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BASIC SEQUENCES AND SUBSPACES IN LORENTZ SEQUENCE SPACES WITHOUT LOCAL CONVEXITY

BY

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ABSTRACT. After some preliminary results (§1), we give in §2 another proof of the result of N. J. Kalton [5] concerning the unicity of the unconditional bases of l_p, $0 < p < 1$.

Using this result we prove in §3 the unicity of certain bounded symmetric block bases of the subspaces of the Lorentz sequence spaces $d(w, p)$, $0 < p < 1$. In §4 we show that every infinite dimensional subspace of $d(w,p)$ contains a subspace linearly homeomorphic to l_p, $0 < p < 1$.

Unlike the case $p \geqslant 1$ there are subspaces of $d(w, p)$, $0 < p < 1$, which contain no complemented subspaces of $d(w,p)$ linearly homeomorphic to l_p. In fact there are spaces $d(w, p)$, $0 < p < 1$, which contain no complemented subspaces linearly homeomorphic to l_p. We conjecture that this is true for every $d(w, p)$, $0 < p < 1$. The answer to the previous question seems to be important: for example we can prove that a positive complemented sublattice E of $d(w, p)$, $0 < p < 1$, with a symmetric basis is linearly homeomorphic either to l_p or to $d(w, p)$; consequently, a positive answer to this question implies that E is linearly homeomorphic to $d(w, p)$. In §5 we are able to characterise the sublattices of $d(w, p)$, $p = k^{-1}$ (however under a supplementary restriction concerning the sequence $(w_n)_{n=1}^\infty$), which are positive and contractive complemented, as being the order ideals of $d(w, p)$.

Finally, in §6, we characterise the Mackey completion of $d(w, p)$ also in the case $p = k^{-1}$, $k \in \mathbf{N}$.

1. Preliminary results. Let X be a real linear space and $0 < p < 1$. A function, denoted by $\| \ \|$, defined on X with the values in $\mathbf{R}_+$, is called a *p-norm* (or briefly a *norm*) if the following conditions are verified.

1. $\|x\| = 0$ if and only if $x = 0$.
2. $\|\alpha x\| = |\alpha|^p \|x\|$ for $x \in X$ and $\alpha \in \mathbf{R}$.
3. $\|x + y\| \leqslant \|x\| + \|y\|$ for $x, y \in X$.

Then the subsets $U_n = \{x \in X: \|x\| \leqslant n^{-1}\}$, for $n \in \mathbf{N}$, constitute a fundamental system of neighbourhoods of zero for a metric linear topology of X. If X is complete with respect to this topology we say that X is a *p-Banach space*.

A sequence $(x_n)_{n=1}^\infty$ in X is called a *basis* if for every $x \in X$ there is a unique sequence of scalars $(a_n)_{n=1}^\infty$ such that $x = \sum_{i=1}^\infty a_i x_i$.

The following lemma is essentially known (see Theorem III.6.1, Theorem 6.5 of **[10]**):

LEMMA 1.1. *Let $(x_i)_{i=1}^\infty$ be a sequence in X. The following assertions are equivalent*:

1. *The series $\sum_{n=1}^\infty x_{\pi(n)}$ converges for every permutation π of the integers.*

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2. *The series* $\sum_{i \in A} x_i$ *converges for every subset* $A \subset \mathbf{N}$.

3. *For every* $\varepsilon > 0$ *there exists an integer n such that* $\|\sum_{i \in H} x_i\| \leqslant \varepsilon$ *for every finite set of integers H which satisfies* $\min\{i \in H\} > n$.

4. *For any bounded sequence of real numbers* $(a_n)_{n=1}^{\infty}$, *the series* $\sum_{i=1}^{\infty} a_i x_i$ *converges, when* $\sum_{i=1}^{\infty} x_i$ *converges.*

The proof will be omitted.

A basis $(x_n)_{n=1}^{\infty}$ in X is said to be *unconditional* if for every $X \ni x = \sum_{i=1}^{\infty} a_i x_i$, the sequence $(a_n x_n)_{n=1}^{\infty}$ verifies one of the equivalent assertions of Lemma 1.1.

If $0 < \inf_n \|x_n\| \leqslant \sup_n \|x_n\| < +\infty$, we say that the basis $(x_n)_{n=1}^{\infty}$ is *bounded.*

The following corollary is also known.

COROLLARY 1.2. *Let* $(x_i)_{i=1}^{\infty}$ *be an unconditional bounded basis of X. Then*

$$|||x||| = \sup_{|b_i| \leqslant 1} \left\| \sum_{i=1}^{\infty} a_i b_i x_i \right\| < +\infty. \tag{1.1}$$

We have also

LEMMA 1.3. *The space X with the p-norm* $|||\ |||$ *is a p-Banach space.*

PROOF. Let $(x^k)_{k=1}^{\infty}$ be a Cauchy sequence in $(X, |||\ |||)$ and let $\varepsilon > 0$. Then there exists the sequence of integers $(n_k)_{k=1}^{\infty}$ so that $|||x^n - x^{n_1}||| \leqslant \varepsilon$ for $n \geqslant n_1$ and $|||x^{n_k} - x^{n_{k+1}}||| < \varepsilon/2^k$ for every $k \in \mathbf{N}$. Since $(X, \|\ \|)$ is a p-Banach space and $\|x\| \leqslant |||x|||$, there exists $x = \sum_{n=1}^{\infty}(x^{n_{k+1}} - x^{n_k}) + x^{n_1} \in X$ and $|||x - x^n||| \leqslant 2\varepsilon$ for $n \geqslant n_1$. □

COROLLARY 1.4. *There exists a constant* $0 < M < +\infty$ *such that*

$$\left\| \sum_{i=1}^{\infty} a_i b_i x_i \right\| \leqslant M \left\| \sum_{i=1}^{\infty} a_i x_i \right\| \sup_{i \in \mathbf{N}} |b_i|^p \tag{1.2}$$

where $(x_i)_{i=1}^{\infty}$ *is an unconditional bounded basis of X and* $x = \sum_{i=1}^{\infty} a_i x_i \in X$.

PROOF. It follows by Lemma 1.3 and by the open mapping theorem. □

Two bases $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ of X are *equivalent* and we write $(x_n) \sim (y_n)$, if for every sequence of scalars $(a_n)_{n=1}^{\infty}$, $\sum_{n=1}^{\infty} a_n x_n$ converges if and only if $\sum_{n=1}^{\infty} a_n y_n$ converges. A basis $(x_n)_{n=1}^{\infty}$ of X is called *symmetric* if every permutation $(x_{\pi(n)})_{n=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$ is a basis of X equivalent to $(x_n)_{n=1}^{\infty}$.

Let $w = (w_i)_{i=1}^{\infty} \in c_0 \setminus l_1$, where $1 = w_1 \geqslant w_2 \geqslant \cdots \geqslant w_n \geqslant \cdots \geqslant 0$. For a fixed p with $0 < p < 1$, let

$$d(w,p) = \left\{ a = (a_i)_{i=1}^{\infty} \in c_0 \colon \|a\|_{p,w} = \sup_{\pi} \sum_{i=1}^{\infty} |a_{\pi(i)}|^p \cdot w_i < +\infty \right\}$$

where π is an arbitrary permutation of the integers.

Then $X = (d(w,p), \|\ \|_{p,w})$ is a p-Banach space and the canonical basis $(x_n)_{n=1}^{\infty}$, is a symmetric basis of X (see [4]). If $p \geqslant 1$ we can define analogously $d(w,p)$, which is a Banach space under the norm

$$\|a\|_{p,w} = \sup_{\pi} \left(\sum_{i=1}^{\infty} |a_{\pi(i)}|^p \cdot w_i \right)^{1/p}, \qquad a = (a_i)_{i=1}^{\infty} \in d(w,p).$$

A sequence $(y_n)_{n=1}^{\infty}$ in a p-Banach space with a basis $(x_n)_{n=1}^{\infty}$ is called a *block basic sequence of* $(x_n)_{n=1}^{\infty}$, if there is an increasing sequence of integers $(p_n)_{n=1}^{\infty}$ such that $y_n = \sum_{i=p_n+1}^{p_{n+1}} a_i x_i$ with $(a_n)_{n=1}^{\infty}$ scalars.

It is known that $(y_n)_{n=1}^{\infty}$ is a basis of $\overline{\mathrm{Sp}}\{y_n: n \in \mathbf{N}\}$ (see II.5.6 of **[10]**).

If E and F are two Banach spaces and $0 < r < +\infty$, we say that the linear bounded operator T: $E \to F$ is *r-absolutely summing* if, for any finite set of elements $(x_i)_{i=1}^{n}$ of E, there is $c > 0$ such that

$$\left(\sum_{i=1}^{n} \|Tx_i\|^r\right)^{1/r} \leqslant c \cdot \sup\left\{\left(\sum_{i=1}^{n} |x'(x_i)|^r\right)^{1/r} : x' \in E', \|x'\| \leqslant 1\right\}. \tag{1.3}$$

Denote the set of all r-absolutely summing operators from E to F by $P_r(E, F)$.

If $T \in P_r(E, F)$ and $r \geqslant 1$ then $\pi_r(T) = \inf\{c > 0: c \text{ verifies (1.3)}\}$ is a norm on $P_r(E, F)$, which is a Banach space with respect to this norm.

We recall here two more theorems which we need:

If A is an infinite matrix of real numbers $(a_{ij})_{i,j=1}^{\infty}$ and $0 < p \leqslant 1$, then we denote by

$$\|A\|_{\infty,p} = \sup_{\|x\|_\infty \leqslant 1} \left(\sum_j \left|\sum_k a_{jk} x_k\right|^p\right)^{1/p}$$

where $\|x\|_\infty = \sup_{k \in \mathbf{N}} |x_k|$.

THEOREM 1.5 (SEE THEOREM 2 OF **[2]**). *Let* $0 < p < 1$. *For every matrix A such that* $\|A\|_{\infty,p} < \infty$, *we have the inequality*

$$\inf\left\{\|d\|_{p/(1-p)} \cdot \|B\|_{\infty,1}: A = (\operatorname{diag} d) \circ B\right\} \leqslant K\|A\|_{\infty,p} \tag{1.4}$$

where

$$d = (d_n)_{n=1}^{\infty} \in l_{p/(1-p)}, \qquad \|d\|_{p/(1-p)} = \left(\sum_i |d_i|^{p/(1-p)}\right)^{(1-p)/p},$$

diag d *is the matrix which has on the diagonal the numbers* d_n, *and K is a positive constant depending only on p.*

THEOREM 1.6 (SEE THEOREM 94 OF **[9]**). *Let* (X, μ) *be a measure space and H a Hilbert space. Then any linear bounded operator from* $L^1(X, \mu)$ *on H is p-absolutely summing for all* $0 < p < \infty$.

(In fact Maurey proved a stronger version of Theorem 1.6.)

2. The unicity of the unconditional bases of l_p, $0 < p < 1$. In this section we give a new proof of Kalton's result **[5]** concerning the unicity of unconditional bases of l_p, $0 < p < 1$. The proof follows the idea of Lindenstrauss and Pełczyński's proof concerning the unicity of unconditional bases of l_1 **[7]**.

THEOREM 2.1. *Any two unconditional bounded bases of* l_p, $0 < p < 1$, *are equivalent.*

PROOF. If $(e_n)_{n=1}^{\infty}$ is the canonical basis of l_p, $0 < p < 1$, let $(x_n)_{n=1}^{\infty}$, where $x_i = \sum_{j=1}^{\infty} b_{ij} e_j$ such that $\sum_{j=1}^{\infty} |b_{ij}|^p = 1$ for every $i \in \mathbf{N}$, another unconditional (assumed normalized) basis and let $y = \sum_{i=1}^{\infty} a_i x_i$ an element of l_p, $0 < p < 1$. For any $x \in l_p$ we denote by $\|x\|_p = \|\sum_{i=1}^{\infty} \alpha_i e_i\|_p = \sum_{i=1}^{\infty} |\alpha_i|^p$, the norm of l_p for $0 < p \leqslant 1$.

Let $A: c_0 \to l_p$ be the operator defined by the infinite matrix $A = (a_i b_{ij})_{i,j=1}^{\infty}$. By (1.2) it follows that A is a continuous linear operator, and moreover

$$\|A\|_{\infty,p} = \sup_{|\lambda_i| \leqslant 1} \left\| \sum_{i=1}^{\infty} \lambda_i a_i x_i \right\|_p^{1/p} \leqslant M \left\| \sum_{i=1}^{\infty} a_i x_i \right\|_p^{1/p}. \tag{2.1}$$

Theorem 1.5 shows us that for every $\varepsilon > 0$ there are $K = K(p) > 0$, the diagonal matrix $D = (d_i \partial_{ij})_{i,j=1}^{\infty}$, where $d = (d_i)_{i=1}^{\infty} \in l_{p/(1-p)}$ and the matrix $C = (c_{ij})_{i,j=1}^{\infty}$ such that

$$\begin{cases} A = D \cdot C, \\ \|C\|_{\infty,1} = \sup_{|\lambda_i| \leqslant 1} \left\| \sum_{j=1}^{\infty} \left(\sum_{i=1}^{\infty} \lambda_i c_{ij} \right) e_j \right\|_1 < +\infty, \\ \|d\|_{p/(1-p)} \cdot \|C\|_{\infty,1} \leqslant K \|A\|_{\infty,p} + \varepsilon. \end{cases} \tag{2.2}$$

Theorem 2.b.7 of **[8]** says that each linear bounded operator $T: c_0 \to l_1$ is 2-absolutely summing, consequently

$$\pi_2(C) \leqslant K_G \|C\|_{\infty,1} \tag{2.3}$$

where C is the operator defined by the matrix C and K_G is a universal constant.

