taking the orthogonal projection of $L^2(\mathbf{R}^k)$ to $\mathrm{cls}(\mathbf{F}_Q^+(s^2 - 2s))$, i.e., with $\mathrm{cls}(\mathbf{F}_Q^+(s^2 - 2s))$, the closure of $\mathbf{F}_Q^+(s^2 - 2s)$ in $L^2(\mathbf{R}^k)$.

We let

$$D = \left[\bigoplus_{s>1} \mathbf{F}_Q^+(s^2 - 2s) \right] \oplus \left[\bigoplus_{|s|>1} \mathbf{F}_Q^-(s^2 + 2s) \right],$$

where $s \equiv \frac{1}{2}k \bmod 1$.

Then D is a $\tilde{G}_2(Q)$ stable subspace in $\mathbf{F}_Q$.

We set $\mathbf{F}_Q^m = \{\varphi \in \mathbf{F}_Q | \pi_{\mathfrak{M}}(k(\theta, \varepsilon))\varphi = e^{\sqrt{-1}\, m\theta}(\operatorname{sgn} \varepsilon)^{2m}\varphi\}$ with $m \in \frac{1}{2}\mathbf{Z}$. Then from §0 (see Theorem 1.3 of **[II]**) we deduce that $\mathbf{F}_Q^m \cap D = 0$ if $m = 0$ or $m = \pm\frac{1}{2}$.

LEMMA 1-4. *Let $m = 0$ or $m = \pm\frac{1}{2}$. Let $\mathfrak{B}$ be the closed subspace in $\mathbf{F}_Q$ generated by all linear translates of $\pi_{\mathfrak{M}}(G)\varphi$ as G varies in $\widetilde{SL}_2$ and φ varies in $\mathbf{F}_Q^m$. Then $\mathfrak{B}$ is a $\tilde{G}_2(Q)$ stable module in $\mathbf{F}_Q$ (relative to $\pi_{\mathfrak{M}}$) and $\mathbf{F}_Q = \mathfrak{B} \oplus D$.*

PROOF. It suffices to show in $L^2(\mathbf{R}^k)$ that $L^2(\mathbf{R}^k) = \operatorname{cls}(\mathfrak{B}) \oplus \operatorname{cls}(D)$ is an orthogonal splitting. The splitting is orthogonal; for if $\psi \in \mathbf{F}_Q^m$ and $\varphi \in D$, then $(\pi_{\mathfrak{M}}(G)\psi|\varphi)_{L^2} = (\psi|\pi_{\mathfrak{M}}(G)^{-1}\varphi)_{L^2} = 0$ for $G \in \widetilde{SL}_2$. Then let H be the $\pi_{\mathfrak{M}}(\widetilde{SL}_2)$ stable subspace in $L^2(\mathbf{R}^k)$ so that $L^2(\mathbf{R}^k) = \operatorname{cls}(\mathfrak{B}) \oplus \operatorname{cls}(D) \oplus H$ with $H = [\operatorname{cls}(\mathfrak{B}) \oplus \operatorname{cls}(D)]^{\perp}$. Then if $H_\infty \neq 0$, there exists an integer w so that $H_\infty \cap \mathbf{F}_Q^{w/2} \neq (0)$. This implies that there exists an integer q so that $\pi_{\mathfrak{M}}(N_+^q)(\psi) \in \mathbf{F}_Q^m$ ($m = 0$ or $m = \pm\frac{1}{2}$) for $\psi \in H_\infty$. Hence if $\pi_{\mathfrak{M}}(N_+^q)(\psi) \neq 0$, then $H_\infty \cap \mathfrak{B} \neq 0$, a contradiction. This means that $\pi_{\mathfrak{M}}(N_+^q)(\psi) = 0$. Assuming that $\pi_{\mathfrak{M}}(N_+)^{q-1}(\psi) \neq 0$, this implies that $\pi_{\mathfrak{M}}(N_+^{q-1})(\psi)$ is an $\widetilde{SL}_2$ extreme vector in $\mathbf{F}_Q$. Thus $\pi_{\mathfrak{M}}(N_+^{q-1})\psi \in D$, which again is a contradiction. Q.E.D.

We recall from §5 of **[I]** that $S(\mathbf{R}^k)$ is dense in $\mathbf{F}_Q$ (in the $\mathbf{F}_Q$ topology). Hence it follows that $S(\mathbf{R}^k) \cap \mathbf{F}_Q^M$ is dense in $\mathbf{F}_Q^m$ for all $m \in \frac{1}{2}\mathbf{Z}$.

THEOREM 1-1. *Let $\varphi \in E_{\mathfrak{M}}(s^2 - 2s, s, s_1, s_2)$ ($E_{\mathfrak{M}}(s^2 + 2s, s, s_1, s_2)$ resp.) with $|s| > \frac{1}{2}k + 1$ be given by* (1-7). *Let (Γ, β) satisfy the (L, η, χ) condition (with ρ_Γ given by* (1-9)). *Let $Z \in \Omega_+$ (Ω_- resp.) so that $Q(Z, Z) = 2n/N_\Gamma > 0$ ($Q(Z, Z) = 2n/N_\Gamma < 0$ resp.). Let $f \in S(\mathbf{R}^k) \cap \mathbf{F}_Q^s$ and K finite. Then*

$$\int_{\widetilde{SL}_2/\psi^{-1}(\Gamma)} U_{\varphi,Z}^L(G, \Gamma, \rho_\Gamma)\, \overline{T_{f,\eta,\chi}^L(G, g)}\, d\sigma(G)$$
$$= \int_{\widetilde{SL}_2/\psi^{-1}(\Gamma)} U_{\varphi,Z}^L(G, \Gamma, \rho_\Gamma)\, \overline{T_{P_s^+(f),\eta,\chi}^L(G, g)}\, d\sigma(G). \tag{1-19}$$

PROOF. By using the fact that $f \in S(\mathbf{R}^k)$, an easy adaptation of the argument of Lemma 1-3 shows that (for $s > \frac{1}{2}k + 1$)

$$\int_{\widetilde{SL}_2/\psi^{-1}(\Gamma)} U_{\varphi,Z}^L(G, \Gamma, \rho_\Gamma)\, \overline{T_{f,\eta,\chi}^L(G, g)}\, d\sigma(G)$$
$$= \sum_{r \in \mathbf{Z}/O(J_L(\eta))\mathbf{Z}} \chi(r) \Bigg\{ \sum_{\{\xi \in L | Q(\xi + r\eta, \xi + r\eta) = 2n/N_\Gamma\}} \int_{\widetilde{SL}_2/N3} \pi_{\mathfrak{M}}(G)^{-1}(\varphi)(Z)$$
$$\cdot\, \overline{\pi_{\mathfrak{M}}((G, g))^{-1}(f)(\xi + r\eta)}\, d\mu_0(G) \Bigg\}. \tag{1-20}$$

Note we can majorize $f(X)$ by $\|X_+\|^{1-k}$ and then use an argument similar to that in (1-16).

We may write

$$f = \sum_{\substack{u \equiv k/2 \\ u > 1}} P_u^+(f) + \sum_{\substack{u \equiv k/2 \\ u < -1}} P_u^-(f) + M_f,$$

where $M_f \in \mathcal{B}$. Then noting the $\tilde{K}$ eigenvalue behavior of $P_u^+(f)$ and $P_u^-(f)$ in Theorem 1.3 of **[II]**, we deduce that

$$\int_{\widetilde{SL}_2/N\cdot 3} \pi_{\mathfrak{M}}(G^{-1})(\varphi)(Z)\,\overline{\pi_{\mathfrak{M}}((G,g))^{-1}(h)(\xi + r\eta)}\; d\mu_0(G) \equiv 0, \tag{1-21}$$

when $h = P_u^+(f)$ with $u > s$ or $h = P_u^-(f)$ for all u. Thus it suffices to show that

$$\int_{\widetilde{SL}_2/N 3} \pi_{\mathfrak{M}}(G^{-1})(\varphi)(Z)\,\overline{\pi_{\mathfrak{M}}((G,g))^{-1}(M_f)(\xi + r\eta)}\; d\mu_0(G) \equiv 0. \tag{1-22}$$

But if $m = 0$ or $\pm\frac{1}{2}$, we note that $\pi_{\mathfrak{M}}(u(\widetilde{SL}_2))(\mathbf{F}_Q^m \cap S(\mathbf{R}^k))$ is dense in $\mathcal{B}$ (where $\mathfrak{U}(\widetilde{SL}_2)$ is the universal enveloping algebra of $\widetilde{SL}_2$). Hence it suffices to prove (1-22) where M_f is replaced by all $\pi_{\mathfrak{M}}(N_-^p)(\psi)$, where $\psi \in \mathbf{F}_Q^m$ and $p + m = s$. But then we note that

$$\pi_{\mathfrak{M}}(G^{-1})(\pi_{\mathfrak{M}}(Z)(\psi))(T) = Z * H_\psi(G, T), \tag{1-23}$$

where $H_\psi(G, T) = \pi_{\mathfrak{M}}(G^{-1})(\psi)(T)$ with $T \in \Omega_+$ and $*$ represents differentiation on the left in the variable G (for $Z \in \mathfrak{U}(\widetilde{SL}_2)$). Thus

$$\begin{aligned}
&\int_{\widetilde{SL}_2/\psi^{-1}(\Gamma)} \pi_{\mathfrak{M}}(G^{-1})(\varphi)(Z)\,\overline{N_-^p * \left\{\pi_{\mathfrak{M}}((G,g))^{-1}(\psi)(\xi + r\eta)\right\}}\; d\mu_0(G)\\
&= \int_{\widetilde{SL}_2/\psi^{-1}(\Gamma)} N_+^p * \left\{\pi_{\mathfrak{M}}(G^{-1})(\varphi)(Z)\right\}\,\overline{\pi_{\mathfrak{M}}((G,g))^{-1}(\psi)(\xi + r\eta)}\; d\mu_0(G)\\
&= \int_{\widetilde{SL}_2/\psi^{-1}(\Gamma)} \pi_{\mathfrak{M}}(G^{-1})(\pi_{\mathfrak{M}}(N_+)^p(\varphi))(Z)\,\overline{\pi_{\mathfrak{M}}((G,g))^{-1}(\psi)(\xi + r\eta)}\; d\mu_0(G).
\end{aligned} \tag{1-24}$$

But φ is an "extreme vector", which means that $\pi_{\mathfrak{M}}(N_+)(\varphi) \equiv 0$.

On the other hand, if $h = P_u^+(f)$ with $1 < u < s$, then we may assume that $h = (N_-)^p(h_1)$ with $p > 0$ and $h_1 \in E_{\mathfrak{M}}(u^2 - 2u, 2, s_1, s_2)$ for some s_1, s_2. By the same reasoning as above the result follows immediately. Q.E.D.

REMARK 1-4. We know that in the Weil representation $\pi_{\mathfrak{M}}^s$ of $\widetilde{SL}_2$ in $L^2(\mathbf{R}^m)$ relative to the positive definite quadratic form $\|\ \|^2$ on $\mathbf{R}^m$, $P_{s_1}(X)e^{-\pi\|X\|^2}$ (with P a harmonic polynomial on $\mathbf{R}^m$ of degree s_1 if $m \geqslant 2$ and

$$P(X) = e^{\pi x^2}(d/dx)^{s_1}e^{-\pi x^2},$$

a Hermite polynomial if $m = 1$) transforms according to the character of $\tilde{K}$: $k(\theta, \varepsilon) \rightsquigarrow e^{\sqrt{-1}\,(s_1 + m/2)\theta}(\operatorname{sgn}\varepsilon)^m$. Then it follows by the tensor product properties of the Weil representation (see §2) that $P_{s_1}(X_+)P_{s_2}(X_-)e^{-\pi[\|X_+\|^2 + \|X_-\|^2]}$ (where $Q(X, X) = \|X_+\|^2 - \|X_-\|^2$) transforms according to the character of $\tilde{K}$: $k(\theta, \varepsilon) \rightsquigarrow e^{\sqrt{-1}\,(s_1 - s_2 + (a-b)/2)\theta}$. Finally we note that there exists $f \in S(\mathbf{R}^k) \cap \mathbf{F}_Q^s$ (where $s_1 - s_2 + (a - b)/2 = s$) so that

$$P_s^+(f) = c\varphi, \qquad c \neq 0, \tag{1-25}$$

where $\varphi \in E_{\mathfrak{M}}(s^2 - 2s, s, s_1, s_2)$. Indeed if φ is given by (1-7), then choose

$$f(X) = P_{s_1}(X_+)P_{s_2}(X_-)e^{-\pi[\|X_+\|^2 + \|X_-\|^2]} \tag{1-26}$$

where we assume that P_{s_1} and P_{s_2} are the unique, up to scalar multiple, spherical harmonics of degree s_1 and s_2 on $\mathbf{R}^a$ and $\mathbf{R}^b$ invariant by $O(\mathbf{R}^a)^{e_1}$ and $O(\mathbf{R}^b)^{e_k}$, respectively (note here that if $b = 1$, then $P_{s_2} = 1$ or x, which are the first two Hermite polynomials). Then we have $\int_{\mathbf{R}^k} f(X)\overline{\varphi(X)}\, dX > 0$. Hence, noting that f transforms according to the tensor representation $[s_1]_a \otimes [s_2]_b$ of K, we deduce that $(f|\varphi)_{L^2} = (P_s^+(f)|\varphi)$ with $P_s^+(f) \in E_{\mathfrak{M}}(s^2 - 2s, s, s_1, s_2)$. Then using the specific form of φ in (1-7) the result (1-25) follows.

2. Zagier correspondence. Let Q be a nondegenerate quadratic form on $\mathbf{R}^k$. Moreover let $Q = \tilde{Q}_1 \oplus \tilde{Q}_2$, where $\mathbf{R}^k = \mathbf{R}^m \oplus \mathbf{R}^n$ and $Q|_{\mathbf{R}^m} = \tilde{Q}_1$ and $Q|_{\mathbf{R}^n} = \tilde{Q}_2$. Then the representation $\pi_{\mathfrak{M}}$ is functorial relative to this splitting in the following way. Let $F_1 = L^2(\mathbf{R}^m)$ and $F_2 = L^2(\mathbf{R}^n)$ and form the Hilbert space tensor product $F_1 \hat{\otimes} F_2$. We define in the usual way the unitary representation of the product $\widetilde{SL}_2 \times \widetilde{SL}_2$ on $F_1 \hat{\otimes} F_2$ by $(G_1, G_2) \rightsquigarrow \pi^m_{\mathfrak{M}}(G_1) \hat{\otimes} \pi^n_{\mathfrak{M}}(G_2)$. Then we know from [**I**, §1] that the representation of $\pi_{\mathfrak{M}}$ of $\widetilde{SL}_2$ in $L^2(\mathbf{R}^k)$ (relative to the quadratic form Q) is unitarily equivalent to the restriction of the tensor product representation $\pi^m_{\mathfrak{M}} \hat{\otimes} \pi^n_{\mathfrak{M}}$ to the diagonal subgroup of the product $\widetilde{SL}_2 \times \widetilde{SL}_2$.

REMARK 2-1. Let $k = 2$ and Q be the hyperbolic plane $\langle v, \tilde{v}\rangle$ with $Q(v, v) = Q(\tilde{v}, \tilde{v}) = 0$ and $Q(v, \tilde{v}) = 1$. The group $SL_2(\mathbf{R})$ operates on $\langle v, \tilde{v}\rangle$ as follows. $\sigma = \left[\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right]$ has the effect $\sigma(v) = av + c\tilde{v}$ and $\sigma(\tilde{v}) = bv + d\tilde{v}$. Then it is possible to linearize the representation $\pi_{\mathfrak{M}}$ of $\widetilde{SL}_2$ in the following manner. Let

$$F_v(\varphi)(rv + s\tilde{v}) = \int_{\mathbf{R}} \varphi(uv + s\tilde{v})e^{2\pi\sqrt{-1}\, ur}\, du \tag{2-1}$$

(for $\varphi \in L^1(\mathbf{R}^2) \cap L^2(\mathbf{R}^2)$). Then F_v extends uniquely to a unitary operator on $L^2(\mathbf{R}^2)$, and we deduce from Theorem 1.1 of [**II**] that

$$F_v^{-1}\pi_{\mathfrak{M}}((M, \varepsilon))F_v(\varphi)(X) = \varphi(M^{-1}(X)) \tag{2-2}$$

with $M \in SL_2(\mathbf{R})$.

Let $L_{(u_1,u_2)}$ be the lattice in $\mathbf{R}^2$ given by $\mathbf{Z}u_1v \oplus \mathbf{Z}u_2\tilde{v}$, where $u_1 \cdot u_2 \in \mathbf{Z}^*$. Thus $L_{(u_1,u_2)}$ is a Q integral lattice in $\mathbf{R}^2$. Moreover the Q integral dual $L_*(Q)$ of $L_{(u_1,u_2)}$ is $\mathbf{Z}(1/u_2)v \oplus \mathbf{Z}(1/u_1)\tilde{v}$.

We say that the function f on $\mathbf{R}^2$ which is both integrable and continuous satisfies the *-Poisson Summation Formula Property relative to $L_{(u_1,u_2)}$ if

(a) f satisfies (A) and (B) of the Poisson Summation Formula Property relative to $L_{(u_1,u_2)}$ (see 11 of §0).

(b) For every $\eta \in L_{(u_1,u_2)} * (Q)$, the function $g_\eta(x) = f(xu_1 \cdot v + \eta)$ satisfies the Poisson Summation Formula Property relative to $\mathbf{Z}$.

(c) For any $\gamma \in SL_2(\mathbf{Z})$ the function $x \to F_v^{-1}(\varphi)[\gamma(xv)/u_2]$ is continuous and integrable. Moreover the series

$$\sum_{(t,n)\in\mathbf{Z}^2} \left| F_v^{-1}(f)\left(\frac{t}{u_1}v + nu_2 + \eta\right)\right| < \infty$$

with $\eta \in L_{(u_1,u_2)} * (Q)$.

(d) For every $M \in SL_2(\mathbf{Z})$, the function $h(x, M) = \pi^2_{\mathfrak{M}}((M, \varepsilon))(f)(xv/u_2)$ satisfies the Poisson Summation Formula Property relative to $\mathbf{Z}$.

(e) The series

$$\sum_{\substack{\gamma \in SL_2(\mathbf{Z})/P \cap SL_2(\mathbf{Z}) \\ j \in \mathbf{Z}}} \left| \pi^2_{\mathfrak{M}}(\gamma^{-1})(f)\left(\frac{j}{u_2} v\right) \right| < \infty.$$

It may seem that these conditions are rather technical and arbitrary. However, their importance is seen in the following lemma.

LEMMA 2-1. *Let f be a continuous L^1 function on $\mathbf{R}^2$. Suppose for each $(G, g) \in \tilde{G}_2(Q)$ the function satisfies the *-Poisson Summation Formula Property relative to $L_{(u_1,u_2)}$.*

Also suppose that for each $(G, g) \in \tilde{G}_2(Q)$ we have $F_v(\pi_{\mathfrak{M}}((G, g)^{-1}))(f)(\vec{0}) \equiv 0$. Then if $\eta = v/u_2$,

$$T^{L_{(u_1,u_2)}}_{f,\eta,X}(G, g) = \sum_{\gamma \in \Gamma_0(u_1u_2)/\Gamma_0(u_1u_2) \cap \psi(P)} \cdot \left\{ \sum_{\substack{s \in \mathbf{Z} \\ a \bmod u_1u_2}} \pi_{\mathfrak{M}}((G(\gamma, 1), g)^{-1})(f)\left[\left(su_1 + \frac{aa_\gamma}{u_2}\right)v\right]\chi(a)\right\}. \quad (2\text{-}3)$$

PROOF. By hypothesis on f we have that (with $X = av/u_2$)

$$\sum_{(m,n) \in \mathbf{Z}^2} f(mu_1v + nu_2\tilde{v} + X) = \frac{1}{u_1} \sum_{(t,n) \in \mathbf{Z}^2} F_v^{-1}(f)\left(\frac{t}{u_1} v + nu_2\tilde{v}\right) \exp\left(2\pi\sqrt{-1}\,(at/u_1u_2)\right). \quad (2\text{-}4)$$

The argument inside $F_v^{-1}(f)$ on the right-hand side of (2-4) is of the form $[tv + nu_1u_2\tilde{v}]/u_1$ with $(t, n) \in \mathbf{Z}^2$.

Then the correspondence $\gamma \to \gamma(v)$ for $\gamma \in SL_2(\mathbf{Z})$ determines a bijection of $SL_2(\mathbf{Z})/SL_2(\mathbf{Z}) \cap \{\left(\begin{smallmatrix} 1 & \mathbf{Z} \\ 0 & 1 \end{smallmatrix}\right)\}$ to the set $\{mv + n\tilde{v} | (m, n) \in \mathbf{Z}^2 - (0, 0)$ and m, n relatively prime$\}$. This implies that the set $A = \{xv + y\tilde{v} | (x, y) \in \mathbf{Z}^2 - (0, 0)$ with x, y relatively prime and $y \equiv 0 \bmod u_1u_2$ and if $y \neq 0$ then $y > 0$ and if $y = 0$ then $x = 1\}$ is in one-one correspondence with $\Gamma_0(u_1u_2)/\Gamma_0(u_1u_2) \cap \psi(P)$. (Note here that $\psi(P) = \left(\begin{smallmatrix} \pm 1 & \mathbf{Z} \\ 0 & \pm 1 \end{smallmatrix}\right)$.) Then we partition the lattice $\mathbf{Z}v \oplus \mathbf{Z}u_1u_2\tilde{v}$ into two sets, i.e., $T = \bigcup_{j \in \mathbf{Z} - \{0\}} jA$ and the complement T^c of T in $\mathbf{Z}v \oplus \mathbf{Z}u_1u_2\tilde{v}$. Now if $w \in T^c$, we assert that $w = t_wv + u_1u_2\mathfrak{z}_w\tilde{v}$ has the property that t_w and u_1u_2 are *not* relatively prime. Indeed if $\gcd(t_w, u_1u_2) = 1$, then let $\alpha_w = \gcd(t_w, u_1u_2\mathfrak{z}_w)$. This means that α_w divides $\mathfrak{z}_w$. Moreover, t_w/α_w and $\mathfrak{z}_wu_1u_2/\alpha_w$ are relatively prime. Hence $w = \alpha_w(t_wv/\alpha_w + u_1u_2\mathfrak{z}_w\tilde{v}/\alpha_w) \in T$, which is a contradiction.

