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ON THE PICARD GROUP OF A CONTINUOUS TRACE C^*-ALGEBRA

BY

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ABSTRACT. Let A be a continuous trace C^*-algebra with paracompact spectrum T, and let $C(T)$ be the algebra of bounded continuous functions on T, so that $C(T)$ acts on A in a natural way. An A-A bimodule X is an $A\text{-}_{C(T)}A$ imprimitivity bimodule if it is an A-A imprimitivity bimodule in the sense of Rieffel and the induced actions of $C(T)$ on the left and right of X agree. We denote by $\operatorname{Pic}_{C(T)} A$ the group of isomorphism classes of $A\text{-}_{C(T)}A$ imprimitivity bimodules under $\otimes_A$. Our main theorem asserts that $\operatorname{Pic}_{C(T)} A \cong \operatorname{Pic}_{C(T)} C_0(T)$. This result is well known to algebraists if A is an n-homogeneous C^*-algebra with identity, and if A is separable it can be deduced from two recent descriptions of the automorphism group $\operatorname{Aut}_{C(T)} A$ due to Brown, Green and Rieffel on the one hand and Phillips and Raeburn on the other. Our main motivation was to provide a direct link between these two characterisations of $\operatorname{Aut}_{C(T)} A$.

The Picard group of a C^*-algebra has recently been introduced and studied by Brown, Green and Rieffel [3]. In the purely algebraic case of an algebra A with identity over a commutative ring R, the Picard group $\operatorname{Pic}_R A$ of A consists of the invertible $A\text{-}_R A$ bimodules under $\otimes_A$. The C^*-algebra version uses instead the imprimitivity bimodules of Rieffel [11]; although he was primarily interested in applications to group representation theory, these bimodules appear to be the correct C^*-analogue of invertible bimodules. The Picard group $\operatorname{Pic} A$ of a C^*-algebra A consists of the A-A imprimitivity bimodules under the operation $\otimes_A$.

Rosenberg and Zelinsky showed that the Picard group $\operatorname{Pic}_R A$ of an algebra is related to the group $\operatorname{Aut}_R A$ of R-automorphisms of A via an exact sequence

$$0 \to \operatorname{Inn} A \to \operatorname{Aut}_R A \to \operatorname{Pic}_R A,$$

where as usual $\operatorname{Inn} A$ is the group of inner automorphisms (cf. [2, Proposition 5.2]). Brown, Green and Rieffel show that for a C^*-algebra A with countable approximate identity there is also an exact sequence

$$0 \to \operatorname{Inn} A \to \operatorname{Aut} A \xrightarrow{\zeta} \operatorname{Pic} A,$$

where now $\operatorname{Inn} A$ is the group of automorphisms implemented by multipliers of A; in addition they proved that ζ is surjective when A is stable. This result bears a striking resemblance to the main theorem of [9], which asserts that for a separable continuous trace C^*-algebra A with spectrum T there is an exact sequence

$$0 \to \operatorname{Inn} A \to \operatorname{Aut}_{C(T)} A \xrightarrow{\eta} H^2(T, \mathbf{Z}),$$

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where $C(T)$ is the algebra of bounded continuous functions on T, and that η is surjective if A is stable. There is a natural action of $C(T)$ on a C^*-algebra A with spectrum T, and if we denote by $\mathrm{Pic}_{C(T)} A$ the subgroup of $\mathrm{Pic}\, A$ consisting of imprimitivity bimodules X such that

$$(af)xb = ax(fb) \qquad (a, b \in A, f \in C(T), x \in X),$$

then it follows from these two results that for a stable continuous trace C^*-algebra with spectrum T, $\mathrm{Pic}_{C(T)} A \cong H^2(T, \mathbf{Z})$. In case T is compact, the Picard group $\mathrm{Pic}\, C(T) = \mathrm{Pic}_{C(T)}\, C(T)$ can be naturally identified with $H^2(T, \mathbf{Z})$, and we therefore have $\mathrm{Pic}_{C(T)} A \cong \mathrm{Pic}\, C(T)$; in fact, since Pic is a stable isomorphism invariant it is possible to deduce this result for all separable continuous trace C^*-algebras. In the case where A is an n-homogeneous C^*-algebra–known to algebraists either as an Azumaya algebra over $C(T)$ or as a central separable $C(T)$-algebra–this result can be proved directly (cf., for example, [**1**, Corollary 4.5, p. 108]). In [**9**, §2.20] we suggested that it should be possible to prove the general result directly, and the object of this paper is to show how this can be done.

Our proof involves a construction rather like that of building vector bundles from transition functions; since imprimitivity bimodules carry a lot of structure the process is a bit more complicated, and we require quite a few technical properties of imprimitivity bimodules which as far as we know have not been made explicit in the literature. We therefore devote our first section to a discussion of imprimitivity bimodules over arbitrary and continuous trace C^*-algebras. The second section contains our main theorem, which asserts that for any continuous trace C^*-algebra with paracompact spectrum T there is an isomorphism

$$\mathrm{Pic}_{C(T)} A \cong \mathrm{Pic}_{C(T)}\, C_0(T).$$

We have included a short appendix which contains some elementary facts about imprimitivity bimodules; these are mostly concerned with relating imprimitivity bimodules to the purely algebraic invertible bimodules. In particular, Proposition A3 and our main theorem show that Theorem 2.1 of [**9**] is a special case of Corollary 3.5 of [**3**]; although this clearly should be true we know of no easier way to do it, and one of our main motivations for the present work was to make the relationship between [**3**] and [**9**] explicit. Conversely, of course, when A is separable our main result follows from [**3**] and [**9**], but we do not know how to adapt this argument to give our theorem in its full generality.

1. Imprimitivity bimodules. Let A and B be C^*-algebras, and let X be an A-B bimodule; by this we mean that X is a left A- and a right B-module satisfying

$$(ax)b = a(xb), \quad a\lambda xb = ax\lambda b \qquad (a \in A, b \in B, x \in X, \lambda \in \mathbf{C}).$$

Suppose there are two pre-inner products on X, one A-valued, conjugate-linear in the second variable, and denoted $\langle \cdot\,,\cdot \rangle_A$, and the other B-valued, conjugate-linear in the first variable, and denoted $\langle \cdot\,,\cdot \rangle_B$. We call X an A-B imprimitivity bimodule if the following axioms are satisfied for all $x, y \in X$, $a \in A$, $b \in B$:

(1) $\langle x, x \rangle_A \geq 0$ in A, $\langle x, x \rangle_B \geq 0$ in B;

(2) $\langle x, y \rangle_A^* = \langle y, x \rangle_A$, $\langle x, y \rangle_B^* = \langle y, x \rangle_B$;

(3) $\langle x, yb\rangle_B = \langle x, y\rangle_B b$, $\langle ax, y\rangle_A = a\langle x, y\rangle_A$;

(4) the ranges of $\langle \cdot, \cdot\rangle_A$ and $\langle \cdot, \cdot\rangle_B$ generate dense subalgebras of A and B respectively;

(5) $\langle x, y\rangle_A z = x\langle y, z\rangle_B$;

(6) $\langle ax, y\rangle_B = \langle x, a^*y\rangle_B$, $\langle x, yb\rangle_A = \langle xb^*, y\rangle_A$.

If there is an A-B imprimitivity bimodule A and B are said to be strongly Morita equivalent.

REMARK. Our definition of imprimitivity bimodule is not quite the same as Rieffel's [**11**, Definition 6.10]; however the two are equivalent provided we restrict attention to modules over (complete) C^*-algebras (cf. [**11**, Lemma 6.12] and the second part of the proof of [**11**, Lemma 4.27]).

An A-B imprimitivity bimodule X carries two natural seminorms defined by

$$\|x\|_B = \|\langle x, x\rangle_B\|^{1/2}, \qquad \|x\|_A = \|\langle x, x\rangle_A\|^{1/2};$$

by [**12**, Proposition 3.1] these agree. The elements of length zero form a closed A-B submodule of X, so we can mod them out to obtain another A-B imprimitivity bimodule X_0 which carries a natural norm; the completion of X_0 in this norm is yet another A-B imprimitivity bimodule which we call the completion of X. By a complete A-B imprimitivity bimodule we shall mean one which has no vectors of length zero and is complete in the natural norm. If X and Y are complete A-B imprimitivity bimodules, then an isomorphism $\alpha\colon X \to Y$ is a module isomorphism of X onto Y which preserves the inner products.

A fundamental result which we shall often use is the following version of the Cauchy-Schwarz inequality: if X is an A-B imprimitivity bimodule and $x, y \in X$ then $\|\langle x, y\rangle_B\|^2 \leqslant \|\langle x, x\rangle_B\|\,\|\langle y, y\rangle_B\|$ [**11**, Proposition 2.9]. A simple corollary of this is that if X is complete $\langle x, z\rangle_B = \langle y, z\rangle_B$ for all $z \in X$ implies that $x = y$.

A simple nontrivial example of an imprimitivity bimodule can be constructed as follows. Let H be a Hilbert space, with inner product $(\cdot|\cdot)$ conjugate linear in the first variable. With the inner products $\langle h, k\rangle_{K(H)} = h \otimes \bar{k}$, $\langle h, k\rangle_{\mathbf{C}} = (h|k)$, H becomes a complete $K(H)$-$\mathbf{C}$ imprimitivity bimodule, where $K(H)$ is the algebra of compact operators on H and $h \otimes \bar{k}$ is the rank one operator sending g to $(k|g)h$. Equivalently, if A is an elementary C^*-algebra and $p \in A$ is a rank one projection, then Ap is a complete A-$\mathbf{C}$ imprimitivity bimodule with the inner products

$$\langle ap, bp\rangle_A = apb^*, \qquad \langle ap, bp\rangle_{\mathbf{C}} = \operatorname{tr}(pa^*bp).$$

Of interest to us later will be the following generalisation of this example: here if A is a C^*-algebra with primitive ideal space T and $a \in A$, then we denote by $A(t)$ the quotient of A by the ideal $t \in T$, and by $a(t)$ the image of a in $A(t)$.

PROPOSITION 1.1. *Suppose that A is the C^*-algebra defined by a continuous field $\mathcal{H}$ of Hilbert spaces over a locally compact space T (cf.* [**7**, 10.9.2]*). Then the space $X = \Gamma_0(\mathcal{H})$ of continuous vector fields (sections) of $\mathcal{H}$ which vanish at infinity is a complete A-$C_0(T)$ imprimitivity bimodule with the inner products*

$$\langle h, k\rangle_A(t) = h(t) \otimes \overline{k(t)}, \qquad \langle h, k\rangle_{C_0(T)}(t) = (h(t)|k(t)),$$

and the module actions

$$(a \cdot h)(t) = a(t)(h(t)), \qquad (h \cdot f)(t) = h(t)f(t) \qquad (a \in A, f \in C_0(T)).$$

PROOF. First we note that the actions of A and $C_0(T)$ do preserve X by 10.7.3 and 10.1.9 of [7], and all we have to do is verify the six axioms. Axiom (1) follows from the observation that $a \in A$ is positive if and only if $a(t) \geq 0$ for every $t \in T$ (Corollary 10.3.4 of [7]), and similarly axioms (2), (3), (5) and (6) reduce to pointwise statements about rank one operators which are easy to check. That (4) holds for $\langle \cdot , \cdot \rangle_{C_0(T)}$ follows from the Stone-Weierstrass theorem, and it only remains to show that the algebra B generated by the range of $\langle \cdot , \cdot \rangle_A$ is dense in A. Since B is clearly a $C_0(T)$ submodule of A, it is enough to show that $B(t)$ is dense in $A(t)$ for each $t \in T$. But $B(t)$ is the set of finite rank operators on $\mathcal{H}(t)$, and so is dense in $A(t) = K(\mathcal{H}(t))$. $\square$

COROLLARY 1.2. *Let A be a continuous trace C^*-algebra with spectrum T, and suppose that p is an element of A such that $p(t)$ is a rank one projection for every $t \in T$. Then Ap is a complete A-$C_0(T)$ imprimitivity bimodule with the inner products*

$$\langle ap, bp \rangle_A = apb^*, \qquad \langle ap, bp \rangle_{C_0(T)}(t) = \operatorname{tr}(p(t)a(t)^*b(t)p(t)).$$

Further, if q is another such element of A and $v \in A$ satisfies

$$v(t)v(t)^* = p(t), \qquad v(t)^*v(t) = q(t) \quad \text{for } t \in T,$$

then right multiplication by v defines an isomorphism r_v of Ap onto Aq.