Note now that by Hölder's inequality we have

$$\|D\| = \sup_{\|b\|_1 \leqslant 1} \left(\sum_{i=1}^{\infty} |d_i b_i|^p \right)^{1/p} \leqslant \|d\|_{p/(1-p)} \tag{2.4}$$

where $D: l_1 \to l_p$ is defined by the matrix D and $b = (b_i)_{i=1}^{\infty} \in l_1$. Hence

$$\begin{aligned} \left(\sum_{i=1}^{\infty} |a_i|^2 \right)^{1/2} &= \left(\sum_{i=1}^{\infty} \|a_i x_i\|_p^{2/p} \right)^{1/2} \\ &= (\text{by } (2.2)) = \left(\sum_{i=1}^{\infty} \|DC(e_i)\|_p^{2/p} \right)^{1/2} \\ &\leqslant \|D\| \left(\sum_{i=1}^{\infty} \|Ce_i\|^2 \right)^{1/2} \leqslant (\text{by (2.3) and (2.4)}) \\ &\leqslant \|d\|_{p/(1-p)} \cdot K_G \|C\|_{\infty,1} \cdot \sup_{\sum_{i=1}^{\infty} |\alpha_i| \leqslant 1} \left(\sum_{i=1}^{\infty} |\alpha_i|^2 \right)^{1/2} \leqslant (\text{by } (2.2)) \\ &\leqslant K_G(K\|A\|_{\infty,p} + \varepsilon) \leqslant (\text{by } (2.1)) \\ &\leqslant KK_G M \left\| \sum_{i=1}^{\infty} a_i x_i \right\|_p^{1/p} + \varepsilon K_G. \end{aligned}$$

Since ε is arbitrarily small it follows that, for every $y = \sum_{i=1}^{\infty} a_i x_i \in l_p$,

$$\left(\sum_{i=1}^{\infty} |a_i|^2 \right)^{1/2} \leqslant MKK_G \left\| \sum_{i=1}^{\infty} a_i x_i \right\|_p^{1/p}. \tag{2.5}$$

Consequently the operator $U\colon l_p \to l_2$ defined by $U(x_i) = e_i$, $i = 1, 2, \ldots,$ is continuous and verifies

$$\|U\|_{p,2} = \sup\left\{ \left\| \sum_{i=1}^{\infty} a_i e_i \right\| : a = \sum_{i=1}^{\infty} a_i x_i,\ \|a\|_p \leqslant 1 \right\} \leqslant KK_G M. \tag{2.6}$$

Denoting by S_q the unit ball of l_q, $0 < q < \infty$, and by $\bar{\Gamma}^{(1)}(S_p)$ the closure for the topology of l_1 of the convex and balanced hull of S_p, it is easy to see that $S_1 \subseteq \bar{\Gamma}^{(1)}(S_p)$. Then, by (2.6), it follows that

$$\begin{aligned} U(l_p \cap S_1) &\subset U\big(\bar{\Gamma}^{(1)}(S_p) \cap l_p\big) \\ &\subseteq (KK_G M)^{-1} \bar{\Gamma}^{(1)}(S_2) \subset (K_G K M)^{-1} S_2, \end{aligned}$$

and this relation implies that U can be extended to an operator $V\colon l_1 \to l_2$ such that we have

$$\|V\| \leqslant K_G K M. \tag{2.7}$$

But Theorem 2.b.6 of [**8**] says that each linear bounded operator $V\colon l_1 \to l_2$ is a 1-absolutely summing operator and $\pi_1(V) \leqslant K_G \|V\| \leqslant K_G^2 K M$.

Applying Theorem 1.6 it follows that $V \in P_p(l_1, l_2)$, hence there is a positive constant M_1 depending only on p such that

$$\pi_p(V) \leqslant K_G^2 K M M_1 = K_1. \tag{2.8}$$

(2.8) implies that

$$\begin{aligned} \left(\sum_{i=1}^{\infty} |a_i|^p \right)^{1/p} &= \left(\sum_{i=1}^{\infty} \|V(a_i x_i)\|^p \right)^{1/p} \\ &\leqslant K_1 \sup_{|\lambda_j| < 1} \left(\sum_{i=1}^{\infty} |a_i|^p \left| \sum_{j=1}^{\infty} \lambda_j b_{ij} \right|^p \right)^{1/p}. \end{aligned} \tag{2.9}$$

On the other hand, denoting by C' the adjoint of the operator C, we have

$$\begin{aligned} &\sup_{|\lambda_j| < 1} \left(\sum_{i=1}^{\infty} |a_i|^p \left| \sum_{j=1}^{\infty} \lambda_j b_{ij} \right|^p \right)^{1/p} = (\text{by } (2.2)) \\ &\qquad = \sup_{|\lambda_j| < 1} \left(\sum_{i=1}^{\infty} |d_i|^p \left| \sum_{j=1}^{\infty} \lambda_j c_{ij} \right|^p \right)^{1/p} \leqslant (\text{by Hölder's inequality}) \\ &\qquad \leqslant \sup_{|\lambda_j| < 1} \left(\sum_{i=l}^{\infty} \left| \sum_{j=1}^{\infty} \lambda_j c_{ij} \right| \right) \cdot \|d\|_{p/(1-p)} \\ &\qquad = \|d\|_{p/(1-p)} \cdot \|C'\|_{\infty,1} \leqslant (\text{by } (2.2)) \leqslant K\|A\|_{\infty,p} + \varepsilon \\ &\qquad \leqslant (\text{by } (2.1)) \leqslant KM \left\| \sum_{i=1}^{\infty} a_i x_i \right\|_p^{1/p} + \varepsilon. \end{aligned} \tag{2.10}$$

(2.9) and (2.10) imply that, for every $y = \sum_{i=1}^{\infty} a_i x_i \in l_p$,

$$\left(\sum_{i=1}^{\infty} |a_i|^p \right)^{1/p} \leqslant K_1 K M \left\| \sum_{i=1}^{\infty} a_i x_i \right\|_p^{1/p} \leqslant K_1 K M \left(\sum_{i=1}^{\infty} |a_i|^p \right)^{1/p}.$$

Thus $(x_i)_i \sim (e_i)_i$. □

3. Symmetric basic sequences in $d(w, p)$, $0 < p < 1$. In this section we prove the analogues of Theorem 3 and Lemma 1 [**1**] for $0 < p < 1$. We show moreover the unicity of the symmetric bases in $d(w, p)$ in the case that this space is not included in l_1. Finally we state some open problems concerning the symmetric basic sequences in $d(w, p)$, $0 < p < 1$.

In the proofs of the following two results the techniques of Altshuler, Cassaza and Lin [**1**] work almost unchanged.

LEMMA 3.1. *Let $(x_n)_{n=1}^{\infty}$ be the canonical basis in $d(w, p)$, $0 < p < 1$. If $y_n = \sum_{i=p_n+1}^{p_{n+1}} a_i x_i$, $n = 1, 2, \ldots$, is a bounded block basic sequence of $(x_n)_{n=1}^{\infty}$ such that $\lim_n a_n = 0$, then there is a subsequence of $(y_n)_{n=1}^{\infty}$ which is equivalent to the canonical basis of l_p.*

PROOF. Since every change of signs and every permutation of the integers induces an isometry in $d(w, p)$ we may assume, by switching to a subsequence if necessary, that $(a_i)_{i=1}^{\infty}$ is a nonincreasing sequence of positive numbers. Moreover we may assume that $\|y_n\|_{p,w} = 1$ for $n \in \mathbf{N}$.

Now let $0 < \varepsilon < 2^2(2^p - 1)^{1/p}(2^{p+1} - 1)^{-1/p}$. Using the facts that $\lim_n a_n = 0$ and $\|y_n\|_{p,w} = 1$ for every $n \in \mathbf{N}$, it is easy to construct by induction two increasing sequences of integers $(n_j)_{j=1}^{\infty}$ and $(r_j)_{j=1}^{\infty}$ such that

$$\begin{cases} p_{n_j} < r_j < p_{n_j+1}, \\ Q_0 = 0, \\ Q_j = \sum_{k=1}^{j} (p_{n_k+1} - p_{n_k}) \leqslant r_{j+1} - p_{n_{j+1}}, \qquad j = 1, 2, \ldots, \\ \left(\sum_{i=p_{n_j}+1}^{r_j} (a_i)^p \cdot w_{i-p_{n_j}} \right)^{1/p} < \varepsilon / 2^{j+1}. \end{cases} \tag{3.1}$$

For the sequence of scalars $(\lambda_j)_{j=1}^{\infty}$ we have

$$\begin{aligned} \left\| \sum_{j=1}^{\infty} \lambda_j y_{n_j} \right\|_{p,w} &= \left\| \sum_{j=1}^{\infty} \lambda_j \left[\left(\sum_{i=p_{n_j}+1}^{r_j} a_i x_i \right) + \left(\sum_{i=r_j+1}^{p_{n_j+1}} a_i x_i \right) \right] \right\|_{p,w} \\ &\geqslant \left\| \sum_{j=1}^{\infty} \lambda_j \left(\sum_{i=r_j+1}^{p_{n_j+1}} a_i x_i \right) \right\|_{p,w} - \left\| \sum_{j=1}^{\infty} \lambda_j \left(\sum_{i=p_{n_j}+1}^{r_j} a_i x_i \right) \right\|_{p,w} \\ &\geqslant \left\| \sum_{j=1}^{\infty} \lambda_j \left(\sum_{i=r_j+1}^{p_{n_j+1}} a_i x_i \right) \right\|_{p,w} - \sum_{j=1}^{\infty} |\lambda_j|^p \left\| \sum_{i=p_{n_j}+1}^{r_j} a_i x_i \right\|_{p,w} \\ &\geqslant \Big(\text{since } \{ r_k - p_{n_k} + 1, \ldots, p_{n_k+1} - p_{n_k} \} \\ &\qquad \cap \{ r_j - p_{n_j} + 1, \ldots, p_{n_j+1} - p_{n_j} \} = \varnothing \text{ and} \\ &\qquad\qquad \{ r_k + 1, \ldots, p_{n_k+1} \} \cap \{ r_j + 1, \ldots, p_{n_j+1} \} = \varnothing \text{ for } k \neq j \Big) \end{aligned}$$

$$\geq \sum_{j=1}^{\infty} |\lambda_j|^p \left(\sum_{i=r_j+1}^{p_{n_j+1}} (a_i)^p \cdot w_{Q_{j-1}-r_j+i} \right) - \sum_{j=1}^{\infty} (\varepsilon/2^{j+1})^p |\lambda_j|^p$$

$$\geq \sum_{j=1}^{\infty} |\lambda_j|^p \left(\sum_{i=r_j+1}^{p_{n_j+1}} (a_i)^p \cdot w_{i-p_{n_j}} \right) - \left[\varepsilon^p 2^{-p} (2^p - 1)^{-1} \right] \cdot \sup_j |\lambda_j|^p$$

$$\geq \sum_{j=1}^{\infty} |\lambda_j|^p \left[1 - \varepsilon^p 2^{-p(j+1)} - \varepsilon^p 2^{-p} (2^p - 1)^{-1} \right]$$

$$\geq \left[1 - \frac{\varepsilon^p (2^{p+1} - 1)}{2^{2p} (2^p - 1)} \right] \sum_{j=1}^{\infty} |\lambda_j|^p.$$

On the other hand we have

$$\left\| \sum_{j=1}^{\infty} \lambda_j y_{n_j} \right\|_{p,w} \leq \sum_{j=1}^{\infty} |\lambda_j|^p \|y_{n_j}\|_{p,w} = \sum_{j=1}^{\infty} |\lambda_j|^p. \quad \square$$

We shall use forward a special block basic sequence. Let $(x_n)_{n=1}^{\infty}$ be a symmetric basis in the p-Banach space X. If $0 \neq a = \sum_{n=1}^{\infty} a_n x_n \in X$ and if $(p_i)_{i=1}^{\infty}$ is an increasing sequence of integers, let $y_n^{(a)} = \sum_{i=p_n+1}^{p_{n+1}} a_{i-p_n} x_i$, $n \in \mathbf{N}$. Then $(y_n^{(a)})_{n=1}^{\infty}$ is a bounded block basic sequence of $(x_n)_{n=1}^{\infty}$, and we shall call it a *block of type* I *of* $(x_n)_{n=1}^{\infty}$.

THEOREM 3.2. *Every bounded block basic sequence of* $(x_n)_{n=1}^{\infty}$ *in* $d(w,p)$, $0 < p < 1$, *has a subsequence equivalent either to the unit vector basis of* l_p *or to a block basic sequence of type* I *of* $(x_n)_{n=1}^{\infty}$.

PROOF. Let $y_n = \sum_{i=p_n+1}^{p_{n+1}} a_i x_i$, $n = 1, 2, \ldots$. We may assume that $\|y_n\|_{p,w} = 1$ and that $a_{p_n+1} \geq \cdots \geq a_{p_{n+1}} \geq 0$ for every $n \in \mathbf{N}$. If $\sup_n(p_{n+1} - p_n) < +\infty$, then it is clear that $(y_n)_n \sim (x_n)_n$, hence $(y_n)_{n=1}^{\infty}$ is equivalent to a block basic sequence of type I of $(x_n)_{n=1}^{\infty}$. Assume now that $\sup_n(p_{n+1} - p_n) = \infty$. Let $b_i = \sup_n |a_{p_n+i}|$ for $i \in \mathbf{N}$. It is easy to prove that $\lim_i b_i = 0$.