Thus taking the right-hand side of (2-4), we sum over $a \bmod u_1u_2$. We consider only those terms in the sum where $tv + nu_1u_2\tilde{v} \in T^c$. However, fixing one such term, we see that the inner sum becomes

$$\sum_{a \bmod u_1u_2} F_v^{-1}(f)\left(\frac{t}{u_1}v + nu_2\tilde{v}\right)\chi(a)e^{2\pi\sqrt{-1}\,(at/u_1u_2)}$$

$$= F_v^{-1}(f)\left(\frac{t}{u_2}v + nu_2\tilde{v}\right)\sum_{a \bmod u_1u_2}\chi(a)e^{2\pi\sqrt{-1}\,(at/u_1u_2)}. \quad (2\text{-}5)$$

But we know that the last sum in (2-5) is a Gauss sum $G(\chi, t, u_1u_2)$ which is zero when t and u_1u_2 are not relatively prime.

Thus it suffices to consider only the sum of the form

$$\sum_{a \bmod u_1u_2}\chi(a)\sum_{\gamma\in\Gamma_0(u_1u_2)/\Gamma_0(u_1u_2)\cap\left\{\begin{pmatrix}\pm 1 & \mathbf{Z}\\ 0 & \pm 1\end{pmatrix}\right\}}$$

$$\cdot\left\{\sum_{j>0} F_v^{-1}(f)(j\gamma(v))\exp\left(2\pi\sqrt{-1}\,\frac{aa_\gamma}{u_1u_2}j\right)\right.$$

$$\left.+ F_v^{-1}(f)(-j\gamma(v))\exp\left(-2\pi\sqrt{-1}\,\frac{aa_\gamma}{u_1u_2}j\right)\right\}. \quad (2\text{-}6)$$

We note that the inner sum is invariant under left multiplication of γ by an element of the form $\left(\begin{smallmatrix}\pm 1 & \mathbf{Z}\\ 0 & \pm 1\end{smallmatrix}\right)$. Moreover we have used here the fact that $F_v(f)(\vec{0}) \equiv 0$.

Then using the hypothesis on f again (specifically the integrability of f on $\mathbf{R}^2$ and the integrability and continuity of $x \rightsquigarrow F_v(f)[\gamma(xv)/u_2]$ for all $\gamma \in SL_2(\mathbf{Z})$), we deduce that $F_v \circ \gamma \circ F_v^{-1}(f)(xv/u_2) = \pi_{\mathfrak{M}}(\gamma)(f)(xv/u_2)$ for all $x \in \mathbf{R}$ and all $\gamma \in SL_2(\mathbf{Z})$. Then we have

$$\sum_{j\in\mathbf{Z}} F_v^{-1}(f)\left[\frac{1}{u_1}j\gamma(v)\right]\exp\left(2\pi\sqrt{-1}\,\frac{aa_\gamma}{u_1u_2}j\right)$$

$$= u_1\sum_{n\in\mathbf{Z}} F_v\circ\gamma^{-1}\circ F_v^{-1}(f)\left[\left(nu_1 + \frac{aa_\gamma}{u_2}\right)v\right]. \quad (2\text{-}7)$$

Then using (2-7) we deduce that the left-hand side of (2-3) equals

$$\sum_{\gamma\in\Gamma_0(u_1u_2)/\Gamma_0(u_1u_2)\cap\left\{\begin{pmatrix}\pm 1 & \mathbf{Z}\\ 0 & \pm 1\end{pmatrix}\right\}}$$

$$\cdot\left\{\sum_{\substack{a \bmod u_1u_2\\ j\in\mathbf{Z}}} F_v\circ\gamma\circ F_v^{-1}(f)\cdot\left[\left(ju_1 + \frac{aa_\gamma}{u_2}\right)v\right]\chi(a)\right\}. \quad (2\text{-}8)$$

Then if we replace f by $\pi_{\mathfrak{M}}((G, g)^{-1})(f)$, the above reasoning remains valid because of the hypothesis on $\pi_{\mathfrak{M}}((G, g)^{-1})(f)$ in the lemma. Finally we use (2-2). Q.E D.

We can extend Lemma 2-1 to higher dimensional spaces in the following way. First we need the analogue of the *-Poisson Summation Property for $\mathbf{R}^k$ space.

Let L be a Q integral lattice in $\mathbf{R}^k$. Assume that L has a Q orthogonal splitting of the form $L = \mathfrak{L} \oplus L_{(u_1,u_2)}$, where $\mathfrak{L}$ is a Type II lattice in $\mathbf{R}^{k-2}$ and $L_{(u_1,u_2)}$ is as before.

Then let f be a continuous and integrable function on $\mathbf{R}^k$. We say that f satisfies the **-Poisson Summation Property relative to L if*

(a′) f satisfies the Poisson Summation Property relative to L (see §2 of **[II]**).

(b′) For each $\eta \in \mathfrak{L}_*(Q)$, the Q integral dual lattice to $\mathfrak{L}$ in $\mathbf{R}^{k-2}$, the function $Q_\eta(X) = f(\eta + X)$ with $X \in \mathbf{R}^2$ satisfies properties (a), (b) and (d) above.

(c′) For any $V \in \mathfrak{L}_*(Q) \oplus L_{(u_1,u_2)^*}$,

$$\sum_{\substack{\xi \in \mathfrak{L} \\ \mu \in L_{(u_1,u_2)}}} |F_V^{-1}(f)(\xi + \mu + V)| < \infty.$$

Also if $\gamma \in SL_2(\mathbf{Z})$, then the function $x \rightsquigarrow F_v^{-1}(f)[\gamma(xv)/u_2 + W]$ is continuous and integrable (for any $W \in \mathbf{R}^{k-2}$).

(d′) For each $\gamma \in SL_2(\mathbf{Z})$ and each $Z \in \mathbf{R}^{k-2}$ the function

$$Z \rightsquigarrow \pi^2_{\mathfrak{M}}(\gamma^{-1})(f)(Z + \mu v/u_2) \quad (\text{with } \mu \in \mathbf{Z})$$

is continuous and square integrable on $\mathbf{R}^{k-2}$, and the series (with $V_1 \in \mathfrak{L}_*(Q)$)

$$\sum_{\substack{\xi \in \mathfrak{L} \\ \mu \in \mathbf{Z}}} \left| \pi^2_{\mathfrak{M}}(\gamma^{-1})(f)\left(\xi + \frac{\mu}{u_2} v + V_1\right)\right| < \infty.$$

(e′) For each $\gamma \in SL_2(\mathbf{Z})$ and $\mu \in \mathbf{Z}$ the function $Z \to \pi_{\mathfrak{M}}(\gamma^{-1})(f)(Z + \mu v/u_2)$ with $Z \in \mathbf{R}^{k-2}$ satisfies the Poisson Summation formula relative to the lattice $\mathfrak{L}$.

Again these conditions seem unmotivated but become essential in the following.

PROPOSITION 2-1. *Let L be a Q integral lattice in $\mathbf{R}^k$. Assume that L has a Q orthogonal splitting in the form $L = \mathfrak{L} \oplus L_{(u_1,u_2)}$, where $\mathfrak{L}$ is a Type* II *lattice in $\mathbf{R}^{k-2}$ and $L_{(u_1,u_2)}$ is as above. Let f be a continuous and integrable function on $\mathbf{R}^k$. Assume that for each $(G, g) \in \tilde{G}_2(Q)$, the function $\pi_{\mathfrak{M}}((G, g)^{-1})(f)$ satisfies the *-Poisson Summation formula relative to L. Also assume that $F_v(\pi_{\mathfrak{M}}((G', g')^{-1}))(f)(T) \equiv 0$ for all $T \in \mathbf{R}^{k-2}$ and for $(G', g') \in \tilde{G}_2(Q)$. Let $\eta \in \mathfrak{L}_*(Q)$. Let $\{\sigma^{\mathfrak{L}}_{ij}(\ )\}$ be the matrix defined in* §0 *relative to the lattice $\mathfrak{L}$ (where, with an abuse of notation, the pair of vectors (x, y) index an entry of $\{\sigma^{\mathfrak{L}}_{ij}(\ )\}$). Then we have*

$$T^L_{\varphi, v+\eta, X}(G', g') = \sum_{\gamma \in \Gamma_0(u_1u_2)/\Gamma_0(u_1u_2) \cap \psi(P)} \cdot \left\{ \sum_{\substack{\xi \in \mathfrak{L} \\ j \in \mathbf{Z} \\ a \bmod u_1u_2}} \left\{ \sum_{\eta_i \in \mathfrak{L}_*(Q)/\mathfrak{L}} \sigma^{\mathfrak{L}}_{\eta,\eta_i}((\gamma, 1)^{-1}) \pi_{\mathfrak{M}}((G'(\gamma, 1), g')^{-1})(f) \cdot \left[\xi + \eta_i + \left(ju_1 + \frac{aa_\gamma}{u_2}\right)v\right]\chi(a)\right\}\right\}. \tag{2-9}$$

(*Note*: *The order of summation above is critical in that the series on the right-hand side of* (2-9) *may not be absolutely convergent*.)

PROOF. Let $\varphi = \pi_{\mathfrak{M}}((G', g')^{-1})(f)$. Then by using properties (b′), (c′), and (d′), we deduce by following similar reasoning as in Lemma 2-1 that

$$\sum_{\substack{\xi\in\mathfrak{L}\\(m,n)\in\mathbf{Z}^2\\a \bmod u_1u_2}} \varphi\left(\xi+\eta+\left(mu_1+\frac{a}{u_2}\right)v+nu_2\tilde{v}\right)\chi(a)$$

$$= \sum_{\gamma\in\Gamma_0(u_1u_2)/\Gamma_0(u_1u_2)\cap\psi(P)} \left\{ \sum_{\substack{\xi\in\mathfrak{L}\\ j\in\mathbf{Z}\\ a \bmod u_1u_2}} \pi^2_{\mathfrak{M}}((\gamma,1)^{-1})(\varphi) \cdot\left[\xi+\eta+\left(ju_1+\frac{aa_\gamma}{u_2}\right)v\right]\chi(a)\right\}. \tag{2-10}$$

We note here that the order of taking the summation on the right-hand side of (2-10) is important. We essentially are using the argument of Lemma 2-1 where the Poisson Summation formula is applied two times. We know that the series on the left-hand side of (2-10) is absolutely convergent. Also we know that

$$\sum_{\gamma\in\Gamma_0(u_1u_2)/\Gamma_0(u_1u_2)\cap\psi(P)} |\{\cdots\}| < \infty$$

(from (c′) above) and that the series in $\{\cdots\}$ is absolutely convergent (from (d′) above).

Then using (e′) above, we deduce that for $\gamma\in\Gamma_0(u_1u_2)$ (here we use 10 of §0) with $\lambda\in\mathfrak{L}_*(Q)$ and

$$\pi_{\mathfrak{M}}((G,1))(\varphi)(Z_1+Z_2) = \pi^{k-2}_{\mathfrak{M}}((G,1))\otimes\pi^2_{\mathfrak{M}}((G,1))(\varphi)(Z_1+Z_2)$$

where $Z_1\in\mathbf{R}^{k-2}$, $Z_2\in\mathbf{R}^2$ (here $\varphi\in L^1(\mathbf{R}^k)$), we have

$$\sum_{\xi\in\mathfrak{L}} \pi^2_{\mathfrak{M}}((\gamma,1)^{-1})(\varphi)\left[\xi+\lambda+\frac{\mu}{u_2}v\right]$$

$$= \sum_{\eta_i\in\mathfrak{L}_*(Q)/\mathfrak{L}} \sigma^{\mathfrak{L}}_{\lambda\eta_i}((\gamma,1)^{-1})\left\{\sum_{\xi\in\mathfrak{L}} \pi_{\mathfrak{M}}((\gamma,1)^{-1})(\varphi)\left[\xi+\eta_i+\frac{\mu}{u_2}v\right]\right\}. \tag{2-11}$$

We note here that

$$\pi^{k-2}_{\mathfrak{M}}((\gamma,1))\pi_{\mathfrak{M}}((\gamma,1)^{-1})(\varphi)[X+\mu v/u_2] = \pi^2_{\mathfrak{M}}((\gamma,1)^{-1})(\varphi)[X+\mu v/u_2]$$

is valid because of the continuity and L^2 assumptions in (d′).

Then we substitute (2-11) on the right-hand side of (2-10). Q.E.D.

At this point we make precise the relationship between the lattice $\mathfrak{L}\oplus L_{(u_1,u_2)}$ *and the* Q *orthobasis defined in* 1 *of* §0. *Namely we assume that* $\mathfrak{L}\subseteq\mathbf{Q}\otimes\{e_2,\ldots,\hat{e}_{a+1},\ldots,e_k\}$ (*i.e.* $\hat{e}_{a+1}$ *denotes* e_{a+1} *is omitted*) *and that* $v=\sqrt{2}\,v_a$ *and* $\tilde{v}=\tilde{v}_a/\sqrt{2}$.

The main problem is to find functions f so that $\pi_{\mathfrak{M}}((G,g)^{-1})(f)$ satisfies the *-Poisson Summation formula relative to L. However this is satisfied by functions given in the following lemma.

LEMMA 2-2. *Let* $|s| > 2k$. *Let* $\varphi \in \mathbf{F}_Q^+(s^2 - 2s)_{\tilde{K}\times K}$ ($\mathbf{F}_Q^-(s^2 + 2s)_{\tilde{K}\times K}$ *resp.*). *Then for every* $(G, g) \in \tilde{G}_2(Q)$, *the function* $X \rightsquigarrow \pi_{\mathfrak{M}}((G, g)^{-1})(\varphi)(X)$ *for* $X \in \mathbf{R}^k$ *satisfies the* *-*Poisson Summation Property relative to* L.

PROOF. See Appendix.

THEOREM 2-1 (ZAGIER IDENTITY). *Let* L *be a* Q *integral lattice having the form* $\mathfrak{L} \oplus L_{(u_1,u_2)}$ *where* $\mathfrak{L}$ *and* $L_{(u_1,u_2)}$ *are as given in Proposition* 2-1. *Let* $\varphi \in E_{\mathfrak{M}}(s^2 - 2s, s, s_1, s_2)$ ($E_{\mathfrak{M}}(s^2 + 2s, s, s_1, s_2)$ *resp.*) *with* $|s| > 2k$ *be given by* (1-7). *Then assume that*

$$\Phi^{FT}_{\pi_{\mathfrak{M}}(g^{-1})(\varphi)}(Y) = \int_{\mathbf{R}} \pi_{\mathfrak{M}}(g^{-1})(\varphi)[Y + tv_a]\, dt \equiv 0 \tag{2-12}$$

for each $Y \in L_1$. *Let* $n_{\mathfrak{L}}$ *be the exponent of* $\mathfrak{L}_*(Q)/\mathfrak{L}$. *Then the exponent* n_L *of* $L_*(Q)/L$ *is the least common multiple of* $n_{\mathfrak{L}}$ *and* u_1u_2.

Assume that $2n_{\mathfrak{L}}|u_1u_2$ (*so that* $n_L = u_1u_2$). *Then we have* (*valid for* g *as given in* (2-17) *and any* $z \in H$)

$$\check{T}^L_{\varphi,v,\chi}(z, g) = 2^{s-1} \sum_{\gamma \in \Gamma_0(u_1u_2)\cap\psi(P)\backslash\Gamma_0(u_1u_2)} \cdot \left\{ \sum_{r\in X^+} |r|^{s-1} \left\{ \sum_{\nu \bmod u_1u_2} \chi(\nu) M_\varphi(g, \mathfrak{L}, r, \nu)\right\} \overline{s^L_{Q,\chi}(\gamma)} \left(\frac{1}{c_\gamma z + d_\gamma}\right)^s e^{2\pi\sqrt{-1}\, r\gamma(z)} \right\}, \tag{2-13}$$

where

$$M_\varphi(g, \mathfrak{L}, r, \nu) = \sum_{\substack{\{\xi\in\mathfrak{L}|Q(\xi,\xi)=2r\}\\ j\in\mathbf{Z}}} h_+\left(g\left(\xi + \left(ju_1 + \frac{\nu}{u_2}\right)v\right)\right) \tag{2-14}$$

and where $X^+ = \{(Q(\mathfrak{L}, \mathfrak{L}))/2\} \cap \mathbf{R}_+$ *and*

$$h_+(X) = \|X_+\|^{-(s+(k/2)+s_1+s_2-2)} P_{s_1}(X_+) P_{s_2}(X_-) \tag{2-15}$$

with $X \in \Omega_+$.

PROOF. We apply *Lemma* 2-2 and *Proposition* 2-1. We note that if $2n_{\mathfrak{L}}|u_1u_2$, then $\sigma_{\vec{0}\eta_i}((\gamma, 1)) = 0$ if $\eta_i \neq \vec{0}$ for $\gamma \in \Gamma_0(u_1u_2)$. Next it suffices to observe that $a_\gamma d_\gamma \equiv 1 \bmod u_1u_2$ implies that aa_γ ranges over a complete set of representatives of integers mod u_1u_2 when a does the same. Thus we deduce that $T^L_{\varphi,v,\chi}(G, g)$ equals

$$\sum_{\gamma\in\Gamma_0(u_1u_2)/\Gamma_0(u_1u_2)\cap\psi(P)} \left\{ \sum_{\nu \bmod u_1u_2} \left\{ \sum_{\substack{j\in\mathbf{Z}\\ \xi\in\mathfrak{L}}} \sigma^{\mathfrak{L}}_{\vec{0}\vec{0}}((\gamma, 1)^{-1}) \pi_{\mathfrak{M}}((G(\gamma, 1), g)^{-1})(\varphi) \cdot \left[\xi + \left(ju_1 + \frac{\nu}{u_2}\right)v\right] \chi(\nu)\chi(d_\gamma) \right\} \right\}. \tag{2-16}$$

Then we let $G = (((\begin{smallmatrix} x & y \\ 0 & x^{-1}\end{smallmatrix}), 1))$ and using an argument similar to *Lemma* 1-2, we obtain

$$\begin{aligned} \sigma_{\bar{0}\bar{0}}^{\mathfrak{L}}((\gamma, 1)^{-1})\pi_{\mathfrak{M}}((G(\gamma, 1), g)^{-1})(\varphi)[\xi + (ju_1 + \nu/u_2)v] \\ = (\operatorname{Im} z)^{s/2}|r|^{s-1}h_+(g(\xi + (ju_1 + \nu/u_2)v))(-c_\gamma z + a_\gamma)^{-s} \\ \cdot e^{\pi\sqrt{-1}\, r\gamma^{-1}(z)}c_{Q_1}^{\mathfrak{L}}((\gamma, 1)^{-1})[\psi_2(\gamma)]^k \end{aligned} \tag{2-17}$$

with $z = -(y/x) + x^{-2}\sqrt{-1}$ and $r = Q(\xi, \xi)$. But then

$$c_{Q_1}^{\mathfrak{L}_1}((\gamma, 1)^{-1})[\psi_2(\gamma)]^k\chi(d_\gamma) = s_{Q_1,\chi}^{\mathfrak{L}}(\gamma)$$

for $\gamma \in \Gamma_0(u_1u_2)$. Finally we note that $s_{Q,\chi}^{L}(\gamma) = s_{Q_1,\chi}^{\mathfrak{L}}(\gamma)$ by using the explicit form of s_Q^L given in §0.

We note here that if $\varphi \in E_{\mathfrak{M}}(s^2 - 2s, s, s_1, s_2)$ and if (2-12) is satisfied, then for any $G \in \widetilde{SL}_2$

$$\int \pi_{\mathfrak{M}}((G, g)^{-1})(\varphi)[X + tv_a]\, dt \equiv 0$$

for all $X \in L_1$. Indeed

$$\pi_{\mathfrak{M}}((G, g)^{-1})(\varphi)(W) = c_1|r_G|^{-k/2}e^{-\pi\sqrt{-1}\, x_G r_G^{-1}Q(W,W)}\pi_{\mathfrak{M}}(g^{-1})(\varphi)[r_G^{-1}W].$$

Then using change of variables in (2-12), we deduce the above statement. Q.E.D.

REMARK 2-1a. At this point we can make certain preliminary remarks about the behavior of the function (with $t = u_1$ and $u_2 = 1$)

$$M_\varphi(g, \mathfrak{L}, r, \nu) = \sum_{\substack{\{\xi \in \mathfrak{L} | Q(\xi,\xi) = 2r\} \\ j \in \mathbf{Z}}} h_+\left(g\left(\xi + (jt + \nu)\frac{v_a}{\sqrt{2}}\right)\right),$$

where h_+ is defined by (2-15). Then we have

$$M_\varphi(g\gamma, \mathfrak{L}, r, \nu) = M_\varphi(g, \mathfrak{L}, r, v)$$

for all $\gamma \in \Gamma^L(Q) \cap O(Q|_{L_1}) \cdot U(F_1)$. Moreover there is another elementary way to write the function M_φ. Indeed let $\xi_1^r, \ldots, \xi_{h(r)}^r$ be a set of representatives of the $O(Q|_{L_1}) \cap \Gamma^L(Q)$ orbits in $\{\xi \in \mathfrak{L} | Q(\xi,\xi) = 2r\}$. Then we have

$$M_\varphi(g, \mathfrak{L}, r, \nu) = \sum_{i=1}^{h(r)} \sum_{\substack{j \in \mathbf{Z} \\ \gamma \in \Gamma^L(Q) \cap O(Q|_{L_1})/(\Gamma^L(Q) \cap O(Q|_{L_1}))^{\xi_i^r}}} \cdot h_+\left(g\gamma N_1\left(\left(\frac{jt + \nu}{2r}\right) \cdot \xi_i^r\right)(-\xi_i^r)\right). \tag{2-18}$$

REMARK 2-2. The order of summation on the right-hand side of (2-13) is important. In general the series is not absolutely convergent. However we note that by "formally" changing the order of summation in the right-hand side of (2-13) we can deduce from Lemma 1-2 the formal identity

$$\check{T}^L_{\varphi,v,\chi}(z,g) = 2^{s-1}\sum_{r\in X^+}|r|^{|s|-1}\left\{\sum_{\nu \bmod u_1u_2}\chi(v)\sum_{\substack{\{\xi\in\mathfrak{L}\mid Q(\xi,\xi)=2r\}\\ j\in\mathbf{Z}}} \cdot h_+\left(g\left(\xi+\left(ju_1+\frac{v}{u_2}\right)v\right)\right)\right\}$$
$$\cdot G_s\left(z, s^L_{Q,\chi}, \Gamma_0(u_1u_2), r\right). \tag{2-19}$$

REMARK 2-3. The condition that $\Phi^{F_1^\dagger}_\psi(Y)\equiv 0$ (for all $Y\in L_1$) is called the *Second Cusp Vanishing Condition* for $\psi\in E_{\mathfrak{M}}(s^2-2s,s,s_1,s_2)$ in **[II]**. We recall from **[II]** that the map $\varphi\to\Phi^{F_1}_\varphi$ is an $\widetilde{SL}_2\times O(Q_1)$ infinitesimal intertwining map of $\mathbf{F}^\pm_Q(s^2-2s)_{\tilde K\times K}$ to $\mathbf{F}^\pm_{Q_1}(s^2-2s)_{\tilde K\times K_1}$ ($K_1=K\cap O(Q_1)$). Thus if $\Phi^{F_1^\dagger}_\varphi(Y)\equiv 0$ (for all $Y\in L_1$), it follows that $\Phi^{F_1^\dagger}_{\pi_{\mathfrak{M}}((G,g)^{-1})(\varphi)}(Y)\equiv 0$ for all $Y\in L_1$, all $G\in\widetilde{SL}_2$ and all $g\in p_{F_1}$. However if $b=2$ with $\varphi\in\mathbf{F}^-_Q(s^2+2s)_{\tilde K\times K}$ ($|s|>\frac{1}{2}k$) and if $a=2$ with $\varphi\in\mathbf{F}^+_Q(s^2-2s)_{\tilde K\times K}$ ($s>\frac12 k$), then $\Phi^{F_1^\dagger}_{\pi_{\mathfrak{M}}((G,g)^{-1})(\varphi)}(Y)\equiv 0$ for all $Y\in L_1$, all $G\in\widetilde{SL}_2$ and all $g\in O(Q)$. The latter cases are precisely the cases when $\check{T}^L_{\varphi,v,\chi}(z,g)$ is a *cusp form* in the $O(Q)$ variable (see **[II**, §5**]**). Thus in the "cusp" cases, (2-13) is valid for *all* $z\in H$ and *all* $g\in O(Q)$.