PROOF. In this case A is the C^*-algebra defined by the continuous field $A(t)p(t)$ of Hilbert spaces (cf. [7, 10.7.6]), and the space of sections of $A(t)p(t)$ which vanish at infinity is Ap. That $ap \to apv$ preserves inner products is a straightforward calculation, and it is an isomorphism since $aq \to aqv^*$ is an inverse. $\square$

If X and Y are, respectively, A-B and B-C imprimitivity bimodules, then their algebraic tensor product $X \odot_B Y$ is an A-C bimodule, and we can define two inner products on $X \odot_B Y$ by

$$\langle x \otimes y, x_1 \otimes y_1 \rangle_C = \langle \langle x_1, x \rangle_B y, y_1 \rangle_C, \quad \langle x \otimes y, x_1 \otimes y_1 \rangle_A = \langle x, x_1 \langle y_1, y \rangle_B \rangle_A.$$

In this way $X \odot_B Y$ becomes an A-C imprimitivity bimodule: the axioms all follow from [**11**, Theorem 5.9], except for (5) which is a simple calculation. We shall denote by $X \otimes_B Y$ the completion of $X \odot_B Y$. If Z is a C-D imprimitivity bimodule then the map $(x \otimes y) \otimes z \to x \otimes (y \otimes z)$ induces an inner product preserving bijection of $(X \odot_B Y) \odot_C Z$ onto $X \odot_B (Y \odot_C Z)$; we identify both with the triple tensor product $X \odot_B Y \odot_C Z$ and let $X \otimes_B Y \otimes_C Z$ denote its completion. It is quite easy to check that this is isomorphic to $(X \otimes_B Y) \otimes_C Z$ and $X \otimes_B (Y \otimes_C Z)$.

We are interested in imprimitivity bimodules because they are a suitable analogue for C^*-algebras of invertible bimodules. The trivial B-B imprimitivity bimodule is B itself, equipped with the obvious module actions and the (left and right) inner products $\langle b_1, b_2 \rangle_B^l = b_1 b_2^*$, $\langle b_1, b_2 \rangle_B^r = b_1^* b_2$. If X is an A-B imprimitivity bimodule, then its inverse is the dual module $\tilde{X}$, which as a linear space is just X, has the same inner products as X, and has the following module actions: $b\tilde{x} = (xb^*)^\sim$, $\tilde{x}a = (a^*x)^\sim$. The map $\tilde{x} \otimes y \to \langle x, y\rangle_B$ defines an inner-product preserving map of $\tilde{X} \odot_A X$ into B with dense range, and so extends to an isomorphism of $\tilde{X} \otimes_A X$ onto B; similarly $X \otimes_B \tilde{X}$ is isomorphic to A. For example, the dual of the imprimitivity bimodule Ap of Corollary 1.2 is isomorphic to pA under the map $(ap)^\sim \to pa^*$. An isomorphism α: $X_1 \to X_2$ of imprimitivity bimodules induces an isomorphism $\tilde{\alpha}$: $\tilde{X}_1 \to \tilde{X}_2$ via $\tilde{\alpha}(\tilde{x}) = \alpha(x)^\sim$; if we identify $(Ap)^\sim$ with pA and $(Aq)^\sim$ with qA, then the dual of the isomorphism r_v of Corollary 1.2 is given by left multiplication by v^*.

The set of isomorphism classes of complete A-A imprimitivity bimodules forms a group under $\otimes_A$, which is called the Picard group of A and denoted by Pic A (this differs slightly from the definition given in §3 of **[3]**, but the two are easily seen to agree). If Z is an A-B imprimitivity bimodule, then the map $X \to \tilde{Z} \otimes_A X \otimes_A Z$ induces an isomorphism of Pic A onto Pic B, and it follows immediately from **[3**, Theorem 1.2**]** that Pic is a stable isomorphism invariant.

If X is a complete A-B imprimitivity bimodule and $m \in M(A)$, the multiplier algebra of A, then m acts naturally on $AX = \{ax: a \in A, x \in X\}$ via the formula $m(ax) = (ma)x$; this action is well defined since if $a_1x_1 = a_2x_2$ and $z \in X$, then

$$\begin{aligned}\langle (ma_1)x_1 - (ma_2)x_2, z\rangle_A &= (ma_1)\langle x_1, z\rangle_A - (ma_2)\langle x_2, z\rangle_A \\ &= m\langle a_1x_1 - a_2x_2, z\rangle_A = 0.\end{aligned}$$

A similar calculation shows that $\|m(ax)\|_A^2 = \|m\langle ax, ax\rangle_A m^*\| \leq \|m\|^2\|ax\|_A^2$ so that the action is continuous. Since AX is dense in X (Lemma 6.13 of **[11]**) it follows that $M(A)$ acts on all of X, and we have $\langle mx, y\rangle_A = m\langle x, y\rangle_A$, $a(mx) = (am)x$, $\langle mx, y\rangle_B = \langle x, m^*y\rangle_B$ for all x, $y \in X$, $m \in M(A)$ and $a \in A$. The Dauns-Hofmann theorem identifies the algebra $C(\text{Prim } A)$ of bounded continuous functions on the primitive ideal space of A with the centre of $M(A)$, and so there is in particular an action of $C(\text{Prim } A)$ on X with the same properties. If X is a complete A-A imprimitivity bimodule then $C(\text{Prim } A)$ acts on both the left and right of X; if

$$fx = xf \quad \text{for all } f \in C(\text{Prim } A), x \in X,$$

then we call X an A-${}_{C(\text{Prim } A)}A$ imprimitivity bimodule. The collection of (classes of) such bimodules forms a subgroup of Pic A which we denote by $\text{Pic}_{C(\text{Prim } A)} A$ or $\text{Pic}_{ZM(A)} A$.

Rieffel **[12**, §3**]** has shown that an A-B imprimitivity bimodule X determines lattice isomorphisms between the (closed two-sided) ideals of A, the closed A-B submodules of X and the ideals of B: if I is an ideal in A then the submodule of X corresponding to I is

$${}_IX = \{x \in X: \langle x, x\rangle_A \in I\} = \{x \in X: \langle x, y\rangle_A \in I \text{ for all } y \in X\},$$

and the ideal of B corresponding to the submodule Y of X is

$${}_YJ = \overline{\text{sp}}\,\{\langle x, y\rangle_B: x \in X, y \in Y\}.$$

If I is an ideal in A we shall write $\alpha_X(I)$ for the ideal in B which X associates to I. Rieffel has also shown that α_X maps primitive ideals to primitive ideals, and that

α_X: Prim $A \to$ Prim B is a homeomorphism [**12**, Corollary 3.8]. For example, if A is as in Corollary 1.2 and we identify Prim A and Prim $C_0(T)$ with T in the usual way, then α_{A_p} is the identity map: $T \to T$.

LEMMA 1.3. *If X is an A-B imprimitivity bimodule and Y is a B-C imprimitivity bimodule, then*

$$\alpha_{X \otimes_B Y} = \alpha_Y \circ \alpha_X.$$

PROOF. Let I be an ideal in A: we have to show that

$$_I(X \otimes_B Y) = (X \otimes_B Y)_{\alpha_Y(\alpha_X(I))}.$$

First we claim that $_I(X \otimes_B Y)$ is the closure of $_I(X \odot_B Y)$ in the A-norm on $X \otimes_B Y$. [Since $X \otimes Y$ is the completion of $(X \odot_B Y)_0$, not $X \odot_B Y$, what we mean by this is the closure of the image of $_I(X \odot_B Y)$ in $(X \odot_B Y)_0$.] To see this we observe that since $_I(X \odot_B Y)^-$ is a closed A-C submodule of $X \otimes_B Y$, by Rieffel's theorem $_I(X \odot_B Y)^- = {}_J(X \otimes_B Y)$ for some ideal J in A. Since the inner products are continuous we have $_I(X \odot_B Y)^- \subset {}_I(X \otimes_B Y)$, so that $J \subset I$. However, if $z \in {}_I(X \odot_B Y)$ then z is a finite tensor in $_J(X \otimes_B Y)$ and so belongs to $_J(X \odot_B Y)$; thus $I \subset J$ and we have verified the claim.

By [**12**, Lemma 3.3], $_I(X \odot_B Y)$ is the closure in $X \odot_B Y$ of $I(X \odot_B Y) = (IX) \odot_B Y$; a simple approximation argument shows that $\overline{IX} = \overline{X\alpha_X(I)}$, and so we have $_I(X \odot_B Y) = [X\alpha_X(I) \odot_B Y]^-$ where the closure is in $X \odot_B Y$. But elements of B pull across the tensor, and so a similar argument shows

$$\left[X\alpha_X(I) \odot_B Y\right]^- = \left[X \odot_B Y\alpha_Y(\alpha_X(I))\right]^-.$$

Unravelling this and using the right-handed version of our claim gives the result. $\square$

Let X be an A-B imprimitivity bimodule, let I be an ideal in A and let $J = \alpha_X(I)$. We shall write ρ_I: $X \to X/{}_IX$, σ_I: $A \to A/I$, τ_J: $B \to B/J$ for the various quotient maps. The submodule $_IX$ of X is an I-J imprimitivity bimodule in the obvious way; further, the quotient $X/{}_IX$ is an A/I-B/J imprimitivity bimodule if we define

$$\langle \rho_I(x), \rho_I(y) \rangle_{A/I} = \sigma_I(\langle x, y \rangle_A), \qquad \sigma_I(a) \cdot \rho_I(x) = \rho_I(ax),$$

and so on–it is straightforward to verify that these actions and inner products are well defined (again this was first noticed by Rieffel in [**12**, §3]). We can identify Prim I with $\{K \in \text{Prim } A\colon K \not\supset I\}$ via $K \cap I \leftrightarrow K$, and Prim A/I with $\{K \in \text{Prim } A\colon K \supset I\}$ via $K/I \leftrightarrow K$.

LEMMA 1.4. *The homeomorphisms induced by $_IX$ and $X/{}_IX$ are given by*

$$\alpha_{{}_IX}(K \cap I) = \alpha_X(K) \cap \alpha_X(I); \qquad \alpha_{X/{}_IX}(K/I) = \alpha_X(K)/\alpha_X(I).$$

PROOF. Suppose $K \in$ Prim A and $K \not\supset I$. Then

$$_{K \cap I}({}_IX) = {}_{K \cap I}X = X_{\alpha_X(K \cap I)};$$

since α_X is a lattice isomorphism, $\alpha_X(K \cap I) = \alpha_X(K) \cap \alpha_X(I)$, and

$$_{K \cap I}({}_IX) = \left(X_{\alpha_X(I)}\right)_{\alpha_X(K) \cap \alpha_X(I)} = ({}_IX)_{\alpha_X(K) \cap \alpha_X(I)}.$$

If now $K \in \operatorname{Prim} A$ and $K \supset I$, then it is easy to check that ${}_K X/{}_I X = {}_{K/I}(X/{}_I X)$, and so

$$ {}_{K/I}(X/{}_I X) = X_{\alpha_X(K)}/X_{\alpha_X(I)} = \left(X/{}_{\alpha_X(I)}X\right)_{\alpha_X(K)/\alpha_X(I)}, $$

which is enough to prove the lemma. □

PROPOSITION 1.5. *Let X be a complete A-B imprimitivity bimodule, and let I, J be ideals in B with $I \cap J = \{0\}$. Then the quotient map ρ_I: $X \to X/X_I$ induces a bijection of X_J onto $(X/X_I)_{(I+J)/I}$.*

PROOF. We first observe that $(X/X_I)_{(I+J)/I} = X_{I+J}/X_I$, and that, since $I \to X_I$ is a lattice isomorphism, we have $X_{I+J} = (X_I + X_J)^-$. These two observations imply that $(X_I + X_J)/X_I$ is dense in $(X/X_I)_{(I+J)/I}$ in the $(I + J)/I$ norm. Now the mapping ρ_I: $X_J \to (X_I + X_J)/X_I$ is surjective, and if τ_I is the quotient map: $B \to B/I$ then for $x \in X_J$

$$ \|x\|_J^2 = \|\langle x, x\rangle_J\| = \|\tau_I(\langle x, x\rangle_{I+J})\| = \|\rho_I(x)\|_{(I+J)/I}^2 $$

since J is isometrically isomorphic to $(I + J)/I$. Thus ρ_I is isometric with dense range, and since X_J is complete we conclude that ρ_I is surjective as claimed. □