Case I. Assume that for every $\varepsilon > 0$ there exists m such that $\|\sum_{i=p_n+m}^{p_{n+1}} a_i x_i\|_{p,w} \leq \varepsilon$ for every n so that $p_{n+1} - p_n \geq m$. Since $\sup_n(p_{n+1} - p_n) = +\infty$, we may assume that $p_{n+2} - p_{n+1} \geq p_{n+1} - p_n$ for every $n \in \mathbf{N}$. Define now $z_n = \sum_{i=1}^{p_{n+1}-p_n} a_{i+p_n} x_i$, $n \in \mathbf{N}$. Then $\|z_n\|_{p,w} = \|y_n\|_{p,w} = 1$ for $n \in \mathbf{N}$. By hypothesis and using the fact that there is a subsequence $(n_k)_{k=1}^{\infty}$ such that the sequences $(a_{i+p_{n_k}})_{k=1}^{\infty}$ converge simultaneously for $i \leq m$, we can find a Cauchy subsequence of $(z_n)_{n=1}^{\infty}$. Hence we may assume that $\lim_n z_n = z = \sum_{i=1}^{\infty} c_i x_i \in d(w,p)$. It is clear that $z \neq 0$.

Since $(y_n)_{n=1}^{\infty}$ is a bounded block basic sequence of $(x_n)_{n=1}^{\infty}$ it is well known that $K = \sup_n \|P_n\| < +\infty$, where $P_n(\sum_{i=1}^{\infty} a_i y_i) = \sum_{i=1}^{n} a_i y_i$ (see III.2.11 of **[10]**). Consequently we can find a subsequence $(z_{n_i})_{i=1}^{\infty}$ such that $\sum_{i=1}^{\infty} \|z_{n_i} - z\|_{p,w} < 1/2K$. Define now

$$u_i = \sum_{k=p_{n_i}+1}^{p_{n_i+1}} c_{k-p_{n_i}} x_k, \qquad i = 1, 2, \ldots.$$

Then $(u_i)_{i=1}^{\infty}$ is a block basic sequence of type I of $(x_n)_{n=1}^{\infty}$ and $\sum_{i=1}^{\infty}\|y_{n_i} - u_i\|_{p,w} \leq \sum_{i=1}^{\infty}\|z_{n_i} - z\|_{p,w} < 1/2K$, hence by the Krein-Milman-Rutman Theorem (see Theorem III.2.13 of [**10**]) it follows that $(y_{n_i})_{i=1} \sim (u_i)_{i=1}$.

Case II. There exists an $\varepsilon > 0$ such that for every $m \in \mathbf{N}$ there exists $n(m)$ such that $p_{n+1} - p_n \geq m$ and $\|\sum_{i=p_n+m}^{p_{n+1}} a_i x_i\|_{p,w} > \varepsilon$. Then there exists an increasing sequence of integers $(n_i)_{i=1}^{\infty}$ such that

$$p_{n_i+1} - p_{n_i} > i \quad \text{and} \quad \left\| \sum_{j=p_{n_i}+i}^{p_{n_i+1}} a_j x_j \right\|_{p,w} > \varepsilon \quad \text{for some } i \in \mathbf{N}.$$

Since $\lim_i b_i = 0$ we may assume that $(a_j)_{j=1}^{\infty}$ is a decreasing sequence. Let

$$z_i = \sum_{j=p_{n_i}+1}^{p_{n_i+1}} a_j x_j \quad \text{for } i \in \mathbf{N}.$$

Then $\varepsilon \leq \|z_i\|_{p,w} \leq \|y_i\|_{p,w} = 1$ for $i \in \mathbf{N}$, also $(z_i)_{i=1}^{\infty}$ is a bounded block basic sequence of $(x_n)_{n=1}^{\infty}$ and the coefficients of z_i converge to zero. By Lemma 3.1 it follows that there exists a subsequence $(t_i)_{i=1}^{\infty}$ of $(z_i)_{i=1}^{\infty}$ which is equivalent to the canonical basis $(e_i)_{i=1}^{\infty}$ of l_p. Since $(y_{n_i})_{i=1}^{\infty}$ dominates $(t_i)_{i=1}^{\infty}$ (i.e. $\|\sum_{i=1}^{\infty} b_i y_{n_i}\|_{p,w} < +\infty$ implies that $\|\sum_{i=1}^{\infty} b_i t_i\|_{p,w} < \infty$ for every sequence $(b_i)_{i=1}^{\infty}$) then $\|\sum_{i=1}^{\infty} d_i y_{n_i}\|_{p,w} < \infty$ implies that $\sum_{i=1}^{\infty}|d_i|^p < \infty$ for every sequence of scalars $(d_i)_{i=1}^{\infty}$. On the other hand, since $\|y_{n_i}\|_{p,w} = 1$ for every $i \in \mathbf{N}$, $\sum_{i=1}^{\infty}|d_i|^p < \infty$ implies that $\|\sum_{i=1}^{\infty} d_i y_{n_i}\|_{p,w} < \infty$, hence $(y_{n_i})_i \sim (e_i)_i$. □

In the remainder of this section we study the unicity of the symmetric bases of the subspaces of $d(w,p)$, $0 < p < 1$. We shall often use the following notion. Let X be a p-Banach space with a separating dual X (this is the case whenever X has a basis). We consider on X the finest locally convex topology weaker than the original one i.e. the *Mackey topology* on X. It is easy to see that the Mackey topology on X is generated by the neighbourhoods $((1/n)\mathrm{co}(S))_{n=1}^{\infty}$, where $S = \{x \in X\colon \|x\| \leq 1\}$. Then the completion of X in the Mackey topology, $\tilde{X}$, is a Banach space. It is interesting that $\widetilde{d(w,p)}$, $0 < p < 1$, may be exactly l_1. The routine proof of the following proposition will be omitted.

PROPOSITION 3.3. *A bounded set $A \subset d(w,p)$, $0 < p < \infty$, is precompact if and only if for every $\varepsilon > 0$ there is $n \in \mathbf{N}$ such that*

$$\sup_{\pi} \sum_{i=n}^{\infty} |a_{\pi(i)}|^p \cdot w_i < \varepsilon \tag{3.2}$$

uniformly for $a = (a_i)_{i=1}^{\infty} \in A$. (The supremum is taken over all permutations of integers.)

PROPOSITION 3.4. *Let $0 < p < 1$. Then $\widetilde{d(w,p)} = l_1$ if and only if $d(w,p) \subset l_1$. Moreover if $d(w,p) \not\subset l_1$ then $d(w,p) \not\approx l_1$ (i.e. $d(w,p)$ is not linearly homeomorphic to l_1).*

PROOF. If $d(w,p) \not\subset l_1$, let $a = (a_i)_{i=1}^{\infty} \in d(w,p) \setminus l_1$, $A = \{x_i\colon i \in \mathbf{N}\}$ and $f \in d(w,p)'$. We shall show that

$$\lim_i f(x_i) = 0. \tag{3.3}$$

Indeed if (3.3) is not true, there exist the sequence $(i_k)_{k=1}^{\infty}$ of integers and $\alpha > 0$ such that $|f(x_{i_k})| \geqslant \alpha$ for every $k \in \mathbf{N}$. We consider $b = (b_i)_{i=1}^{\infty}$, where

$$b_i = \begin{cases} a_j \operatorname{sign} f(x_{i_j}), & i = i_j, j \in \mathbf{N}, \\ 0, & i \neq i_j, j \in \mathbf{N}. \end{cases}$$

It is clear that $b \in d(w,p) \setminus l_1$. But $\infty = \alpha.\ \Sigma_{i=1}^{\infty}|a_i| \leqslant |\Sigma_{j=1}^{\infty} f(x_j)b_j| < \infty$, which is contradictory. Then (3.3) is true and A is weakly relatively compact. On the other hand (3.2) is not verified for A and $p = 1$, hence A is not a relatively compact subset of $d(w, 1)$. Since the canonical mapping i: $d(w,p) \to d(w, 1)$, $0 < p < 1$, is clearly continuous, it is obvious that on $d(w,p)$ the topology induced by $\widetilde{d(w,p)}$ is stronger than that induced by $d(w, 1)$, consequently A is not a relatively compact subset of $\widetilde{d(w,p)}$.

If $\widetilde{d(w,p)} \approx l_1$, then, A being weakly relatively compact in $\widetilde{d(w,p)}$ (since clearly $d(w,p)' = \widetilde{d(w,p)}'$) it is also relatively compact in $\widetilde{d(w,p)}$, which is a contradiction. Thus $\widetilde{d(w,p)} \not\approx l_1$. Conversely, if $d(w,p) \subset l_1$, we denote by I: $d(w,p) \to l_1$ the canonical mapping. We shall show that I is continuous, that is there exists $K > 0$ such that, for every $a \in d(w,p)$, we have

$$\sum_{i=1}^{\infty} |a_i| \leqslant K\|a\|_{p,w}^{1/p}. \tag{3.4}$$

It is clear that it suffices to prove (3.4) only for positive decreasing sequences $(a_i)_{i=1}^{\infty} \in d(w,p)$. If (3.4) is not true, then for every $n \in \mathbf{N}$ there is the positive decreasing sequence $a^{(n)} = (a_i^{(n)})_{i=1}^{\infty} \in d(w,p)$, such that

$$2^n\|a^{(n)}\|_{p,w}^{1/p} < \sum_{i=1}^{\infty} a_i^{(n)}. \tag{3.5}$$

Denote by $b^{(n)} = 2^{-n}\|a^{(n)}\|_{p,w}^{-1/p}a^{(n)}$. Then $\|\Sigma_{n=1}^{\infty} b^{(n)}\|_{p,w} \leqslant \Sigma_{n=1}^{\infty}\|b^{(n)}\|_{p,w} \leqslant \Sigma_{n=1}^{\infty} 1/2^{np} < \infty$, consequently it follows that $b = \Sigma_{n=1}^{\infty} b^{(n)} \in l_1$. But (3.5) implies that $1 < \Sigma_{i=1}^{\infty} b_i^{(n)}$ for every $n \in \mathbf{N}$, hence $\|\Sigma_{n=1}^{\infty} b^{(n)}\|_1 = \Sigma_{n=1}^{\infty} \Sigma_{i=1}^{\infty} b_i^{(n)} = +\infty$, which is a contradiction. Hence I is continuous. On the other hand the canonical mapping J: $l_p \to d(w,p)$ is continuous and consequently the extensions $\tilde{I}$: $\tilde{l}_p \to \widetilde{d(w,p)}$ and $\tilde{J}$: $\widetilde{d(w,p)} \to l_1$ are continuous. Since clearly $\tilde{l}_p = l_1$ for $0 < p < 1$, and $\tilde{J} \circ \tilde{I} = \mathrm{id}_{l_1}$, it follows that $d(w,p) = l_1$. □

We shall give an example of space $d(w,p)$, $0 < p < 1$, for which $d(w,p) \subset l_1$. (It is known [4] that $d(w,p) \not\subset l_1$ for $p \geqslant 1$.)

EXAMPLE 3.5. Let $w_n = (1 + |\log n|)^{-1}$ for $n \geqslant 1$ and $0 < p < 1$. Then $d(w,p) \subset l_1$.

PROOF. Our proof is indirect. We denote by

$$M(t) = \begin{cases} \dfrac{t^p}{1 + |\log t|}, & t \in (0, 1], \\ 0, & t = 0. \end{cases}$$

Then $M(t)$ is a continuous nondecreasing function on $[0, 1]$, with $\sup_{t\in(0,1]} M(2t)/M(t) < \infty$, that is an Orlicz function (see [5]). We consider now the locally convex Orlicz sequence space $l_M = \{a = (a_i)_{i=1}^{\infty}\colon \|a\|_M = \sum_{i=1}^{\infty} M(|a_i|) < +\alpha\}$. We shall show that

$$l_M = d(w, p). \tag{3.6}$$

It is clear that $l_M = l_{M_p}$, and that $d(w, 1) = l_{M_1}$ implies that $d(w, p) = l_{M_p}$, where

$$M_q(t) = \begin{cases} \dfrac{t^q}{1 + |\log t^q|}, & t \in (0, 1], \\ & \qquad\qquad 0 < q < +\infty. \\ 0, & t = 0, \end{cases}$$

Consequently it is sufficient to show that

$$d(w, 1) = l_{M_1} = \left\{a = (a_i)_{i=1}^{\infty}\colon \sum_{i=1}^{\infty} \frac{|a_i|}{1 + |\log|a_i||} < +\infty\right\}.$$

Theorem 4.e.2 of **[8]** says that $d(w, 1) = l_{M_1}$ if and only if there exists $\gamma > 0$ such that

$$\sum_{n=1}^{\infty} 1/W^{-1}(\gamma w_n) < +\infty \tag{3.7}$$

where $W(x) = (1 + \log x)^{-1}$ for $x \geqslant 1$.