The main point of (2-13) (and the formal formula (2-19)) is to prove that $\langle\check{T}^L_{\varphi,v,\chi}(\ ,g)|f(\ )\rangle_{\mathfrak{D}_{\Gamma_0(u_1u_2)}}$ is given by the formula

$$\sum_{r\in X^+}\left\{\sum_{\nu \bmod u_1u_2}\chi(\nu)\sum_{\substack{\{\xi\in\mathfrak{L}\mid Q(\xi,\xi)=2r\}\\ j\in\mathbf{Z}}} h_+\left(g\left(\xi+\left(ju_1+\frac{\nu}{u_2}\right)v\right)\right)\right\}\overline{a_r(f)}, \tag{2-20}$$

where $a_r(f)$ is the rth Fourier coefficient of f at $\{\infty\}$, i.e. $f(z)=\sum_{r\geq 1}a_r(f)e^{2\pi\sqrt{-1}\,rz}$. Here we use the reproducing formula $\langle G_s(\ ,s^L_{Q,\chi},\Gamma_0(u_1u_2),r)|f(\ )\rangle = c\cdot\overline{a_r(f)}r^{-|s|+1}$, c a nonzero constant depending only on s.

We can prove, by the methods of §§4 and 5, the equality between (2.20) and $\langle\check{T}^L_{\varphi,v,\chi}(\ ,g)|f(\ )\rangle_{\mathfrak{D}_{\Gamma_0(u_1u_2)}}$ for the cases when $\check{T}^L_{\varphi,v,\chi}(\ ,\ )$ is a cusp form in the $O(Q)$ variable (see Remark 2-3 above).

We may ask what sort of modification is required to obtain an analog of this formula for general $\check{T}^L_{\varphi,v,\chi}(\ ,\ )$. One possible way is to decompose $g=k\cdot p$ with $k\in K\cap O(Q_1)$ and $p\in p_{F_1}$. Then for $\varphi\in E_{\mathfrak{M}}(s^2-2s,s,s_1,s_2)$, we know that there exist *linearly independent* elements φ_i in $E_{\mathfrak{M}}(s^2-2s,s,s_1,s_2)$ so that $\pi_Q(g^{-1})(\varphi)(X)=\sum_i\alpha_i(k)\varphi_i(pX)$, with α_i certain analytic functions on K. Moreover the φ_i can be chosen in a special way. Namely let $\varphi_1,\dots,\varphi_m$ span the unique subspace of $E_{\mathfrak{M}}(s^2-2s,s,s_1,s_2)$ which, via the map $\psi\to\Phi^{F_1^\dagger}_\psi$, is mapped bijectively onto $E^{k-2}_{\mathfrak{M}}(s^2-2s,s,s_1,s_2)$ (the corresponding $\tilde K\times K_1$ eigenspace in $L^2(\mathbf{R}^{k-2})$). Then let $\varphi_{m+1},\dots,\varphi_M$ span the kernel of the map $\psi\to\Phi^{F_1^\dagger}_\psi$ ($\psi\in E_{\mathfrak{M}}(s^2-2s,s,s_1,s_2)$).

Let f be a holomorphic cusp form which satisfies $\langle\check{T}^L_{\varphi_i,v,\chi}(\ ,p)|f(\ )\rangle_{\mathfrak{D}_{\Gamma_0(u_1u_2)}}\equiv 0$ for all $p\in p_{F_1}$ and $i=1,\dots,m$. Then for such f we have

$$\langle\check{T}^L_{\varphi,v,\chi}(\ ,g)|f(\ )\rangle=\sum_{i\geq m+1}^{M}\alpha_i(k)\langle\check{T}^L_{\varphi_i,v,\chi}(\ ,p)|f(\ )\rangle.$$

Then we can try to apply (2-20) to each summand $\langle \check{T}^L_{\varphi_i, v, \chi}(\ , p)|f(\ )\rangle$ with $p \in p_{F_1}$. Hence it should be possible to derive a formula similar to (2-20) for general $g \in O(Q)$ provided f satisfies certain orthogonality conditions given above.

3. Dirichlet series attached to automorphic forms on $\mathfrak{R}_+$. First we summarize the results of §7 in **[II]**. Here $b = 2$, i.e., the signature of Q is $(a, 2)$. Let $\mathfrak{R}_+ = \{X + \sqrt{-1}\ Y | X,\ Y \in L_1$ and $Q(Y, Y) < 0$ and $Q(Y, e_{a+2}) < 0\}$ be the tube domain associated to the forward light cone, $W = \{Y \in L_1 | Q(Y, Y) < 0$ and $Q(Y, e_{a+2}) < 0\}$. We recall that the group $O(Q)_0$, the connected component of $O(Q)$, operates as analytic automorphisms on $\mathfrak{R}_+$. Thus if $g \in O(Q)_0$, we let $\mathfrak{d}(g, Z) = \det(\partial(g \cdot Z)/\partial(Z))$ where $\partial(gZ)/\partial(Z)$ is the Jacobian matrix of the analytic map $Z \to g(Z)$ of $\mathfrak{R}_+$ to itself. We note that $\mathfrak{d}(g, Z)$ is a holomorphic automorphy factor on $\mathfrak{R}_+$. Moreover there exists a unique automorphy factor $\underset{\sim}{\mathfrak{d}}$ on $\mathfrak{R}_+$ so that $\underset{\sim}{\mathfrak{d}}^{2-k} = \mathfrak{d}$ (see §7 of **[II]** for relevant definitions, etc.).

We note here that for $g \in P_{F_1} \cap O(Q)_0$ having the form $h_1 A_a(r) N_1(Y)$ with $h_1 \in O(Q_1)$, $r \in \mathbf{R}^*$ and $Y \in L_1$ we have that

$$g(U + \sqrt{-1}\ V) = r\left[h_1(U + Y) + \sqrt{-1}\ h_1(V)\right]$$

(with $U + \sqrt{-1}\ V \in \mathfrak{R}_+$). Then $\underset{\sim}{\mathfrak{d}}(h_1 A_a(r) N_1(Y), \mathbf{Z}) = r^{-1}$.

Also let c_a be the element in $O(Q)_0$ so that

$$c_a(U + \sqrt{-1}\ V) = \frac{1}{Q(U + \sqrt{-1}\ V, U + \sqrt{-1}\ V)}\{U + \sqrt{-1}\ V\}. \tag{3-1}$$

Then we deduce easily that $\underset{\sim}{\mathfrak{d}}(c_a, Z) = Q(Z, Z)$.

We recall that a holomorphic function for $\mathfrak{R}_+$ is an *integral automorphic form of weight ν* for an arithmetic subgroup Γ_* of $O(Q)_0$ if

(i) $f(\gamma \cdot Z) = \underset{\sim}{\mathfrak{d}}(\gamma, Z)^\nu f(Z)$ for all $Z \in \mathfrak{R}_+$ and all $\gamma \in \Gamma_*$.

(ii) For each $\sigma \in C(\Gamma_*)$, the commensurability group of Γ_* in $O(Q)_0$, there is a positive number $d(\sigma)$ so that $(f|\sigma)(Z) = f(\sigma(Z))\underset{\sim}{\mathfrak{d}}(\sigma, Z)^{-\nu}$ is bounded on the subdomain $\mathfrak{R}_+(c(\sigma)) = \{X + \sqrt{-1}\ Y|\ |Q(Y, Y)| \geqslant c(\sigma)\}$.

We note that $C(\Gamma_*)$ is exactly the set of $\mathbf{Q}$ rational points in $O(Q)_0$, i.e., $C(\Gamma_*) = O(Q)_0 \cap O(Q)_{\mathbf{Q}}$. Moreover we say that an integral automorphic form is a parabolic or cusp form if $f(\sigma(Z))\underset{\sim}{\mathfrak{d}}(\sigma, Z)^{-\nu} \to 0$ as $|Q(Y, Y)| \to +\infty$ in $\mathfrak{R}_+(c(\sigma))$ (for each $\sigma \in C(\Gamma_*)$).

We recall from §7 in **[II]** that for any $\gamma \in C(\Gamma_*)$ there exists a lattice $\mathfrak{N}(\gamma) \subset L_1$ so that $U(F_1^*) \cap \gamma\Gamma_*\gamma^{-1} = \{N_1(\xi) \in U(F_1^*) | \xi \in \mathfrak{N}(\gamma)\}$. Then let $\mathfrak{N}(\gamma)^* = \{\eta \in L_1 | Q(\eta, \mathfrak{N}(\gamma)) \subseteq \mathbf{Z}\}$.

EXAMPLE 3-1. If $\Gamma_* = \Gamma^L(Q)$, the stabilizer of the lattice $L = \mathfrak{L} \oplus L_{(t,1)}$ in $\mathbf{R}^k$ then as an easy exercise we see that $\mathfrak{N}(e) = \sqrt{2}\ \{\xi \in \mathfrak{L} | Q(\xi, \xi) \equiv 0 \bmod 2t,\ Q(\mathfrak{L}, \xi) \equiv 0 \bmod t\}$ for e, the identity element of $O(Q)_0$.

Then using the invariance of f relative to $U(F_1^*) \cap \gamma\Gamma_*\gamma^{-1}$ we deduce that

$$(f|\gamma)(Z) = \sum_{\Omega \in \mathfrak{N}(\gamma)^*} a_\Omega(f|\gamma) e^{-2\pi\sqrt{-1}\ Q(Z, \Omega)} \tag{3-2}$$

where

$$a_\Omega(f|\gamma) = \int_{\mathbf{R}^{k-2}/\mathfrak{N}(\gamma)^*} (f|\gamma)(X + \sqrt{-1}\ Y) e^{2\pi\sqrt{-1}\, Q(X+\sqrt{-1}\ Y,\Omega)}\, dX \tag{3-3}$$

with the series in (3-2) being absolutely and uniformly convergent on $\mathfrak{R}_+$. We recall that $a_\Omega(f|\gamma) = 0$ for all $\Omega \in \mathfrak{N}(\gamma)^*$ satisfying $Q(\Omega, \Omega) > 0$ (if f is a cusp form then $a_\Omega(f|\gamma) = 0$ for all $\Omega \in \mathfrak{N}(\gamma)^*$ satisfying $Q(\Omega, \Omega) \geqslant 0$).

Furthermore, for $\gamma \in C(\Gamma_*)$, we know that $\gamma\Gamma_*\gamma^{-1} \cap O(Q_1)_0$ is an arithmetic subgroup of $O(Q_1)_0$ and we have that $a_{g\cdot\Omega}(f|\gamma) = a_\Omega(f|\gamma)$ for all $g \in \gamma\Gamma_*\gamma^{-1} \cap O(Q_1)_0$.

We then observe that the subgroup $\{g \in \gamma\Gamma_*\gamma^{-1}O(Q_1)_0 | g(\Omega) = \Omega\}$ (for $\Omega \in \mathfrak{N}(\gamma)^*$ and $Q(\Omega, \Omega) < 0$) of $\gamma\Gamma_*\gamma^{-1} \cap O(Q_1)_0$ is a finite subgroup. Indeed since $Q(\Omega, \Omega) < 0$ we deduce that $O(Q_1)_0^\Omega$ is a compact group. Then we let (for $\Omega \in \mathfrak{N}(\gamma)^*$ so that $Q(\Omega, \Omega) < 0$)

$$\varepsilon(\Omega|\gamma) = \#\left\{ g \in \gamma\Gamma_*\gamma^{-1} \cap O(Q_1)_0 | g(\Omega) = \Omega \right\}. \tag{3-4}$$

Then following **[6]** we can associate a Dirichlet series to a cusp form of weight ν (with respect to the arithmetic group Γ_*) by

$$D(\mathfrak{s}, f, \gamma) = \sum_{\Omega \in \{\mathfrak{N}(\gamma)^*\}_{\Gamma_*}} a_\Omega(f|\gamma) \frac{1}{\varepsilon(\Omega|\gamma)} |Q(\Omega, \Omega)|^{-\mathfrak{s}} \tag{3-5}$$

where $\{\mathfrak{N}(\gamma)^*\}_{\Gamma_*}$ represents a complete set of representatives of the orbit space

$$\left\{\Omega \in \mathfrak{N}(\gamma)^*/\gamma\Gamma_*\gamma^{-1} \cap O(Q_1)_0 | Q(\Omega, \Omega) < 0\right\}.$$

Then following the ideas in **[6]** we can prove

THEOREM 3-1. *$D(\mathfrak{s}, f, \gamma)$ is an absolutely convergent series for* $\mathrm{Re}(\mathfrak{s}) > \nu/2$ (*with* $\nu > k - 4$). *Then we let*

$$D^*(\mathfrak{s}, f, \gamma) = \left\{\pi^{-2\mathfrak{s}}\Gamma\left(\mathfrak{s} - \tfrac{1}{2}k + 2\right)\Gamma(\mathfrak{s})\right\} D(\mathfrak{s}, f, \gamma). \tag{3-6}$$

Then $D^*(\mathfrak{s}, f, \gamma)$ can be analytically continued to the whole $\mathfrak{s}$-plane. Moreover it satisfies a functional equation

$$D^*(\mathfrak{s}, f, \gamma) = D^*(\nu - \mathfrak{s}, f, \gamma c_a). \tag{3-7}$$

PROOF. We let $F(\Gamma_*^\gamma)$ be a fundamental domain for $\gamma\Gamma_*\gamma^{-1} \cap O(Q_1)_0$ in W. Let $d^*(Y)$ be the measure on W given by $dY/|Q(Y, Y)|^{(k-2)/2}$. Then we decompose $d(Y) = dr/r \otimes d\sigma_{-1}$ where $d\sigma_{-1}$ is an $O(Q_1)_0$ invariant measure on the + hyperboloid $\tilde{\Gamma}_{-1} \cap L_1 = \{X \in L_1 | Q(X, X) = -1,\ Q(X, e_k) < 0\}$. Then we can also write $F(\Gamma_*^\gamma) = \mathbf{R}_+ \times F_{-1}(\Gamma_*^\gamma)$ where $F_{-1}(\Gamma_*^\gamma)$ is a fundamental domain for $\gamma\Gamma_*\gamma^{-1} \cap O(Q_1)_0$ in $\tilde{\Gamma}_{-1} \cap L_1$.

First we need an auxiliary lemma.

LEMMA 3-1. *Let $\mathfrak{s} \in \mathbf{C}$ so that* $\mathrm{Re}(\mathfrak{s}) > k/2 - 2$. *Then for $T \in W$,*

$$\int_W |Q(Z, Z)|^{\mathfrak{s}} e^{2\pi Q(T,Z)} d^*(Z) = \tfrac{1}{2}\left\{\pi^{-2\mathfrak{s}+(k-4)/2}\Gamma(\mathfrak{s} - k/2 + 2)\Gamma(\mathfrak{s})\right\} |Q(T, T)|^{-\mathfrak{s}}. \tag{3-8}$$

PROOF. We refer to Hilfsatz 1 of **[13]**. Q.E.D.

Then we define for f, a cusp form of weight ν relative to the group Γ_*,

$$R(\mathfrak{s}, f, \gamma) = \int_{F(\Gamma_*^\gamma)} (f|\gamma)(\sqrt{-1}\ Y)|Q(Y, Y)|^{\mathfrak{s}}\, d^*(Y). \tag{3-9}$$

But using the comments above

$$R(\mathfrak{s}, f, \gamma) = \int_0^\infty r^{2\mathfrak{s}-1}\left\{\int_{F_{-1}(\Gamma_*^\gamma)} (f|\gamma)(\sqrt{-1}\ r\cdot\xi)\, d\sigma_{-1}(\xi)\right\} dr. \tag{3-10}$$

The first problem is to determine the convergence of (3-10). Following the usual method we write the integral in (3-10) as

$$\int_0^1 r^{2\mathfrak{s}-1}\{\ \cdots\ \}\, dr + \int_1^\infty r^{2\mathfrak{s}-1}\{\ \cdots\ \}\, dr.$$

Since f is a cusp form of weight ν we have that

$$\left|(f|\gamma)(\sqrt{-1}\ r\cdot\xi)\right| \leqslant r^{-\nu} \sup_{\xi\in\Gamma_{-1}\cap W} \left|(f|\gamma)(\sqrt{-1}\ \xi)\right|.$$

However $(f|\gamma)(\sqrt{-1}\cdot\xi)$ is bounded on $F_{-1}(\Gamma_*^\gamma)$; but $F_{-1}(\Gamma_*^\gamma)$ has finite volume relative to the measure $d\sigma_{-1}$. Hence

$$\int_0^1 r^{2\mathfrak{s}-1}\left|\left\{\int_{F_{-1}(\Gamma_*^\gamma)} (f|\gamma)(\sqrt{-1}\ r\xi)\, d\sigma_{-1}(\xi)\right\}\right| dr$$

$$\leqslant \left\{\sup_{\xi\in F_{-1}(\Gamma_*^\gamma)} \left|f(\sqrt{-1}\cdot\xi)\right|\right\}\left\{\mathrm{vol}\left(F_{-1}(\Gamma_*^\gamma)\right)\right\}\left\{\int_0^1 r^{2\mathfrak{s}-\nu-1}\, dr\right\}. \tag{3-11}$$

The latter integral in (3-11) converges if $\mathrm{Re}(\mathfrak{s}) > \nu/2$. On the other hand we note that

$$\left|(f|\gamma)(\sqrt{-1}\ r\cdot\xi)\right| \leqslant \sum_{\Omega\in\mathfrak{N}(\gamma)^*} |a_\Omega(f|\gamma)| e^{2\pi r Q(\Omega,\xi)}. \tag{3-12}$$

But since f is a cusp form on $\mathfrak{R}_+$ of weight ν we know from **[6]** that $|a_\Omega(f|\gamma)| \leqslant C_f|Q(\Omega, \Omega)|^{\nu/2}$ for all $\Omega\in\mathfrak{N}(\gamma)^*$ (C_f, a positive constant depending only on f). Thus the series on the right-hand side of (3-12) is majorized by

$$\sum_{\Omega\in\mathfrak{N}(\gamma)^*} |Q(\Omega, \Omega)|^{\nu/2} e^{2\pi r Q(\Omega,\xi)} \tag{3-13}$$

with $\xi\in F_{-1}(\Gamma_*^\gamma)$.

But we know that (see **[II**, §5**]**) there is a positive scalar c_γ so that $c_\gamma\cdot\mathfrak{N}(\gamma)^*\subseteq (L_1)_{\mathbf{Z}}$ (with $(L_1)_{\mathbf{Z}}$, the $\mathbf{Z}$ span of a basis $\{U_i\}$ of L_1 so that Q_1 on L_1 becomes $\alpha_1 U_1^2 + \cdots + \alpha_{k-3}U_{k-3}^2 - U_{k-2}^2 = \|U_+\|^2 - \|U_-\|^2$, with $\alpha_1, \ldots, \alpha_{k-3}$ positive rational numbers). Hence it suffices to study (A, some positive constant)

$$\sum_{\substack{\Omega_+\in\mathbf{Z}^{k-3}\\ n\in\mathbf{Z}_+\\ \|\Omega_+\|<n}} \left(n^2 - \|\Omega_+\|^2\right)^{\nu/2} e^{2\pi A r\{[\Omega_+,\xi_+] - n\xi_-\}} \tag{3-14}$$

where $\xi = \xi_+ + \xi_- e_k$ with $\xi_+\in\langle e_2, \ldots, e_{k-2}\rangle$ and $\xi\in F_{-1}(\Gamma_*^\gamma)$. But (3-14) is majorized by the series

$$\sum_{n>0}\left\{\sum_{\substack{\Omega_+\in\mathbf{Z}^{k-3}\\ \|\Omega_+\|<n}} n^{\nu}e^{2\pi Ar[\Omega_+,\xi_+]}\right\}e^{-2\pi Ar\xi_- n}. \tag{3-15}$$

But then

$$\sum_{\substack{\Omega_+\in\mathbf{Z}^{k-3}\\ \|\Omega_+\|<n}} n^{\nu}e^{2\pi Ar[\Omega_+,\xi_+]} \leqslant \left\{\sum_{\substack{\Omega_+\in\mathbf{Z}^{k-3}\\ \|\Omega_+\|<n}} 1\right\} n^{\nu}e^{2\pi Arn\|\xi_+\|}. \tag{3-16}$$

Then noting **[II**, Appendix] we find that (3-16) is majorized by

$$\sum_{n>0} n^{k-3+\nu}e^{-2\pi Ar[\xi_- - \|\xi_+\|]n} \tag{3-17}$$

(where $\xi_- > \|\xi_+\|$ by hypothesis). But using the classical Maclaurin integral test we deduce that (3-17) is majorized by

$$\left(\frac{k-3+\nu}{\pi Ar[\xi_- - \|\xi_+\|]}\right)^{k-3+\nu}\left\{\frac{e^{-\pi Ar[\xi_- - \|\xi_+\|]}}{1-e^{-\pi Ar[\xi_- - \|\xi_+\|]}}\right\}. \tag{3-18}$$

But we recall that $e^{-x}/(1-e^{-x}) \leqslant e^{-x/2}/x$ for $x > 0$. Hence (3-18) is majorized by

$$\left(\pi Ar[\xi_- - \|\xi_+\|]\right)^{-(k-2)-\nu}e^{-\pi(Ar/2)[\xi_- - \|\xi_+\|]}. \tag{3-19}$$

But then

$$\int_1^{+\infty} r^{2\mathfrak{s}-1-\nu-(k-2)}e^{-\pi Ar[\xi_- - \|\xi_+\|]/2}\,dr$$

$$= \left\{\frac{\pi A[\xi_- - \|\xi_+\|]}{2}\right\}^{-2\mathfrak{s}+\nu+(k-2)}\Gamma\big((\pi A/2)[\xi_- - \|\xi_+\|], 2\mathfrak{s}-\nu-(k-2)\big) \tag{3-20}$$

where $\Gamma(\ ,\ )$ is the incomplete gamma function. We note that the above identity holds for all $\mathfrak{s}\in\mathbf{C}$.