If A and B are two C^*-algebras whose primitive ideal spaces have both been identified with the same topological space T, then an A-${}_TB$ imprimitivity bimodule X is one for which the induced homeomorphism α_X is the identity. To each closed subset K of T correspond closed ideals A_K in A and B_K in B, and since α_X is the identity these correspond to the same submodule X_K of X. We can identify both $\operatorname{Prim} A_K$ and $\operatorname{Prim} B_K$ with $T \setminus K$, and Lemma 1.4 then implies that α_{X_K} is the identity, so that X_K is an A_K-${}_{T\setminus K}B_K$ imprimitivity bimodule. Similarly we can identify the primitive ideal spaces of the quotients $A^K = A/A_K$ and $B^K = B/B_K$ with K, and $X^K = X/X_K$ is an A^K-${}_KB^K$ imprimitivity bimodule. If $x \in X$ then we shall write x^K for its image in X^K, and similarly for elements of A and B. In this notation, the imprimitivity bimodule structure is defined as follows:

$$ a^K x^K = (ax)^K; \qquad \langle x^K, y^K\rangle_{A^K} = (\langle x, y\rangle_A)^K, $$
$$ x^K b^K = (xb)^K; \qquad \langle x^K, y^K\rangle_{B^K} = (\langle x, y\rangle_B)^K, $$

and Proposition 1.5 takes the following form:

COROLLARY 1.6. *Let A, B be C^*-algebras with the same primitive ideal space T, and let X be a complete A-${}_TB$ imprimitivity bimodule. If K is a closed subset of T and U is an open subset of K, then the quotient map ρ^K: $X \to X^K$ induces a bijection of $X_{T\setminus U}$ onto $(X^K)_{K\setminus U}$.*

We shall now specialise to the case of imprimitivity bimodules over continuous trace C^*-algebras. If A is a continuous trace C^*-algebra with spectrum (and primitive ideal space) T, we denote by $A(t)$ the quotient of A by the ideal A_t corresponding to $t \in T$. The field $\mathcal{Q} = \{A(t)\}$ is a continuous field of elementary C^*-algebras over the locally compact space T, and A can be regarded as the algebra $\Gamma_0(\mathcal{Q})$ of continuous sections of $\mathcal{Q}$ which vanish at infinity. The ideal A_K corresponding to the closed set $K \subset T$ is then the space of sections in $\Gamma_0(\mathcal{Q})$ which vanish on K, and the quotient A^K is canonically isomorphic to the space $\Gamma(\mathcal{Q}|_K)$ of sections of $\mathcal{Q}$ over K. If X is an A-${}_TA$ imprimitivity bimodule, then identifying A^K with $A|_K$ we see that the inner products on X^K are defined by $\langle y^K, z^K\rangle_{A^K} = \langle y, z\rangle_{A|_K}$. In particular, if $t \in K \subset T$, then

$$\langle y^t, z^t\rangle_{A(t)} = \langle y, z\rangle_A(t) = \langle y^K, z^K\rangle_{A^K}(t);$$

we shall frequently use this observation without comment.

PROPOSITION 1.7. *Let A, B be continuous trace C^*-algebras with paracompact spectrum T, let X be an A-${}_TB$ imprimitivity bimodule and let $K \subset T$ be closed. Then the quotient norm of X^K inherited from the B-norm on X coincides with the B^K-norm.*

PROOF. The B^K-norm and the quotient norms are given respectively by

$$\|y^K\|^2_{B^K} = \|\langle y, y\rangle_{B|_K}\|,$$

$$\|y^K\|^2_{X/X_K} = \inf_{z \in X_K} \|\langle y, y\rangle_B + \langle z, y\rangle_B + \langle y, z\rangle_B + \langle z, z\rangle_B\|.$$

If $z \in X_K$, then $\langle y, z\rangle_B$, $\langle z, y\rangle_B$ and $\langle z, z\rangle_B$ are all in B_K, so that

$$\|y^K\|^2_{X/X_K} \geqslant \inf_{z \in X_K} \|(\langle y, y\rangle_B + \cdots)|_K\| = \|\langle y, y\rangle_B|_K\|.$$

To see the converse, let $\varepsilon > 0$, choose a neighbourhood U of K such that

$$\|\langle y, y\rangle_B(t)\| \leqslant \|\langle y, y\rangle_B|_K\| + \varepsilon \quad \text{for } t \in U,$$

and let $\rho\colon T \to [0, 1]$ be a continuous function such that $\rho \equiv 0$ on K and $\rho \equiv 1$ off U (which exists since T is paracompact and hence normal). Then $z = -y\rho \in X_K$, and expanding out the left-hand side shows that

$$\|\langle y - z, y - z\rangle_B(t)\| = (1 - \rho(t))^2\|\langle y, y\rangle_B(t)\|.$$

Since this is zero off U, the result follows. □

COROLLARY 1.8. *Under the same hypotheses as the proposition, X^K is a complete A^K-${}_KB^K$ imprimitivity bimodule.*

PROOF. Since X_K is a closed submodule of X, $X|X_K$ is complete in the quotient norm. □

REMARK. We have been unable to decide whether Proposition 1.7 is true for imprimitivity bimodules over arbitrary C^*-algebras. It does, however, hold for imprimitivity bimodules over a C^*-algebra A which is a maximal full algebra of operator fields (see the remark following Corollary 2.7).

In §3 we shall frequently localise by passing to quotient imprimitivity bimodules; the following observations show that this operation commutes with other procedures, such as tensoring. The first lemma is an easy consequence of the definition of the quotient structure:

LEMMA 1.9. *Suppose that X_i $(i = 1, 2, 3)$ are complete A-${}_TB$ imprimitivity bimodules and that $\alpha\colon X_1 \to X_2$ is an isomorphism. If K is a closed subset of T then $\alpha^K(x^K) = \alpha(x)^K$ defines an isomorphism $\alpha^K\colon X_1^K \to X_2^K$. If $\beta\colon X_2 \to X_3$ is another isomorphism, then $(\beta \circ \alpha)^K = \beta^K \circ \alpha^K$.*

Let A, B, C be continuous trace C^*-algebras with spectrum T, and let X, Y respectively be complete A-${}_T B$, B-${}_T C$ imprimitivity bimodules. Then Lemma 1.3 shows that $X \otimes_B Y$ is an A-${}_T C$ imprimitivity bimodule, so that we can form the quotients $(X \otimes_B Y)^K$; it is natural to ask whether these are the same as we would have got by taking quotients first and then tensoring:

Lemma 1.10. *Let K be a closed subset of T. Then the map ϕ defined on elementary tensors by $\phi(x \otimes y) = x^K \otimes y^K$ induces an isomorphism σ^K of $(X \otimes_B Y)^K$ onto $X^K \otimes_{B^K} Y^K$. Further, if α: $X_1 \to X_2$ and β: $Y_1 \to Y_2$ are isomorphisms then the following diagram commutes*:

$$\begin{array}{ccc} (X_1 \otimes_B Y_1)^K & \stackrel{\sigma_1^K}{\to} & X_1^K \otimes_{B^K} Y_1^K \\ (\alpha \otimes \beta)^K \downarrow & & \downarrow \alpha^K \otimes \beta^K \\ (X_2 \otimes_B Y_2)^K & \stackrel{\sigma_2^K}{\to} & X_2^K \otimes_{B^K} Y_2^K. \end{array}$$

Proof. A simple computation shows that

$$\langle x^K \otimes y^K, w^K \otimes z^K \rangle_{C^K} = (\langle x \otimes y, w \otimes z \rangle_C)^K,$$

so that $\ker \phi = X_K$ and the induced map σ^K preserves inner products. Every finite tensor is in the range of σ^K, so that σ^K has dense range and extends to an isomorphism of the complete space $(X \otimes_B Y)^K$ onto the completed tensor product. That the diagram commutes is a routine calculation. $\square$

This lemma implies that identifying the isomorphisms $(\alpha \otimes \beta)^K$ and $\alpha^K \otimes \beta^K$ will not cause any problems. In the same way if α: $X_1^K \to X_2^K$ and β: $Y_1^K \to Y_2^K$ are isomorphisms we can define an isomorphism $\alpha \otimes \beta$: $(X_1 \otimes Y_1)^K \to (X_2 \otimes Y_2)^K$ by

$$\alpha \otimes \beta(x^K \otimes y^K) = (w \otimes z)^K \text{ where } w, z \text{ satisfy } w^K = \alpha(x^K),\ z^K = \beta(y^K),$$

and similar arguments show that confusing this with the composition

$$\left(\sigma_2^K\right)^{-1} \circ \alpha \otimes \beta \circ \left(\sigma_1^K\right)$$

will not cause any problems either. We shall later do this without comment.

If A and B are any two C^*-algebras with the same primitive ideal space T and X is a complete A-${}_T B$ imprimitivity bimodule then there are actions of $C(T)$ on the left and right of X induced by the actions of $C(\operatorname{Prim} A)$ and $C(\operatorname{Prim} B)$. In fact, these two actions are the same–in other words, $fx = xf$ for $x \in X$ and $f \in C(T)$. To see this, note that the action of $C(T)$ on A is defined by $fa - f(t)a \in A_t$ for all $t \in T$ (cf. for example, [6]), from which it follows that

$$fx - f(t)x \in {}_tX \quad \text{for all } t \in T.$$

Similarly we have

$$xf(t) - xf \in X_t \quad \text{for all } t;$$

since ${}_tX = X_t$ and $\lambda x = x\lambda$ for $\lambda \in \mathbf{C}$, we deduce that $fx - xf \in X_t$ for each $t \in T$. This implies that $\|fx - xf\|_B^2 = 0$, and we conclude that $fx = xf$ as claimed. If A and B are continuous trace C^*-algebras, we have the following converse:

PROPOSITION 1.11. *Let A and B be continuous trace C^*-algebras, and let X be a complete A-B imprimitivity bimodule. Suppose that h*: Prim $A \to$ Prim B *is a homeomorphism, and define an action of* $C(\text{Prim } A)$ *on the right of X by $x \cdot f = x(f \circ h^{-1})$. If $fx = x \cdot f$ for all $x \in X$ and $f \in C(\text{Prim } A)$, then $\alpha_X = h$.*

LEMMA 1.12. *Let A be a continuous trace C^*-algebra, $t \in$ Prim A and $a \in A_t$. Then there is a continuous function $f \in C(\text{Prim } A)$ and $c \in A$ such that $f(t) = 0$ and $a = fc$.*

PROOF. Define $f \in C(\text{Prim } A)$ by $f(t) = \|a(t)\|^{1/2}$ and $c \in A$ by

$$c(t) = \begin{cases} a(t)/\|a(t)\|^{1/2} & \text{if } \|a(t)\| \neq 0, \\ 0 & \text{if } \|a(t)\| = 0. \end{cases} \quad \square$$

PROOF OF PROPOSITION 1.11. Since the space $AXB = \{axb\colon a \in A,\ x \in X,\ b \in B\}$ is dense in X, [**12**, Lemma 3.3] implies that

$$\alpha_X(A_t) = \overline{\mathrm{sp}}\,\{\langle x, ayb\rangle_B\colon x, y \in X, a \in A_t, b \in B\}.$$

By Lemma 1.12 this is the same as

$$\begin{aligned} &\overline{\mathrm{sp}}\,\{\langle x, cfyb\rangle_B\colon x, y \in X, c \in A, b \in B, f \in C(\text{Prim } A), f(t) = 0\} \\ &\quad = \overline{\mathrm{sp}}\,\big\{\langle x, cy\big[(f \circ h^{-1})b\big]\rangle_B\colon \\ &\qquad\qquad x, y \in X, c \in A, b \in B, f \in C(\text{Prim } A), f(t) = 0\big\}. \end{aligned}$$

Using Lemma 1.12 again shows that this is just $B_{h(t)}$ and the result is proved. $\square$

REMARK. Proposition 1.11 is false without some restriction on the C^*-algebras A, B. For example, take $A = \{f \in C([0, 1], M_2(\mathbf{C}))\colon f_{21}(0) = f_{12}(0) = 0\}$, $X = \{f \in C([0, 1], M_2(\mathbf{C}))\colon f_{11}(0) = f_{22}(0) = 0\}$ where of course $f_{ij}(t)$ denotes the (i, j)th entry in the matrix $f(t)$. Then X is an A-A imprimitivity bimodule with the actions of pointwise matrix multiplication and the inner products $\langle f, g\rangle_A^r(t) = f(t)^* g(t)$, $\langle f, g\rangle_A^l(t) = f(t)g(t)^*$. The primitive ideal space of A is $]0, 1] \cup \{0_1, 0_2\}$, topologised so that sequences converging to 0 in $]0, 1]$ converge to both 0_1 and 0_2 in Prim A, where for $t \in\,]0, 1]$, A_t is the ideal of functions vanishing at t and for $i = 1, 2$, $0_i = \{f \in A\colon f_{ii}(0) = 0\}$. Then $C(\text{Prim } A) = C([0, 1])$, and so the natural actions of $C(\text{Prim } A)$ on the right and left of X agree. However, the homeomorphism α_X is not the identity: it interchanges 0_1 and 0_2.