In our case $W^{-1}(\gamma w_n) = e^{1/(\gamma - 1)} n^{1/\gamma}$, hence, for every $0 < \gamma < 1$, (3.7) and also (3.6) are true. But Theorem 3.3 of **[5]** shows us that $\widetilde{d(w, p)} = l_{\hat{M}}$, where $\hat{M}$ is the largest Orlicz convex function on [0, 1] such that $\hat{M}(x) \leqslant M(x)$, $x \in [0, 1]$. Since $K(p)t \leqslant t^p/(1 + |\log t|) \leqslant t^p$ for $t \in (0, 1]$, where $K(p) > 0$ depends only on p, it follows that $\hat{M}$ is equivalent to the function $N(t) = t$, hence

$$\widetilde{d(w, p)} = l_{\hat{M}} = l_1. \tag{3.8}$$

By Proposition 3.4 it follows that $d(w, p) \subset l_1$. □

REMARK 3.6.1. Since $M(t)$ is not equivalent to $N(t) = t^p$ (i.e. there are not the relations $0 < \inf_{t \in (0, 1]} M(t)/\mathrm{N}(t) \leqslant \sup_{t \in (0, 1]} M(t)/\mathrm{N}(t) < +\infty$) the previous space $d(w, p)$ is a p-Banach space, other than l_p, whose dual is l_∞.

2. This space is moreover an example of a p-Banach space X with a unique unconditional basis, other than l_p, for which $\tilde{X}$ has a unique unconditional basis. Indeed $\lim_{x \to 0} M(x)/x = +\infty$, hence by Theorem 7.6 of **[5]**, $d(w, p) = l_M$ has a unique unconditional basis. □

There are spaces $d(w, p)$, $0 < p < 1$, for which $d(w, p) \not\subset l_1$.

PROPOSITION 3.7. *Let $(w_i)_{i=1}^{\infty} \in c_0 \setminus l_1$ be a decreasing sequence of positive numbers such that there exist an increasing sequence of integers $(n_j)_{j=1}^{\infty}$, the scalars $b > 0$ and $0 \leqslant \gamma < p/(1 - p)$ (where $0 < p < 1$) such that*

$$n_{j+1} - n_j \leqslant n_{j+2} - n_{j+1}, \qquad j \in \mathbf{N}. \tag{3.9}$$

$$n_{j+1} - n_j \leqslant bj^\gamma, \qquad j \in \mathbf{N}. \tag{3.10}$$

$$w_n \leqslant 1/j \quad \text{if } n \geqslant n_j, j \in \mathbf{N}. \tag{3.11}$$

Then $d(w, p) \not\subset l_1$.

PROOF. Let

$$a_i = \begin{cases} 1 & \text{if } i \leqslant n_1, \\ j^{-\beta/p}(n_{j+1} - n_j)^{-1/p} & \text{if } n_j < i \leqslant n_{j+1}, \end{cases}$$

where

$$0 < \beta \leqslant p - (1-p)\gamma. \tag{3.12}$$

By (3.9) it follows that $(a_i)_{i=1}^{\infty}$ is decreasing to zero and (3.11) implies that

$$\|a\|_{p,w} = \sum_{i=1}^{n_1} w_i + \sum_{j=1}^{\infty} \sum_{i=n_j+1}^{n_{j+1}} \frac{w_i}{j^{\beta}(n_{j+1} - n_j)}$$
$$\leqslant \sum_{i=1}^{n_1} w_i + \sum_{j=1}^{\infty} \frac{1}{j^{1+\beta}} < +\infty.$$

On the other hand

$$\|a\|_1 = \sum_{i=1}^{n_j} 1 + \sum_{j=1}^{\infty} \sum_{i=n_j+1}^{n_{j+1}} \frac{1}{j^{\beta/p}(n_{j+1} - n_j)^{1/p}}$$
$$= n_1 + \sum_{j=1}^{\infty} \frac{1}{j^{\beta/p}(n_{j+1} - n_j)^{(1/p)-1}}$$
$$\geqslant (\text{by } (3.10)) \geqslant n_1 + b^{1-1/p} \sum_{j=1}^{\infty} \frac{1}{j^{\beta/p} \cdot j^{\gamma(1/p-1)}}$$
$$\geqslant (\text{by } (3.12)) \geqslant n_1 + b^{1-1/p} \sum_{j=1}^{\infty} \frac{1}{j} = \infty,$$

thus $a = (a_i)_{i=1}^{\infty} \notin l_1$. □

The space $d(w,p)$, $0 < p < 1$, with $w_n = 1/n$ for $n \geqslant 1$, satisfies the conditions of Proposition 3.7. In Theorem 4 of [**1**] it is proved that every two symmetric bounded bases of a subspace of $d(w,p)$, $p \geqslant 1$, are equivalent. Similarly we can state the following still open problem.

Problem 1. Let $X \subset d(w,p)$, $0 < p < 1$, be a subspace and $(y_n)_{n=1}^{\infty}$ and $(z_n)_{n=1}^{\infty}$ two symmetric bounded bases of X. Are $(y_n)_{n=1}^{\infty}$ and $(z_n)_{n=1}^{\infty}$ equivalent?

We are unable to give an answer to Problem 1, however the following theorem is true:

THEOREM 3.8. *Let $d(w,p)$, $0 < p < 1$, such that $d(w,p) \not\subset l_1$. Then $d(w,p)$ has a unique bounded symmetric basis.*

PROOF. Let $(y_n)_{n=1}^{\infty}$ be a bounded symmetric basis of l_p other than $(x_n)_{n=1}^{\infty}$. By Proposition 3.4 the hypothesis implies that $\widehat{d(w,p)} \not\approx l_1$, hence

$$\lim_m x'_n(y_m) = 0 \quad \text{for every } n \in \mathbf{N}, \tag{3.13}$$

where $(x'_n)_{n=1}^{\infty}$ is the biorthogonal sequence in $d(w,p)'$ associated to $(x_n)_{n=1}^{\infty}$ (i.e. $x'_n(x_m) = \partial_{mn}$ for every $m, n \in \mathbf{N}$). Indeed if (3.13) is not true for $n_0 \in \mathbf{N}$, there exist the subsequence $(y_{m_i})_{i=1}^{\infty}$ of $(y_n)_{n=1}^{\infty}$ and $\alpha > 0$ such that $\alpha \leqslant |x'_{n_0}(y_{m_i})|$ for every $i \in \mathbf{N}$. Then for every scalar $(a_i)_{i=1}^{n}$, there are $\varepsilon_i = \pm 1$, $i = 1, 2, \ldots, n$ and

$0 < M < \infty$, such that

$$\alpha \sum_{i=1}^{n} |a_i| \leqslant x'_{n_0}\left(\sum_{i=1}^{n} a_i \varepsilon_i y_{m_i} \right) \leqslant \left\| \sum_{i=1}^{n} a_i \varepsilon_i y_{m_i} \right\|_{p,w}^{\sim}$$
$$\leqslant \sum_{i=1}^{n} |a_i| \, \|y_{m_i}\|_{p,w}^{\sim} \leqslant M \sum_{i=1}^{n} |a_i|,$$

where $\|x\|_{p,w}^{\sim}$ is the norm of the element $x \in \widetilde{d(w,p)}$. This inequality shows us that $\widetilde{d(w,p)} \approx l_1$, which is a contradiction. (3.13) implies by Proposition 3.1 of [5] that there exists a subsequence $(y_{n_i})_{i=1}^{\infty}$ of $(y_n)_{n=1}^{\infty}$ which is equivalent to a block basic sequence of $(x_n)_{n=1}^{\infty}$.

The basis $(y_n)_{n=1}^{\infty}$ is a symmetric basis, then $(y_{n_i})_{i=1}^{\infty} \sim (y_n)_{n=1}^{\infty}$ and consequently we may assume that $y_m = \sum_{i=p_m+1}^{p_{m+1}} b_i x_i$, $m \in \mathbf{N}$. Moreover by Lemma 3.1 it follows that $\lim\sup_n |b_n| \neq 0$. (If not then $(y_n)_n \sim (e_n)_n$.)

Since $(x_n)_{n=1}^{\infty}$ is symmetric, we may also assume that there is $\varepsilon > 0$ such that for every $m \in \mathbf{N}$ there is $p_m + 1 \leqslant k_m \leqslant p_{m+1}$ so that $|b_{k_m}| > \varepsilon$. If $\sum_{n=1}^{\infty} a_n y_n$ converges in $d(w,p)$, then for every $n \in \mathbf{N}$,

$$\left\| \sum_{i=1}^{n} a_i x_{k_i} \right\|_{p,w} \leqslant \frac{1}{\varepsilon^p} \left\| \sum_{i=1}^{n} b_{k_i} a_i x_{k_i} \right\|_{p,w} \leqslant \frac{1}{\varepsilon^p} \left\| \sum_{i=1}^{n} a_i y_i \right\|_{p,w},$$

hence $\sum_{n=1}^{\infty} a_n x_n$ converges in $d(w,p)$. If we interchange the roles of $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ we deduce the equivalence of these two bases. □

We can state a weaker version of Problem 1.

Problem 1a. Let $d(w,p) \subset l_1$, $0 < p < 1$. Is the canonical basis of $d(w,p)$ the unique bounded symmetric basis of $d(w,p)$? Finally we have

THEOREM 3.9. *Let $X \subset d(w,p)$, $0 < p < 1$, be a subspace which has a bounded symmetric basis $(y_n)_{n=1}^{\infty}$ verifying the equality* (3.13). *Then any other basis $(z_n)_{n=1}^{\infty}$ of X, which has the same properties as $(y_n)_{n=1}^{\infty}$ is equivalent to this.*

PROOF. If $X \approx l_p$, $0 < p < 1$, then by Theorem 2.1 it follows that $(y_n)_n \sim (z_n)_n$. Otherwise, Proposition 3.1 of [5] implies that $(y_n)_{n=1}^{\infty}$ is equivalent to a block basic sequence $(u_n)_{n=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$, $u_n = \sum_{i=q_n+1}^{q_{n+1}} a_i x_i$, $n = 1, 2, \ldots,$ and $(z_n)_{n=1}^{\infty}$ is equivalent to a block basic sequence $v_m = \sum_{n=r_m+1}^{r_{m+1}} b_n y_n$, $m = 1, 2, \ldots,$ of $(y_n)_n^{\infty}$. Moreover we may assume that $\lim\sup_{n\to\infty} b_n \neq 0$ (otherwise Lemma 3.1 implies that $(z_n)_n \sim (e_n)_n$ which contradicts our assumption). Reasoning as in Theorem 3.8 we obtain that $(z_n)_{n=1}^{\infty}$ domainates $(y_n)_{n=1}^{\infty}$. By interchanging the roles of $(y_n)_{n=1}^{\infty}$ and $(z_n)_{n=1}^{\infty}$ we deduce the conclusion. □

Let us mention the following problem.

Problem 2. Is there a subspace $X \subset d(w,p)$, $0 < p < 1$, different from l_p and $d(w,p)$, such that $\tilde{X} = l_1$?

Remark that a negative answer to Problem 2 and a positive one to Problem 1a imply that Theorem 3.9 is true without any restriction concerning the basis $(y_n)_{n=1}^{\infty}$.

4. Complemented subspaces and sublattices of $d(w,p)$, $0 < p < 1$. We prove now the version for $0 < p < 1$ of Theorem 1 of [1]. By $(x_n)_{n=1}^{\infty}$ we mean forward the canonical basis of $d(w,p)$.

LEMMA 4.1. *For every bounded block basic sequence $(y_n)_{n=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$ there is a block basic sequence of $(y_n)_{n=1}^{\infty}$ which is equivalent to the canonical basis of l_p.*

PROOF. Let $y_n = \sum_{i=q_n+1}^{q_{n+1}} a_i x_i$, $n = 1, 2, \ldots$. Let us remark that, since $\inf_n \|y_n\|_{p,w} > 0$, $\sum_{i=1}^{\infty} a_i x_i$ does not converge in $d(w,p)$. On the other hand, if $\sup_n \|\sum_{i=1}^{n} a_i x_i\|_{p,w} < \infty$, then $\sum_{i=1}^{\infty} a_i x_i$ converges in $d(w,p)$. Hence $\sup_{k \leqslant m} \|\sum_{i=k}^{n} y_i\|_{p,w} = +\infty$.

Let $(p_k)_{k=1}^{\infty}$ be an increasing sequence of integers such that

$$\sup_n \left\| \sum_{i=p_n+1}^{p_{n+1}} y_i \right\|_{p,w} = \infty.$$

Also let

$$z_n = \left\| \sum_{i=p_n+1}^{p_{n+1}} y_i \right\|_{p,w}^{-1/p} \cdot \left(\sum_{i=p_n+1}^{p_{n+1}} y_i \right), \qquad n \in \mathbf{N}.$$

Then the block basic sequence $(z_n)_{n=1}^{\infty}$ of $(y_n)_{n=1}^{\infty}$ satisfies the conditions of Lemma 3.1 and, consequently, there is a subsequence $(z_{n_j})_{j=1}^{\infty}$ of $(z_n)_{n=1}^{\infty}$, which is equivalent to $(e_n)_{n=1}^{\infty}$. □

Using Lemma 4.1 we can prove

THEOREM 4.2. *Let $X \subset d(w,p)$, $0 < p < 1$, be a (closed) subspace of infinite dimension. Then there is a closed subspace $Y \subset X$ such that $Y \approx l_p$.*

PROOF. By Proposition III.2.15 of **[10]** it follows that X contains a bounded basic sequence $(y_n)_{n=1}^{\infty}$ which is equivalent to a block basic sequence $(z_n)_{n=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$. Then Lemma 4.1 gives us a subspace of $\overline{\mathrm{Sp}}\{z_n\colon n \in \mathbf{N}\}$ which is linearly homeomorphic to l_p. Consequently X contains a subspace $Y \approx l_p$. □

REMARK 4.3. In Corollary 17 of **[3]** it is shown that every (closed) subspace of infinite dimension X of $d(w,p)$, $1 \leqslant p < \infty$, contains a subspace Y complemented in $d(w,p)$ and linearly homeomorphic to l_p.