On the other hand we recall from [3] that $F_{-1}(\Gamma_*^{\gamma})$ is contained in a set of the following form (for some t and C positive numbers).

$$\bigcup_{i=1}^{i=j} \gamma_i(S_{t,C}) \tag{3-21}$$

where

$$S_{t,T} = \{A_{a+1}(r)N_1(S)[e_k] \mid r \leqslant t \text{ and } S \in T_{r,C}\}$$

with $T_{r,C} = \{U \in L_2 \mid \|U\| \leqslant r\cdot C\}$ and where $\{\gamma_1,\dots,\gamma_j\}$ is a finite set in $O(Q_1)_0$ having the property that

$$\bigcup_{i=1}^{i=j}\{\gamma\Gamma_*\gamma^{-1}\cap O(Q_1)\}\gamma_i\cdot\{O(Q_1)_0\cap P_a\}_{\mathbf{Q}} = (O(Q_1)_0)_{\mathbf{Q}}.$$

We note here that

$$A_{a+1}(r)N_1(S)[e_k] = A_{a+1}(r)N_1(S)\left[(1/\sqrt{2})(\tilde{v}_{a+1} - v_{a+1})\right]$$
$$= (1/\sqrt{2})\left[r^{-1}\tilde{v}_{a+1} - \left[\frac{1+Q(S,S)}{2}\right]rv_{a+1} + S\right].$$

If $\xi = A_{a+1}(r)N_1(S)[e_k]$ then $\xi_- = e^x\|\omega\|^2/4 + \cosh(x)$ and $\|\xi_+\| = \sqrt{\xi_-^2 - 1}$ with $r = e^x$ and $\omega = S$.

We note that $O(Q_1)$ invariant measure $d\sigma_{-1}$ on $S_{t,T}$ is given by $dr/r \otimes dS$.

Then we let $u(x, R) = e^x R^2/4 + \cosh x$. Using the change of variables $r \to r^{-1}$, then there exist positive constants t_1 and T_1 so that ($\mathfrak{s}_0 = \mathrm{Re}(\mathfrak{s})$)

$$\int_{S_{t,T}} [\xi_- - \|\xi_+\|]^{-2\mathfrak{s}_0}\Gamma\left(\frac{\pi A}{2}(\xi_- - \|\xi_+\|), 2\mathfrak{s}_0 - \nu - k + 2\right) d\sigma_{-1}(\xi)$$
$$\leqslant \int_{R=0}^{R=T_1} R^{k-5}\left\{\int_{x=\log t_1}^{+\infty} \Gamma\left(\frac{\pi A}{2}\left(u(x,R) - \sqrt{u(x,R)^2 - 1}\right), 2\mathfrak{s}_0 - \nu - k + 2\right)\right.$$
$$\left.\cdot \left\{u(x,R) - \sqrt{u(x,R)^2 - 1}\right\}^{-2\mathfrak{s}_0} dx\right\} dR. \tag{3-22}$$

An easy argument shows that for fixed x the function

$$R \to V(x,R) = u(x,R) - \sqrt{u(x,R)^2 - 1}$$

for $R \in [0, T_1]$ is a decreasing function. Thus

$$\int_{\log t_1}^{+\infty} \Gamma\left(\frac{\pi A}{2} V(x, T_1), 2\mathfrak{s}_0 - \nu - k + 2\right) H(x)^{-2\mathfrak{s}_0}\, dx \tag{3-23}$$

with

$$H(x) = \begin{cases} V(x, T_1) & \text{if } \mathfrak{s}_0 > 0, \\ V(x, 0) & \text{if } \mathfrak{s}_0 \leqslant 0, \end{cases}$$

majorizes the inner integral on the right-hand side of (3-22). However for fixed R, the function $x \to u(x, R)$ achieves its minimum at x_0 when $R^2 = 2(e^{-2x_0} - 1)$. Then decomposing the interval $[\log t_1, \infty)$ into at most two parts consisting of $[\log t_1, x_0]$ and $[x_0, +\infty)$ and making the change of variables $x \to V(x, T_1)$ in (3-23) we deduce that (3-23) is majorized by an integral of the form

$$\int_M^{+\infty} \Gamma\left(\frac{\pi A}{2} V, 2\mathfrak{s}_0 - \nu - k + 2\right)\psi_{\mathfrak{s}_0}(V)$$
$$\cdot \left|\frac{1}{4}(V^2 + V^{-2}) - \left(\frac{T_1^2 + 1}{2}\right)\right|^{-1/2} \left|V - \frac{1}{V}\right| \frac{dV}{|V|} \tag{3-24}$$

where

$$\psi_{\mathfrak{s}_0}(V) = \begin{cases} |V|^{-2\mathfrak{s}_0} & \text{if } \mathfrak{s}_0 > 0, \\ \left(\dfrac{V^2 + 1}{|V|}\right)^{-2\mathfrak{s}_0} & \text{if } \mathfrak{s}_0 \leqslant 0, \end{cases}$$

with M a positive constant depending only on t.

Then we recall that for $v \geqslant M' > 0$ there exists a constant $C_{M'}$ so that $|\Gamma(v, \alpha)| \leqslant C_{M'}e^{-v}v^{\alpha-1}$. Hence (3-24) is majorized by

$$\int_M^{+\infty} e^{-\pi AV/2}Q_{\mathfrak{s}_0}(V)\left|\frac{1}{4}V^4 - \left(\frac{T_1^2+1}{2}\right)V^2 + \frac{1}{4}\right|^{-1/2} |V^2 - 1|\, dV \quad (3\text{-}25)$$

with

$$Q_{\mathfrak{s}_0}(V) = \begin{cases} |V|^{-(\nu+k)} & \text{if } \mathfrak{s}_0 > 0, \\ |V|^{4\mathfrak{s}_0-(\nu+k)}(V^2+1)^{-2\mathfrak{s}_0} & \text{if } \mathfrak{s}_0 \leqslant 0. \end{cases}$$

But the factoring $V^4 - 2(T_1^2 + 1)V^2 + 1$ into a product of four terms and noting the local integrability of each term (in V) we deduce that the integral in (3-25) is finite. Moreover for $\mathfrak{s}_0 \geqslant M''$ (3-25) is bounded by a constant depending only on M'' and M.

Thus using (3-12), (3-20) and the immediately preceding arguments we deduce that the function

$$\mathfrak{s} \to \int_1^{+\infty} r^{2\mathfrak{s}-1}\left\{\int_{S_{t,T}} (f|\gamma)(\sqrt{-1}\, r\cdot\xi)\, d\sigma_{-1}(\xi)\right\} dr \quad (3\text{-}26)$$

is bounded uniformly in every half space $\{\mathfrak{s} \in \mathbf{C}|\mathrm{Re}(\mathfrak{s}) \geqslant M''\}$. Moreover the function defined by (3-26) is holomorphic in the complex $\mathfrak{s}$-plane.

On the other hand if we replace $(f|\gamma)$ by $(f|\gamma\gamma_i)$ (with γ_i given by (3-21) and note that $\underset{\sim}{\mathfrak{d}}(\gamma_i, Z) = 1$ ($\gamma_i \in O(Q_1)_0$) then by using the same arguments as above

$$\mathfrak{s} \to \int_1^{+\infty} r^{2\mathfrak{s}-1}\left\{\int_{\gamma_i(S_{t,T})} (f|\gamma)(\sqrt{-1}\, r\xi)\, d\sigma_{-1}(\xi)\right\} dr \quad (3\text{-}27)$$

is holomorphic in the complex $\mathfrak{s}$-plane. Thus

$$\mathfrak{s} \to \int_1^{+\infty} r^{2\mathfrak{s}-1}\left\{\int_{F_{-1}(\gamma_*^{\gamma})} (f|\gamma)(\sqrt{-1}\, r\cdot\xi)\, d\sigma_{-1}(\xi)\right\} dr \quad (3\text{-}28)$$

defines a holomorphic function in $\mathfrak{s}$.

Then starting with (3-19) and adapting easily the above arguments we also show that in

$$\int_{F_{-1}(\Gamma_*^{\gamma})}\left\{\sum_{\Omega\in\mathfrak{N}(\gamma)^*} a_\Omega(f|\gamma)e^{2\pi rQ(\Omega,\xi)}\right\} d\sigma_{-1}(\xi) \quad (3\text{-}29)$$

the summation and integration can be interchanged. Thus (3-29) equals

$$\sum_{\Omega\in\mathfrak{N}(\gamma)^*} a_\Omega(f|\gamma)\left\{\int_{F_{-1}(\Gamma_*^{\gamma})} e^{2\pi rQ(\Omega,\xi)}\, d\sigma_{-1}(\xi)\right\}. \quad (3\text{-}30)$$

Then we integrate (3-30) against $r^{2\mathfrak{s}-1}dr$ over $[0, \infty)$ (this is possible by the arguments above) and deduce immediately from Lemma 3-1 that for $\mathrm{Re}(\mathfrak{s}) > \max(\nu/2, k/2 - 2)$

$$R(\mathfrak{s}, f, \gamma) = \tfrac{1}{2}\pi^{-2\mathfrak{s}+(k-4)/2}\Gamma(\mathfrak{s} - k/2 + 2)\Gamma(\mathfrak{s}) \cdot\left\{\sum_{\Omega\in\{\mathfrak{N}(\gamma)^*\}_{\Gamma_*}} a_\Omega(f|\gamma)\frac{1}{\varepsilon(\Omega|\gamma)}|Q(\Omega,\Omega)|^{-\mathfrak{s}}\right\} \quad (3\text{-}31)$$

where the series is absolutely convergent for $\operatorname{Re}(\mathfrak{s}) > \max(\nu/2, k/2 - 2)$.

Then to prove the functional equation and analytic continuation of $R(\mathfrak{s}, f, \gamma)$ we let $h = (f|\gamma c_a)$. Then by change of variables

$$\int_0^1 r^{2\mathfrak{s}-1}\left\{\int_{F_{-1}(\Gamma_*^\gamma)} (f|\gamma)(\sqrt{-1}\, r\cdot\xi)\, d\sigma_{-1}(\xi)\right\} dr$$
$$= \int_1^{+\infty} r^{2\nu-2\mathfrak{s}-1}\left\{\int_{F_{-1}(\Gamma_*^\gamma)} h(\sqrt{-1}\, r\cdot\xi)\, d\sigma_{-1}(\xi)\right\} dr \qquad (3\text{-}32)$$

and

$$\int_1^{+\infty} r^{2\mathfrak{s}-1}\left\{\int_{F_{-1}(\Gamma_*^\gamma)} (f|\gamma)(\sqrt{-1}\, r\cdot\xi)\, d\sigma_{-1}(\xi)\right\} dr$$
$$= \int_0^1 r^{2\nu-2\mathfrak{s}-1}\left\{\int_{F_{-1}(\Gamma_*^\gamma)} h(\sqrt{-1}\, r\cdot\xi)\, d\sigma_{-1}(\xi)\right\} dr.$$

Thus we deduce easily the analytic continuation and functional equation of $R(\mathfrak{s}, f, \gamma)$. Q.E.D.

We recall that if f is an integral form of weight ν on $\mathfrak{R}_+$ that we can define $f^\#$: $O(Q)_0 \to \mathbf{C}$ by

$$f^\#(g^{-1}) = f\left(g\left(\sqrt{-1}\,\sqrt{2e_k}\right)\right)\left[\underset{\sim}{\mathfrak{b}}\left(g, \sqrt{-1}\, e_k\right)\right]^{-\nu}. \qquad (3\text{-}33)$$

Then we observe that $f^\#((gq(\theta))^{-1}) = f^\#(g^{-1})e^{\sqrt{-1}\,\nu\theta}$ (where $g(\theta)$ is the rotation group of the plane $\langle e_{k-1}, e_k\rangle$). Then for $\mathfrak{s}$ so that $\operatorname{Re}(\mathfrak{s}) > \nu/2$,

$$R(\mathfrak{s}, f, \gamma) = 2^{\mathfrak{s}}\int_0^{+\infty} r^{2\mathfrak{s}-\nu}\left\{\int_{O(Q_1)_0/O(Q_1)_0\cap\gamma\Gamma_*\gamma^{-1}} f^\#\left(gA_a(r^{-1})\gamma^{-1}\right) d\lambda(g)\right\}\frac{dr}{r} \qquad (3\text{-}34)$$

where $d\lambda$ is some $O(Q_1)_0$ invariant measure on $O(Q_1)_0/O(Q_1)_0 \cap \gamma\Gamma_*\gamma^{-1}$ and dr/r is the Haar measure on $A_a(r)$.

We are going to determine $R(\mathfrak{s}, f, \gamma)$ for special choices of f, i.e., for those given by (1-3). For such a computation we use (3-34) as the starting point.

EXAMPLE 3-2. We consider a special case of the Dirichlet series defined above. We know that if $k = 5$ then $O(Q)_0$ is isomorphic to the symplectic group in two variables, i.e., $Sp_2(\mathbf{R})$. We make this correspondence explicit. We recall that $Sp_2(\mathbf{R})$ is defined as the subgroup of $q \in SL_4(\mathbf{R})$ satisfying $gJg^t = J$, where J is the 4×4 skew symmetric matrix

$$\left(\begin{array}{c|c} 0 & I_2 \\ \hline -I_2 & 0 \end{array}\right)$$

with I_2, the 2×2 identity matrix. This implies that relative to the standard basis $\vec{e}_1, \ldots, \vec{e}_4$ of $\mathbf{R}^4$ the alternating form $\vec{e}_1 \wedge \vec{e}_3 + \vec{e}_2 \wedge \vec{e}_4$ (an element of $\Lambda^2(\mathbf{R}^4)$) is invariant by $g \in Sp_2(\mathbf{R})$ when the natural linear action of $SL_4(\mathbf{R})$ is extended to a linear action in $\Lambda^2\mathbf{R}^4$. We know that for any two elements α, β of $\Lambda^2\mathbf{R}^4$ the product $\alpha \wedge \beta$ is a multiple of $\varepsilon_0 = \vec{e}_1 \wedge \vec{e}_2 \wedge \vec{e}_3 \wedge \vec{e}_4$ in $\Lambda^4(\mathbf{R}^4)$. Thus $\alpha \wedge \beta = \phi(\alpha, \beta)\varepsilon_0$ and it is easy to see that ϕ defines a nondegenerate quadratic form in $\Lambda^2(\mathbf{R}^4)$.

Moreover for $g \in SL_4(\mathbf{R})$ we deduce that $\phi(g\alpha, g\beta) = \det(g)\phi(\alpha, \beta)$. Hence $SL_4(\mathbf{R})$ (modulo its center) via the representation σ_2 on $\Lambda^2(\mathbf{R}^4)$ is a subgroup of $O(\phi)_0$. But by dimensional considerations (i.e., $\dim SL_4(\mathbf{R}) = \dim O(\phi)_0$) we have that $\sigma_2(SL_4(\mathbf{R})) = O(\phi)_0$.

By simple computation we deduce that $\langle \vec{e}_1 \wedge \vec{e}_3, \vec{e}_2 \wedge \vec{e}_4 \rangle$, $\langle \vec{e}_1 \wedge \vec{e}_4, \vec{e}_2 \wedge \vec{e}_3 \rangle$ and $\langle \vec{e}_1 \wedge \vec{e}_2, \vec{e}_3 \wedge \vec{e}_4 \rangle$ span hyperbolic planes relative to ϕ. Thus ϕ has signature type (3, 3). On the other hand since $\sigma_2(Sp_2)$ leaves $\vec{e}_1 \wedge \vec{e}_3 + \vec{e}_2 \wedge \vec{e}_4$ invariant we deduce that $\sigma_2(Sp_2(\mathbf{R}))$ operates on $(\vec{e}_1 \wedge \vec{e}_3 + \vec{e}_2 \wedge \vec{e}_4)^{\perp}$ in $\Lambda^2(\mathbf{R}^4)$ as orthogonal transformations relative to ϕ. Again by dimensional considerations we deduce that $\sigma_2(Sp_2(\mathbf{R}))$ (restricted to $(\vec{e}_1 \wedge \vec{e}_3 + \vec{e}_2 \wedge \vec{e}_4)^{\perp}$) equals $O(\phi^*)_0$ where ϕ^* is ϕ restricted to $(\vec{e}_1 \wedge \vec{e}_3 + \vec{e}_2 \wedge \vec{e}_4)^{\perp}$. Thus $Sp_2(\mathbf{R})$ modulo $\pm I_4$ is isomorphic to $O(\phi^*)_0$ where ϕ^* has signature type (3, 2).

Then following the notation used in §0 we let $v_3 = \vec{e}_1 \wedge \vec{e}_4$, $\tilde{v}_3 = (\vec{e}_2 \wedge \vec{e}_3)$, $v_2 = \vec{e}_1 \wedge \vec{e}_2$, $\tilde{v}_2 = (\vec{e}_3 \wedge \vec{e}_4)$ and $e_3 = (1/\sqrt{2})(\vec{e}_1 \wedge \vec{e}_3 - \vec{e}_2 \wedge \vec{e}_4)$. Then by computation we deduce that

$$\sigma_2\left(\begin{array}{c|c} I & M \\ \hline 0 & I \end{array}\right) = N_1\left(av_3 + (-c)\tilde{v}_3 + (-b\sqrt{2})e_3\right)$$

where $M = \left[\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right]$, a 2×2 real symmetric matrix. Thus it follows that $\det(M) = -\frac{1}{2}\phi^*(X_M, X_M)$ with $X_M = av_3 - c\tilde{v}_3 + (-b\sqrt{2})e_3$ (with M as above). Moreover we have that

$$\sigma_2\left(\left(\begin{array}{c|c} I & gMg^t \\ \hline 0 & I \end{array}\right)\right) = N_1(\sigma_2(g) \cdot (X_M))$$

with $g \in SL_2(\mathbf{R})$ being identified with the matrix

$$\left[\begin{array}{c|c} g & 0 \\ \hline 0 & (g^t)^{-1} \end{array}\right]$$

in $Sp_2(\mathbf{R})$.

Let L be the lattice in $(\vec{e}_1 \wedge \vec{e}_3 + \vec{e}_2 \wedge \vec{e}_4)^{\perp}$ given by $\sqrt{2}\ \{\mathbf{Z}v_2 \oplus \mathbf{Z}\tilde{v}_2 \oplus \mathbf{Z}v_3 \oplus \mathbf{Z}\tilde{v}_3 \oplus \mathbf{Z}e_3/\sqrt{2}\}$. Then it is easy to see that $\sigma_2(Sp_2(\mathbf{Z}))$ leaves L stable.

Then we recall from {**1**} that a modular parabolic form of genus 2 and degree ν is a holomorphic function f on $H_2 = \{X + \sqrt{-1}\ Y | X, Y$ real symmetric 2×2 matrices with $Y > 0\}$ so that for all $Z \in H_2$

$$f\left(\frac{AZ + B}{CZ + D}\right) = \{\det(CZ + D)\}^{+\nu} f(Z)$$

with

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \in Sp_2(\mathbf{Z}).$$

The associated Fourier expansion of f is given by

$$f(Z) = \sum_{N > 0} s_f(N) e^{2\pi\sqrt{-1}\,\mathrm{Tr}(N \cdot Z)} \tag{3-35}$$

where N in the summation above runs over all symmetric matrices $N = \left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)$ so that $a, c \in \mathbf{Z}$ and $b \in \frac{1}{2}\mathbf{Z}$. And in **[6]** the associated Dirichlet series to f is given by

$$\sum_{\{N\}} \frac{1}{\varepsilon(N)} s_f(N)(\det N)^{-\mathfrak{s}} \tag{3-36}$$

where the summation is over $SL_2(\mathbf{Z})$ equivalence classes of semi-integral symmetric matrices (via the action $N \rightsquigarrow gNg^t$) and $\varepsilon(N) =$ the number of elements $U \in SL_2(\mathbf{Z})$ satisfying $UNU^t = N$.

Then via the identification $U + \sqrt{-1}\, V \to X_U + \sqrt{-1}\, X_V$ with $U + \sqrt{-1}\, V \in H_2$, we have an analytic isomorphism of H_2 onto $\mathfrak{R}_+$ which commutes with the $Sp_2(\mathbf{R})$ action in the following manner. If $g \in Sp_2(\mathbf{R})$ then $\sigma_2(g)(X_{U+\sqrt{-1}\,V}) = X_{g(U+\sqrt{-1}\,V)}$.

Moreover we deduce by an easy computation (see *Example* 3-1) that $\mathfrak{N} = \{\xi \in L_1 | N_1(\xi)(L) = L\} = \mathbf{Z}v_3 \oplus \mathbf{Z}\tilde{v}_3 \oplus \mathbf{Z}\sqrt{2}\, e_3$. Thus $\mathfrak{N}^* = \mathbf{Z}v_3 \oplus \mathbf{Z}\tilde{v}_3 \oplus \mathbf{Z}e_3/\sqrt{2}$.

Then via the identification of a holomorphic f on $\mathfrak{R}_+$ satisfying (3-1) with a holomorphic f^* on H_2 (i.e. $f^*(U + \sqrt{-1}\, V) = f(X_U + \sqrt{-1}\, X_V)$), we deduce that $s_{f^*}(N) = a_{X_N}(f)$. Indeed we note here the relationship $\mathrm{Tr}(W \cdot M) = -Q(X_W, X_{M'})$ with

$$M' = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} M \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then we equate (3-36) to (3-2) and use the invariance of $s_f(\ )$ under the action of $GL_2(\mathbf{Z})$.

4. Mellin transform of F_f. In this section we are going to determine $R(\mathfrak{s}, f, e)$ for $f^{\#}$ given by (1-3) (see (3-34)). We first recall the following data in the problem.