2. The Picard group of a continuous trace C^*algebra. We are now ready to prove our main theorem.

THEOREM 2.1. *Let A be a continuous trace C^*-algebra with paracompact spectrum T. Then*

$$\mathrm{Pic}_{C(T)}\, A \cong \mathrm{Pic}_{C(T)}\, C_0(T).$$

If A is defined by a continuous field of Hilbert spaces then there is an A-${}_T C_0(T)$ imprimitivity bimodule X (cf. Proposition 1.1) and $Y \to \tilde{X} \otimes_A Y \otimes_A X$ gives the required isomorphism. Since a continuous trace C^*-algebra A is locally given by a field of Hilbert spaces, we can find a cover $\{N_i\}$ of T and $A^{\overline{N}_i}$-${}_{\overline{N}_i}C(\overline{N}_i)$ imprimitivity bimodules X_i, and to each A-${}_T A$ imprimitivity bimodule Y we can associate a family $\{\tilde{X}_i \otimes_{A^{\bar{N}_i}} Y^{\bar{N}_i} \otimes_{A^{\bar{N}_i}} X_i\}$ of $C(\bar{N}_i)$-${}_{\bar{N}_i}C(\bar{N}_i)$ imprimitivity bimodules. Our problem is to prove that we can construct imprimitivity bimodules from this sort of local data, and, of course, that our construction does give an isomorphism of Picard groups. Throughout this section A will be a continuous trace C^*-algebra with paracompact spectrum T.

We denote by L_A the collection of all triples $\{N_i, Y_i, \phi_{ij}: i, j \in I\}$ consisting of an open cover $\{N_i\}$ of T, complete $A^{\bar{N}_i}$-${}_{\bar{N}_i}A^{\bar{N}_i}$ imprimitivity bimodules Y_i, and isomorphisms $\phi_{ij}: Y_j^{\bar{N}_{ij}} \to Y_i^{\bar{N}_{ij}}$ satisfying

$$\phi_{ij}^{\bar{N}_{ijk}} \circ \phi_{jk}^{\bar{N}_{ijk}} = \phi_{ik}^{\bar{N}_{ijk}}.$$

(Here $\bar{N}_{ij} = \bar{N}_i \cap \bar{N}_j$, and so on.) We shall call the ϕ_{ij}'s transition isomorphisms. We define an equivalence relation $\sim$ on L_A as follows

$$\{N_i, Z_i, \phi_{ij}: i, j \in I\} \sim \{M_k, Y_k, \psi_{kl}: k, l \in K\}$$

if and only if there is an open cover $\{L_\lambda: \lambda \in \Lambda\}$ of T with maps $r: \Lambda \to I$, $s: \Lambda \to K$ such that $L_\lambda \subset N_{r(\lambda)} \cap M_{s(\lambda)}$, and imprimitivity bimodule isomorphisms $\rho_\lambda: Z_{r(\lambda)}^{\bar{L}_\lambda} \to Y_{s(\lambda)}^{\bar{L}_\lambda}$ such that for each $\lambda, \mu \in \Lambda$ the following diagram commutes:

$$\begin{array}{ccc} Z_{r(\mu)}^{\bar{L}_{\lambda\mu}} & \xrightarrow{\rho_\mu^{\bar{L}_{\lambda\mu}}} & Y_{s(\mu)}^{\bar{L}_{\lambda\mu}} \\ \phi_{r(\lambda)r(\mu)}^{\bar{L}_{\lambda\mu}} \downarrow & & \downarrow \psi_{s(\lambda)s(\mu)}^{\bar{L}_{\lambda\mu}} \\ Z_{r(\lambda)}^{\bar{L}_{\lambda\mu}} & \xrightarrow{\rho_\mu^{\bar{L}_{\lambda\mu}}} & Y_{s(\lambda)}^{\bar{L}_{\lambda\mu}}. \end{array}$$

That this is in fact an equivalence relation follows at once from elementary properties of the quotient isomorphisms (cf. Lemma 1.9). Notice that if Z is a complete A-${}_TA$ imprimitivity bimodule then $Z \in L_A$ in the sense that $\{T, Z, \mathrm{id}\} \in L_A$, and that if $\{N_i\}$ is any open cover of T, then each $Z^{\bar{N}_i}$ is complete by Proposition 1.7 and $Z \sim \{N_i, Z^{\bar{N}_i}, \mathrm{id}_{ij}: i, j \in I\}$ where $\mathrm{id}_{ij}: Z^{\bar{N}_{ij}} \to Z^{\bar{N}_{ij}}$ is the identity for each i, j. In the same way we can always replace $\{N_i\}$ by a refinement without changing the class of $\{N_i, Y_i, \phi_{ij}\}$.

We now describe our basic construction of imprimitivity bimodules from local data. Let $\{N_i, Y_i, \phi_{ij}\} \in L_A$. Since each ϕ_{ij} preserves the inner products, for every $y = \{y_i\}_{i \in I} \in \prod_{i \in I} Y_i$ satisfying $\phi_{ij}(y_j^{\bar{N}_{ij}}) = y_i^{\bar{N}_{ij}}$ and every $t \in T$, the formula

$$\langle y, y\rangle_A^r(t) = \langle y_i, y_i\rangle_{A^{\bar{N}_i}}^r(t) \quad (\text{if } t \in N_i)$$

gives a well-defined element of $A(t)$. We set

$$Y = \left\{ \{y_i\} \in \prod_{i \in I} Y_i \;\middle|\; \begin{array}{ll} \phi_{ij}^{\bar{N}_{ij}}\left(y_j^{\bar{N}_{ij}}\right) = y_i^{\bar{N}_{ij}} & \text{for all } i, j \in I, \text{ and} \\ t \to \|\langle y, y\rangle_A^r(t)\| & \text{vanishes at infinity} \end{array} \right\}.$$

If $y, z \in Y$ we define

$$\langle y, z\rangle_A^r(t) = \langle y_i, z_i\rangle_{A^{\bar{N}_i}}(t) \quad \text{for } t \in N_i;$$

the same argument as above shows that $\langle y, z\rangle_A^r$ is well defined and the Cauchy-Schwarz inequality implies that $\|\langle y, z\rangle_A^r(t)\|$ vanishes at infinity. Since $\langle y, z\rangle_A^{r\,\bar{N}_i} = \langle y_i, z_i\rangle_{A^{\bar{N}_i}}^r = a|_{\bar{N}_i}$ for some $a \in A$, and since A is closed under local uniform approximation, we conclude that $\langle y, z\rangle_A^r$ is in fact an element of A. Similarly, we set

$$\langle y, z\rangle_A^l(t) = \langle y_i, z_i\rangle_{A^{\bar{N}_i}}^l(t) \quad \text{if } t \in N_i.$$

The equality of the left and right norms in the $A(t)$-$A(t)$ imprimitivity bimodule Y^t implies that $\|\langle y, z\rangle_A^l(t)\|$ vanishes at infinity, and as before we see that $\langle y, z\rangle_A^l \in A$. We define module actions of A on Y by $a \cdot \{y_i\} = \{a|_{\bar{N}_i} y_i\}$, $\{y_i\} \cdot a = \{y_i(a|_{\bar{N}_i})\}$; since the ϕ_{ij} are $A^{\bar{N}_{ij}}$-bimodule homomorphisms and

$$\|\langle a|_{\bar{N}_i} y_i, a|_{\bar{N}_i} y_i\rangle_{A^{\bar{N}_i}}^r(t)\| \leqslant \|a(t)\|^2 \|\langle y_i, y_i\rangle_{A^{\bar{N}_i}}^r(t)\|,$$

these actions do map Y back into itself. With these actions Y becomes an A-A bimodule, which we shall denote by $\Phi\{N_i, Y_i, \phi_{ij}\}$. We shall show that in fact it is an imprimitivity bimodule, but first we need a lemma.

LEMMA 2.2. *Let $i \in I$ be fixed and suppose that U is an open subset of N_i and $y_i \in (Y_i)_{\bar{N}_i\setminus U}$. Then for each $j \neq i$ we can choose $y_j \in (Y_j)_{\bar{N}_j\setminus U}$ such that*

$$\phi_{jk}\left(y_k^{\bar{N}_{jk}}\right) = y_j^{\bar{N}_{jk}} \quad \textit{for all } j, k \in I.$$

PROOF. We define $z_j \in Y_j^{\bar{N}_{ij}}$ by $z_j = \phi_{ji}(y_i^{\bar{N}_{ij}})$ $(j \neq i)$; note that $z_j \in (Y_j^{\bar{N}_{ij}})_{\bar{N}_{ij}\setminus U}$ since $\phi_{ji} = \phi_{ij}^{-1}$ preserves the inner products. By Corollary 1.6 for each $j \neq i$ we can find $y_j \in (Y_j)_{\bar{N}_j\setminus U}$ such that $y_j^{\bar{N}_{ij}} = z_j$, and the condition in the lemma is trivially satisfied if one of j, k is i. So suppose that neither of j, k is i. It is enough to show that if $w \in Y^{\bar{N}_{jk}}$ then

$$\left\langle \phi_{jk}\left(y_k^{\bar{N}_{jk}}\right), w\right\rangle_{A^{\bar{N}_{jk}}}(t) = \left\langle y_j^{\bar{N}_{jk}}, w\right\rangle_{A^{\bar{N}_{jk}}}(t) \tag{1}$$

for all $t \in \bar{N}_{jk}$. If $t \notin \bar{N}_i$, then $t \notin U$ so that $\langle y_j, y_j\rangle_{A^{\bar{N}_j}}(t) = 0 = \langle y_k, y_k\rangle_{A^{\bar{N}_k}}(t)$ and the Cauchy-Schwarz inequality implies that both sides of (1) are zero. If $t \in \bar{N}_i$, then

$$\begin{aligned}
\left\langle \phi_{jk}\left(y_k^{\bar{N}_{jk}}\right), w\right\rangle_{A^{\bar{N}_{jk}}}(t) &= \left\langle \phi_{jk}^{\bar{N}_{ijk}}\left(y_k^{\bar{N}_{ijk}}\right), w^{\bar{N}_{ijk}}\right\rangle_{A^{\bar{N}_{ijk}}}(t) \\
&= \left\langle \phi_{jk}^{\bar{N}_{ijk}}\phi_{ki}^{\bar{N}_{ijk}}\left(y_i^{\bar{N}_{ijk}}\right), w^{\bar{N}_{ijk}}\right\rangle_{A^{\bar{N}_{ijk}}}(t) \\
&= \left\langle \phi_{ji}^{\bar{N}_{ijk}}\left(y_i^{\bar{N}_{ijk}}\right), w^{\bar{N}_{ijk}}\right\rangle_{A^{\bar{N}_{ijk}}}(t) \\
&= \left\langle y_j^{\bar{N}_{ijk}}, w^{\bar{N}_{ijk}}\right\rangle_{A^{\bar{N}_{ijk}}}(t)
\end{aligned}$$

which is equal to the right-hand side of (1), and the lemma is proved. □

PROPOSITION 2.3. *If $\{N_i, Y_i, \phi_{ij}\} \in L_A$, then $Y = \Phi\{N_i, Y_i, \phi_{ij}\}$ is a complete A-${}_TA$ imprimitivity bimodule.*

PROOF. We first verify the six axioms. The first follows from the observation that $a \in A$ is positive if and only if $a(t) \geqslant 0$ for all $t \in T$ (cf. Corollary 10.3.4 of [7]). Routine calculations show that axioms (2), (3), (5) and (6) are satisfied, and it remains to check (4). We observe that there is an action of $C(T)$ on the right of Y given by $\{y_i\} \cdot f = \{y_i(f|_{\bar{N}_i})\}$, which satisfies

$$\langle\{y_i\}, \{z_i\}f\rangle_A^r(t) = \langle\{y_i\}, \{z_i\}\rangle_A^r(t)f(t);$$

this follows from the corresponding property of the action of $C(\overline{N}_i)$ on Y_i. Hence the span B of the range of $\langle \cdot, \cdot \rangle_A^r$ is a $C(T)$-submodule of A, and to prove that B is dense in A it is enough to show that $B(t)$ is dense in $A(t)$ for each $t \in T$.