This assertion is not true in the case $0 < p < 1$. Indeed in **[12]** it is shown that the subspace of l_p, $0 < p < 1$, $Y = \overline{\mathrm{Sp}}\{u_n\colon n \in \mathbf{N}\} \approx l_p$, where $u_n = n^{-1/p} \sum_{i=n(n-1)/2+1}^{n(n+1)/2} e_i$, $n = 1, 2, \ldots$, does not contain any infinite dimensional subspace which is complemented in l_p. Consequently, let X be a subspace of $d(w,p)$ linearly homeomorphic to l_p and let $(z_i)_{i=1}^{\infty}$ be a bounded basis of X. We consider the block basic sequence $u_n = n^{-1/p} \sum_{j=n(n-1)/2+1}^{n(n+1)/2} z_j$, $n = 1, 2, \ldots$. Then $Y = \overline{\mathrm{Sp}}\{u_n\colon n \in \mathbf{N}\} \approx l_p$ does not contain any complemented infinite dimensional subspace of $d(w,p)$. □

Remark 4.3 shows us that there are subspaces linearly homeomorphic to l_p (in fact isometric to l_p) which are not complemented in $d(w,p)$ for $0 < p < 1$. We can prove moreover that there are examples of spaces $d(w,p)$, $0 < p < 1$, without any complemented subspace linearly homeomorphic to l_p.

EXAMPLE 4.4. Let $w_n = (1 + |\log n|)^{-1}$ for every $n \in \mathbf{N}$ and let $0 < p < 1$. Then any complemented subspace of $d(w,p)$ with an unconditional basis is linearly homeomorphic to $d(w,p) \not\approx l_p$.

PROOF. By Example 3.5 and Remark 3.6 it follows that $d(w,p)l_M \not\approx l_p$, where

$$M(t)=\begin{cases}\dfrac{t}{1+|\log t|} & \text{if } t\in(0,1],\\ 0 & \text{if } t=0.\end{cases}$$

Moreover, the canonical basis of $d(w,p)$ is the unique unconditional bounded basis of $d(w,p)$. But a theorem of Kalton (see Theorem 7.2 of [5]) shows us that if l_F is a nonlocally convex Orlicz sequence space, then any two unconditional bounded bases of l_F are equivalent if and only if any complemented subspace with an unconditional basis is isomorphic to l_F. We conclude applying this result. $\square$

Example 4.4 motivates the following question:

Problem 3. Let $d(w,p)$, $0<p<1$. It is true that there does not exist a complemented subspace of $d(w,p)$ which is linearly homeomorphic to l_p?

First we give some results which seem to indicate an affirmative answer to Problem 3. First we study the positive complemented sublattices of $d(w,p)$, $0<p<1$, which have a symmetric basis. Let X be a p-Banach space, $0<p<1$, which is simultaneously a vector lattice verifying the condition

$$|a|\leqslant|b| \text{ implies } \|a\|\leqslant\|b\| \text{ for every } a,b\in X. \tag{4.1}$$

We call such a space X a *p-Banach lattice*. (Analogously it is defined a Banach lattice.) It is clear that, extending the order relation to $\tilde{X}$ (whenever the last space exists), X is a sublattice of $\tilde{X}$. Let X be a vector lattice and $Y\subset X$ a sublattice of X (i.e. $x\in Y$ implies that $|x|\in Y$) which has the property that $x\in X$, $y\in Y$ and $|x|\leqslant|y|$ imply that $x\in Y$. Then we call Y to be an *order ideal* of X. If $A\subset X$ is a subset of X, then we denote by $I_X(A)=\{x\in X\colon \exists a\in A \text{ and } 0<\lambda\in\mathbf{R}, \text{ such that } |x|\leqslant\lambda|a|\}$, the order ideal generated by A.

It is clear now that $d(w,p)$, $0<p<1$, with the canonical order relation is a p-Banach lattice and, moreover, an order ideal of the space ω of all sequences of real numbers. Now we can extend (under certain conditions) Lemma 2.a.11 of [8] to the p-Banach lattices.

LEMMA 4.5. *Let X be a p-Banach lattice which is an order ideal of ω, $(\nu_n)_{n=1}^\infty\in c_0$ a sequence of positive real numbers, $0\leqslant y_n=\sum_{i\in\sigma_n}\alpha_i x_i$, and $0\leqslant z_n=\sum_{j\in\psi_n}\beta_j x_j$, $n\in\mathbf{N}$, two bounded basic sequences of X (where $x_n=(\partial_{ni})_{i=1}^\infty$ for $n\in\mathbf{N}$) such that $\sigma_n\cap\psi_m=\varnothing$ for every $n,m\in\mathbf{N}$. If there exists a positive and continuous projection P from X onto $\overline{\mathrm{Sp}}\{\nu_n y_n+z_n\colon n\in\mathbf{N}\}$, then the sequence $(z_n)_n$ dominates $(\nu_n y_n)_n$.*

PROOF. Let $P(y_i)=\sum_{j=1}^\infty c_j^{(i)}(\nu_j y_j+z_j)$ and $P(z_i)=\sum_{j=1}^\infty d_j^{(i)}(\nu_j y_j+z_j)$, $i=1,2,\ldots,$ where $c_j^{(i)}\geqslant 0$ and $d_j^{(i)}\geqslant 0$. Since the basis $(x_n)_{n=1}^\infty$ is clearly an unconditional basis of X, there is a positive and continuous projection Q such that $Q(x_n)=x_n$ for $n\in\sigma_i$, where $i\in\mathbf{N}$, and $Q(x_n)=0$ otherwise. Then $QP(z_i)=\sum_{j=1}^\infty d_j^{(i)}\nu_j y_j$, $i\in\mathbf{N}$, and QP may be considered as an operator from $\overline{\mathrm{Sp}}\{z_i\colon i\in\mathbf{N}\}$ to $\overline{\mathrm{Sp}}\{y_n\colon n\in\mathbf{N}\}$ defined by the infinite matrix with positive entries $(d_j^{(i)}\nu_j)_{i,j=1}^\infty$. Then the diagonal matrix defines an operator

$$D\colon \overline{\mathrm{Sp}}\{z_i\colon i\in\mathbf{N}\}\to\bar{I}_\omega(\overline{\mathrm{Sp}}\{y_i\colon i\in\mathbf{N}\})=\bar{I}_X(\overline{\mathrm{Sp}}\{y_i\colon i\in\mathbf{N}\}).$$

D is obviously positive and $0 \leqslant D(x) \leqslant QP(x)$ for every $0 \leqslant x \in \overline{\mathrm{Sp}}\{y_i: i \in \mathbf{N}\}$. Since X is a p-Banach lattice we have

$$\|Dx\| \leqslant \|QPx\| \leqslant M\|x\| \quad \text{for every } 0 \leqslant x \in \overline{\mathrm{Sp}}\{y_i: i \in \mathbf{N}\},$$

consequently D is a continuous operator.

Assume now that $\sum_{n=1}^{\infty} a_n z_n$ converges in $\overline{\mathrm{Sp}}\{z_i: i \in \mathbf{N}\}$. Then $D(\sum_{n=1}^{\infty} a_n z_n) = \sum_{n=1}^{\infty} a_n d_n^{(n)} \nu_n y_n$ converges in $\bar{I}_X(\overline{\mathrm{Sp}}\{y_i: i \in \mathbf{N}\})$. Since $0 \leqslant c_n^{(n)}(\nu_n y_n + z_n) \leqslant P(y_n)$ for every $n \in \mathbf{N}$, then $|c_n^{(n)}| \leqslant M_1$ for $n \in \mathbf{N}$. On the other hand $\nu_n c_n^{(n)} + d_n^{(n)} = 1$, $n = 1, 2, \ldots,$ hence $\lim_n d_n^{(n)} = 1$ and, consequently, $\sum_{n=1}^{\infty} a_n \nu_n y_n$ converges. □

We can now prove

THEOREM 4.6. *Let X be an order ideal of ω, which is a p-Banach lattice and assume that its canonical basis $(x_n)_{n=1}^{\infty}$ is symmetric. If $(y_n)_{n=1}^{\infty}$ is a positive block of type* I *of $(x_n)_{n=1}^{\infty}$ then $(y_n)_n \sim (x_n)_n$ if and only if $E = \overline{\mathrm{Sp}}\{y_n: n \in \mathbf{N}\}$ is a positive complemented sublattice of X (i.e. there exists a positive and continuous projection P from X onto E).*

PROOF. Let $\sum_{n=1}^{\infty} a_n x_n \in X$, $a_1 \neq 0$ and $y_n = \sum_{i=p_n+1}^{p_{n+1}} a_{i-p_n} x_i$ for every $n \in \mathbf{N}$, where $a_i \geqslant 0$. We may assume that all $a_i \leqslant 1$. Since $(x_n)_{n=1}^{\infty}$ is a symmetric basis of X, there exists $M > 1$ such that

$$\frac{1}{M}\left\|\sum_n b_n x_{p_n+1}\right\| \leqslant \left\|\sum_{n=1}^{\infty} b_n x_n\right\| \leqslant M\left\|\sum_{n=1}^{\infty} b_n x_{p_n+1}\right\|$$

for every $\sum_n b_n x_n \in X$. Suppose now that $(y_n)_n \sim (x_n)_n$. Let $K > 0$ such that $\|\sum_{n=1}^{\infty} b_n y_n\| \leqslant K\|\sum_{n=1}^{\infty} b_n x_n\|$ for every $\sum_n b_n x_n \in X$. Define $P(\sum_{n=1}^{\infty} b_n x_n) = \sum_{n=1}^{\infty}(b_{p_n+1}/a_1) y_n$ for $\sum_n b_n x_n \in X$. Since $(y_n)_n \sim (x_n)_n$, P is well defined, $0 \leqslant P$ and $\|P\| \leqslant K a_1^{-p}$.

Conversely, let $P\colon K \to E$ be a positive and continuous projection. If $x = \sum_n b_n x_n \in X$ and $\|x\| \leqslant 1$, we choose the sequence of integers $1 = n_1 < n_2 < \ldots$ such that $\|\sum_{j=n_i}^{\infty} b_j x_j\| \leqslant 1/2^i$, $i = 1, 2, \ldots$. For $n_i \leqslant m < n_{i+1}$, $i = 1, 2, \ldots,$ we put

$$z_m = \begin{cases} \sum_{j=1}^{i} a_j x_{p_m+j} & \text{if } p_m + i \leqslant p_{m+1}, \\ y_m & \text{if } p_m + i > p_{m+1}, \end{cases}$$

and

$$w_m = \begin{cases} (y_m - z_m)/\|y_m - z_m\| & \text{if } y_m \neq z_m, \\ 0 & \text{if } y_m = z_m. \end{cases}$$

Let $\nu_n = \|y_n - z_n\|$ for $n \in \mathbf{N}$. Then ν_m, w_m, z_m are positive, $y_m = \nu_m w_m + z_m$ for $m \in \mathbf{N}$, $(\nu_n)_{n=1}^{\infty} \in c_0$ and $\inf(z_n, w_m) = 0$ for $m, n \in \mathbf{N}$. Since $0 \leqslant a_i \leqslant 1$, $i = 1, 2, \ldots,$ it follows that

$$\left\|\sum_{n=1}^{\infty} b_n z_n\right\| = \left\|\sum_{i=1}^{\infty} \sum_{m=n_i}^{n_{i+1}-1} b_m z_m\right\|$$

$$\leqslant \left\|\sum_{i=1}^{\infty} \sum_{m=n_i}^{n_{i+1}-1} b_m \sum_{j=1}^{p_{m+1}-p_m} x_{p_m+j}\right\|$$

$$\leqslant (\text{since } (x_i)_{i=1}^{\infty} \text{ is a symmetric basis}) \leqslant M\left\|\sum_{i=1}^{\infty} \sum_{m=n_i}^{n_{i+1}-1} b_m x_{p_m+1}\right\|$$

$$\leqslant M\sum_{i=1}^{\infty}\left\|\sum_{j=n_i}^{\infty} b_j x_{p_j+1}\right\| \leqslant M^2 \sum_{i=1}^{\infty}\left\|\sum_{j=n_i}^{\infty} b_j x_j\right\|$$

$$\leqslant M^2 \sum_{i=1}^{\infty} \frac{1}{2^i} < +\infty.$$

Consequently $\sum_{n=1}^{\infty} b_n z_n$ converges and, applying Lemma 4.5, $\sum_{n=1}^{\infty} b_n \nu_n w_n$ converges, therefore $\|\sum_{n=1}^{\infty} b_n y_n\| < \infty$. Conversely if $\sum_{n=1}^{\infty} b_n y_n$ converges, then $\sum_{n=1}^{\infty} a_1 b_n x_{p_n+1}$ converges. Since $(x_n)_{n=1}^{\infty}$ is a symmetric basis, then $a_1(\sum_{n=1}^{\infty} b_n x_n)$ converges. Consequently $(y_n)_n \sim (x_n)_n$. □

PROPOSITION 4.7. *Let X be a separable p-Banach lattice with the property that every order interval $[0, x]$, where $X \ni x > 0$, is compact. Then there is in X a normalized sequence $(x_n)_{n=1}^{\infty}$ of positive pairwise disjoint (i.e. $\inf(x_i, x_j) = x_i \wedge x_j = 0$, $i \neq j$) elements such that $(x_n)_{n=1}^{\infty}$ is a basis simultaneously of X and of $\tilde{X}$. Particularly $(x_n)_{n=1}^{\infty}$ is an unconditional basis of X. Moreover X is an order ideal of $\tilde{X}$.*

PROOF. We prove first the second assertion. Let $y \in X$ and $z \in \tilde{X}$ such that $0 \leqslant z \leqslant y$. If $0 \leqslant z_n \in X$ and $\lim_n z_n = z$ in $\tilde{X}$, then, since $\tilde{X}$ is a p-Banach lattice, $\lim_n z_n \wedge y = z \wedge y = z$ in $\tilde{X}$. On the other hand $z_n \wedge y \in [0, y] \subset X$ and, by hypothesis, there is a subsequence $(z_{n_k} \wedge y)_{k=1}^{\infty}$ which converges in X (and consequently in $\tilde{X}$). Hence $\lim_n z_{n_k} \wedge y = z \in X$ and X is an order ideal in $\tilde{X}$.