Let Q_1 be a quadratic form on $\mathbf{R}^{k-2}$ and $\mathfrak{L}$ a Q_1 integral lattice in $\mathbf{R}^{k-2}$. Let t be a positive integer so that $4|t$. Let $L_{(t,1)}$ be the lattice in $\mathbf{R}^2$ constructed in §2. Then we consider the lattice in $\mathbf{R}^k = \mathbf{R}^{k-2} \oplus \mathbf{R}^2$ given by the direct sum $\mathfrak{L} \oplus L_{(t,1)}$. Here $\mathfrak{L} \oplus L_{(t,1)}$ is a Q integral lattice, where $Q = Q_1 \oplus 2xy$ and

$$2xy = Q(xv + y\tilde{v}, xv + y\tilde{v})$$

(see §2). Let $N_L = \mathrm{lcm}(t, N_{\mathfrak{L}})$.

Also recall here that $\mathfrak{L} \subseteq \mathbf{Q} \otimes \{e_2, \ldots, \hat{e}_{a+1}, \ldots, e_k\}$ (i.e., $\hat{e}_{a+1}$ denotes e_{a+1} is omitted) and that $v = \sqrt{2}\, v_a$, $\tilde{v} = \tilde{v}_a/\sqrt{2}$.

Let $f \in [\Gamma_0(N_L), |s|, s^L_{Q,\chi}]^*_0$ (see §0 for this definition). Noting that $4|t$, then the space $[\Gamma_0(N_L), |s|, s^L_{Q,\chi}]^*_0$ coincides with $S_{2|s|}(N_L, \lambda^L_{Q,|s|} \otimes \chi)^*$, which consists of all holomorphic cusp forms $f\colon \overline{H} \to \mathbf{C}$ satisfying $f(\gamma(z)) = (\lambda^L_{Q,|s|} \otimes \chi)(d_\gamma) j(\gamma, z)^{2|s|} f(z)$ with

$$\lambda^L_{Q,s}(\delta) = \left(\frac{2}{\delta}\right)^k \left(\frac{D_{Q(L)}}{\delta}\right) \left(\frac{-1}{\delta}\right)^{|s| - k/2}$$

and χ a Dirichlet character mod N_L and $\gamma \in \Gamma_0(N_L)$.[1] We note here that $S_{2|s|}(N_L, \lambda^L_{Q,|s|} \otimes \chi)^*$ is the isomorphic image of the space

$$S_{2|s|}\left(N_L, \lambda^L_{Q,|s|} \otimes \chi \cdot \left(\frac{-1}{*}\right)^{2|s|}\right)$$

via the map $g(w) \rightsquigarrow \overline{g(\overline{w})}$ (see 7 of §0).

Thus if $h \in S_{2|s|}(N_L, \beta)$ (for any Dirichlet character β mod N_L) then $h_1(w) = \overline{h(\overline{w})}$ belongs to $S_{2|s|}(N_L, \lambda^L_{Q,|s|} \otimes \chi_\beta)^*$, with

$$\chi_\beta(x) = \overline{\beta(x)} \otimes \left(\frac{2}{x}\right)^k \left(\frac{D_{Q(L)}}{x}\right) \cdot \left(\frac{-1}{x}\right)^{s+k/2}.$$

In particular we note that

$$\chi_\beta(-1) = \beta(-1)\left(\frac{D_{Q(L)}}{-1}\right)(-1)^{s_2}$$

(here $s_2 = s + \frac{1}{2}k - 2$). But we note here that $S_{2|s|}(N_L, \beta) = \{0\}$ if $\beta(-1) = -1$. Hence we can assume that $\beta(-1) = 1$. Also $\operatorname{sgn}(D_{Q(L)}) = (-1)^b$ (b = number of $-$ signs = 2). Hence $\chi_\beta(-1) = (-1)^{s_2}$.

Let φ be given by (1-7) with $|s| > \frac{1}{2}k$. Then we consider (1-3).

$$F_f(g|\varphi, L, \eta_1, \chi, \Gamma_0(N_L)) = \left\langle \check{T}^L_{\varphi,\eta_1,\chi}(z, g) | f(z)\right\rangle_{\mathfrak{D}^*_{\Gamma_0(N_L)}} \tag{4-1}$$

where $\langle \mid \rangle$ represents the Petersson inner product over the fundamental domain $\mathfrak{D}^*_{\Gamma_0(N)}$ in $\overline{H}$.

To be consistent with the notation set forth in *Proposition* 2-1, we see that $\eta_1 = v$.

Thus the problem is to give an explicit determination of

$$\int_0^{+\infty} r^{2\mathfrak{s} - s_2}\left\{\int_{O(Q_1)_0/O(Q_1)_0 \cap \Gamma^L(Q)} F_f\big(A_q(r^{-1})m|\varphi, L, v, \chi, \Gamma_0(N_L)\big)\, d\lambda(m)\right\} \frac{dr}{r} \tag{4-2}$$

for $\mathfrak{s}$ so that $\operatorname{Re}(\mathfrak{s}) > \frac{1}{2}s_2$. We note here $F_f(g|\cdots)$ has weight s_2 relative to $\mathfrak{R}_+$ (see [**II**, §7]).

The computation of (4-2) will be accomplished in a series of steps starting with (2-19). Then substituting in (4-2) and assuming that we can change the various orders of integration, we deduce that (4-2) equals

$$2^{|s|-1} \sum_{r \in X^-} |r|^{|s|-1}\left\{\sum_{\nu \bmod t} \chi(\nu)\Phi(\nu, \mathfrak{s}, r)\right\}\langle G_{|s|}(\ , s^L_{Q,\chi}, \Gamma_0(N_L), r)|f\rangle_{\mathfrak{D}_{\Gamma_0(N_L)}}, \tag{4-3}$$

with

$$\phi(\nu, \mathfrak{s}, r) = \int_0^{+\infty} p^{2\mathfrak{s} - s_2}\left\{\int_{O(Q_1)_0/O(Q_1)_0 \cap \Gamma^L(Q)} \sum_{\substack{\{\xi \in \mathfrak{L} | Q(\xi,\xi) = 2r\} \\ j \in \mathbf{Z}}} \cdot h_-\big(mA_a(p^{-1})[\xi + (jt + \nu)v]\big)\, d\lambda(m)\right\} \frac{dp}{p}. \tag{4-4}$$

The procedure is first to evaluate (4-4) and then to show that the various changes in order of integration can be done.

We recall that $h_-(X) = Q(X, e_{a+1} + \sqrt{-1}\, e_{a+2})^{-s_2}$ (on Ω_-) is used in *Theorem*

2-1. Then we have

$$\sum_{\substack{\{\xi \in \mathfrak{L} \mid Q(\xi,\xi)=2r\} \\ j \in \mathbf{Z}}} h_{-}(mA_a(p)[\xi + (jt+\nu)v])$$

$$= (pt)^{-s_2} \sum_{\substack{\{\xi \in \mathfrak{L} \mid Q(\xi,\xi)=2r\} \\ j \in \mathbf{Z}}} \left\{ j + \frac{1}{t}\left(\nu + \frac{1}{p}\sqrt{-1}\, Q(m\xi, e_{a+2})\right)\right\}^{-s_2}. \quad (4\text{-}5)$$

But then we recall the formula

$$\sum_{l \in \mathbf{Z}} \left(\frac{1}{w+l}\right)^m = \frac{(2\pi\sqrt{-1}\,)^m}{(m-1)!} \sum_{r=1}^{+\infty} r^{m-1} e^{2\pi\sqrt{-1}\, wr}, \quad (4\text{-}6)$$

with $w \in \mathbf{C}$ so that $\operatorname{Im}(w) > 0$. Hence we deduce from (4-5) and (4-6) that (4-5) equals

$$(pt)^{-s_2} \frac{(2\pi\sqrt{-1}\,)^{s_2}}{(s_2-1)!} \sum_{l > 1} \left\{ \sum_{\{\xi \in \mathfrak{L} \mid Q(\xi,\xi)=2r,\, Q(\xi,e_k)<0\}} e^{(2/pt)l\pi Q(m\xi, e_{a+2})} \right\}$$

$$\cdot\, l^{s_2-1}\left(e^{2\pi\sqrt{-1}\, l\nu/t} + (-1)^{s_2} e^{-2\pi\sqrt{-1}\, l\nu/t}\right). \quad (4\text{-}7)$$

At this point we make precise the normalization of measures used in the ensuing discussion. First we fix a Haar measure $d\sigma$ on $O(Q_1)_0$. Then we choose $O(Q_1)_0$ invariant measures on the hyperboloid

$$\tilde{\Gamma}_w \cap L_1 = \{Z \in L_1 \mid Q(Z,Z) = w,\, Q(Z, e_k) < 0\} \cong O(Q_1)_0 / O(Q_1)_0^{T_w},$$

with $T_w = \tilde{v}_{a+1} + \frac{1}{2} w v_{a+1}$, in the following manner. We recall from [**I**, §5] that the map

$$(p, Y) \rightsquigarrow A_{a+1}(p) N_1(Y)\left[\tilde{v}_{a+1} + \tfrac{1}{2} w v_{a+1}\right]$$

is a diffeomorphism from $\mathbf{R}_*^+ \times \mathbf{R}^{k-4}$ to $\tilde{\Gamma}_w \cap L_1$. Then we let $d\mu_w$ be the $O(Q_1)_0$ invariant measure on $\tilde{\Gamma}_w \cap L_1$ so that $d\mu_w = dp/p \otimes dY$ (see [**I**, §5]). Then for T_w we let dv_w be the Haar measure on $O(Q_1)_0^{T_w}$ so that $d\sigma = dv_w \otimes d\mu_w$.

Let $d\lambda$ be the $O(Q_1)_0$ invariant measure on $O(Q_1)_0 / O(Q_1)_0 \cap \Gamma^L(Q)$ deduced from $d\sigma$ and the counting measure on $O(Q_1)_0 \cap \Gamma^L(Q)$.

LEMMA 4-1. *Let*

$$G_w(p, m) = \sum_{l > 1} \left\{ \sum_{\{\xi \in \mathfrak{L} \mid Q(\xi,\xi)=w,\, Q(\xi,e_k)<0\}} e^{2\pi(lp/t)Q(m\xi, e_{a+2})} \right\} l^{s_2-1}$$

for $m \in O(Q_1)_0$ and $p > 0$. If $p \geqslant D > 0$, then

$$\int_{O(Q_1)_0/O(Q_1)_0 \cap \Gamma^L(Q)} G_w(p, m)\, d\lambda(m)$$

$$\leqslant C \operatorname{vol}\left(O(Q_1)_0^{T_w}\right) M(Q_1, \mathfrak{L}, w) p^{-(s_2+1)} S_k(p, w) e^{-\alpha p w^{1/2}}, \quad (4\text{-}8)$$

where

$$S_k(p, w) = \begin{cases} w^{(k-5-s_2)/2} & \text{if } k \geqslant 6, \\ A w^{-s_2/s} p + B w^{-s_2/2+1/2} & \text{if } k = 5, \end{cases}$$

with vol( ) *taken relative to the measure* v_w, $M(Q_1, \mathfrak{L}, w)$ *the Siegel mass number defined in* §0, *and* A, B, α *and* C *certain positive constants depending only on* D.

PROOF. For fixed $\xi \in \mathfrak{L}$ we consider the sum

$$\sum_{l>1} l^{s_2-1} e^{2\pi(lp/t)Q(m\xi, e_{a+2})}. \tag{4-9}$$

But by using classical arguments (Maclaurin integral test), we deduce that (4-9) is majorized by

$$p^{-s_2}|Q(m\xi, e_{a+2})|^{-s_2} e^{(-1/2)\pi(p/t)|Q(m\xi, e_{a+2})|}.$$

Then we consider the sum

$$p^{-s_2} \sum_{\{\xi \in \mathfrak{L} | Q(\xi,\xi)=w,\ Q(\xi,e_k)<0\}} |Q(m\xi, e_{a+2})|^{-s_2} e^{(-1/2)\pi(p/t)|Q(m\xi, e_{a+2})|}. \tag{4-10}$$

Then we deduce

$$\int_{O(Q_1)_0/O(Q_1)_0 \cap \Gamma^L(Q)} G_w(p, m)\, d\lambda(m)$$
$$\leq p^{-s_2}\left\{ \sum_{i=1}^{h(w)} \mathrm{vol}\big(O(Q_1)_0^{\xi_i^w} / O(Q_1)_0^{\xi_i^w} \cap \Gamma^L(Q)\big)\right\}$$
$$\cdot \left\{ \int_{O(Q_1)_0/O(Q_1)_0^{T_w}} |Q(mT_w, e_{a+2})|^{-s_2} e^{(-1/2)\pi(p/t)|Q(mT_w, e_{a+2})|}\, d\mu_w(m)\right\}, \tag{4-11}$$

where vol(/) is taken relative to the measure induced on the homogeneous space

$$O(Q_1)_0^{\xi_i^w}/O(Q_1)_0^{\xi_i^w} \cap \Gamma^L(Q) = O(Q_1)_0^{T_w}/O(Q_1)_0^{T_w} \cap \delta_w \Gamma^L(Q)\delta_w^{-1}$$

(with $\delta_w \in O(Q_1)_0$ so that $\delta_w(\xi_i^w) = T_w$) by dv_w and the counting measure on $O(Q_1)_0^{T_w} \cap \delta_w \Gamma^L(Q)\delta_w^{-1}$.

We then note that

$$\mathrm{vol}\big(O(Q_1)_0^{\xi_i^w} / O(Q_1)_0^{\xi_i^w} \cap \Gamma^L(Q)\big) = \mathrm{vol}\big(O(Q_1)_0^{T_w}\big)\frac{1}{\#\{O(Q_1)_0^{\xi_i^w} \cap \Gamma^L(Q)\}}.$$

Thus we deduce that the right-hand side of (4-11) equals

$$M(Q_1, \mathfrak{L}, w)\mathrm{vol}\big(O(Q_1)_0^{T_w}\big)\int_{O(Q_1)_0/O(Q_1)_0^{T_w}} \psi(Q(m \cdot T_w, e_{a+2}))\, d\mu_w(m), \tag{4-12}$$

where $\psi(x) = x^{-s_2}e^{(-1/2)\pi(p/t)x}$ for $x > 0$.

Then the integral in (4-12) equals

$$\int_{\mathbf{R}^{k-4}\times\mathbf{R}_+} \{S(w, W, x)\}^{-s_2} \exp\left(-\frac{1}{2}\pi\frac{p}{t}\frac{1}{x}|S(w, W, x)|\right)x^{s_2}\, dW\, \frac{dx}{x}, \tag{4-13}$$

with $S(w, W, x) = \frac{1}{2}(w - Q(W, W))x^2 - 1$. But since Q is positive definite on $\mathbf{R}^{k-4}$ ($\cong L_2$), we obtain, by using polar coordinates in $\mathbf{R}^{k-4}$, that (4-13) equals

$$\int_{\substack{z>-w\\ x>0}} (zx^2 + 1)^{-s_2} \exp\left(-\frac{1}{2}\pi\frac{p}{t}\frac{1}{x}(zx^2 + 1)\right)x^{s_2}(z + w)^{(k-6)/2}\, dz\, \frac{dx}{x}. \tag{4-14}$$

Then, letting $v = zx^2$ above, we have that (4-14) is majorized by

$$\int_{\substack{z > -w \\ v > 0}} (v + 1)^{-s_2} v^{s_2/2} \exp\left(-\frac{1}{2}\pi\frac{p}{t} z^{1/2}(v^{1/2} + v^{-1/2})\right) z^{-s_2/2}(z + w)^{(k-6)/2} \frac{dv}{v}\, dz. \tag{4-15}$$

Then since $v^{1/2} + v^{-1/2} \geqslant 1$, (4-15) is majorized by

$$\left\{\int_{z > -w} \exp\left(-\frac{1}{2}\pi\frac{p}{t} z^{1/2}\right) z^{-s_2/2}(z + w)^{(k-6)/2}\, dz\right\} \cdot \left\{\int_{v > 0} (v + 1)^{-s_2} v^{s_2/2 - 1}\, dv\right\}. \tag{4-16}$$

However $s_2 > 0$ and $k \geqslant 5$. Thus both integrals above are convergent. By using the asymptotic properties of the incomplete gamma function, we deduce that the first integral in (4-16) is majorized by

$$\begin{cases} w^{(k-5-s_2)/2} p^{-1} e^{-\alpha p w^{1/2}} & \text{for } \alpha \text{ some positive constant if } k \geqslant 6, \\ (A w^{-s_2/2} p + B w^{-s_2/2 + 1/2}) p^{-1} e^{-\alpha p w^{1/2}} & (\text{if } k = 5) \text{ with } A, B, \alpha \text{ constants.} \end{cases} \tag{4-17}$$

Thus we deduce (4-8). Q.E.D.

PROPOSITION 4-1. *Let the same hypotheses hold as in Theorem* 2-1. *Then for each* $p > 0$

$$\int_{O(Q_1)_0 / O(Q_1)_0 \cap \Gamma^L(Q)} \check{T}^L_{\varphi,v,\chi}(z, A_a(p^{-1})m)\, d\lambda(m)$$
$$= \left(\frac{p}{t}\right)^{s_2} \frac{(2\pi\sqrt{-1}\,)^{s_2}}{(s_2 - 1)!} \{1 + (-1)^{s_2}\chi(-1)\} 2^{|s| + 1 - (k/2)}$$
$$\cdot \left\{ \sum_{\{r \in \mathbf{Z} | r < 0\}} |r|^{|s| + 1 - k/2} G_{|s|}(z, s^L_{Q,\chi}, \Gamma_0(N_L), r) M(Q_1, \mathfrak{L}, 2r) \cdot \left\{\left\{ \sum_{l \geqslant 1} l^{s_2 - 1} G(\chi, l, t) \psi(l, p, r) \right\}\right\}\right\}, \tag{4-18}$$

where $G(\chi, l, t)$ *is the Gauss sum, given after* (2-11), *and*

$$\psi(l, p, r) = 2\int_r^{+\infty} K_0\left(2\sqrt{2}\, \pi \frac{lp}{t} \sqrt{u}\,\right)(u - r)^{(k-6)/2}\, du \tag{4-19}$$

in terms of the K_0 *Bessel function.*

PROOF. To show that the integral in (4-1) exists, we must prove from (4-8) with $p \geqslant D > 0$ that

$$\sum_{\{r \in \mathbf{Z}, r < 0\}} |r|^{|s| - 1} |G_{|s|}(z, s^L_{Q,\chi}, \Gamma_0(N_L), r)| M(Q_1, \mathfrak{L}, 2r) \cdot \operatorname{vol}(O(Q_1)^{T_{2r}}) e^{-\alpha p |2r|^{1/2}} S_k(p, 2r) < \infty. \tag{4-20}$$

But for $z \in \overline{H}$, the lower half-plane, we know that $G_{|s|}(z, s^L_{Q,\chi}, \Gamma_0(N_L), r)$ is majorized by a series of the form

$$Q(z, |s|) = \sum_{(m_1,m_2)\in \mathbf{Z}^2 - \{0,0\}} |m_1 z + m_2|^{-|s|}.$$

Hence (4-20) is majorized by the product

$$\left\{ \sum_{m \geq 1} m^{|s|-1} M(Q_1, \mathfrak{L}, -m) \operatorname{vol}\left(O(Q_1)_0^{T_{-m}}\right) e^{-\alpha p m^{1/2}} S_k(m, p) \right\} Q(z, |s|). \quad (4\text{-}21)$$

As a simple application of the definition of v_{-m} we can show easily that $\operatorname{vol}(O(Q_1)_0^{T_{-m}}) = |m|^{2-k/2} \operatorname{vol}(O(Q_1)_0^{T_{-1}})$. With this in mind we deduce that the series in (4-21) is convergent. It is also evident that $Q(z, |s|)$ is a convergent series. Thus the integral in (4-18) is finite. Then we deduce that the left-hand side of (4-18) equals

$$\left(\frac{p}{t}\right)^{s_2} \frac{(2\pi\sqrt{-1}\,)^{s_2}}{(s_2 - 1)!} \{1 + (-1)^{s_2}\chi(-1)\} 2^{|s|-1}$$

times

$$\sum_{r \geq 1}^{+\infty} |r|^{|s|-1} G_{|s|}\left(z, s_{Q,\chi}^L, \Gamma_0(N_L), r\right)$$

$$\cdot \left\{ \sum_{l \geq 1} l^{s_2 - 1} G(\chi, l, t) \int_{O(Q_1)_0 / O(Q_1)_0 \cap \Gamma^L(Q)} \right.$$

$$\left. \cdot \left\{ \left\{ \sum_{\{\xi \in \mathfrak{L} \mid Q(\xi,\xi) = 2r,\ Q(\xi, e_k) < 0\}} \exp\left(2\pi \frac{lp}{t} Q(m\xi, e_{a+2})\right) \right\} \right\} d\lambda(m) \right\}. \quad (4\text{-}22)$$

However using an argument similar to the one in (4-11), we deduce that the integral in (4-22) has the form

$$M(Q_1, \mathfrak{L}, 2r) \operatorname{vol}\left(O(Q_1)_0^{T_{-2r}}\right) \left\{ \int_{O(Q_1)_0 / O(Q_1)_0^{T_{-2r}}} \exp\left(2\pi \frac{lp}{t} Q(mT_{-2r}, e_{a+2})\right) d\mu_{2r}(m) \right\}. \quad (4\text{-}23)$$

Thus it suffices to evaluate the integral in (4-23). By using an argument similar to the one in Lemma 4-1, we deduce that the integral in (4-23) equals

$$\int_r^{+\infty} \left\{ \int_0^{+\infty} \exp\left(- \sqrt{2}\, \pi \frac{lp}{t} (x + u^{-1} x)\right) \frac{dx}{x} \right\} (u - r)^{(k-6)/2}\, du. \quad (4\text{-}24)$$

By using the integral representation of the K_0 Bessel function we have the desired result. Q.E.D.