If $a \in A$ and $t_0 \in N_i$, then we can approximate $a^{\overline{N}_i}$ by an element of the form $\sum_{k=1}^n \langle y_i^k, z_i^k \rangle_{A^{\overline{N}_i}}$. Multiplying each y_i^k and z_i^k by a continuous function $\rho: T \to [0, 1]$ such that $\operatorname{supp} \rho \subset N_i$ and $\rho(t_0) = 1$ does not change the value of $\sum_{k=1}^n \langle y_i^k, z_i^k \rangle_{A^{\overline{N}_i}}(t_0)$; hence we can assume that $y_i^k, z_i^k \in (Y_i)_{\overline{N}_i \setminus N_i}$. By Lemma 2.2 these extend to $\{y_j^k\}, \{z_j^k\} \in Y_{T \setminus N_i}$; then

$$\sum_k \langle y_i^k, z_i^k \rangle_{A^{\overline{N}_i}} = \sum_k \langle \{y_j^k\}, \{z_j^k\} \rangle_A \Big|_{\overline{N}_i} \in B \Big|_{\overline{N}_i},$$

so that we have approximated a at t_0 by an element of B. We conclude that Y is an A-A imprimitivity bimodule.

To see that α_Y is the identity on T note that if $t \in N_i$ then

$$Y_{A_t} = \{\{y_j\} \in Y: \langle y_i, y_i \rangle_{A^{\overline{N}_i}}^r(t) = 0\},$$

which is $\{\{y_i\} \in Y: \langle y_i, y_i \rangle_{A^{\overline{N}_i}}^l(t) = 0\} = {}_{A_t}Y$ since α_{Y_i} is the identity on $\overline{N}_i$. It only remains to check that Y is complete. But if $\{y^n\}_{n=1}^\infty$ is a Cauchy sequence in Y and $y^n = \{y_i^n\}$, then, for each i, $\{y_i^n\}_{n=1}^\infty$ is a Cauchy sequence in Y_i and hence converges to some element $y_i \in Y_i$. It is not hard to see that $\{y_i\} \in Y$ and that $y^n \to \{y_i\}$, which completes the proof. □

PROPOSITION 2.4. *If* $\{N_i, Y_i, \phi_{ij}: i \in I\} \in L_A$, *then* $\Phi\{N_i, Y_i, \phi_{ij}\} \sim \{N_i, Y_i, \phi_{ij}\}$.

PROOF. Again we write $Y = \Phi\{N_i, Y_i, \phi_{ij}\}$. Since T is paracompact we can suppose that $\{N_i\}$ is locally finite, and we can choose an open cover $\{M_i: i \in I\}$ of T such that $\overline{M}_i \subset N_i$. For each i we define $\delta_i: Y \to Y^{\overline{M}_i}$ by $\delta_i(\{y_i\}) = y_i^{\overline{M}_i}$. We note that the kernel of δ_i is $Y_{\overline{M}_i}$, so that δ_i induces a map, also denoted by δ_i, of $Y^{\overline{M}_i}$ into $Y_i^{\overline{M}_i}$. Since the inner products on the quotients are the restrictions of the usual inner products, each δ_i preserves the inner products, and to show δ_i is an isomorphism it will be enough to show that it has dense range. Because $(\overline{N}_i \setminus N_i) \cap M_i = \varnothing$, we have $(A^{\overline{N}_i})_{\overline{N}_i \setminus N_i} + (A^{\overline{N}_i})_{\overline{M}_i} = A^{\overline{N}_i}$, which by Theorem 3.2 of **[12]** implies that

$$Y_i = \overline{(Y_i)_{\overline{N}_i \setminus N_i} + (Y_i)_{\overline{M}_i}}$$

and

$$Y_i^{\overline{M}_i} = Y_i / (Y_i)_{\overline{M}_i} = \overline{(Y_i)_{\overline{N}_i \setminus N_i} + (Y_i)_{\overline{M}_i}} / (Y_i)_{\overline{M}_i}.$$

Thus we can approximate any $y \in Y_i^{\overline{M}_i}$ by an element $z_i^{\overline{M}_i}$ such that $z_i \in (Y_i)_{\overline{N}_i \setminus N_i}$. By Lemma 2.2 we can extend this to an element $\{z_j\} \in Y_{T \setminus N_i}$; applying δ_i to this gives us $z_i^{\overline{M}_i}$, so that the range of δ_i is dense. Since both $Y^{\overline{M}_i}$ and $(Y_i)^{\overline{M}_i}$ are complete, δ_i is therefore an isomorphism of $Y^{\overline{M}_i}$ onto $(Y_i)^{\overline{M}_i}$. To show that the δ_i's define an equivalence we have to show that the following diagram commutes:

$$\begin{tikzcd}
 & Y_j^{\overline{M}_{ij}} \arrow[dd, "\phi_{ij}^{\overline{M}_{ij}}"] \\
\Phi\{N_i, Y_i, \phi_{ij}\}^{\overline{M}_{ij}} \arrow[ur, "\delta_i^{\overline{M}_{ij}}"] \arrow[dr, "\delta_i^{\overline{M}_{ij}}"'] & \\
 & Y_i^{\overline{M}_{ij}}
\end{tikzcd}$$

However, this is a routine calculation using the definition of the quotient isomorphisms (Lemma 1.9), and the proof is complete. $\square$

We have now shown that each equivalence class in $L_A/\sim$ contains an A-${}_TA$ imprimitivity bimodule and our next goal is to prove that each class contains exactly one up to isomorphism. Now it is clear that if $Z \cong Y$ then $Z \sim Y$; our task is to prove the converse. The following lemma will help:

LEMMA 2.5. *If $\{N_i\}$ is an open covering of T and Y is an A-${}_TA$ imprimitivity bimodule, then $Y \cong \Phi\{N_i, Y^{\overline{N}_i}, \mathrm{id}_{ij}\}$, where id_{ij}: $Y^{\overline{N}_{ij}} \to Y^{\overline{N}_{ij}}$ is the identity isomorphism.*

PROOF. Define γ: $Y \to \Phi\{N_i, Y^{\overline{N}_{ij}}, \mathrm{id}_{ij}\}$ by $\gamma(y) = \{y^{\overline{N}_i}\}$; then γ preserves the inner products and all we have to do is show that the range of γ is dense. Let $\{y_i\} \in \Phi\{N_i, Y^{\overline{N}_{ij}}, \mathrm{id}_{ij}\}$. Without loss of generality we may assume that $\|\langle\{y_i\}, \{y_i\}\rangle_A^r(t)\| = 0$ off some compact set K, for if not we can multiply by a continuous function of compact support which is identically one except where $\|\langle y_i, y_i\rangle(t)\|$ is small. We choose $N_1, \ldots, N_n$ such that $K \subset \bigcup_{i=1}^n N_i$, and a partition of unity $\{p_i\}$ for K subordinate to $\{N_i\}$. By Corollary 1.6 we can find $z_i \in Y_{T\setminus N_i}$ such that $z_i^{\overline{N}_i} = p_i y_i$. We set $y = \sum_{k=1}^n z_k$; we shall prove that $\gamma(y) = \{y_i\}$.

We first note that if $t \notin K$, then $\langle z_i, z_i\rangle_A^r(t) = 0$ for all i; this is trivial if $t \notin N_i$ and if $t \in N_i$ then

$$\langle z_i, z_i\rangle_A^r(t) = \left\langle z_i^{\overline{N}_i}, z_i^{\overline{N}_i}\right\rangle_{A^{\overline{N}_i}}^r(t) = p_i(t)^2\langle y_i, y_i\rangle_{A^{\overline{N}_i}}^r(t) = 0.$$

The Cauchy-Schwarz inequality implies that $\langle y, y\rangle_A$ vanishes off K. We shall show that $y^{\overline{N}_j} = y_j$ for $j \in I$ by proving that if $w \in A^{\overline{N}_j}$ and $t \in N_j$, then

$$\langle y_j, w\rangle_{A^{\overline{N}_j}}(t) = \left\langle y^{\overline{N}_j}, w\right\rangle_{A^{\overline{N}_j}}(t) = \sum_k \left\langle z_k^{\overline{N}_j}, w\right\rangle_{A^{\overline{N}_j}}(t). \tag{2}$$

Now if $t \in \overline{N}_k$, then

$$\begin{aligned}\left\langle z_k^{\overline{N}_j}, w\right\rangle_{A^{\overline{N}_j}}(t) &= \left\langle z_k^{\overline{N}_{jk}}, w^{\overline{N}_{jk}}\right\rangle_{A^{\overline{N}_{jk}}}(t)\\ &= p_k(t)\left\langle y_k^{\overline{N}_{jk}}, w^{\overline{N}_{jk}}\right\rangle_{A^{\overline{N}_{jk}}}(t) = p_k(t)\langle y_j, w\rangle_{A^{\overline{N}_j}}(t)\end{aligned}$$

since $y^{\overline{N}_{jk}} = y_j^{\overline{N}_{jk}}$, and if $t \notin \overline{N}_k$ then

$$\langle z_k^{\overline{N}_j}, w\rangle_{A^{\overline{N}_j}}(t) = 0 = p_k(t)\langle y_j, w\rangle_{A^{\overline{N}_j}}(t).$$

Thus for $t \in K \cap N_j$

$$\sum_k \left\langle z_k^{\bar{N}_j}, w \right\rangle_{A^{\bar{N}_j}}(t) = \sum_k p_k(t)\langle y_j, w\rangle_{A^{\bar{N}_j}}(t) = \langle y_j, w\rangle_{A^{\bar{N}_j}}(t).$$

We have already seen that both sides of (2) are zero if $t \notin K$, so we have now shown $y^{\bar{N}_j} = y_j$, or, in other words, that $\gamma(y) = \{y_j\}$, as claimed. □

PROPOSITION 2.6. *If Y and Z are A-${}_TA$ imprimitivity bimodules and $Y \sim Z$ as elements of L_A, then Y is isomorphic to Z.*

PROOF. Since $Y \sim Z$ there are an open cover $\{N_i\}$ and isomorphisms δ_i: $Z^{\bar{N}_i} \to Y^{\bar{N}_i}$ such that $\delta_i^{\bar{N}_{ij}} = \delta_j^{\bar{N}_{ij}}$ for all $i, j \in I$. If $\{z_i\} \in \Phi\{N_i, Z^{\bar{N}_i}, \mathrm{id}_{ij}\}$ then it is easy to check that $\{\delta_i z_i\} \in \Phi\{N_i, Y^{\bar{N}_i}, \mathrm{id}_{ij}\}$ and that δ: $\{z_i\} \to \{\delta_i z_i\}$ is an inner-product preserving map. Since δ has an obvious inverse, we conclude that δ: $\Phi\{N_i, Z^{\bar{N}_i}, \mathrm{id}_{ij}\} \to \Phi\{N_i, Y^{\bar{N}_i}, \mathrm{id}_{ij}\}$ is an isomorphism, and the result follows from the lemma. □

COROLLARY 2.7. *$L_A/\sim$ is in one-to-one correspondence with* $\mathrm{Pic}_{C(T)} A$.

REMARK. The referee has suggested to us that the argument used to establish Corollary 2.7 should work equally well in the case where A is a maximal full algebra of operator fields over a locally compact and paracompact Hausdorff space T (or, equivalently, when there is a continuous open map of Prim A onto a paracompact Hausdorff space T [**8**, Theorem 4]). This is indeed the case, provided that when K is a closed subset of T we interpret an A^K-${}_KA^K$ imprimitivity bimodule as one such that

$$(af)xb = ax(fb) \quad \text{for } a, b \in A^K, f \in C(K), x \in X^K$$

(cf. Proposition 1.11 and the subsequent example). The proof of 2.7 goes through as we have given it.