Remark now that the hypothesis implies that $[0, y]$ is compact in $\tilde{X}$ for $0 \leqslant y \in \tilde{X}$. But a Walsh's result (see **[13]**) asserts that a Banach lattice X such that every order interval $[0, x]$ is compact, has a normalized basis of positive pairwise disjoint elements. Consequently there exists a subsequence $(x_n)_{n=1}^{\infty}$ of $\tilde{X}$ of positive pairwise disjoint elements, with $0 < \inf_n \|x_n\|^{\sim} \leqslant \sup_n \|x_n\|^{\sim} < \infty$ (where $\|x\|^{\sim}$ is the norm of the element $x \in \tilde{X}$), such that we have a unique expansion $x = \sum_{n=1}^{\infty} a_n x_n$ with $0 \leqslant a_n$, for every $0 \leqslant x \in \tilde{X}$.

Since X is an order ideal of $\tilde{X}$, there is a subset $A \subseteq N$ such that $x_i \in X$ for every $i \in A$. Let $0 \leqslant x \in X$. Since $[0, x]$ is compact in X, it follows that $x = \sum_{n \in A} a_n x_n$, the convergence being with respect to the topology of X. Hence $(x_n)_{n \in A}$ is a basis of X. It is easy to see that $(x_n)_{n \in A}$ is a bounded basis (see also Proposition 3.2 of **[5]**). Consequently we may assume that $\|x_n\| = 1$ for every $n \in A$. Since X is dense in $\tilde{X}$ and $x_i \wedge x_j = 0$ for $i \neq j$, it follows that $A = \mathbf{N}$. The remaining assertion is a consequence of Lemma 1.1. □

Remark now that in view of Proposition 3.3 it follows that $[0, x]$ is a compact set of $d(w, p)$, $0 < p < 1$, for every $0 \leqslant x \in d(w, p)$. Consequently, by Proposition 4.7, every sublattice of $d(w, p)$ has a basis of positive pairwise disjoint elements.

We can now state the analogue of Corollary 14 of [3].

THEOREM 4.8. *Let E be a positive complemented sublattice of* $d(w, p)$, $0 < p < 1$, *which has a symmetric basis. Then E is linearly homeomorphic either to* $d(w, p)$ *or to* l_p.

PROOF. Let $(y_n)_{n=1}^{\infty}$ be the symmetric basis of E whose elements are all positive and pairwise disjoint. If $y_n = \sum_{i=1}^{\infty} t_{ni} x_i$ for $n \in \mathbf{N}$, then $\lim_n t_{ni} = 0$ for every $i \in \mathbf{N}$ and, by Proposition 3.1 of [5], there is a subsequence $(y_{n_k})_{k=1}^{\infty}$ of $(y_n)_{n=1}^{\infty}$ which is equivalent to a bounded block basic sequence $(z_n)_{n=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$. Moreover repeating the proof of Proposition 1.a.9 of [8], it follows that we can choose $z_n \geqslant 0$ for $n \in \mathbf{N}$ such that $\overline{\mathrm{Sp}}\{z_n\colon n \in \mathbf{N}\}$ is a positive complemented sublattice of $d(w, p)$. $(y_n)_{n=1}^{\infty}$ being a symmetric basis it follows that $(y_n)_n \sim (z_n)_n$. Then we can apply Theorem 3.2, consequently if $E \not\approx l_p$, there exists a positive block of type I $(u_n)_{n=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$ such that $(y_n)_n \sim (u_n)_n$. Using again the proof of Proposition 1.a.9 of [8], we may assume moreover that $\overline{\mathrm{Sp}}\{u_n\colon n \in \mathbf{N}\}$ is a positive complemented sublattice of $d(w, p)$. By Theorem 4.6, we obtain now that $(u_n)_{n=1}^{\infty}$, also and $(y_n)_{n=1}^{\infty}$, is equivalent to $(x_n)_{n=1}^{\infty}$. □

In connection with Theorem 4.8 we can state a weaker version of Problem 3.

Problem 3a. Let $d(w, p)$, $0 < p < 1$. Is there a positive complemented sublattice $E \subset d(w, p)$ linearly homeomorphic to l_p?

If we have a negative answer to Problem 3a, then Theorem 4.8 shows us that the positive complemented sublattices with a symmetric basis of $d(w, p)$, $0 < p < 1$, are linearly homeomorphic to $d(w, p)$. We gave only a partially (negative) answer to Problem 3a, when we assume, supplementarily, that the positive projection P: $d(w, p) \to E$ is a *contraction*, i.e. $\|P\| \leqslant 1$.

5. Positive and contractive complemented sublattices of $d(w, 1/k)$, $k \in \mathbf{N}$. In this section we shall characterise, under certain conditions, the positive and contractive complemented sublattices of $d(w, 1/k)$, $k \in \mathbf{N}$. We prove first an inequality:

LEMMA 5.1. *Let* $k \in \mathbf{N}$, $(w_i)_{i=1}^{\infty} \in c_0 \setminus l_1$ *such that* $w_1 \geqslant w_2 \geqslant \cdots \geqslant 0$ *and* $\alpha_1 = \cdots = \alpha_l > \alpha_{l+1} \geqslant \alpha_{l+2} \geqslant \cdots \geqslant \alpha_m > 0$ *for some* $l < m$. *Then denoting by* $s_n = \sum_{i=1}^{n} w_i$, *by*

$$C(l) = \left(\frac{1}{k}\right)(\alpha_{l+1})^{1/k}\left(\alpha_l^{1-1/k} - \alpha_{l+1}^{1-1/k}\right) w_{l+1} s_l^{k-1}$$

and by

$$\partial_l(n) = \begin{cases} 1 & \text{for } n \geqslant l+1, \\ 0 & \text{for } n < l+1, \end{cases}$$

we have the following inequality

$$A_n = \sum_{i=1}^{n-1} s_i^k(\alpha_i - \alpha_{i+1}) + s_n^k \alpha_n + \partial_l(n) C(l)$$

$$\leqslant \left(\sum_{i=1}^{n} \alpha_i^{1/k} \cdot w_i\right)^k = B_n \quad \text{for every } n \leqslant m. \tag{5.1}$$

Proof. We use the induction for n. If $n \leqslant l$ it is nothing to prove. Now let $n = l + 1$. Then

$$A_{l+1} = \alpha_l s_l^k + (s_{l+1}^k - s_l^k)\alpha_{l+1} + \binom{1}{k}\alpha_{l+1}^{1/k}(\alpha_l^{1-1/k} - \alpha_{l+1}^{1-1/k})w_{l+1}s_l^{k-1}$$

$$= \alpha_l s_l^k + \alpha_{l+1}\left(\binom{2}{k}w_{l+1}^2 s_l^{k-2} + \cdots + w_{l+1}^k\right) + \binom{1}{k}\alpha_{l+1}^{1/k}\alpha_l^{1-1/k}\cdot w_{l+1}s_l^{k-1}$$

$\leqslant$ (since $\alpha_{l+1} \leqslant \alpha_i$, $i = 1, \ldots, l$)

$$\leqslant \alpha_l s_l^k + \binom{1}{k}\alpha_{l+1}^{1/k}\alpha_l^{1-1/k}\cdot w_{l+1}s_l^{k-1}$$

$$+ \left(\binom{2}{k}\alpha_{l+1}^{2/k}\alpha_l^{1-2/k}\cdot w_{l+1}^2 s_l^{k-2} + \cdots + \alpha_{l+1}w_{l+1}^k\right)$$

$$= \left(\sum_{i=1}^{l+1}\alpha_i^{1/k}\cdot w_i\right)^k = B_{l+1}.$$

Thus (5.1) is proved in this case.

Assume now that (5.1) is true for $n - 1 \geqslant l + 1$. We have

$$A_n = A_{n-1} + (s_n^k - s_{n-1}^k)\alpha_n$$

$$= A_{n-1} + \left(\binom{1}{k}s_{n-1}^{k-1}\cdot w_n + \binom{2}{k}s_{n-1}^{k-2}\cdot w_n^2 + \cdots + w_n^k\right)\alpha_n$$

$\leqslant$ (since $\alpha_i \geqslant \alpha_{i+1}$ for $i \leqslant n$)

$$\leqslant A_{n-1} + \left[\binom{1}{k}\left(\sum_{i=1}^{n-1}\alpha_i^{1/k}\cdot w_i\right)^{k-1}\alpha_n^{1/k}\cdot w_n + \cdots + \alpha_n w_n^k\right]$$

$\leqslant$ (by induction hypothesis)

$$\leqslant \left(\sum_{i=1}^{n-1}\alpha_i^{1/k}\cdot w_i\right)^k + \left[\binom{1}{k}\left(\sum_{i=1}^{n-1}\alpha_i^{1/k}\cdot w_i\right)^{k-1}\alpha_n^{1/k}\cdot w_n + \cdots + \alpha_n w_n^{1/k}\right]$$

$$= \left(\sum_{i=1}^{n}\alpha_i^{1/k}\cdot w_i\right)^k = B_n. \quad \square$$

Lemma 5.2. *With the notations of Lemma* 5.1, *if* $(\beta_i)_{i=1}^m$ *are positive numbers such that*

$$\sum_{i=1}^{n}\beta_i \leqslant s_n^k \quad \textit{for every } n \leqslant m, \tag{5.2}$$

then

$$\sum_{i=1}^{n}\alpha_i\beta_i + \partial_l(n)C(l) \leqslant \left(\sum_{i=1}^{n}\alpha_i^{1/k}\cdot w_i\right)^k \quad \textit{for } n \leqslant m. \tag{5.3}$$

PROOF. If $n \leqslant m$, we have

$$\sum_{i=1}^{n} \alpha_i\beta_i + \partial_l(n)C(l) = \alpha_n\left(\sum_{i=1}^{n} \beta_i\right) + (\alpha_{n-1} - \alpha_n)\left(\sum_{i=1}^{n-1} \beta_i\right) + \cdots + (\alpha_1 - \alpha_2)\beta + \partial_l(n)C(l)$$
$$\leqslant (\text{by } (5.2))$$
$$\leqslant \alpha_n s_n^k + (\alpha_{n-1} - \alpha_n)s_{n-1}^k + \cdots + (\alpha_1 - \alpha_2)s_1^k + \partial_l(n)C(l)$$
$$\leqslant (\text{by } (5.1)) \leqslant \left(\sum_{i=1}^{n} \alpha_i^{1/k} \cdot w_i\right)^k. \quad \square$$

Using Lemma 5.2 we can prove now

PROPOSITION 5.3. *If $p = 1/k$, $1 < k \in \mathbf{N}$, and E is a positive and contractive complemented sublattice of $d(w, p)$, then there is a sequence of finite pairwise disjoint subsets $(\sigma_n)_{n=1}^{\infty}$ of $\mathbf{N}$ such that $u_n = (\sum_{i=1}^{\bar{\sigma}_n} w_i)^{-k}(\sum_{i\in\sigma_n} x_i)$ for $n \in \mathbf{N}$, constitute a normalized basis of E, $\bar{\sigma}_n$ being the cardinal number of σ_n.*

PROOF. By Proposition 4.7, E has a normalised basis $(u_n)_{n=1}^{\infty}$ of positive pairwise disjoint elements of E. Let $u_n = \sum_{i\in\sigma_n} \alpha_{in}x_i$, where $(\sigma_n)_{n=1}^{\infty}$ is a sequence of pairwise disjoint subsets of $\mathbf{N}$ and $\alpha_{in} > 0$ for $i \in \sigma_n$, $n \in \mathbf{N}$.