THEOREM 4-1. *Let* $k \geqslant 5$. *Suppose the same hypotheses hold as in Proposition* 4-1. *Let* $f \in S_{2|s|}(N_L, \lambda_{Q,|s|}^L \otimes \chi)^*$ (*with* χ *a Dirichlet character* mod t). *Then consider the Fourier expansion at* ∞ *of* f *given by*

$$f(z) = \sum_{n \leqslant -1} a_n(f) e^{2\pi\sqrt{-1}\, nz}.$$

Then there exists a positive number μ_{k,s_2} (depending only on s_2 and k) so that for $\mathrm{Re}(\mathfrak{s}) \geqslant \mu_{k,s_2}$

$$\int_0^{+\infty} p^{2\mathfrak{s}-s_2}\left\{\int_{O(Q_1)_0/O(Q_1)_0\cap\Gamma^L(Q)} F_f(A_a(p^{-1})m|\varphi, L, v, \chi, \Gamma_0(N_L))\, d\lambda(m)\right\}\frac{dp}{p}$$
$$= c_1\left\{t^{2\mathfrak{s}-s_2}\Gamma(\mathfrak{s})\Gamma\left(\mathfrak{s}+2-\tfrac{1}{2}k\right)\pi^{-2\mathfrak{s}}2^{-\mathfrak{s}}\right\}G(\chi, 1, t)L(\bar{\chi}, 2\mathfrak{s}+1-s_2)$$
$$\sum_{\{n\in\mathbf{Z}, n<-1\}}\overline{a_n(f)}\, M(Q_1, \mathfrak{L}, 2n)|n|^{-\mathfrak{s}}, \tag{4-25}$$

where c_1 is a nonzero constant independent of $\mathfrak{s}$ and t.

Proof. For $p \geqslant D > 0$, we deduce from (4-8) and (4-18) that

$$\int_D^{+\infty}\left\{\sum_{\{r\in\mathbf{Z},\, r<-1\}}|r|^{|s|+1-k/2}|G_{|s|}(z, s^L_{Q,\chi}, \Gamma_0(N_L), r)|\right.$$
$$\left.\cdot M(Q_1, \mathfrak{L}, 2r)e^{-\alpha p|r|^{1/2}}S_k(p, 2r)\right\}p^{2\mathfrak{s}_0-(1+s_2)}\frac{dp}{p}$$
$$\leqslant Q(z, |s|)\left\{\int_D^{+\infty}\left\{\sum_{\{r\in\mathbf{Z},\, r<-1\}}|r|^{|s|+1-(k/2)}\right.\right.$$
$$\left.\left.\cdot M(Q_1, \mathfrak{L}, 2r)e^{-\alpha p|2r|^{1/2}}S_k(p, 2r)\right\}p^{2\mathfrak{s}_0-(1+s_2)}\frac{dp}{p}\right\}, \tag{4-26}$$

where $\mathfrak{s}_0 = \mathrm{Re}(\mathfrak{s})$. Then using the explicit form of $S_k(p, 2r)$, we deduce that there is a positive constant δ_{k,s_2} so that

$$\sum_{r\in\mathbf{Z}, r<-1}|r|^{|s|+1-k/2}M(Q_1, \mathfrak{L}, 2r)$$
$$\cdot\left\{\int_D^{+\infty} e^{-\alpha p r^{1/2}}S_k(p, 2r)\cdot p^{2\mathfrak{s}_0-(1+s_2)}\frac{dp}{p}\right\}$$
$$\leqslant\left\{\sum_{\{r\in\mathbf{Z},\, r<-1\}}|r|^{-\mathfrak{s}_0+\delta_{k,s_2}}M(Q_1, \mathfrak{L}, 2r)\right\}\Gamma(2\mathfrak{s}_0-s_2). \tag{4-27}$$

Then the series above converges if $\mathfrak{s}_0 > \delta'_{k,s_2}$ (δ'_{k,s_2} some positive constant), and the Γ function is defined if $\mathfrak{s}_0 > \frac{1}{2}(1+s_2)$.

On the other hand if $0 \leqslant p \leqslant D$, then we cannot apply (4-8) directly. However adapting the argument in Lemma 4-1, it suffices to study

$$\int_0^D\left\{\sum_{\{r\in\mathbf{Z},\, r<-1\}}|r|^{|s|+1-k/2}M(Q_1, \mathfrak{L}, 2r)\cdot\left\{\sum_{l>1} l^{s_2-1}\psi(l, p, r)\right\}\right\}p^{2\mathfrak{s}_0}\frac{dp}{p}. \tag{4-28}$$

Moreover from [7] we know that for $\mathrm{Re}(\mathfrak{z}) > k - 4$

$$\int_0^{+\infty} \psi(l, p, r) p^{\mathfrak{z}} \frac{dp}{p}$$

$$= 2\int_r^{+\infty} (u - r)^{(k-6)/2} \left\{ \int_0^{+\infty} K_0\left(2\sqrt{2}\, \pi \frac{l}{t} \sqrt{u}\, p\right) p^{\mathfrak{z}} \frac{dp}{p} \right\} du$$

$$= 2^{\mathfrak{z}-1} \left(\frac{2\sqrt{2}\, \pi l}{t} \right)^{-\mathfrak{z}} \left\{ \Gamma\left(\tfrac{1}{2}\mathfrak{z}\right) \right\}^2 \left\{ \int_r^{+\infty} u^{-\mathfrak{z}/2} (u - r)^{(k-6)/2}\, du \right\}$$

$$= 2^{\mathfrak{z}-1} \left(\frac{2\sqrt{2}\, \pi l}{t} \right)^{-\mathfrak{z}} \left\{ \Gamma\left(\tfrac{1}{2}\mathfrak{z}\right) \right\}^2 \cdot B\left(\tfrac{1}{2}\mathfrak{z} + 2 - \tfrac{1}{2}k, \tfrac{1}{2}k - 2\right) (r)^{-\mathfrak{z}/2 + k/2 - 2}, \quad (4\text{-}29)$$

where B is the usual Beta function.

Then using (4-29) we deduce that (4-28) is majorized by

$$g(\mathfrak{s}_0) \left\{ \sum_{\{r \in \mathbf{Z},\, r < -1\}} |r|^{s_2 + 1 - k/2 - \mathfrak{s}_0} M(Q_1, \mathfrak{L}, 2r) \right\} \left\{ \sum_{l > 1} l^{s_2 - 1 - \mathfrak{s}_0} \right\} \quad (4\text{-}30)$$

where g is a function dependent only on $\mathfrak{s}_0$. However as above, we see that there exists δ''_{k,s_2}, a positive number, so that if $\mathfrak{s}_0 \geqslant \delta''_{k,s_2}$, the series in (4-30) are convergent.

Summarizing the arguments above, it follows that there exists a positive number μ_{k,s_2} so that if $\mathfrak{s}_0 = \mathrm{Re}(\mathfrak{s}) \geqslant \mu_{k,s_2}$, then

$$\int_0^{+\infty} \left\{ \int_{O(Q_1)_0 / O(Q_1)_0 \cap \Gamma^L(Q)} |\check{T}^L_{\varphi,v,\chi}(z, A_a(p^{-1})m)|\, d\lambda(m) \right\} p^{2\mathfrak{s}_0 - s_2} \frac{dp}{p}$$
$$\leqslant Q(z, |s| \cdot \beta(\mathfrak{s}_0)), \quad (4\text{-}31)$$

where β is a function which depends only on $\mathfrak{s}_0$.

It is well known that for f a cusp form of weight $|s|$

$$\int_{\mathfrak{D}^*_{\Gamma_0(N_L)}} |f(z) Q(z, |s|)|\, |\mathrm{Im}\, z|^{s-2}\, dx\, dy < \infty \quad (4\text{-}32)$$

where $\mathfrak{D}^*_{\Gamma_0(N_L)}$ is the fundamental domain of $\Gamma_0(N_L)$ in $\overline{H}$.

Thus (4-31) and (4-32) allow us to conclude that integration of the function

$$\overline{f(z)}\, \check{T}^L_{\varphi,v,\chi}(z, A_a(p^{-1})m) \quad (4\text{-}33)$$

can be done in any order over the domains in question.

Then we let $\mathfrak{z} = 2\mathfrak{s}$ in (4-29). Finally we note that

$$\left\langle G_{|s|}\left(\ , s^L_{Q,\chi}, \Gamma_0(N_L), r\right) | f(\ ) \right\rangle_{\mathfrak{D}^*_{\Gamma_0(N_L)}} = d \cdot |r|^{1-|s|}\, \overline{a_r(f)}, \quad (4\text{-}34)$$

with d a nonzero constant independent of f and r.

Thus from the above arguments, (4-25) is a valid formula. Q.E.D.

5. Fourier coefficient of F_f. Using *Theorem* 2-1 we can also determine certain Fourier coefficients of the function $F_f(\ |\varphi, L, v, \chi, \Gamma_0(N_L))$. We again have the same hypotheses in effect as in §4. Then if $F_f(\ |\varphi, L, v, \chi, \eta)$ is given by (4-1) we let Υ_f be the corresponding function on the tube domain $\mathfrak{R}_+$ given by the relationship in (3-33). Then we know that for $\Omega \in \mathfrak{N}(e)^*$

$$a_\Omega(\Upsilon_f|e) = e^{-2\sqrt{2}\,\pi Q(e_{a+2})}\int_{F_{\mathfrak{N}}} F_f(N_1(S)|\varphi, L, v, \chi, \Gamma_0(N_L))e^{-2\pi\sqrt{-1}\,Q(\Omega,S)}\,dS \tag{5-1}$$

where $F_{\mathfrak{N}}$ is a fundamental domain for the lattice $\mathfrak{N}$ in L_1. As in §4 we start with (2-13) and substitute in (5-1). Then assuming integration can be interchanged with summation we deduce that

$$a_\Omega(\Upsilon_f|e) = 2^{|s|-1}\sum_{r\in X^-}|r|^{|s|-1} \cdot\left\{\sum_{\nu \bmod t}\chi(\nu)\Psi(\nu,\Omega,r)\right\}\left\langle G_{|s|}\left(\,,s^L_{Q,\chi},\Gamma_0(N_L),r\right)|f\right\rangle_{\mathfrak{D}^*_{\Gamma_0(N_L)}} \tag{5-2}$$

where

$$\Psi(\nu,\Omega,r) = e^{-2\sqrt{2}\,\pi Q(\Omega,e_{a+2})}\int_{F_{\mathfrak{N}}}\left\{\sum_{\{\xi\in\mathfrak{L}\,|\,Q(\xi,\xi)=2r\}} h_-(N_1(S)[\xi+(jt+\nu)v])\right\} \cdot e^{-2\pi\sqrt{-1}\,Q(\Omega,S)}\,dS \tag{5-3}$$

with $h_-(X) = Q(X, e_{a+1} + \sqrt{-1}\,e_{a+2})^{-s_2}$ (on Ω_-).

Recall the explicit form of $\mathfrak{N}$ given in *Example* 3-1. Then an easy argument shows that for fixed $\xi \in \mathfrak{L}$ the set

$$\{\xi + (jt+\nu)v|j\in\mathbf{Z}\} = \left\{N_1\left(\left(\frac{\rho t+\nu}{r}\right)(-\sqrt{2}\,)\cdot\xi+\mu\right)[\xi]\,|\,\mu\in\mathfrak{N},\ \rho=0,\dots,d_\xi-1\right\}$$

where $Q(\xi,\xi) = r < 0$ and d_ξ is the smallest positive integer such that $(t\cdot d_\xi)\mathbf{Z} = \{Q(\mu,\xi)|\,\mu\in\mathfrak{N}/\sqrt{2}\,\}$. We note at this point that the number d_ξ also represents the number of $N_1\cap\Gamma^L(Q)$ orbits in the set $\{\xi+(jt+\nu)v|j\in\mathbf{Z}\}$.

Then following the discussion in [**II**, §5] we let $l_\xi = \xi/Q(\xi,\xi)$. We recall that $N_1^\xi = \{N_1(W)|Q(W,\xi)=0\}$. Then we choose Haar measure dW_1 on N_1^ξ so that $dS = dW_{1_{\{u\in L_1|Q(u,\xi)=0\}}}\otimes dl_\xi$. Then it follows by the discussion in [**II**, §5] that volume$(F_{\mathfrak{N}}) = t\cdot$ volume$(N_1^\xi|N_1^\xi\cap\Gamma^L(Q))\cdot d_\xi$, where the first volume is taken relative to the measure dS on $F_{\mathfrak{N}}$ and the second volume is taken relative to dW_1 discussed above (here counting measure is used on discrete groups).

Hence we have that

$$\int_{F_{\mathfrak{N}}}\left\{\sum_{\substack{\mu\in\mathfrak{N}\\ \rho=0,\dots,d_\xi-1}} h_-\left(N_1\left(S+\left(\frac{\rho t+\nu}{r}\right)(-\sqrt{2}\,)\xi+\mu\right)[\xi]\right)\right\}\cdot e^{-2\pi\sqrt{-1}\,Q(S,\Omega)}\,dS$$

$$= \begin{cases} 0 \quad \text{if } \sqrt{2}\cdot\xi \text{ is not a rational multiple of } \Omega; \\ \text{volume}(N_1^\xi/N_1^\xi\cap\Gamma^L(Q)) \\ \cdot\displaystyle\sum_{\rho=0}^{d_\xi-1}\left\{\int_{-\infty}^{+\infty} h_-\left(N_1\left(ul_\xi+\left(\frac{\rho t+\nu}{r}\right)(-\sqrt{2}\,)\xi\right)[\xi]\right)e^{-2\pi\sqrt{-1}\,uQ(l_\xi,\Omega)}\,du\right\} \\ \text{if } \sqrt{2}\,\xi \text{ is a rational multiple of } \Omega. \end{cases} \tag{5-4}$$

Then if $\sqrt{2}\cdot\xi$ is a rational multiple of Ω we make the change of variables $w = u - \sqrt{2}(\rho t + \nu)$ in each summand in (5-4) and deduce that (with $\xi = \alpha\Omega/\sqrt{2}$) the right-hand side of (5-4) equals

$$\operatorname{volume}(N_1^\xi/N_1^\xi \cap \Gamma^L(Q)) \cdot \left\{ \sum_{j=0}^{d_\xi - 1} \exp\left(-2\pi\sqrt{-1}\left(\frac{jt+\nu}{r}\right)\sqrt{2}\, Q(\xi, \Omega)\right)\right\} \cdot \left\{\int_{-\infty}^{+\infty} h\_(\xi + u\cdot v_a)^{2\pi\sqrt{-1}\,(u/r)Q(\xi,\Omega)}\, du\right\}. \tag{5-5}$$

However if $\xi = \alpha\Omega/\sqrt{2}$ with $\alpha \in \mathbf{Q}$ then the summation in (5-5) becomes

$$\exp\left(-2\pi\sqrt{-1}\,\nu\frac{\alpha}{r}Q(\Omega,\Omega)\right)\left\{\sum_{j=0}^{d_\xi-1}\exp\left(-2\pi\sqrt{-1}\,\frac{j}{d_\xi}\{\pm Q(\Omega,\mathfrak{R})\}\right)\right\} \tag{5-6}$$

where $Q(\Omega, \mathfrak{R})$ is the smallest positive integer so that $Q(\Omega,\mathfrak{R})\cdot\mathbf{Z} = \{Q(\mu,\Omega) \mid \mu \in \mathfrak{R}\}$.

Then the sum in (5-6) is 0 if d_ξ does not divide $Q(\Omega, \mathfrak{R})$ and is equal to d_ξ if d_ξ divides $Q(\Omega, \mathfrak{R})$.

We know that every element $\Omega \in \mathfrak{R}^*$ can be expressed in the form $\beta\cdot\Omega_0$ with $\beta \in \mathbf{Z}$ and Ω_0 a *primitive* element in $\mathfrak{R}^*$ (i.e., if $\Omega_0/m \in \mathfrak{R}^*$, then $m = \pm 1$). Then let $\xi = (\alpha/\sqrt{2})\cdot(\beta\Omega_0)$ with $\alpha \in \mathbf{Q}$ and $\xi \in \mathfrak{L}$. If $\alpha = m/n$ with m and n relatively prime integers then, since Ω_0 is primitive, it follows that n divides β. On the other hand we recall that

$$(t\cdot d_{\alpha\Omega/\sqrt{2}})\mathbf{Z} = \left\{Q(\mu, \alpha\beta\Omega_0) \mid \mu \in \mathfrak{R}/\sqrt{2}\right\} = \alpha\cdot\left\{Q(\mu,\beta\Omega_0)\mid \mu\in\mathfrak{R}/\sqrt{2}\right\};$$

if $d_{\alpha\Omega/\sqrt{2}}$ divides $Q(\beta\Omega_0, \mathfrak{R})$ then m divides $2t$.

LEMMA 5-1. *Let* $\Omega \in \mathfrak{R}^*$. *Let* $\xi \in \mathfrak{L}$ *with* $Q(\xi,\xi) = r$. *Also assume that* $\sqrt{2}\cdot\xi = m\Omega/n$ *with* m, n *relatively prime integers. If* d_ξ *divides* $Q(\Omega, \mathfrak{R})$, *then*

$$\Psi(\nu,\Omega,r) = c_1(m/n)^{1-s_2}d_\xi c(L_\xi) \cdot\left[\exp\left(-2\pi\sqrt{-1}\,(m\nu/nr)Q(\Omega,\Omega)\right) + (-1)^{s_2}\exp\left(2\pi\sqrt{-1}\,(m\nu/nr)Q(\Omega,\Omega)\right)\right] \tag{5-7}$$

where $c(L_\xi) = \operatorname{volume}(N_1^\xi/N_1^\xi \cap \Gamma^L(Q))$ *and where* c_1 *is a nonzero constant depending only on* s_2.

PROOF. The argument follows from the considerations above and a statement identical to (5-7) in [**II**, §7]. Q.E.D.

COROLLARY TO LEMMA 5-1. *Let the same conditions hold as in Lemma* 5-1. *Then*

$$\sum_{\nu=0}^{\nu=t-1}\chi(\nu)\Psi(\nu,\Omega,r) = \begin{cases} 0 \quad \textit{if } m \neq 2t, \\ \overline{\chi(n)}\,G(\chi,1,t)c_1\{\chi(-1)+(-1)^{s_2}\} \\ \cdot c\left(L_t\Omega/\sqrt{2}\,n\right)(2t/n)^{1-s_2}d_t\Omega/\sqrt{2}\,n \quad \textit{if } m = 2t. \end{cases} \tag{5-8}$$

Proof. We essentially evaluate a Gauss sum and use the comments preceding Lemma 5-1. Q.E.D.

We note here that from comments in the beginning of §4, $\chi(-1) = \chi_\beta(-1) = (-1)^{s_2}$. Hence $\chi(-1) + (-1)^{s_2} = \pm 2 \neq 0$ above.

Since every $\mu \in \mathfrak{N}/\sqrt{2}$ satisfies $Q(\mu, \mathfrak{L}) \equiv 0 \bmod t$ (see Example 4-1) it follows that the lattice $\mathfrak{L}/t$ is a subset of $\sqrt{2}\,\mathfrak{N}^*$.

THEOREM 5-1. (i) *Let* $\Omega \in \mathfrak{N}^* \cap W$ *and assume that* $\Omega \notin \mathfrak{L}/\sqrt{2}\, t$. *Then* $a_\Omega(\Upsilon_f|e) \equiv 0$.

(ii) *Let* $\Omega = m\xi/\sqrt{2}\, t$ *with* ξ *a primitive element in the lattice* $\mathfrak{L}$. *Then*

$$a_\Omega(\Upsilon_f|e) = c_1' G(\chi, 1, t) t^{-s_2} \left\{ \sum_{\{\nu|\nu|m\}} \overline{\chi(v)}\, \nu^{s_2-1}\, \overline{a_{(m^2/\nu^2)(Q(\xi,\xi)/2)}(f)} \right\} \tag{5-9}$$

where c_1' *is a nonzero constant independent of* Ω, t, χ, *and* m.

PROOF. Part (i) follows directly from the Corollary to Lemma 5-1. Then in (ii) with $\Omega = (m/\sqrt{2}\, t) \cdot \xi$, with ξ primitive in $\mathfrak{L}$, it follows that $\xi' = 2t\Omega/\sqrt{2}\,\nu \in \mathfrak{L}$ if and only if $\nu | m$. Then we apply (5-8) and the comments preceding this theorem.

In order to justify changing orders of integration above, we must show that

$$\int_{F_{\mathfrak{N}}} \sum_{\omega \in \Gamma_0(u_1u_2) \cap \psi(\tilde{P}) \backslash \Gamma_0(u_1u_2)} \cdot \left\{ \sum_{r \in X^-} |r|^{|s|-1} \left| \left\{ \sum_{\nu \bmod u_1u_2} \chi(\nu) M_\varphi(N_1(S), \mathfrak{L}, r, \nu) \right\} \cdot \overline{s^L_{Q,\chi}(\omega)} \left(\frac{1}{c_\omega z + d_\omega} \right)^{|s|} e^{2\pi\sqrt{-1}\, r\omega(z)} \right| \right\} dS < \infty. \tag{5-10}$$

But we can use (4-6) again and substitute the right-hand side of (4-5) for $M_\varphi(N_1(S), \mathfrak{L}, r, v)$ and deduce that (5-10) is majorized by

$$t^{-s_2} \int_{F_{\mathfrak{N}}} \left\{ \left\{ \sum_{\omega \in \cdots} \sum_{r \in X^-} |r|^{|s|-1} \left\{ \sum_{\substack{\{\xi \in \mathfrak{L} | Q(\xi,\xi) = 2r\} \\ l > 1}} l^{s_2-1} e^{-l\alpha|Q(\xi, e_{a+2})|} \right\} \left| \frac{1}{c_\omega z + d_\omega} \right|^{|s|} \right\} \right\} dS. \tag{5-11}$$

Thus we have by the same reasoning as in (4-9) that (5-11) is majorized by

$$t^{-s_2} \operatorname{vol}(F_{\mathfrak{N}}) Q(z, |s|) \sum_{r \in X^-} |r|^{|s|-1} \left\{ \sum_{\{\xi \in \mathfrak{L} | Q(\xi,\xi) = 2r\}} \left| \frac{1}{Q(\xi, e_{a+2})} \right|^{s_2} \cdot e^{-\alpha'|Q(\xi, e_{a+2})|} \right\}. \tag{5-12}$$

Thus we can consider the double series in (5-12). However we may assume that $Q(\xi, \xi) = \|\xi_+\|^2 - \|\xi_-\|^2$, with $\|\xi_+\|^2$ a positive definite diagonal form with positive rational entries and $\|\xi_-\|^2$ a positive square. Hence we have that the inner series of

(5-12) is majorized by

$$\sum_{\{\xi\in\mathbf{Z}^{k-2}|\ \|\xi_+\|^2-\|\xi_-\|^2=r\}}\left(\frac{1}{\|\xi_-\|}\right)^{s_2}e^{-\alpha'\|\xi_-\|}. \tag{5-13}$$

Hence (5-13) is majorized by

$$\sum_{\{\xi|\xi\in\mathbf{Z}^{k-3}\}}\left[\frac{1}{\|\xi_+\|^2+|r|}\right]^{s_2/2}\exp\left(-\alpha'\sqrt{\|\xi_+\|^2+|r|}\right). \tag{5-14}$$

But (5-14) in turn is majorized by

$$|r|^{-s_2/2}e^{-\alpha' r}+\int_{\mathbf{R}^{k-3}}\left(\frac{1}{\|X\|^2+|r|}\right)^{s_2/2}\exp\left(-\alpha'\sqrt{\|X\|^2+|r|}\right)dX. \tag{5-15}$$

Then the integral in (5-15) is dominated by

$$e^{-\alpha'|r|^{1/2}}\int_0^{+\infty}\left(\frac{1}{w^2+|r|}\right)^{s_2/2}w^{k-4}\,dw, \tag{5-16}$$

which, in turn, is majorized by

$$r^{-(s_2-(k-3))/2}\exp\left(-\alpha'\sqrt{|r|}\right). \tag{5-17}$$

Thus in any case, (5-13) is dominated by

$$|r|^q\exp\left(-\alpha'\sqrt{|r|}\right),$$

for q some number independent of r. Finally the double series in (5-12) is majorized by

$$\sum_{r\in X_-}|r|^{q+|s|-1}\exp\left(-\alpha'\sqrt{|r|}\right),$$

which is clearly convergent.