Our plan is to build a map from $\mathrm{Pic}_{C(T)} A$ to $\mathrm{Pic}_{C(T)} C_0(T)$ by defining a map from L_A to $L_{C_0(T)}$ and showing that it respects $\sim$. The construction uses the "local" imprimitivity bimodules we described at the beginning of the section; more precisely, we require:

PROPOSITION 2.8. *There is an open cover $\{N_i: i \in I\}$ of T by relatively compact sets such that there are complete $A^{\bar{N}_i}$-${}_{\bar{N}_i}C(\bar{N}_i)$ imprimitivity bimodules X_i and isomorphisms α_{ij}: $X_j^{\bar{N}_{ij}} \to X_i^{\bar{N}_{ij}}$ satisfying*:

(1) *If L is a closed subset of $\bar{N}_{ijk}$, Z_m $(m = 1, 2, 3)$ are A^L-${}_LA^L$ imprimitivity bimodules and ϕ: $Z_1 \to Z_2$, ψ: $Z_2 \to Z_3$ are isomorphisms, then*

$$\left(\tilde{\alpha}_{ij}^L \otimes \psi \otimes \alpha_{ij}^L\right) \circ \left(\tilde{\alpha}_{ij}^L \otimes \phi \otimes \alpha_{jk}^L\right) = \tilde{\alpha}_{ik}^L \otimes (\psi \circ \phi) \otimes \alpha_{ik}^L.$$

(2) *If L is a closed subset of $\bar{N}_{ijk}$, Y_m $(m = 1, 2, 3)$ are $C(L)$-${}_LC(L)$ imprimitivity bimodules and ϕ: $Y_1 \to Y_2$, ψ: $Y_2 \to Y_3$ are isomorphisms, then*

$$\left(\alpha_{ij}^L \otimes \psi \otimes \tilde{\alpha}_{ij}^L\right) \circ \left(\alpha_{jk}^L \otimes \phi \otimes \tilde{\alpha}_{jk}^L\right) = \alpha_{ik}^L \otimes (\psi \circ \phi) \otimes \tilde{\alpha}_{ik}^L.$$

(*Here $\tilde{\alpha}_{ij}$ is the dual of the isomorphism α_{ij}; cf.* §1, *following Corollary* 1.2.)

PROOF. Since A is a continuous trace C^*-algebra, for each $t \in T$ there is a neighbourhood N of t and an element $p \in A$ such that $p(s)$ is a rank one projection for each $s \in \overline{N}$ [7, Proposition 4.5.3], and Corollary 1.2 tells us that $A^{\overline{N}}p^{\overline{N}} = (Ap)|_{\overline{N}}$ is a complete $A^{\overline{N}}$-${}_{\overline{N}}C(\overline{N})$ imprimitivity bimodule. If p, q are two such elements of A, then the first part of the proof of Lemma 2.5 of **[10]** shows that there is a smaller neighbourhood M of t and $a \in A$ such that $a(s)a(s)^* = p(s)$, $a(s)^*a(s) = q(s)$ for $s \in \overline{M}$. The argument of Lemma 10.7.11 of [7] with the preceding remarks replacing Proposition 10.7.7 and Lemma 10.7.9 of [7] shows that there are a covering $\{N_i: i \in I\}$ of T by relatively compact open sets, for each $i \in I$ an element $p_i \in A$ such that $p_i(t)$ is a rank one projection for $t \in \overline{N_i}$, and for each pair $i,j \in I$ an element $a_{ij} \in A$ such that

$$a_{ij}(t)a_{ij}(t)^* = p_j(t), \qquad a_{ij}(t)^*a_{ij}(t) = p_i(t) \quad \text{for } t \in \overline{N}_{ij}.$$

We take for X_i the imprimitivity bimodules $(Ap_i)|_{\overline{N}_i}$, and for our isomorphisms α_{ij} the maps

$$\alpha_{ij}(a|_{\overline{N}_{ij}}p_j|_{\overline{N}_{ij}}) = (a|_{\overline{N}_{ij}}p_j|_{\overline{N}_{ij}})a_{ij}|_{\overline{N}_{ij}},$$

so that the dual modules $\tilde{X}_i$ can be identified with $(p_iA)|_{\overline{N}_i}$ and the dual isomorphisms $\tilde{\alpha}_{ij}$ with left multiplication by a_{ij}^*.

We must now check (1) and (2); clearly they are similar so we shall only verify (1). We first note that if L is a compact subset of $\overline{N}_{ijk}$, then the ideal $(X_i)_L$ is $\{(ap_i)|_{\overline{N}_i}: a(t)p_i(t) = 0 \text{ for } t \in L\}$, and we can identify $(X_i)^L$ with $(Ap_i)|_L$; under this identification α_{ij}^L becomes right multiplication by $a_{ij}|_L$. Now let

$$w = (p_k|_L a|_L) \otimes z \otimes (b|_L p_k|_L) \in \tilde{X}_k^L \odot_{A^L} Z \odot_{A^L} X_k^L$$

be an elementary tensor. Then

$$\left(\tilde{\alpha}_{ij}^L \otimes \psi \otimes \alpha_{ij}^L\right) \circ \left(\tilde{\alpha}_{jk}^L \otimes \phi \otimes \alpha_{jk}^L\right)(w) = (a_{ij}^* a_{jk}^* p_k a)|_L \otimes \psi \circ \phi(z) \otimes (bp_k a_{jk} a_{ij})|_L.$$

For each $t \in L$, $a_{jk}(t)a_{ij}(t)$ is a partial isometry sending $p_i(t)$ to $p_k(t)$, and so must differ from $a_{ik}(t)$ by a constant of modulus 1; hence there is a continuous function $f: L \to S^1$ such that $(a_{jk}a_{ij})|_L = f(a_{ik}|_L)$. It follows at once that $(a_{ij}^* a_{jk}^*)|_L = \bar{f}(a_{ik}^*|_L)$. A simple approximation argument shows that continuous functions pull across tensors, and since all our imprimitivity bimodules induce the identity homeomorphism on L they also pull across elements of $\tilde{X}_j^L$, Z and X_j^L (cf. the argument before Proposition 1.12). Hence

$$\begin{aligned}\left(\tilde{\alpha}_{ij}^L \otimes \psi \otimes \alpha_{ij}^L\right) \circ \left(\alpha_{jk}^L \otimes \phi \otimes \alpha_{jk}^L\right)(w) &= \bar{f}f(a_{ik}^* p_k a)|_L \otimes \psi \circ \phi(z) \otimes (bp_k a_{ik})|_L \\ &= \left(\tilde{\alpha}_{ik}^L \otimes (\psi \circ \phi) \otimes \alpha_{ik}^L\right)(w)\end{aligned}$$

since $|f(t)| = 1$ for all t. This implies that (1) is satisfied, and the result is proved. □

We now fix $\{N_i\}$, X_i and α_{ij} as in Proposition 2.8 and define a map γ: $L_A \to L_{C_0(T)}$ as follows:

Given $\{M_n, Y_n, \phi_{nm}: n, m \in N\} \in L_A$, we choose a common refinement $\{L_p: p \in P\}$ of $\{M_n\}$ and $\{N_i\}$, together with restriction maps $\rho: P \to N$, $\sigma: P \to I$ such that $L_p \subset M_{\rho(p)}$, $L_p \subset N_{\sigma(p)}$. Then $\gamma\{M_n, Y_n, \phi_{nm}\}$ will be

$$\{L_p: \tilde{X}^{\bar{L}_p}_{\sigma(p)} \otimes_{A^{\bar{L}_p}} Y^{\bar{L}_p}_{\rho(p)} \otimes_{A^{\bar{L}_p}} X^{\bar{L}_p}_{\sigma(p)}, \tilde{\alpha}^{\bar{L}_{pq}}_{\sigma(p)\sigma(q)} \otimes \phi^{\bar{L}_{pq}}_{\rho(p)\rho(q)} \otimes \alpha^{\bar{L}_{pq}}_{\sigma(p)\sigma(q)}\},$$

where the transition isomorphisms make sense by Lemma 1.10.

That $\gamma\{M_n, Y_n, \phi_{nm}\}$ is actually an element of $L_{C_0(T)}$ is a consequence of Lemma 1.10 and Proposition 2.8(1).

LEMMA 2.9. *If $\{M_n, Y_n, \phi_{nm}\} \in L_A$ then the equivalence class of $\gamma\{M_n, Y_n, \phi_{nm}\}$ in $L_{C_0(T)}/\sim$ is independent of any choice made and depends only on the class of $\{M_n, Y_n, \phi_{nm}\}$ in $L_A/\sim$.*

PROOF. Suppose we had chosen a different refinement $\{K_\delta\}_{\delta\in\Delta}$ of $\{M_n\}$ and $\{N_i\}$; since we can always take a common refinement of $\{K_\delta\}$ and $\{L_p\}$ we may as well assume that $\{K_\delta\}$ is a refinement of $\{L_p\}$, so that there is a map $\pi: \Delta \to P$ with $K_\delta \subset L_{\pi(\delta)}$. By Lemma 1.10 the quotient maps induce isomorphisms

$$\xi_\delta: \left(\tilde{X}^{\bar{L}_{\pi(\delta)}}_{\sigma(\pi(\delta))} \otimes Y^{\bar{L}_{\pi(\delta)}}_{\rho(\pi(\delta))} \otimes X^{\bar{L}_{\pi(\delta)}}_{\sigma(\pi(\delta))}\right)^{\bar{K}_\delta} \to \tilde{X}^{\bar{K}_\delta}_{\sigma(\pi(\delta))} \otimes Y^{\bar{K}_\delta}_{\rho(\pi(\delta))} \otimes X^{\bar{K}_\delta}_{\sigma(\pi(\delta))}$$

and these implement an equivalence in $L_{C_0(T)}$. The same argument shows that we can replace $\{M_n\}$ by a finer cover, and to prove γ respects $\sim$ it remains to show that if $\{M_n, Z_n, \psi_{nm}\} \in L_A$ and there are isomorphisms $\theta_n: Z_n \to Y_n$ satisfying $\psi_{nm} \circ \theta_m^{\bar{M}_{nm}} = \theta_n^{\bar{M}_{nm}} \circ \phi_{nm}$, then $\gamma\{M_n, Z_n, \psi_{nm}\} \sim \gamma\{M_n, Y_n, \phi_{nm}\}$. But a standard calculation on elementary tensors shows that

$$\begin{aligned}(\tilde{\alpha}^{\bar{L}_{pq}}_{\sigma(p)\sigma(q)} \otimes \psi^{\bar{L}_{pq}}_{\rho(p)\rho(q)} \otimes \alpha^{\bar{L}_{pq}}_{\sigma(p)\sigma(q)}) &\circ (\mathrm{id} \otimes \theta^{\bar{L}_{pq}}_{\rho(q)} \otimes \mathrm{id}) \\ &= (\mathrm{id} \otimes \theta^{\bar{L}_{pq}}_{\rho(p)} \otimes \mathrm{id}) \circ (\tilde{\alpha}^{\bar{L}_{pq}}_{\sigma(p)\sigma(q)} \otimes \phi^{\bar{L}_{pq}}_{\rho(p)\rho(q)} \otimes \alpha^{\bar{L}_{pq}}_{\sigma(p)\sigma(q)}),\end{aligned}$$

which is enough to prove the lemma. $\square$

In exactly the same way, we define a map $\beta: L_{C_0}(T) \to L_A$:

If $\{M_n, Y_n, \phi_{nm}\} \in L_{C_0(T)}$ and $\{L_p\}_{p\in P}$ is a refinement of $\{M_n\}$ and $\{N_i\}$ as before, then $\beta\{M_n, Y_n, \phi_{nm}\} \in L_A$ is

$$\{L_p, X^{\bar{L}_p}_{\sigma(p)} \otimes Y^{\bar{L}_p}_{\rho(p)} \otimes \tilde{X}^{\bar{L}_p}_{\sigma(p)}, \alpha^{\bar{L}_{pq}}_{\sigma(p)\sigma(q)} \otimes \phi^{\bar{L}_{pq}}_{\rho(p)\rho(q)} \otimes \tilde{\alpha}^{\bar{L}_{pq}}_{\sigma(p)\sigma(q)}\}.$$

Proposition 2.8(2) shows that this is indeed an element of L_A, and exactly as before we can see that β is well defined and respects the equivalence relation $\sim$. Thus β and γ define maps

$$\tilde{\beta}: \mathrm{Pic}_{C(T)}\, C_0(T) = L_{C_0(T)}/\sim \to L_A/\sim = \mathrm{Pic}_{C(T)}\, A,$$
$$\tilde{\gamma}: \mathrm{Pic}_{C(T)}\, A = L_A/\sim \to L_{C_0(T)}/\sim = \mathrm{Pic}_{C(T)}\, C_0(T);$$

our next result (and its obvious analogue) shows that $\tilde{\beta}$ is an inverse for $\tilde{\gamma}$.