If $\bar{\sigma}_n = +\infty$, then we write $\sigma_n = \{i_j : j \in \mathbf{N}\}$, and, since $(u_n)_{n=1}^{\infty}$ is normalised, it follows that $(\alpha_{i_j n})_{j=1}^{\infty} \in d(w, p)$. Consequently there exists a permutation of integers π_n such that $\pi_n(\sigma_n) = \sigma_n$, $(\alpha_{\pi_n(i_j)n})_{j=1}^{\infty}$ is a decreasing sequence, and $\pi_n(i) = i$ for every $i \notin \sigma_n$. If $\bar{\sigma}_n < \infty$, such a permutation π_n exists obviously. Since the norm $\| \ \|_{p,w}$ is invariant under permutations of integers, we can find an isometry T: $d(w,p) \to d(w,p)$ such that $T(u_m) = t_m$, where $t_m = \sum_{j=1}^{\infty} \alpha_{\pi_m(i_j)m}$ for every $m \in \mathbf{N}$. Then $Q = TPT^{-1}$: $d(w,p) \to \overline{\mathrm{Sp}}\{t_m : m \in \mathbf{N}\}$ is a positive and contractive projection whenever P is a positive and contractive projection onto E. Consequently, we may assume that the coefficients $(\alpha_{im})_{i\in\sigma_m}$ of u_m are decreasingly ordered for every $m \in \mathbf{N}$. Since P is a positive projection, we have

$$P(x_i) = \beta_{in}u_n \quad \text{where } \beta_{in} \geqslant 0 \text{ for every } i \in \sigma_n,\ n \in \mathbf{N}, \tag{5.4}$$

and moreover

$$\sum_{i\in\sigma_n} \alpha_{in}\beta_{in} = 1 = \left(\sum_{j=1}^{\infty} (\alpha_{i_j n})^{1/k} \cdot w_j\right)^k \quad \text{for every } n \in \mathbf{N}. \tag{5.5}$$

By (5.4) and using the fact that $\|P\| \leqslant 1$ we have

$$\left\|\sum_{n=1}^{\infty}\left(\sum_{i\in\sigma_n} \gamma_i\beta_{in}\right)u_n\right\|_{p,w} \leqslant \sup_{\pi} \sum_{i=1}^{\infty} |\gamma_{\pi(i)}|^{1/k} \cdot w_i \tag{5.6}$$

for every element $(\gamma_i)_{i=1}^{\infty} \in d(w,p)$. Consequently, taking $\gamma_{i_1} = \gamma_{i_2} = \cdots = \gamma_{i_m} = 1$ and $\gamma_j = 0$ for $j \neq i_1, \ldots, i_m$, where $m \leqslant \bar{\sigma}_n$, we obtain

$$\sum_{j=1}^{m} \beta_{i_j n} \leqslant s_m^k \quad \text{for } m \leqslant \bar{\sigma}_n,\ n \in \mathbf{N}. \tag{5.7}$$

If $\bar{\bar{\sigma}}_n = \infty$, then, since $\lim_j \alpha_{i_j n} = 0$, there is an integer $l \in \mathbf{N}$ such that $\alpha_{i_l n} > \alpha_{i_{l+1} n}$ and $\alpha_{i_j n} = \alpha_{i_{j+1} n}$ for $j \leqslant l - 1$.

By (5.3) we have

$$\sum_{j=1}^{m} \alpha_{i_j n}\beta_{i_j n} + C(l) \leqslant \left(\sum_{j=1}^{m} \alpha_{i_j n}^{1/k} \cdot w_j\right)^k \quad \text{for every } m \geqslant l+1.$$

Now, by passing to the limit over $m \to \infty$ in the previous inequality, we obtain, in view of (5.5).

$$1 = \sum_{j=1}^{\infty} \alpha_{i_j n}\beta_{i_j n} + C(l) \leqslant \left(\sum_{j=1}^{\infty} \alpha_{i_j n}^{1/k} \cdot w_j\right)^k = 1.$$

Since $C(l) > 0$, this is a contradiction, hence

$$\bar{\bar{\sigma}}_n < \infty \quad \text{for every } n \in \mathbf{N}. \tag{5.8}$$

Then, reasoning as above, we get that $\alpha_{in} = \alpha_{jn}$ for every $i, j \in \sigma$, $n \in \mathbf{N}$, consequently $\alpha_{in} = (\sum_{i=1}^{\bar{\bar{\sigma}}_n} w_i)^{-k}$ for every $i \in \sigma_n$, $n \in \mathbf{N}$. $\square$

We study now the existence of positive and contractive projections onto a one dimensional sublattice of $d(w, p)$, $p = 1/k$.

Proposition 5.4. (a) *Let $p = 1/k$, $1 < k \in \mathbf{N}$, and $u = \sum_{i\in\sigma} \alpha_i x_i$, where $\|u\|_{p,w} = 1$ and $\alpha_i \geqslant 0$ for every $i \in \sigma$. Then there exists a positive and contractive projection P: $d(w, p) \to \mathrm{Sp}\{u\}$ if and only if the following conditions are satisfied: $\bar{\bar{\sigma}} = n$, $\alpha_{i_1} = \cdots = \alpha_{i_n} = (\sum_{i=1}^{n} w_i)^{-k}$, and there exist the positive numbers $\beta_1 \geqslant \cdots \geqslant 0$ such that*

$$\sum_{i=1}^{m} \beta_i \leqslant \left(\sum_{i=1}^{m} w_i\right)^k = s_m^k \quad \textit{for } m < n, \tag{5.9}$$

and

$$\sum_{i=1}^{n} \beta_i = s_n^k. \tag{5.10}$$

Then $P(\sum_{i=1}^{\infty} \alpha_i x_i) = (\sum_{i\in\sigma} \alpha_i\beta_i)u$ for every $\sum_{i=1}^{\infty} \alpha_i x_i \in d(w, p)$.

(b) *If $u = s_n^{-k}(\sum_{i=1}^{n} x_i)$ and $t_n = (1/n)s_n^k \leqslant t_m$ for every $1 \leqslant m \leqslant n$, then the operator defined by*

$$P\left(\sum_{i=1}^{\infty} \alpha_i x_i\right) = \frac{1}{n}\left(\sum_{i=1}^{n} \alpha_i\right)\left(\sum_{i=1}^{n} x_i\right)$$

is a positive and contractive projection onto $\mathrm{Sp}\{u\}$.

(c) *If $w_1^k < t_n$ for every $n > 1$ and u is as at the point* (a), *then there is a positive and contractive projection P: $d(w, p) \to \mathrm{Sp}\{u\}$ if and only if $\bar{\bar{\sigma}} = 1$.*

Proof. (a) If P: $d(w, p) \to \mathrm{Sp}\{u\}$ is a positive and contractive projection then, by Proposition 5.3, $\bar{\bar{\sigma}} = n < \infty$ and $\alpha_{i_1} = \cdots = \alpha_{i_n} = s_n^{-k}$. Denote now by $(\beta_j)_{j=1}^{n}$ the coefficients of $(P(x_{i_j}))_{j=1}^{n}$ decreasingly ordered. Then

$$\sum_{j=1}^{m} \beta_j = \left\|\left(\sum_{j=1}^{m} \beta_j\right)u\right\|_{p,w}^k = \left\|P\left(\sum_{j\in H} x_{i_j}\right)\right\|_{p,w}^k \leqslant \left\|\sum_{j\in H} x_{i_j}\right\|_{p,w}^k = s_m^k,$$

where $H \subseteq \sigma$ with $\overline{\overline{H}} = m \leqslant n$. If $m = n$, then

$$\sum_{j=1}^{n} \beta_j = \left\| P\left(\sum_{j=1}^{n} x_{i_j}\right)\right\|_{p,w}^{k} = \left\|\sum_{j=1}^{n} x_{i_j}\right\|_{p,w}^{k} = s_n^k.$$

Consequently all the conditions are satisfied and obviously $P(\sum_{i=1}^{\infty} \alpha_i x_i) = (\sum_{i\in\sigma} \alpha_i\beta_i)u$ for every $\sum_{i=1}^{\infty} \alpha_i x_i \in d(w,p)$.

Conversely, by (5.3), $\sum_{i=1}^{n} \alpha_i\beta_i \leqslant (\sum_{i=1}^{n} \alpha_i^{1/k} \cdot w_i)^k$ for every $\alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant 0$, and, consequently, $|\sum_{i=1}^{n} \alpha_i\beta_i| \leqslant \sup_\pi(\sum_{i=1}^{n} |\alpha_{\pi(i)}|^{1/k}w_i)^k$ for all scalars $(\alpha_i)_{i=1}^{n}$. Thus the operator defined by $P(\sum_{i=1}^{\infty} \alpha_i x_i) = (\sum_{i\in\sigma} \alpha_i\beta_i)u$ is well defined, $\|P\| \leqslant 1$ and $P \geqslant 0$. Since $P(u) = s_n^{-k}(\sum_{i=1}^{n} \beta_i)u = u$ (by (5.10)), it follows moreover that P is a projection.

(b) If $t_n \leqslant t_m$ for every $m \leqslant n$, then, putting $\beta_1 = \cdots = \beta_n = t_n$, the relations (5.9) and (5.10) are verified, consequently $P(\sum_{i=1}^{\infty} \alpha_i x_i) = (1/n)(\sum_{i=1}^{n} \alpha_i)(\sum_{i=1}^{n} x_i)$ is a positive and contractive projection.

(c) If there exists P: $d(w,p) \to \mathrm{Sp}\{u\}$ a positive and contractive projection and if $\overline{\overline{\sigma}} > 1$, then by (a) it follows that $(1/n)s_n^k = (1/n)\sum_{i=1}^{n} \beta_i \leqslant \beta_1 \leqslant w_1^k$ for $n > 1$, which is a contradiction. □

We can now state the main result of this section:

THEOREM 5.5. *If $p = 1/k$, $1 < k \in \mathbf{N}$, and $w_1^k < (1/n)(\sum_{i=1}^{n} w_i)^k$ for every $n > 1$, then the positive and contractive complemented sublattices E of $d(w,p)$ coincide with the (closed) order ideals of $d(w,p)$. In particular any positive and contractive complemented sublattice of $d(w,p)$ cannot be linearly homeomorphic to l_p.*

PROOF. If E is a closed order ideal of $d(w,p)$, then $E = \overline{\mathrm{Sp}}\{x_i\colon i \in A \subset \mathbf{N}\}$, hence there is a positive and contractive projection P: $d(w,p) \to E$. Conversely, if E is a positive and contractive complemented sublattice of $d(w,p)$, by Proposition 5.3, it follows that $E = \overline{\mathrm{Sp}}\{u_n\colon n \in \mathbf{N}\}$, where $u_n = \sum_{i\in\sigma_n} \alpha_i x_i > 0$ for $n \in \mathbf{N}$ and $(\sigma_n)_{n=1}^{\infty}$ is a sequence of finite pairwise disjoint subsets of $\mathbf{N}$.

By Proposition 5.4(c) it follows that $\overline{\overline{\sigma}}_n = 1$ for every $n \in \mathbf{N}$. Consequently E is a closed ideal of $d(w,p)$.

The second assertion is an obvious corollary of the first. □

The conditions of Theorem 5.5 are verified for example by the spaces $d(w,p)$, $p = 1/k$, $1 < k \in \mathbf{N}$, with $w_i = 1/i^\alpha$, $0 < \alpha < \frac{1}{2}$. Indeed,

$$\sum_{i=1}^{n} w_i \geqslant \int_1^{n+1} \frac{dx}{x^\alpha} = \frac{1}{1-\alpha}\left[(1+n)^{1-a} - 1\right] > n^{1/k} \quad \text{for every } n > 1.$$

$d(w, 1/k)$, where $1 < k \in \mathbf{N}$ and $w_n = 1/n^\alpha$ for $1 > \alpha > 1 - 1/k$ and $n \in \mathbf{N}$, are examples for spaces $d(w,p)$ for which there is a one dimensional positive and contractive complemented sublattice which is not an order ideal of $d(w,p)$.

More precisely, for a fixed $m \in \mathbf{N}$, there are $n > m$ and a positive and contractive projection onto $\mathrm{Sp}\{u\}$, where $u = (\sum_{i=1}^{n} w_i)^{-1}(\sum_{i=1}^{n} x_i)$. Indeed, using the notations of Lemma 5.4(b) we have

$$t_n \leqslant \frac{1}{n}\left(\int_0^n \frac{dx}{x^\alpha}\right)^k = \frac{n^{k(1-\alpha)}}{(1-\alpha)^k \cdot n} \leqslant \frac{1}{(1-\alpha)^k \cdot n^{1-k+\alpha k}} \quad \text{for every } n \in \mathbf{N}.$$

Consequently, for $0 < \varepsilon < \frac{1}{2}\min_{1 \leq i \leq m} t_i$, there is a lowest $n > m$ with $t_n < \varepsilon$, hence $t_n \leq t_j$ for every $1 \leq j \leq n$. We conclude applying Proposition 5.4(b). □

Theorem 5.5 suggests the following version of Problem 3.

PROBLEM 3b. Let $d(w, p)$, $0 < p < 1$. Is there a positive and contractive complemented sublattice of $d(w, p)$ linearly homeomorphic to l_p?

6. The representation of $d(w, 1/k)$ **for** $1 < k \in \mathbf{N}$. In this section we give a representation of the dual of $d(w, 1/k)$, $1 < k \in \mathbf{N}$, and also a representation of the Mackey completion of this space. These representations seem to be useful in the study of the $d(w, p)$'s structure. Remark first that the dual of $d(w, p)$, $0 < p < 1$, is

$$d(w,p)' = \left\{ (b_i)_{i=1}^{\infty} : \sum_{i=1}^{\infty} |a_i|\,|b_{\pi(i)}| < \infty \text{ for every permutation } \pi \right.$$
$$\left. \text{of the integers and for every } a + (a_i)_{i=1}^{\infty} \in d(w,p) \right\}. \tag{6.1}$$

Then we have

PROPOSITION 6.1. *Let* $d(w, p)$, *where* $p = 1/k$, $k > 1$. *Then*

$$d(w,p)' \cap c_0 = c_0 \cap \left\{ (b_i)_{i=1}^{\infty} : \sup_n \left(\sum_{i=1}^{n} b_i^* \right) \Big/ \left(\sum_{i=1}^{n} w_i \right)^k < \infty \right\} = E,$$
$$\|b\|'_{p,w} = \sup_n \left(\sum_{i=1}^{n} b_i^* \right) \Big/ \left(\sum_{i=1}^{n} w_i \right)^k, \tag{6.2}$$

where $b = (b_i)_{i=1}^{\infty} \in c_0$ *and* $b^* = (b_i^*)_{i=1}^{\infty}$ *is a decreasingly rearrangement of* b.