Then following the same type of arguments as in §4, we deduce that summation and integration can be changed in (5-1). Q.E.D.

REMARK 5-1. If $\Omega_1=(m/\sqrt{2}\ t)\xi_1$ and $\Omega_2=(m/\sqrt{2}\ t)\xi_2$ with ξ_1 and ξ_2, $\Gamma^L(Q)\cap O(Q_1)$ inequivalent primitive lattice points in $\mathfrak{L}$ with $Q(\xi_1,\xi_1)=Q(\xi_2,\xi_2)$, we have that $a_{\Omega_1}=(\Upsilon_f|e)=a_{\Omega_2}(\Upsilon_f|e)$.

REMARK 5-2. By using the expression for $a_\Omega(\Upsilon_f|e)$ in (5-9) we deduce that

$$\sum_{\Omega\in\{\mathfrak{N}^*\}_{\Gamma^L(Q)\cap O(Q_1)}}a_\Omega(\Upsilon_f|e)\frac{1}{\varepsilon(\Omega|e)}Q(\Omega,\Omega)^{-\mathfrak{s}}$$
$$=c_1't^{2\mathfrak{s}-s_2}\cdot G(\chi,1,t)L(\bar{\chi},2\mathfrak{s}+1-s_2)\sum_{\{n\in\mathbf{Z}|n<-1\}}\overline{a_n(f)}\,M(Q_1,\mathfrak{L},2n)|n|^{-\mathfrak{s}}. \tag{5-18}$$

This verifies the computation made in Theorem 4-1.

6. Examples of Euler products associated with Dirichlet series. We have constructed in *Theorem* 4-1 a Dirichlet series of the type discussed in **[10]**. The special feature of this series, aside from the Γ factors, is that it has the form of the product of an L function and a Dirichlet series, which can be expressed essentially as the Rankin convolution of two other Dirichlet series. That is

$$\sum_{n \leq -1} \overline{a_n(f)}\, M(Q_1, \mathfrak{L}, 2n)|n|^{-\mathfrak{s}}$$

is the Rankin convolution of $\sum_{n \leq -1} \overline{a_n(f)}|n|^{-\mathfrak{s}}$, the Dirichlet series $D(\mathfrak{s}, f)$ associated to f, and of $\sum_{n \leq -1} M(Q_1, \mathfrak{L}, 2n)|n|^{-\mathfrak{s}}$ Siegel's zeta function, $\zeta_{-}(Q_1, \mathfrak{L}, \mathfrak{s})$ associated to the quadratic form Q_1 and lattice $\mathfrak{L}$.

A very basic question is to determine what type of Euler product property the series

$$\sum_{n \leq -1} \overline{a_n(f)}\, M(Q_1, \mathfrak{L}, 2n)|n|^{-\mathfrak{s}}$$

has, given certain Euler product properties of $D(\mathfrak{s}, f)$ and $\zeta_{-}(Q_1, \mathfrak{L}, \mathfrak{s})$.

The dimension of the space $\mathbf{R} \otimes \mathfrak{L}$ is critical at this point. The reason is that $D(\mathfrak{s}, f)$ has an Euler product when the weight of f is even, i.e., the number $|s|$ is integral or $s \equiv k/2 \bmod 1$ for k even. However if $|s|$ is an odd integer divided by 2 then f corresponds to a half integral automorphic form; then we must use the modified theory of Euler products developed in **[10]**. It is interesting to note that the parity of k is reflected in the functional equation of $\zeta_{-}(Q_1, \mathfrak{L}, \mathfrak{s})$. For instance when k is odd and $k > 5$ (recall $\dim(\mathbf{R} \otimes \mathfrak{L}) = k - 2$) then $\zeta_{-}(Q_1, \mathfrak{L}, \mathfrak{s})$ satisfies a functional equation of the following form. Let $\varphi(Q_1, \mathfrak{L}, \mathfrak{s}) = \pi^{-\mathfrak{s}}\Gamma(\mathfrak{s})\zeta_{-}(Q_1, \mathfrak{L}, \mathfrak{s})$. Then

$$\varphi(Q_1, \mathfrak{L}, \mathfrak{s}) = (-1)^{(k-3)/2}|D_{Q_1(\mathfrak{L})}|^{-1/2}\varphi(Q_1, \mathfrak{L}^*, k/2 - 1 - \mathfrak{s}).$$

However if k is even, the functional equation is more complicated involving the zeta function ζ_{+} on the hyperbolic lattice points (see Satz 2 of **[13]**).

The first point to investigate is what sort of Euler product property the Dirichlet series $\zeta_{-}(Q_1, \mathfrak{L}, \mathfrak{s})$ has. We start with the example in the case $k = 5$ (see Example 3-2).

We then consider the space $\mathbf{R}^5$, the orthogonal complement to $\vec{e}_1 \wedge \vec{e}_3 + \vec{e}_2 \wedge \vec{e}_4$ in $\Lambda^2(\mathbf{R}^4)$ relative to the quadratic form ϕ defined in Example 3-2. We then let Q be the quadratic form on $\mathbf{R}^5$ defined by $Q(\alpha, \beta) = \phi^*(\alpha, \beta)$, i.e., ϕ^* is ϕ restricted to $(\vec{e}_1 \wedge \vec{e}_3 + \vec{e}_2 \wedge \vec{e}_4)^{\perp}$.

Then we consider in $\mathbf{R}^5$ the lattice given by $L_{N,M} = \mathfrak{L} \oplus L_{(N,1)}$, with $\mathfrak{L} = \sqrt{2}\, M[\mathbf{Z}v_3 \oplus \mathbf{Z}\tilde{v}_3 \oplus \mathbf{Z}e_3/\sqrt{2}\,]$ and $L_{(N,1)} = \mathbf{Z}Nv \oplus \mathbf{Z}\tilde{v}$ (where $\tilde{v} = \tilde{v}_2/\sqrt{2}$ and $v = \sqrt{2}\, v_2$), and where M, N are positive integers such that M *divides* N. Then $L_{N,M}$ is always a Q integral lattice and is a Type II lattice if and only if M is even. Then we note that

$$\sigma_2(\{\gamma \in Sp_2(\mathbf{Z})|\gamma \equiv I_4 \bmod R\})(L_{N,M}) = L_{N,M} \quad \text{if } N|R.$$

Also we note by direct computation that

$$\sigma_2\left(\left\{\left(\begin{array}{c|c} g & 0 \\ \hline 0 & (g')^{-1} \end{array}\right)\middle| g \in SL_2(\mathbf{Z})\right\}\right)(L_{N,M}) = L_{N,M}.$$

In fact we know that the symmetric matrix

$$\mathfrak{T} = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$$

with $(a, b, c) \in \mathbf{Z}^3$ can be identified to the lattice element

$$X_{\mathfrak{T}} = \{av_2 - c\tilde{v}_2 + be_3/\sqrt{2}\}$$

in such a way that via this identification

$$g\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}g^t$$

(with $g \in SL_2(\mathbf{Z})$) goes to

$$\left\{\sigma_2\left[\left[\begin{array}{c|c} g & 0 \\ \hline 0 & (g^t)^{-1} \end{array}\right]\right]\{av_2 - c\tilde{v}_2 + be_3/\sqrt{2}\}\right\}.$$

Thus we can classify the $\sigma_2(SL_2(\mathbf{Z}))$ orbits in

$$\mathfrak{L} = \sqrt{2}\, M[\mathbf{Z}v_3 \oplus \mathbf{Z}\tilde{v}_3 \oplus \mathbf{Z}e_3/\sqrt{2}].$$

The question just reduces to a classical problem in the theory of binary quadratic forms.

First we know that the discriminant of the matrix

$$X = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$$

is given by $b_2 - 4ac = \delta(X)$. Then if $(a, b, c) \in \mathbf{Z}^3$ it follows that $\delta(X) \equiv 0$ or $\delta(X) \equiv 1 \bmod 4$. Moreover such an X is called *primitive* if the greatest common divisor of a, b and c equals 1. Then the classical theory of binary quadratic forms says that for X primitive and $\delta(X) < 0$, $SL_2(\mathbf{Z})^X$ = the isotropy group of X in $SL_2(\mathbf{Z})$ is

$$\begin{cases} \text{a group of order 6} & \text{if } \delta(X) = -3 \\ \text{a group of order 4} & \text{if } \delta(X) = -4 \\ \text{a group of order 2} & \text{if } \delta(X) \neq -3, -4. \end{cases}$$

PROPOSITION 6-1. (i) *Let N be divisible by* 4. *Let $L = L_{N,M}$. Then n_L, the exponent of $L_{N,M}$, equals the least common multiple of $2M^2$ and N. Also the discriminant $D_{Q(L)}$ of $L_{N,M}$ is $4M^6N^2$. If M is even and $2M^2|N$, then $L_{N,M}$ is a Type* II* *lattice.*

(ii) *Let M be even and $2M^2|N$ so that $n_L = N$.*

Let $f \in S_{2|s|}(n_L, \lambda^L_{Q,|s|} \otimes \chi)^*$ with $s > 2k$ (see §4) *and with χ a Dirichlet character* mod n_L. *Let F_f be given by* (4-1) *and let $F_f^*(U + \sqrt{-1}\, V) = \Upsilon_f(X_U + \sqrt{-1}\, X_V) = F_f(g^{-1})[\underset{\sim}{\mathfrak{d}}(g, \sqrt{-1}\, e_k)]^{s_2}$ where the point $U + \sqrt{-1}\, V$ in H_2, the Siegel space of genus* 2, *corresponds to the point $X_U + \sqrt{-1}\, X_V = g(\sqrt{-1}\, e_k)$ in $\mathfrak{R}_+$ (where the symmetric matrix*

$$U = \begin{pmatrix} r_1 & r_2 \\ r_2 & r_3 \end{pmatrix}$$

corresponds to the vector $X_U = r_1v_3 - r_3\tilde{v}_3 - r_2\sqrt{2}\, e_3$). Then F_f^* satisfies the functional equation

$$F_f^*((AZ+B)(CZ+D)^{-1}) = (\det(CZ+D))^{s_2} F_f^*(Z) \tag{6-1}$$

for any

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$

belonging to the group in $Sp_2(\mathbf{Z})$ *generated by* $Sp_2(\mathbf{Z})_R$ (*with* $N \mid R$) *and*

$$SL_2(\mathbf{Z}) = \left\{ \left[\begin{array}{c|c} g & 0 \\ \hline 0 & (g^t)^{-1} \end{array} \right] \middle| \, g \in SL_2(\mathbf{Z}) \right\} \qquad (\textit{with } s_2 = s + 1/2).$$

PROOF. The proof is direct from the remarks in Example 3-2 and (3-33). Q.E.D.

REMARK 6-1. The Dirichlet character (mod n_L)

$$\lambda^L_{Q,|s|}(\delta) = (-1)^{((\delta^2-1)/8)\cdot 5}\left(\frac{-1}{\delta}\right)^{|s|-5/2}$$

(see the beginning of §4).

From *Proposition* 6-1 we also deduce that $F_f^*(Z + N \cdot U) = F_f^*(Z)$ for any $Z \in H_2$, $U = \left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)$, a symmetric matrix with $(a, b, c) \in \mathbf{Z}$. Thus we have the expansion

$$F_f^*(W) = \sum_{U > 0} S_{F_f^*}(U) \exp\left(2\pi\sqrt{-1}\,\frac{1}{N}\,\mathrm{Tr}(U \cdot W)\right) \tag{6-2}$$

where U runs over all symmetric positive semidefinite matrices

$$\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$$

with $(a, b, c) \in \mathbf{Z}^3$.

THEOREM 6-1. (i) *Let* F_f^* *be given as in* (ii) *of Proposition* 6-1. *Then* $S_{F_f^*}(U) = 0$ *if* $\det(U) = 0$. *Thus* F_f^* *is a cusp form on* H_2 *relative to the group* $Sp_2(\mathbf{Z})_R$ (*discussed in* (ii) *of Proposition* 6-1).

(ii) *Let*

$$U = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$$

with $\det(U) > 0$. *If either* $a \not\equiv 0 \bmod M$ *or* $c \not\equiv 0 \bmod M$ *or* $b \not\equiv 0 \bmod M$, *then* $s_{F_f^*}(U) = 0$. *On the other hand, suppose that* $U = MjU_0$, *where*

$$U_0 = \begin{pmatrix} a_0 & b_0/2 \\ b_0/2 & c_0 \end{pmatrix}$$

with $\gcd(a_0, b_0, c_0) = 1$ *and* j *a positive integer. Then* (*up to a nonzero scalar*)

$$S_{F_f^*}(U) = N^{-s_2} G(\chi, 1, N)\left\{ \sum_{\{\nu \mid \nu \mid j\}} \overline{\chi(\nu)\nu^{s_2-1} a_{(j^2/\nu^2)(M^2/2)\delta(U_0)}(f)} \right\}. \tag{6-3}$$

PROOF. The first observation is that if

$$W = \begin{pmatrix} w_1 & w_2 \\ w_2 & w_3 \end{pmatrix} \in H_2, \quad \mathfrak{T} = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$$

(a semi-integral symmetric matrix), then $\mathrm{Tr}(W \cdot \mathfrak{T}) = -Q(X_W, X_{\mathfrak{T}'})$, *where*

$$\mathfrak{T}' = \begin{pmatrix} a & -b/2 \\ -b/2 & c \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \mathfrak{T} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then equating (6-2) to (3-2) and using Theorem 5-1, we deduce the results of *Theorem* 6-1.

REMARK 6-2. We note that if $M = 2$ and $N = 8$, then

$$\lambda^L_{Q,|s|}(x) = \left(\frac{2}{x}\right)\left(\frac{-1}{x}\right)^{|s|-5/2}.$$

If we let $g \in S_{2|s|}(\Gamma_0(8))$, then

$$f(w) = \overline{g(\overline{w})} \in S_{2|s|}\left(\Gamma_0(8), \left(\frac{-1}{*}\right)^{2|s|}\right)^*$$

(see the beginning of §4). Then we apply *Theorem* 6-1 to f. We see that

$$\chi(x) = \left(\frac{-1}{x}\right)^{2|s|} \otimes \lambda^L_{Q,s}(x) = \left(\frac{2}{x}\right)\left(\frac{-1}{x}\right)^{|s|+5/2};$$

hence $G(\chi, 1, 8) \neq 0$. Moreover $S_{2|s|}(\Gamma_0(8))$ is a nonzero space (for almost all positive odd integers $2|s|$). Thus there is a nontrivial example satisfying *Theorem* 6-1.

We recall that a number d is called a *fundamental discriminant* if d is nonsquare, is $\equiv 0$ or $1 \bmod 4$, is not divisible by the square of any odd prime and is either odd or $\equiv 8$ or $12 \bmod 16$. We know then that any number, which is not a square and $\equiv 0$ or $1 \bmod 4$, can be written uniquely in the form dm^2 with $m > 0$ and d a fundamental discriminant.

We let $h(m)$ = the number of $SL_2(\mathbf{Z})$ equivalence classes of primitive semi-integral symmetric matrices with discriminant m.

Let

$$w(d) = \begin{cases} 3 & \text{if } d = -3, \\ 2 & \text{if } d = -4, \\ 1 & \text{if } d \neq -3, -4. \end{cases}$$

PROPOSITION 6-2. *Let $L_{N,2}$ be given where* 2 *and N satisfy the hypotheses in* (ii) *of Theorem* 6-1. *Let d be a fundamental discriminant and define for* $\mathrm{Re}(\mathfrak{s})$ *sufficiently large*

$$A(f, d, L_{N,2}, \mathfrak{s}) = \sum_{\substack{\{U\} \text{ with } \delta(U) = dm^2 \\ m > 1}} S_{F_f^*}(2U) \frac{1}{\varepsilon(U)} |\delta(U)|^{-\mathfrak{s}} \tag{6-4}$$

where { } *is over all $SL_2(\mathbf{Z})$ equivalence classes of semi-integral symmetric matrices* (*via the action $U \to gUg^t$*) *satisfying $\delta(U) = df^2$.*

Then for $\mathrm{Re}(\mathfrak{s})$ *sufficiently large we have the equality*

$$A(f, d, L_{N,2}, \mathfrak{s}) = G(\chi, 1, N) | d|^{-\mathfrak{s}} N^{-s_2} \cdot L(\overline{\chi}, 2\mathfrak{s} + 1 - s_2) \cdot \sum_{\{n \in \mathbf{Z},\, n < -1\}} \overline{a_{2dn^2}(f)}\, M(Q_1, \mathfrak{L}, 4dn^2)|n|^{-2\mathfrak{s}} \tag{6-5}$$

where $M(Q_1, \mathfrak{L}, 4dn^2)$ is the Siegel mass number given by

$$\sum_{\{q, q|n\}} h(dq^2)\frac{1}{w(dq^2)}.$$

PROOF. We apply (5-8). It then becomes a matter of explicitly determining $M(Q_1, \mathfrak{L}, 4dn^2)$ in this case. Then we note that the number of distinct $SL_2(\mathbf{Z})$ orbits in the quadric of semi-integral symmetric matrices of discriminant equal to dn^2 is $\sum_{\{q|q|n\}} h(dq^2)$. However we must assign a mass number $1/w(dq^2)$ to those belonging to the partition determined by $h(dq^2)$. Hence the result follows. Q.E.D.

We now investigate the Euler product property of $A(f, d, L_{N,2}, \mathfrak{s})$ in (6-5). For this we need an explicit representation of Siegel's zeta function.

$$\zeta_-(Q_1, L_{N,2}, \mathfrak{s}) = \sum_{\{r \in \mathbf{Z}, r < -1\}} M(Q_1, L_{N,2}, r)|r|^{-\mathfrak{s}}. \tag{6-6}$$

Let

$$R(Q_1, L_{N,2}, d, \mathfrak{s}) = \sum_{m > 1} M(Q_1, L_{N,2}, 4dm^2)m^{-\mathfrak{s}}.$$

However we recall the relationship

$$\frac{h(dq^2)}{w(dq^2)} = \frac{h(d)}{w(d)} \cdot q \prod_{\{p \text{ prime}, p|q\}} \left\{1 - \left(\frac{d}{p}\right)\frac{1}{p}\right\}.$$

Hence

$$M(Q_1, L_{N,2}, 4dn^2) = \frac{h(d)}{w(d)} \sum_{\{q, q|n\}} \left\{ \prod_{\{p \text{ prime}, p|q\}} \left(1 - \left(\frac{d}{p}\right)\frac{1}{p}\right)\right\} q.$$

Thus we have the following Euler product formula for $R(Q_1, L_{N,2}, d, \mathfrak{s})$

$$R(Q_1, L_{N,2}, d, \mathfrak{s}) = \frac{h(d)}{w(d)} \prod_p \frac{\left(1 - \left(\frac{d}{p}\right)p^{-\mathfrak{s}}\right)}{(1 - p^{-\mathfrak{s}})(1 - p^{1-\mathfrak{s}})}. \tag{6-7}$$

Hence we have that

$$R(Q_1, L_{N,2}, d, \mathfrak{s}) = (h(d)/w(d))\zeta(\mathfrak{s})\zeta(\mathfrak{s} - 1)\left(1/L\left(\left(\frac{d}{*}\right), \mathfrak{s}\right)\right).$$

THEOREM 6-2. *Let $f \in S_{2|s|}(N, \sigma)^*$ with $\sigma = \lambda^L_{Q,|s|} \otimes \chi$. Then $f(w) = \overline{h_f(\overline{w})}$ with $h_f \in S_{2|s|}(N, \omega)$, where*

$$\omega = \bar{\sigma} \otimes \left(\frac{-1}{*}\right)^{2|s|}.$$

Suppose that (d a fundamental discriminant)

$$\sum_{n > 1} \overline{a_{2dn^2}(f)}\, n^{-\mathfrak{s}} = \overline{a_{2d}(f)} \prod_p \frac{\left\{1 - \omega_1(p)\left(\frac{-2d}{p}\right)p^{\lambda_{|s|} - 1 - \mathfrak{s}}\right\}}{\left\{1 - U_p p^{-\mathfrak{s}} + \omega(p)^2 p^{2|s| - 2 - 2\mathfrak{s}}\right\}} \tag{6-8}$$

where $\lambda_{|s|} = |s| - \frac{1}{2}$, ω_1 *is the Dirichlet series given by*

$$\omega_1(x) = \omega(x)\left(\frac{-1}{x}\right)^{\lambda_{|s|}}.$$

Then for $\mathrm{Re}(\mathfrak{s})$ *sufficiently large, the function* $A(f, d, L_{N,2}, \mathfrak{s})$ *has the following Euler product.*

$$A(f, d, L_{N,2}, \mathfrak{s}) = \overline{a_{2d}(f)}\,(h(d)/w(d))G(\chi, 1, N)N^{-s_2}|d|^{-\mathfrak{s}}$$
$$\cdot \prod_p \frac{\left\{1 - \lambda_p p^{-2\mathfrak{s}} + \omega(p)^2 p^{2|s|-2-4\mathfrak{s}}\right\}}{\left\{1 - U_p p^{1-2\mathfrak{s}} + \omega(p)^2 p^{2|s|-4\mathfrak{s}}\right\}\left\{1 - U_p p^{-2\mathfrak{s}} + \omega(p)^2 p^{2|s|-2-4\mathfrak{s}}\right\}} \tag{6-9}$$

where

$$\lambda_p = \left(\frac{2}{p}\right)\left(\frac{-1}{p}\right)^{s_2}\omega(p)p^{|s|-(3/2)}(p+1)\left(\left(\frac{d}{p}\right) - 1\right) + \left(\frac{d}{p}\right)U_p.$$

PROOF. From (6-5) we know that $A(f, d, L_{N,2}, \mathfrak{s})$ is, aside from L functions, the Rankin convolution of the Dirichlet series given by the left-hand side of (6-8) and by $R(Q_1, L_{N,2}, d, \mathfrak{s})$. Thus the problem is purely an algebraic one. Namely suppose

$$g_1(X) = \prod_p \frac{(1 - l_p \cdot X)}{(1 - m_p \cdot X)(1 - n_p \cdot X)}$$

and

$$g_2(X) = \prod_p \frac{(1 - q_p \cdot X)}{(1 - r_p X)(1 - t_p \cdot X)}$$

and $g_3(X)$ is the *Hadamard* product of g_1 and g_2, i.e., $g_3(X) = \sum_{n\geq 0} c_n d_n X^n$ with

$$g_1(X) = \sum_{n\geq 0} c_n X^n, \qquad g_2(X) = \sum_{n \geq 1} d_n X^n.$$

The algebraic problem is to express g_3 as a product.