LEMMA 2.10. *If Y is an A-${}_TA$ imprimitivity bimodule, then $\beta \circ \gamma(Y) \sim Y$ in L_A.*

PROOF. Again using the notation of Proposition 2.8, $\gamma(Y)$ is given by

$$\left\{N_i, \tilde{X}_i \otimes_{A^{\bar{N}_i}} Y^{\bar{N}_i} \otimes_{A^{\bar{N}_i}} X_i, \tilde{\alpha}_{ij} \otimes \mathrm{id}_{ij} \otimes \alpha_{ij}\right\}$$

and so $\beta \circ \gamma(Y)$ is given by

$$\left\{N_i, X_i \otimes_{C(\bar{N}_i)} \tilde{X}_i \otimes_{A^{\bar{N}_i}} \otimes Y^{\bar{N}_i} \otimes_{A^{\bar{N}_i}} X_i \otimes_{C(\bar{N}_i)} \tilde{X}_i, \alpha_{ij} \otimes \tilde{\alpha}_{ij} \otimes \mathrm{id}_{ij} \otimes \alpha_{ij} \otimes \tilde{\alpha}_{ij}\right\}.$$

We define bimodule homomorphisms π_i: $X_i \odot \tilde{X}_i \odot Y^{\bar{N}_i} \odot X_i \odot \tilde{X}_i \to Y^{\bar{N}_i}$ by

$$\pi_i(x_1 \otimes \tilde{x}_2 \otimes y \otimes x_3 \otimes \tilde{x}_4) = \langle x_1, x_2 \rangle^l_{A^{\bar{N}_i}} y \langle x_3, x_4 \rangle^l_{A^{\bar{N}_i}}.$$

Routine (but messy) calculations using the module properties of the inner products show that the π_i preserve the inner products and have dense range, so we obtain isomorphisms π_i: $X_i \otimes_{C(\bar{N}_i)} \tilde{X}_i \otimes_{A^{\bar{N}_i}} Y^{\bar{N}_i} \otimes_{A^{\bar{N}_i}} X_i \otimes_{C(\bar{N}_i)} \tilde{X}_i \to Y^{\bar{N}_i}$. To show that $\beta \circ \gamma(Y) \sim Y$ we have to check that

$$\begin{array}{ccc} \left(X_j \otimes X_j \otimes Y^{\bar{N}_j} \otimes X_j \otimes X_j\right)^{\bar{N}_{ij}} & \overset{\pi_j^{\bar{N}_{ij}}}{\rightarrow} & Y^{\bar{N}_{ij}} \\ \alpha_{ij} \otimes \tilde{\alpha}_{ij} \otimes \mathrm{id} \otimes \alpha_{ij} \otimes \tilde{\alpha}_{ij} \downarrow & & \downarrow \mathrm{id} \\ \left(X_i \otimes \tilde{X}_i \otimes Y^{\bar{N}_i} \otimes X_i \otimes \tilde{X}_i\right)^{\bar{N}_{ij}} & \overset{\pi_j^{\bar{N}_{ij}}}{\rightarrow} & Y^{\bar{N}_{ij}} \end{array}$$

commutes, where the columns are defined by Lemma 1.10. This is another straightforward calculation using the fact that each α_{ij} is an inner-product preserving isomorphism, and the lemma is proved. □

We have now proved that $\tilde{\gamma}$: $\mathrm{Pic}_{C(T)} A \to \mathrm{Pic}_{C(T)} C_0(T)$ is a bijection, and to complete the proof of the theorem all we need to do is show that $\tilde{\gamma}$ is a group homomorphism. We shall do this in Proposition 2.12, but first we need a lemma:

LEMMA 2.11. *Let Z, Y be A-${}_TA$ imprimitivity bimodules, and suppose that $Z \sim \{N_i, Z_i, \phi_{ij}\}$ and $Y \sim \{N_i, Y_i, \psi_{ij}\}$ in L_A. If $\{M_i\}$ is an open cover of X such that $\bar{M}_i \subset N_i$, then*

$$Z \otimes_A Y \sim \left\{M_i, Z^{\bar{M}_i} \otimes_{A^{\bar{M}_i}} Y^{\bar{M}_i}, \phi_{ij}^{\bar{M}_{ij}} \otimes \psi_{ij}^{\bar{M}_{ij}}\right\}$$

(*where, as usual, $\phi_{ij}^{\bar{M}_{ij}} \otimes \psi_{ij}^{\bar{M}_{ij}}$ is interpreted in the sense of Lemma* 1.10).

PROOF. First we note that since, by Proposition 2.4, $\{N_i, Z_i, \phi_{ij}\} \sim \Phi\{N_i, Z_i, \phi_{ij}\}$, Proposition 2.6 implies that $Z \cong \Phi\{N_i, Z_i, \phi_{ij}\}$. In the proof of Proposition 2.4 we saw that there are isomorphisms δ_i: $\Phi\{N_i, Z_i, \phi_{ij}\}^{\bar{M}_i} \to Z_i^{\bar{M}_i}$ which commute with the ϕ_{ij}; composing with the isomorphism $Z \cong \Phi\{N_i, Z_i, \phi_{ij}\}$ gives us isomorphisms ξ_i: $Z^{\bar{M}_i} \to Z_i^{\bar{M}_i}$ satisfying $\phi_{ij}^{\bar{M}_{ij}} \circ \xi_j^{\bar{M}_{ij}} = \xi_i^{\bar{M}_{ij}}$. Similarly, there are isomorphisms η_i: $Y^{\bar{M}_i} \to Y_i^{\bar{M}_i}$ satisfying $\psi_{ij}^{\bar{M}_{ij}} \circ \eta_j^{\bar{M}_{ij}} = \eta_i^{\bar{M}_{ij}}$. We define σ_i: $(Z \otimes Y)^{\bar{M}_i} \to Z_i^{\bar{M}_i} \otimes Y_i^{\bar{M}_i}$ by

$$\sigma_i\left((z \otimes y)^{\bar{M}_i}\right) = \xi_i\left(z^{\bar{M}_i}\right) \otimes \eta_i\left(y^{\bar{M}_i}\right).$$

The usual sort of computations show that

$$\begin{tikzcd} & \left(Z_j^{\bar{M}_j} \otimes Y_j^{\bar{M}_j}\right)^{\bar{M}_{ij}} \arrow[dd, "\phi_{ij}^{\bar{M}_{ij}} \otimes \psi_{ij}^{\bar{M}_{ij}}"] \\ (Z \otimes Y)^{\bar{M}_{ij}} \arrow[ur, "\sigma_j^{\bar{M}_{ij}}"] \arrow[dr, "\sigma_i^{\bar{M}_{ij}}"'] & \\ & \left(Z_i^{\bar{M}_i} \otimes Y_i^{\bar{M}_i}\right)^{\bar{M}_{ij}} \end{tikzcd}$$

commutes, and the lemma follows. □

PROPOSITION 2.12. *If Y, Z are A-${}_TA$ imprimitivity bimodules, then*

$$\tilde{\gamma}(Y \otimes_A Z) \cong \tilde{\gamma}(Y) \otimes_{C_0(T)} \tilde{\gamma}(Z).$$

PROOF. By Lemma 2.11 the imprimitivity bimodule $\tilde{\gamma}(Y) \otimes \tilde{\gamma}(Z)$ is represented by

$$\Big\{ M_i, \tilde{X}_i^{\overline{M}_i} \otimes Y^{\overline{M}_i} \otimes X_i^{\overline{M}_i} \otimes \tilde{X}_i^{\overline{M}_i} \otimes Z^{\overline{M}_i} \otimes X_i^{\overline{M}_i}, \quad \tilde{\alpha}_{ij}^{\overline{M}_{ij}} \otimes \mathrm{id} \otimes \alpha_{ij}^{\overline{M}_{ij}} \otimes \tilde{\alpha}_{ij}^{\overline{M}_{ij}} \otimes \mathrm{id} \otimes \alpha_{ij}^{\overline{M}_{ij}} \Big\}. \tag{3}$$

We define τ_i: $\tilde{X}_i^{\overline{M}_i} \odot Y^{\overline{M}_i} \odot X_i^{\overline{M}_i} \odot \tilde{X}_i^{\overline{M}_i} \odot Z^{\overline{M}_i} \odot X_i^{\overline{M}_i} \to X_i^{\overline{M}_i} \odot Y^{\overline{M}_i} \odot Z^{\overline{M}_i} \odot X_i^{\overline{M}_i}$ by

$$\tau_i(\tilde{x}_1 \otimes y \otimes x_2 \otimes \tilde{x}_3 \otimes z \otimes x_4) = \tilde{x}_1 \otimes y \otimes \langle x_2, x_3 \rangle^l_{A^{\overline{M}_i}} z \otimes x_4.$$

Then each τ_i has dense range and a messy calculation shows that the τ_i preserve the inner products–and so extend to isomorphisms on the completions. More work reveals that the τ_i intertwine the transition isomorphisms of (3) and those of

$$\gamma(Y \otimes Z) = \Big\{ M_i, \tilde{X}_i^{\overline{M}_i} \otimes (Y \otimes Z)^{\overline{M}_i} \otimes X_i^{\overline{M}_i}, \tilde{\alpha}_{ij}^{\overline{M}_{ij}} \otimes \mathrm{id} \otimes \alpha_{ij}^{\overline{M}_{ij}} \Big\},$$

and so define an equivalence between $\tilde{\gamma}(Y) \otimes \tilde{\gamma}(Z)$ and $\tilde{\gamma}(Y \otimes Z)$, which is what we wanted. □

This completes the proof of Theorem 2.1.

Appendix. (A) *Imprimitivity bimodules over a commutative C^*-algebra.* In this section we shall show that an imprimitivity bimodule over a commutative C^*-algebra can always be regarded as the space of sections of a complex line bundle over the spectrum of the algebra.

First we consider the case where T is compact. It is then a well-known consequence of the equivalence between the categories of vector bundles over T and finitely generated projective $C(T)$-modules that every invertible $C(T)$-module is isomorphic to one of the form $\Gamma(L)$ for some complex line bundle L over T. Since every line bundle can be given a Hermitian structure, we might as well assume that L is a Hermitian line bundle, or in other words that there is a continuous choice of inner products $(\cdot \mid \cdot)_t$ on the fibres L_t (which we take to be conjugate linear in the first variable). With the inner products

$$\langle f, g \rangle^r_{C(T)}(t) = (f(t) | g(t))_t, \qquad \langle f, g \rangle^l_{C(T)}(t) = (g(t) | f(t))_t$$

$\Gamma(L)$ becomes a $C(T)$-${}_TC(T)$ imprimitivity bimodule–the axioms can all be easily verified, except perhaps for (4), which follows since the range of $\langle \cdot, \cdot \rangle^r_{C(T)}$ is an ideal of $C(T)$ which is contained in no maximal ideal. Conversely, if X is a $C(T)$-${}_TC(T)$ imprimitivity bimodule, then the range of the map ϕ: $\tilde{x} \otimes y \to \langle x, y \rangle^r_{C(T)}$: $\tilde{X} \otimes_{C(T)} X \to C(T)$ is a dense ideal in $C(T)$ and hence all of $C(T)$, and it follows that ϕ is an isomorphism. For if $\phi(\sum_j \tilde{v}_j \otimes w_j) = 0$ and $\phi(\sum \tilde{x}_i \otimes y_i) = 1$, then

$$\sum_j \tilde{v}_j \otimes w_j = \Big(\sum_j \tilde{v}_j \otimes w_j\Big)\Big(\sum_i \langle x_i, y_i\rangle^r_{C(T)}\Big)$$
$$= \sum_{i,j} \tilde{v}_j \otimes \langle w_j, x_i\rangle^l_{C(T)} y_i$$
$$= \sum_{i,j} \big(\langle x_i, w_j\rangle^l_{C(T)} v_j\big)^{\sim} \otimes y_i$$
$$= \sum_{i,j} \langle v_j, w_j\rangle^r_{C(T)} \tilde{x}_i \otimes y_i = 0.$$

Thus X is an invertible $C(T)$-module in the algebraic sense, and we have shown that the $C(T)$-$_TC(T)$ imprimitivity bimodules are in one-to-one correspondence with the invertible $C(T)$-modules. The next proposition now follows easily:

PROPOSITION A1. *The Picard group* $\mathrm{Pic}_{C(T)}\, C(T)$ *of the* C^**-algebra* $C(T)$ *is isomorphic to the usual algebraic Picard group* $\mathrm{Pic}\, C(T)$.