PROOF. Let $b \in d(w,p)' \cap c_0$, $t_n = \sum_{i=1}^{n} b_i^*$ and $s_n = \sum_{i=1}^{n} w_i$. Suppose first that

$$\sup_n \frac{t_n}{s_n^k} = +\infty. \tag{6.3}$$

Since $\lim_n s_n = +\infty$, it follows that $\lim_n t_n = +\infty$.

Let $n_0 = 0$ and $s_{n_0} = 0$. If $m \in \mathbf{N}$ is fixed, then (6.3) implies that

$$\sup_n \frac{t_n - t_m}{(s_n - s_m)^k} = +\infty. \tag{6.4}$$

Choosing n_1 arbitrarily we can find $n_2 \in \mathbf{N}$, $n_2 > n_1$ such that

$$s_{n_2} - s_{n_1} \geq s_{n_1} - s_{n_0} \quad \left(\text{since } \lim_n s_n = \infty\right),$$
$$t_{n_2} - t_{n_1} \geq 2^{2k-1}(s_{n_2} - s_{n_1})^k \quad (\text{by } (6.4)).$$

By induction we get an increasing sequence of integers $(n_j)_{j=1}^{\infty}$ such that

$$s_{n_{j+1}} - s_{n_j} \geq s_{n_j} - s_{n_{j-1}} \tag{6.5}$$
$$\text{for } j = 1, 2, \ldots.$$
$$t_{n_{j+1}} - t_{n_j} \geq (j+1)^{2k-1}(s_{n_{j+1}} - s_{n_j})^k \tag{6.6}$$

Now let $n_{j-1} < i \leqslant n_j$ and $y_i = j^{-2k}(s_{n_j} - s_{n_{j-1}})^{-k}$. Then $y = (y_i)_{i=1}^{\infty}$ is a decreasing sequence (by (6.5)) and moreover

$$\sum_{i=1}^{\infty} y_i^{1/k} \cdot w_i = \sum_{j=1}^{\infty} \sum_{i=n_{j-1}+1}^{n_j} \frac{w_i}{j^2(s_{n_j} - s_{n_{j-1}})} = \sum_{j=1}^{\infty} l/j^2 < \infty,$$

that is $y \in d(w,p)$. On the other hand

$$\sum_{i=1}^{\infty} y_i b_i^* = \sum_{j=1}^{\infty} \sum_{i=n_{j-1}+1}^{n_j} \frac{t_{n_j} - t_{n_{j-1}}}{j^{2k}(s_{n_j} - s_{n_{j-1}})^k} \geqslant (\text{by } (6.6)) \geqslant \sum_{j=1}^{\infty} \sum_{i=n_{j-1}+1}^{n_j} \frac{1}{j} = +\infty,$$

which contradicts the fact that $b \in d(w,p)'$. Consequently (6.3) is not true and then

$$\lambda_k = \sup_n \frac{t_n}{s_n^k} < +\infty. \tag{6.7}$$

Hence $d(w,p)' \cap c_0 \in E$.

Now let $b \in E$. If the decreasing sequence $y = (y_i)_{i=1}^{\infty} \in d(w,p)$, then

$$\begin{aligned}\sum_{i=1}^{n} y_i b_i^* &= \sum_{i=1}^{n-1} t_i(y_i - y_{i+1}) + t_n y_n \\ &\leqslant (\text{by } (6.7)) \leqslant \lambda_k \left[\sum_{i=1}^{n-1} s_i^k (y_i - y_{i+1}) + s_n^k y_n \right] \\ &\leqslant (\text{by } (5.1)) \leqslant \lambda_k \left(\sum_{i=1}^{n} y_i^{1/k} \cdot w_i \right)^k.\end{aligned}$$

Hence $\|b\|'_{p,w} = \|b^*\|'_{p,w} \leqslant \sup_n t_n/s_n^k$ and $b \in d(w,p)' \cap c_0$. On the other hand, for every $\varepsilon > 0$, there is $n \in \mathbf{N}$ with $t_n/s_n^k > \lambda_k - \varepsilon$. Let

$$y_i = \begin{cases} s_n^{-k} & \text{if } i \leqslant n, \\ 0 & \text{if } i > n. \end{cases}$$

Then $\|y\|_{p,w} = 1$ and $\|b\|'_{p,w} = \|b^*\|'_{p,w} \geqslant \sum_{i=1}^{n} y_i b_i = t_n/s_n^k > \lambda_k - \varepsilon$. ε being arbitrarily small, it follows that $\|b\|'_{p,w} = \sup_n t_n/s_n^k$, consequently the equalities (6.2) are satisfied. □

COROLLARY 6.2. *Let* $p = 1/k$, $k > 1$.

(a) *If* $d(w,p) \not\subset l_1$ *then*

$$\begin{aligned} d(w,p)' &= \left\{ (b_i)_{i=1}^{\infty} \in c_0 \colon \sup_n \left(\sum_{i=1}^{n} b_i^* \right) \Big/ \left(\sum_{i=1}^{n} w_i \right)^k < \infty \right\} \\ &= \left\{ (b_i)_{i=1}^{\infty} \colon \sup_n \left(\sum_{i=1}^{n} b_i^* \right) \Big/ \left(\sum_{i=1}^{n} w_i \right)^k < \infty \right\}. \end{aligned}$$

(b) *If* $d(w,p) \subset l_1$, *then* $d(w,p)' = l_\infty$.

PROOF. The case (b) is clear by Proposition 3.4.

(a) Let $b \in d(w,p)'$. If $b \notin c_0$, then there are $\varepsilon > 0$ and $(n_j)_{j=1}^{\infty}$ such that $|b_{n_j}| \geqslant \varepsilon$ for every $j \in \mathbf{N}$. Reasoning as in the proof of (3.3) we get a contradiction. Hence

$d(w, p)' \subset c_0$, consequently, by (6.2), it follows the first equality. By the last part of the proof of Proposition 6.1 it follows the second equality. □

We state now the main result of this section:

THEOREM 6.3. *Let* $p = 1/k < 1$ *and* $d(w, p) \not\subset l_1$. *If there is a positive decreasing sequence* $(v_n)_{n=1}^{\infty} \in c_0$ *such that*

$$\sum_{i=1}^{n} v_i \sim \left(\sum_{i=1}^{n} w_i \right)^k \quad \textit{for every } n \in \mathbf{N} \tag{6.8}$$

(*i.e. there are constants* A, $B > 0$ *such that* $A(\sum_{i=1}^{n} v_i) \leqslant (\sum_{i=1}^{n} w_i)^k \leqslant B(\sum_{i=1}^{n} v_i)$ *for every* $n \in \mathbf{N}$), *then*

$$\widetilde{d(w, p)} \approx d(v, 1). \tag{6.9}$$

PROOF. Theorem 11 of [**4**] says that $d(v, 1)' = \{(b_i)_{i=1}^{\infty}\colon \sup_n(\sum_{i=1}^{n} b_i^*)/(\sum_{i=1}^{n} v_i) < \infty\}$. Then by the hypothesis and by Corollary 6.2(a) it follows that

$$d(w, p)' = d(v, 1)'. \tag{6.10}$$

It is easy to see that the canonical basis $(x_n)_{n=1}^{\infty}$ of $d(w, p)$ is a symmetric basis of $\widetilde{d(w, p)}$ too. Then, denoting by $\widetilde{d(w, p)}^{x}$ the Köthe dual of $\widetilde{d(w, p)}$, that is the space $\{(b_i)_{i=1}^{\infty}\colon \sum_{i=1}^{\infty}|a_i b_i| < +\infty$ for every $a = (a_i)_{i=1}^{\infty} \in \widetilde{d(w, p)}\}$, it follows that $\widetilde{d(w, p)}^{x} = d(w, p)^{x} = d(w, p)' = \widetilde{d(w, p)}'$. Consequently $\widetilde{d(w, p)}^{xx} = \widetilde{d(w, p)}'^{x} =$ (by (6.10)) $= d(v, 1)'^{x} =$ (by Corollary of Theorem 10 of [**4**]) $= d(v, 1)^{xx}$. We shall show that $d(v, 1)^{xx} = d(v, 1)$.

A Köthe result (see §30, p. 5 of [**6**]) says that for a Banach sequence space E, $E^{xx} = E$ if and only if E is weakly sequentially complete. Hence, it suffices to show that $d(w, 1)$ is weakly sequentially complete. By Theorem 1.c.10 of [**8**] it follows that we must show that $c_0 \not\subset d(v, 1)$. If $c_0 \subset d(v, 1)$, Corollary 17 of [**3**] implies that c_0 contains a complemented subspace linearly homeomorphic to l_1, which is a contradiction. Hence $\widetilde{d(w, p)}^{xx} = d(v, 1)^{xx} = d(v, 1)$, consequently $\widetilde{d(w, p)} \subset d(v, 1) \subset \widetilde{d(w, p)}^{xx}$. By Proposition 3.3 it follows that every order interval $[0, x] \subset \widetilde{d(w, p)}$ is compact. On the other hand Theorem II. 5.10 of [**11**] shows us that in a Banach lattice E the following assertions are equivalent:

(1) The norm of E is order continuous (i.e. if $(y_\alpha)_{\alpha \in A}$ is a downward directed set in E with $\inf_\alpha y_\alpha = 0$, then $\lim_\alpha \|y_\alpha\| = 0$).

(2) Every order interval $[0, x] \subset E$ is $\sigma(E, E')$-compact.

(3) E is an order ideal in E''.

Finally, a theorem of Ando (see Proposition IV.11.1 of [**11**]) says that a Banach lattice E has an order continuous norm if and only if every closed order ideal of E is the range of a positive projection from E.

Consequently $\widetilde{d(w, p)}$ is a complemented subspace, with a symmetric basis $(y_n)_n$, of the space $\widetilde{d(w, p)}'' = \widetilde{d(w, p)}^{xx} = d(v, 1)$. Since $\widetilde{d(w, p)} \not\approx l_1$, by the Corollary 14 of [**3**] it follows that $\widetilde{d(w, p)} \approx d(v, 1)$. □

REMARK 6.4. (1) $d(w, 1/k)$ with $k > 1$ and $w_n = 1/n^\alpha$, where $(k - 1)/k < \alpha \leqslant 1$, are examples of the spaces $d(w, 1/k)$ which verify the conditions of Theorem 6.3. Indeed, if $\alpha = 1$, we take in Proposition 3.7, $\gamma = 0$, $b = 1$ and $n_j = j$ for every $j \in \mathbf{N}$. Hence $d(w, 1/k) \not\subset l_1$.

If $(k-1)/k < \alpha < 1$, let $0 < \gamma < 1/(k-1)$, $n_j = j^{1/\alpha}$ for every $j \in \mathbf{N}$, and $b = \sup_j[(j+1)^{1/\alpha} - j^{1/\alpha}]/j^\gamma < +\infty$. Then, by Proposition 3.7, it follows that $d(w, 1/k) \not\subset l_1$. Suppose now again that $\alpha = 1$ and put

$$v_i = \begin{cases} 1 & \text{if } i \leqslant [e^{k-1}] + 1, \\ \dfrac{(\log i)^{k-1}}{i} & \text{if } i > [e^{k-1}] + 1 \end{cases}$$

where $[e^{k-1}]$ is the entire part of e^{k-1}. Then

$$\sum_{i=1}^n v_i \sim \int_3^n \frac{(\log x)^{k-1}}{x}\, dx \sim (\log n)^k \sim \left(\sum_{i=1}^n w_i\right)^k \quad \text{for every } n.$$

Moreover $\lim_n v_n = 0$ and $v_n \geqslant v_{n+1}$ for every $n \in \mathbf{N}$.

If $(k-1)/k < \alpha < 1$, then we put $v_n = 1/n^\beta$, where $\beta = k(\alpha - (k-1)/k) < 1$ for every $n \in \mathbf{N}$. Clearly $\lim_n v_n = 0$ and $(v_n)_{n=1}^\infty$ is a decreasing sequence. Moreover

$$\sum_{i=1}^n v_i \sim \int_1^n \frac{dx}{x^\beta} \sim n^{1-\beta} \sim n^{k(1-\alpha)} \sim \left(\sum_{i=1}^n w_i\right)^k \quad \text{for every } n \in \mathbf{N}.$$

(2) We do not know if the condition (6.8) is superfluous.

Let us mention the following problem:

Problem 4. Let $p = q/k$, $1 < q < k$. Is there a positive decreasing sequence $(v_i)_{i=1}^\infty \in c_0 \setminus l_1$ such that $\widetilde{d(w,p)} \approx d(v, q)$?

More generally:

Problem 4a. Let $0 < p < 1$, $p \neq 1/k$ for any $k \in \mathbf{N}$. Is $\widetilde{d(w,p)}$ reflexive?

Remark that a positive answer to Problem 4a implies a positive answer to Problem 3 for $p \neq 1/k$.

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