By following the same arguments as in **[11]** we deduce that

$$c_{p^j} d_{p^j} = \left\{\frac{m_p^{j+1} - n_p^{j+1}}{m_p - n_p} - l_p \frac{m_p^j - n_p^j}{m_p - n_p}\right\}\left\{\frac{r_p^{j+1} - t_p^{j+1}}{r_p - t_p} - q_p \frac{r_p^j - t_p^j}{r_p - t_p}\right\}.$$

Using this remark we then deduce easily that

$$\sum_{j\geq 0} c_{p^j} d_{p^j} X^j = \frac{N_p(X)}{D_p(X)}$$

where

$$\begin{aligned} N_p(X) = 1 &+ \left\{l_p q_p - q_p(m_p + n_p) - l_p(r_p + t_p)\right\}X \\ &+ \left\{-m_p n_p r_p t_p + q_p m_p n_p(r_p + t_p) + l_p r_p t_p(m_p + n_p)\right\}X^2 \\ &+ \left\{-l_p q_p m_p n_p r_p t_p\right\}X^3 \end{aligned}$$

and

$$D_p(X) = (1 - r_p m_p X)(1 - r_p n_p X)(1 - t_p m_p X)(1 - t_p n_p X).$$

Then substituting the required values of r_p, t_p, m_p and n_p, we have that $N_p(X) = 1 + a_pX + b_pX^2 + c_pX^3$ with

$$a_p = \left(\frac{d}{p}\right)\left[\omega_1(p)\left(\frac{-2d}{p}\right)p^{|s|-(3/2)}\right] - \omega_1(p)\left(\frac{-2d}{p}\right)p^{|s|-(3/2)}(p+1) - U_p\left(\frac{d}{p}\right),$$

$$b_p = -\omega(p)^2p^{2|s|-1} + \omega_1(p)\left(\frac{-2d}{p}\right)p^{|s|-(1/2)}U_p + \left(\frac{d}{p}\right)\omega(p)^2p^{2|s|-2}(p+1),$$

and

$$c_p = -\left(\frac{-2}{p}\right)\omega_1(p)\omega(p)^2p^{3|s|-(5/2)}.$$

Then by a simple and long computation we show that

$$N_p(X) = \left(1 - \left(\frac{2}{p}\right)\left(\frac{-1}{p}\right)^{s_2}\omega(p)p^{|s|-(1/2)}X\right)\left(1 - \lambda_pX + \omega(p)^2p^{2|s|-2}X^2\right)$$

with λ_p as given above. Q.E.D.

REMARK 6-3. The hypothesis on f in *Theorem* 6-2 is very important. Indeed we refer to Theorem 1.9 of **[10]** when (6-8) expresses the fact that h_f is an eigenfunction of the Hecke operator $T_\chi(p^2)$ with eigenvalue U_p (for each prime p). We consider this matter more carefully in a future paper.

REMARK 6-4. In the *Main Theorem* of **[10]** it is shown that the Euler product

$$\prod_p\left\{1 - U_pp^{-\mathfrak{s}} + \omega(p)^2p^{2|s|-2-2\mathfrak{s}}\right\}^{-1}$$

is the Mellin transform $D(\mathfrak{s}, F)$ of a cusp form F of integral weight (relative to an appropriate subgroup of $SL_2(\mathbf{Z})$). Thus it follows, with (6-8) holding, that

$$A(f, d, L_{N,2}, \mathfrak{s}) = \overline{a_{2d}(f)}\,(h(d)/w(d))G(\chi, 1, N)|\,d|^{-\mathfrak{s}}$$
$$\cdot N^{-s_2}D(2\mathfrak{s}, F)D(2\mathfrak{s} - 1, F)\prod_p\left\{1 - \lambda_pp^{-2\mathfrak{s}} + \omega(p)^2p^{2|s|-2-4\mathfrak{s}}\right\}. \quad (6\text{-}10)$$

Appendix.

(PROOF OF LEMMA 2-2). We give the proof in steps (I) to (V).

(I) We let

$$l(x, t, Z) = \pi_{\mathfrak{M}}((G, g)^{-1})(\varphi)(xv + t\tilde{v} + Z).$$

By the continuity property of φ we see that l is a continuous function. Moreover we know from **[II**, §1**]** that

$$|l(x, t, Z)| \leqslant (2xt + Q(Z, Z))^{|s|-1}e^{-\alpha[2xt+Q(Z,Z)]}\left(\frac{1}{\|Z\|^2 + x^2 + t^2}\right)^{\sigma} \quad \text{(A-1)}$$

with $\sigma = s/2 + k/4 - 1$ and α a constant which depends only on (G, g) (here we recall that $X = xv + t\tilde{v} + Z \in \Omega_+$ and that $\frac{1}{2}\|X_+\| \leqslant \|X\| \leqslant 2\|X_+\|$). Thus we easily deduce that l is an integrable function on $\mathbf{R}^k$.

Also we know from Lemma 1-4 of **[II]** that

$$|l(x, t, Z)| \leqslant \left(\alpha_1^2 + x^2 + t^2 + \|Z\|^2\right)^{-\sigma} \quad \text{(A-1')}$$

with α_1 a constant depending only on (G, g).

From Theorem (1-5) of [**II**], we recall that $l(x, t, Z)$ satisfies the Poisson Summation formula (see 11 of §0) for the lattice L if $s > \frac{1}{2}k$.

Moreover using (A-1′) we see that for fixed Z, the function $(x, t) \rightsquigarrow l(x, t, Z)$ is integrable when $s > \frac{1}{2}k + 1$.

Thus (a′) of *-Poisson Summation Property is satisfied. Moreover for $\eta \in \mathfrak{L}_*(Q)$, the function $Q_\eta(X) = l(x, t, \eta)$ (with $X = xv + y\tilde{v}$) satisfies (a) of the *-Poisson Summmation Property relative to $L_{(u_1,u_2)}$.

(II) On the other hand we know that $G = w_0^{-1}G'w_0$ in $\tilde{SL}_2$. By assuming that $\varphi \in E_{\mathfrak{M}}(s^2 - 2s, s, s_1, s_2)$ we see that

$$\pi_{\mathfrak{M}}((G, g)^{-1})(\varphi)[uv + r\tilde{v} + Z] = c_1[\pi_{\mathfrak{M}}((G', g)^{-1})(\varphi)]^{\wedge}[rv + u\tilde{v} + M_Q(Z)]$$

for a nonzero c. By Fubini's Theorem

$$l(x, t, Z) = \int_{\mathbf{R}} e^{2\pi\sqrt{-1}\,xy} j(y, t, Z)\, dy \tag{A-2}$$

where

$$j(y, t, Z) = \int \pi_{\mathfrak{M}}((G', g)^{-1})(\varphi)[uv + y\tilde{v} + W] e^{2\pi\sqrt{-1}\,[ut + Q(W,Z)]}\, du\, dW.$$

Thus $l(\ , t, Z)$ is the Fourier transform of $j(\ , t, Z)$. From the preceding material it is clear that $l(\ , t, Z)$ is integrable in x (when $s > k/2$) and hence we deduce that

$$F_v^{-1}(\pi_{\mathfrak{M}}((G, g)^{-1}))(\varphi)[uv + r\tilde{v} + Z] = \int_{\mathbf{R}} l(x, r, Z) e^{2\pi\sqrt{-1}\,ux}\, dx = j(u, r, Z) \tag{A-3}$$

at all points $u \in \mathbf{R}$ where $j(\ , r, Z)$ is a continuous function. But using (1-13) of [**II**, §1] we see that for $x > 0$

$$\begin{aligned} |j(x, r, Z)| &\leqslant \int_{\{(y,Y) \in \mathbf{R} \times \mathbf{R}^{k-2} | 2xy + Q(Y,Y) > 0\}} \left[\frac{1}{\|Y\|^2 + y^2 + x^2}\right]^\sigma e^{-\alpha[2xy + Q(Y,Y)]}\, dy dY \\ &= \int_{\mathbf{R}^{k-2}} \left\{ \int_{y = -Q(Y,Y)/2x}^{+\infty} \left[\frac{1}{\|Y\|^2 + y^2 + x^2}\right]^\sigma e^{-\alpha[2xy + Q(Y,Y)]}\, dy \right\} dY. \end{aligned} \tag{A-4}$$

But the inner integral above by the change of variables $y = y' - Q(Y, Y)/2x$ becomes ($m = Q(Y, Y)$)

$$\int_0^{+\infty} \left[\frac{1}{\|Y\|^2 + x^2 + (y' - m/2x)^2}\right]^\sigma e^{-2\alpha x y'}\, dY. \tag{A-5}$$

Hence $|j(x, r, Z)|$ is majorized by

$$\left\{\int_0^{+\infty} e^{-2\alpha x y'}\, dy'\right\}\left\{\int_{\mathbf{R}^{k-2}} \left[\frac{1}{\|Y\|^2 + x^2}\right]^\sigma dY\right\}. \tag{A-6}$$

Then by the change of variable $u = 2\alpha x y'$ in the first integral and $Y = xZ$ in the second integral it follows that (A-6) is majorized by $C_1 x^{-s+k/2-1}$ (where C_1 depends only on (G, g)). Thus

$$|j(x, r, Z)| \leqslant C_1 |x|^{-s+k/2-1} \tag{A-7}$$

for x away from zero.

Thus we deduce that $j(\ , r, Z)$ is continuous away from the origin. By the growth property in (A-7) and (A-1) we deduce that the function

$$x \rightsquigarrow \pi_{\mathfrak{M}}((G, g)^{-1})(\varphi)((xu_1 + \delta)v + W)$$

with $W \in L_1 \oplus \mathbf{R}\tilde{v}$ and with $\delta \in \mathbf{Q}$ satisfies (b) of the *-Poisson Summation Formula Property relative to $L_{(u_1, u_2)}$.

(III) But by using (A-1) and (A-3) we deduce that if $|s| > k/2$, then

$$\begin{aligned} |F_v^{-1}(\cdots)(uv + t\tilde{v} + Z)| &\leqslant \int_{\mathbf{R}} \left[\frac{1}{\|Z\|^2 + t^2 + x^2} \right]^{\sigma} dx \\ &\leqslant \left[\frac{1}{\|Z\|^2 + t^2} \right]^{(s/2 + k/4) - 3/2} . \end{aligned} \tag{A-8}$$

Thus we deduce from (A-7) and (A-8) that (for $|x| \geqslant \frac{1}{2}$)

$$|F_v^{-1}(\cdots)(xv + t\tilde{v} + Z)|^2 \leqslant g(x) \left[\frac{1}{\|Z\|^2 + t^2} \right]^{(s/2 + k/4) - 3/2} \tag{A-9}$$

where g is a continuous function of x, equal to $|x|^{-s+k/2-1}$ when $|x| \geqslant \frac{1}{2}$. Then taking square roots of both sides of (A-9), we find that (aside from a finite number of terms)

$$\sum_{\substack{\xi \in \mathfrak{L} \\ \mu \in L_{(u_1, u_2)}}} |F_v^{-1}(\cdots)(\xi + \mu + v)| \tag{A-10}$$

is majorized by a series of the form

$$\begin{aligned} &\sum_{\substack{n \in \mathbf{Z} - \{0\} \\ (m, \xi) \in \mathbf{Z}^{k-1} - \{0, \vec{0}\}}} \sqrt{g(n)} \left[\frac{1}{m^2 + \|\xi\|^2} \right]^{(s + k/2 - 3)/4} \\ &\quad + \sum_{(m, \xi) \in \mathbf{Z}^{k-1} - \{0, \vec{0}\}} \left[\frac{1}{m^2 + \|\xi\|^2} \right]^{(s + k/2 - 3)/2} + \sum_{n \in \mathbf{Z} - \{0\}} g(n). \end{aligned} \tag{A-11}$$

However (A-11) is a convergent series when $s > 3k/2 + 1$.

Then if

$$\gamma^{-1} = \begin{bmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{bmatrix} \in SL_2(\mathbf{Z}),$$

we see that $g(x) = F_v^{-1}(f)[(1/u_2)\gamma^{-1}(xv) + W]$ is integrable with $W \in \mathbf{R}^{k-2}$ (using (A-7) if $a_\gamma \neq 0$ and (A-8) if $a_\gamma = 0$ with $s > \frac{1}{2}k + 1$). Also using (A-1′), (A-3), and an argument similar to that in (A-8), we see that g is a continuous function.

Thus (c') of the *-Poisson Summation Formula Property holds for $\pi_{\mathfrak{M}}((G, g)^{-1})(f)$ (when $s > 3k/2 + 1$).

(IV) Then we consider the function ($Z \in \mathbf{R}^{k-2}$)

$$\pi^2_{\mathfrak{M}}(\gamma^{-1})\big(\pi_{\mathfrak{M}}((G,g)^{-1})(\varphi)\big)(Z+\mu v/u_2)$$
$$= \frac{r_\gamma}{|c_\gamma|}\int_{\mathbf{R}^2}\pi_{\mathfrak{M}}((G,g)^{-1})(\varphi)[tv+u\tilde{v}+Z] \cdot \exp\left(2\pi\sqrt{-1}\left[\frac{d_\gamma}{c_\gamma}tu-\frac{\mu}{u_2}\frac{u}{c_\gamma}\right]\right)dt\,du \qquad \text{(A-12)}$$

where

$$\gamma^{-1}=\begin{bmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma\end{bmatrix}\in SL_2(\mathbf{Z})$$

with r_γ, a number of modulus 1 depending only on γ ($c_\gamma \neq 0$ here).

Then using (A-1′) we deduce that the left-hand side of (A-12) is majorized by (when $s > k/2+2$)

$$\frac{1}{|c_\gamma|}\int_{\mathbf{R}^2}\left[\frac{1}{\alpha_1+\|Z\|^2+t^2+u^2}\right]^{(s+k/2-2)/2}dt\,du. \qquad \text{(A-13)}$$

The latter integral is convergent when $s > k/2+1$. Moreover using the majoration given by (A-13) we see that $Z \rightsquigarrow \pi^2_{\mathfrak{M}}(\gamma^{-1})(\pi_{\mathfrak{M}}((G,g)^{-1})(\varphi))[Z+\mu v/u_2]$ is a continuous function in Z. Also by applying (A-13) we deduce that

$$\left|\pi^2_{\mathfrak{M}}(\gamma^{-1})\big(\pi_{\mathfrak{M}}((G,g)^{-1})(\varphi)\big)[Z+\mu v/u_2]\right| \leq (1/|c_\gamma|)\big(1/(\alpha_1+\|Z\|^2)\big)^{2-(s+k/2)/2}. \qquad \text{(A-14)}$$

Thus the function $Z \rightsquigarrow \pi^2_{\mathfrak{M}}(\gamma^{-1})(\pi_{\mathfrak{M}}((G,g)^{-1})(\varphi))(Z+\mu v/u_2)$ is an L^2 function on $\mathbf{R}^{k-2}$ (when $s > k/2+1$).

On the other hand we note that the integral in (A-12) can also be written in the form

$$\exp\left(-2\pi\sqrt{-1}\,\frac{d_\gamma}{c_\gamma}Q(Z,Z)\right)\int_{\mathbf{R}^2}\pi_{\mathfrak{M}}\left(n\left(\frac{2d_\gamma}{c_\gamma}\right)(G,g)^{-1}\right) \cdot (\varphi)[tv+u\tilde{v}+Z]\exp\big(-2\pi\sqrt{-1}\,\mu u/u_2c_\gamma\big)\,dt\,du. \qquad \text{(A-15)}$$

The integral expression in (A-15) is simply the Fourier transform of the function

$$(t,u)\rightsquigarrow \pi_{\mathfrak{M}}(\cdots)(\varphi)[tv+u\tilde{v}+Z] \quad \text{at } (0,-\mu/u_2c_\gamma).$$

Then using the same argument as above we note that

$$\pi_{\mathfrak{M}}\big(n(2d_\gamma/c_\gamma)(G,g)^{-1}\big)(\varphi)[tv+u\tilde{v}+Z]$$
$$= c_1\pi_{\mathfrak{M}}\left(\left(\begin{pmatrix}1 & 0\\ -2d_\gamma/c_\gamma & 1\end{pmatrix},1\right)(G',g)^{-1}\right)(\varphi)^{\wedge}[uv+t\tilde{v}+M_Q(Z)]$$

with c_1 some nonzero constant. Then we let

$$U(x,y,Z)=\int_{\mathbf{R}^{k-2}}\pi_{\mathfrak{M}}\left(\left(\begin{pmatrix}1 & 0\\ -2d_\gamma/c_\gamma & 1\end{pmatrix},1\right)\cdot(G',g)^{-1}\right) \cdot(\varphi)[xv+y\tilde{v}+W]e^{2\pi\sqrt{-1}\,Q(W,Z)}\,dW$$

and observe that

$$\pi_{\mathfrak{M}}\big(n(2d_\gamma/c_\gamma)(G,g)^{-1}\big)(\varphi)\big[tv + u\tilde{v} + Z\big]$$
$$= c_1 \int U(xv + y\tilde{v} + Z)e^{2\pi\sqrt{-1}\,[xu+yt]}\,dx\,dy. \tag{A-16}$$

Thus if

$$xv + y\tilde{v} + Z = -\mu\tilde{v}/u_2 c_\gamma + Z$$

is a point of continuity of $U(\ ,\ ,\ )$ it follows that

$$U(0, -\mu/u_2 c_\gamma, Z) = \int_{\mathbf{R}^2} \pi_{\mathfrak{M}}\big(n(2d_\gamma/c_\gamma)(G,g)^{-1}\big)$$
$$\cdot(\varphi)\big[tv + u\tilde{v} + Z\big]e^{-2\pi\sqrt{-1}\ \mu u/u_2 c_\gamma}\,dt\,du. \tag{A-17}$$

However we note from (A-1′) that for $|s| > k/2$

$$|U(x,y,Z)| \leqslant \int_{\mathbf{R}^{k-2}}\left[\frac{1}{\alpha_1 + x^2 + y^2 + \|W\|^2}\right]^{\sigma} dW \tag{A-18}$$

with $\sigma = s/2 + k/4 - 1$. This argument also shows that $U(x,y,Z)$ is continuous at all points (x,y,Z). Moreover we deduce if $s > k/2$ that

$$|U(0,y,Z)| \leqslant C_2\big(1/\left(\alpha_1 + |y|^2\right)\big)^{k/4-s/2}, \tag{A-19}$$

with C_2 a positive constant.

Then using (A-12), (A-17) and (A-19), we deduce that (for $|s| > 3k/2 + 1$) property (d) of *-Poisson Summation formula (relative to $L_{(u_1,u_2)}$) is satisfied. Moreover we have that

$$\sum_{\substack{\xi\in\mathfrak{L}\\ \mu\in\mathbf{Z}}}\left|\pi_{\mathfrak{M}}^2(\gamma^{-1})\big(\pi_{\mathfrak{M}}((G,g)^{-1})(\varphi)\big)\left(\xi + \frac{\mu}{u_2}v + V_1\right)\right| \tag{A-20}$$

is majorized by a series of the form

$$\sum_{\substack{\mu\in\mathbf{Z}\\ \xi\in\mathbf{Z}^{k-2}}}\sqrt{g_1(\mu)}\,\sqrt{h(\xi)} \tag{A-21}$$

where

$$g_1(\mu) = \big(1/\left(\alpha_1 + |\mu|^2\right)\big)^{k/4-s/2} \quad\text{and}\quad h(\xi) = \left(\alpha_1 + \|\xi\|^2\right)^{2-(s+k/2)/2}.$$

The series in (A-21) is convergent when $s > 3k/2$. Thus property (d′) is valid in *-Poisson Summation Property relative to L. (We note here that if $\gamma \in SL_2(\mathbf{Z})$ and $c_\gamma = 0$ then (d′) works trivially in such a case.)

(V) Then to prove (e′) we consider the function $\pi_{\mathfrak{M}}((G,g)^{-1})(\varphi)(Z + \mu v/u_2) = \lambda(Z)$ (with $\mu \in \mathbf{Z}$) and must show that λ satisfies the Poisson Summation formula relative to $\mathfrak{L}$. First λ is continuous in Z and using (A-1′) we deduce that for $|s| > k/2$

$$|\lambda(Z)| \leqslant \|Z\|^{2-(k/2+s)}. \tag{A-22}$$

Again by using arguments as above, the function

$$T \rightsquigarrow \int_{\mathbf{R}^2} \pi_{\mathfrak{M}}((G, g)^{-1})(\varphi)\left[T + xv + y\tilde{v}\right] e^{2\pi\sqrt{-1}\, y\mu/u_2}\, dx\, dy \tag{A-23}$$

is a continuous function in T (if $|s| > k/2 + 1$) and hence we have

$$\hat{\lambda}(W) = c_2 \cdot \int_{\mathbf{R}^2} \pi_{\mathfrak{M}}((G', g)^{-1})(\varphi)\left[M_Q(W) + xv + y\tilde{v}\right] e^{2\pi\sqrt{-1}\, y\mu/u_2}\, dx\, dy \tag{A-24}$$

with c_2 a nonzero constant. Hence (for $|s| > k/2 + 1$)

$$|\hat{\lambda}(W)| \leqslant \|W\|^{4-(s+k/2)} \tag{A-25}$$

for W away from zero. Hence λ satisfies the Poisson Summation formula relative to $\mathfrak{L}$. Q.E.D.

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Department of Mathematics, Ohio State University, Columbus, Ohio 43210

Institut de Recherche Mathematique Avancée, Université Louis Pasteur, 7, Rue René Descartes, 67084-Strasbourg, France