We now suppose that T is a locally compact space. If L is a Hermitian line bundle over T, then as before the space $\Gamma_0(L)$ of sections which vanish at infinity is a $C_0(T)$-$_TC_0(T)$ imprimitivity bimodule; it is still true that they all arise this way but we can no longer prove it by appealing to an algebraic analogue. Let X be a fixed $C_0(T)$-$_TC_0(T)$ imprimitivity bimodule.

LEMMA A2. *Each point* $t \in T$ *has a relatively compact neighbourhood* N *such that* $X^{\overline{N}}$ *and* $C(\overline{N})$ *are isomorphic as imprimitivity bimodules.*

PROOF. If K is any compact neighbourhood of t, then X^K is a $C(K)$-$_KC(K)$ imprimitivity bimodule, and hence by our previous analysis has the form $\Gamma(L)$ for some Hermitian line bundle L over K. The result follows since L is locally trivial. □

Let $\{N_i\}$ be an open cover of T by relatively compact sets such that there are isomorphisms ϕ_i: $X^{\overline{N}_i} \to C(\overline{N}_i)$. Then for each pair i, j we define isomorphisms λ_{ij}: $C(\overline{N}_{ij}) \to C(\overline{N}_{ij})$ by $\lambda_{ij} = \phi_i^{\overline{N}_{ij}} \circ (\phi_j^{-1})^{\overline{N}_{ij}}$. It is not hard to see that the λ_{ij} must take the form of multiplication by continuous maps s_{ij}: $\overline{N}_{ij} \to S^1$, and Lemma 1.9 implies that $\lambda_{ij}^{\overline{N}_{ijk}} \circ \lambda_{jk}^{\overline{N}_{ijk}} = \lambda_{ik}^{\overline{N}_{ijk}}$, so that the s_{ij} form a 1-cocycle with coefficients in the sheaf $\mathcal{S}$ of germs of continuous S^1-valued functions. These are the transition functions of a Hermitian line bundle L, and the space $\Gamma_0(L)$ of sections can be regarded as

$$\left\{\{f_i\} \in \prod_i C(\overline{N}_i):\quad \begin{array}{l} f_i(t) = s_{ij}(t) f_j(t) \ \big(t \in \overline{N}_{ij}\big), \\ t \to (f_i(t) \mid f_i(t)) \text{ vanishes at } \infty \end{array}\right\}.$$

It is routine to check that the isomorphism class of L does not depend on any of the choices we have made. For $x \in X$ we set $\phi(x) = \{\phi_i(x^{\overline{N}_i})\} \in \prod_i C(\overline{N}_i)$; straightforward calculations show that ϕ defines an inner-product preserving isomorphism of X onto a closed $C(T)$-submodule of $\Gamma_0(L)$. Since these all consist of the sections which vanish on some compact set $K \subset T$, it is easy to see that ϕ must be onto. We have proved most of the following result:

PROPOSITION A3. *Let T be a locally compact space. Then every $C_0(T)$-${}_TC_0(T)$ imprimitivity bimodule has the form $\Gamma_0(L)$ for some complex line bundle L over T, and there is an isomorphism* $\mathrm{Pic}_{C(T)}\, C_0(T) \cong H^1(T, \mathbb{S})$. *If in addition T is paracompact, then*

$$\mathrm{Pic}_{C(T)}\, C_0(T) \cong H^2(T, \mathbf{Z}).$$

PROOF. The group $H^1(T, \mathbb{S})$ is isomorphic to the group of Hermitian line bundles over T under $\otimes_{C(T)}$, and the first assertion will follow if we check that $L \to \Gamma(L)$ is multiplicative: this is an easy calculation. The last statement follows from the exact sequence of sheaf cohomology associated to the covering map $t \to \exp 2\pi i t$: $\mathbf{R} \to S^1$ (cf., for example, [9, §1.16]); note that we need to assume T paracompact to ensure that the sheaf of $\mathbf{R}$-valued functions on T is fine. □

(B) *Imprimitivity bimodules over an n-homogeneous C^*-algebra.* If A is a unital n-homogeneous C^*-algebra with (compact) spectrum T, then Proposition A1, Theorem 2.1 and the algebraic result $\mathrm{Pic}_{C(T)} A \cong \mathrm{Pic}\, C(T)$ imply that the C^*-algebraic Picard group $\mathrm{Pic}_{C(T)} A$ is the same as the usual algebraic one. In this case, however, more should be true: imprimitivity bimodules and invertible bimodules both make sense for such an algebra and they should be the same thing. In this section we show that this is true: we begin with a simple lemma.

LEMMA B1. *Let A be a C^*-algebra whose primitive ideal space T is compact Hausdorff, and let X be a $C(T)$-${}_TC(T)$ imprimitivity bimodule. Then $A \odot_{C(T)} X$ is an A-${}_TA$ imprimitivity bimodule with the module actions and inner products defined on elementary tensors by*

$$a(b \otimes x) = (ab) \otimes x, \quad (a \otimes x)b = (ab) \otimes x,$$

$$\langle a \otimes x, b \otimes y\rangle_A^l = ab^*\langle x, y\rangle_{C(T)}^l, \quad \langle a \otimes x, b \otimes y\rangle_A^r = a^*b\langle x, y\rangle_{C(T)}^r.$$

PROOF. The purely algebraic axioms (2), (3), (5) and (6) are easily verified. The range of $\langle\cdot,\cdot\rangle_{C(T)}^l$: $X \otimes X \to X$ is all of $C(T)$, so that the algebra generated by the range of $\langle\cdot,\cdot\rangle_A^l$ contains the ideal A^2 of A consisting of all elements of the form ab. Now A^2 is dense in A–since otherwise $A/(A^2)^-$ would be a C^*-algebra with zero multiplication–and so axiom (4) is satisfied; it only remains to check (1). Let $\sum_{i=1}^n a_i \otimes x_i \in A \odot X$. Then for each $t \in T$, $\langle\cdot,\cdot\rangle_{C(T)}^l(t)$ is a complex-valued pre-inner product on X, and so the $n \times n$ matrix with (i,j)th entry $\langle x_i, x_j\rangle_{C(T)}^l(t)$ is positive semidefinite. It follows that the element $(\{x_i, x_j\}_{C(T)}^l)$ of the C^*-algebra $M_n(C(T))$ is positive, and so has the form TT^* for some $T = (t_{ij}) \in M_n(C(T))$. If we now set $b_k = \sum_i a_i t_{ik}$, then

$$\langle \Sigma\, a_i \otimes x_i, \Sigma\, a_i \otimes x_i\rangle_A^l = \sum_{i,j} a_i\langle x_i, x_j\rangle_{C(T)}^l a_j^*$$

$$= \sum_k \Big(\sum_i a_i t_{ik}\Big)\Big(\sum_j t_{kj} a_j^*\Big) = \sum_k b_k b_k^*,$$

which is a sum of positive elements and hence positive in A. This completes the proof. □

Let A be an n-homogeneous C^*-algebra with compact spectrum T and let M be an invertible $A\text{-}_{C(T)}A$ bimodule. Then by [**4**, Corollary II. 3.6] and [**1**, Corollary 2.6], M is isomorphic to $A \odot_{C(T)} M^A$, where M^A is the invertible $C(T)$-module $\{m \in M\colon ma = am\ \forall a\}$. Since M^A carries an imprimitivity bimodule structure, Lemma B1 shows how to give $M = A \odot_{C(T)} M^A$ the structure of an $A\text{-}_TA$ imprimitivity bimodule; further, if $M_1 \cong M_2$ as A-A bimodules, then the corresponding invertible $C(T)$-modules are isomorphic and so this structure depends only on the isomorphism class of M. If X is an $A\text{-}_TA$ imprimitivity bimodule, then the map $a \otimes x \to ax$ induces an A-A bimodule isomorphism $A \odot_{C(T)} X^A \cong X$ [**4**, Corollary II. 3.6] and since $Z(A) = C(T)$ the A-valued inner products on X drop to $C(T)$-valued inner products on X^A. It is routine to check that X^A is a $C(T)\text{-}_TC(T)$ imprimitivity bimodule–the only nonobvious fact is that axiom (4) is satisfied, and this follows from the fact that every $x \in X$ has the form $\Sigma\, a_i x_i$ for some $x_i \in X^A$, so that if $\langle x, x\rangle_A(t) \neq 0$ there are i, j such that $\langle x_i, x_j\rangle_{C(T)}(t) \neq 0$. In particular, X^A is an invertible $C(T)$-module so that X is an invertible $A\text{-}_{C(T)}A$ bimodule; further, the map $a \otimes x \to ax$ carries the inner products of Lemma B1 into the usual ones for X. We have therefore proved:

PROPOSITION B2. *Let A be an n-homogeneous C^*-algebra with spectrum T. Then the invertible $A\text{-}_{C(T)}A$ bimodules are the $A\text{-}_TA$ imprimitivity bimodules, in the sense that each invertible bimodule can be given the structure of an imprimitivity bimodule and that each imprimitivity bimodule arises this way.*

(C) *C^*-algebras which are strongly Morita equivalent to a commutative C^*-algebra.* We have already seen in Proposition 1.1 that the C^*-algebra defined by a continuous field of Hilbert spaces over a locally compact space T is strongly Morita equivalent to $C_0(T)$; our goal here is to show that these are the only C^*-algebras with this property. We understand this result is known to Philip Green,[1] but we have already done most of the work so we include a short proof.

PROPOSITION C1. *Let A be a C^*-algebra and T a locally compact space. If A is strongly Morita equivalent to $C_0(T)$ then A is the C^*-algebra defined by a continuous field of Hilbert spaces over T.*

PROOF. First of all we note that if Y is a complete A-$\mathbf{C}$ imprimitivity bimodule then Y with the $\mathbf{C}$-inner product is a Hilbert space, and if $x, y \in Y$ then $z \to \langle x, y\rangle_A z$ is by axiom (5) the rank one operator $x \otimes \bar{y}$. The elements of A act as bounded operators on Y (cf. condition (3) of [**11**, Definition 6.10] which is equivalent to ours) and if we define a $*$-representation $\pi\colon A \to B(Y)$ by $\pi(a)y = ay$, then π is norm decreasing. Thus $\pi(A)$ contains the finite rank operators as a dense subset, and since $\pi(A)$ is closed it must be all of $K(Y)$; further, it is easy to see that π is an isomorphism.

Now suppose that X is a complete A-$C_0(T)$ imprimitivity bimodule; we prove first that A is a continuous trace C^*-algebra. We identify the spectra of A and

[1]This is part of his work on "Morita equivalence of C^*-algebras" which we understand will be written up soon.

$C_0(T)$ via X [**11**, Corollary 6.27], so that in particular $\hat{A}$ is Hausdorff and X becomes an A-${}_TC_0(T)$ bimodule. For $t \in T$, X^t is a complete A^t-$C_0(T)^t$ imprimitivity bimodule, so that X^t is a Hilbert space and A^t acts as the compact operators on X^t. If $x \in X$ satisfies $\|x^t\|_{A^t} > 0$, then let N be a neighbourhood of t such that

$$\|x^s\|^2_{A^s} = \left\|\langle x^s, x^s\rangle_{C_0(T)^s}\right\|^2 = \langle x, x\rangle_{C_0(T)}(s) > 0 \quad \text{for } s \in N,$$

and choose ρ: $T \to [0, 1]$ such that $\rho \equiv 1$ near t and $\rho \equiv 0$ outside N. We define $f \in C_0(T)$ by

$$f(s) = \begin{cases} \rho(s)/\|x^s\|_{A^s} & \text{if } s \in N, \\ 0 & \text{if } s \notin N; \end{cases}$$

then $xf \in X$ and, for each s near t, $\langle (xf)^s, (xf)^s\rangle_{A^s}$ is a rank one projection. It now follows from [**7**, 4.5.4] that A is a continuous trace C^*-algebra.

By (the easy direction of) the main theorem of [**3**], $A \otimes K(H)$ is isomorphic to $C_0(T, K(H))$. Since $C_0(T, K(H))$ is the C^*-algebra defined by the trivial field $T \times H$, the Dixmier-Douady class $\delta(C_0(T, K(H)))$ in $H^3(T, \mathbf{Z})$ is zero; hence $\delta(A \otimes K(H)) = 0$ also. It follows from [**5**, Théorème 1] and [**9**, Lemma 1.11] that $\delta(A) = 0$, and so by [**7**, 10.9.3] A is the C^*-algebra defined by a field of Hilbert spaces. □

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