TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 264, Number 2, April 1981

THE $\aleph_2$-SOUSLIN HYPOTHESIS

BY

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ABSTRACT. We prove the consistency with CH that there are no $\aleph_2$-Souslin trees.

The $\aleph_2$-Souslin hypothesis, $SH_{\aleph_2}$, is the statement that there are no $\aleph_2$-Souslin trees. In Mitchell's model [5] from a weakly compact the stronger statement holds (Mitchell and Silver) that there are no $\aleph_2$-Aronszajn trees, a property which implies that $2^{\aleph_0} > \aleph_1$.

THEOREM. Con(ZFC + *there is a weakly compact cardinal*) *implies*

$$\mathrm{Con}(ZFC + 2^{\aleph_0} = \aleph_1 + SH_{\aleph_2}).$$

In the forcing extension, $2^{\aleph_1}$ is greater than $\aleph_2$, and can be arbitrarily large. Analogues of this theorem hold with $\aleph_2$ replaced by the successor of an arbitrary regular cardinal. Strengthenings and problems are given at the end of the paper.

Let $\mathfrak{M}$ be a ground model in which κ is a weakly compact cardinal. The extension which models $SH_{\aleph_2}$ and CH is obtained by iteratively forcing $\geqslant \kappa^+$ times with certain κcc, countably closed partial orders, taking countable supports in the iteration. For $\alpha \geqslant 1$, $(\mathcal{P}_\alpha, \leqslant)$ is the ordering giving the first α steps in the iteration. $\mathcal{P}_\alpha$ is a set of functions with domain α.

Let $L_{\aleph_1\kappa}$ be the Levy collapse by countable conditions of each $\beta \in [\aleph_1, \kappa)$ to $\aleph_1$ (so κ is the new $\aleph_2$). Then $\mathcal{P}_1$ (isomorphic to $L_{\aleph_1\kappa}$) is $\{f\colon \operatorname{dom} f = 1, f(0) \in L_{\aleph_1\kappa}\}$, ordered by $f \leqslant g$ iff $f(0) \leqslant g(0)$. To define $\mathcal{P}_{\beta+1}$, choose a term A_β in the forcing language of $\mathcal{P}_\beta$ for a countably closed partial ordering (to be described later) and let $\mathcal{P}_{\beta+1} = \{f\colon \operatorname{dom} f = \beta + 1,\ f \restriction \beta \in \mathcal{P}_\beta,\ f \restriction \beta \Vdash_{\mathcal{P}_\beta} f(\beta) \in A_\beta\}$, ordered by $f \leqslant g$ iff $f \restriction \beta \leqslant g \restriction \beta$ and $g \restriction \beta \Vdash_{\mathcal{P}_\beta} f(\beta) \leqslant g(\beta)$. For α a limit ordinal, $\mathcal{P}_\alpha = \{f\colon \operatorname{dom} f = \alpha,\ f \restriction \beta \in \mathcal{P}_\beta$ for all $\beta < \alpha$, and $f(\beta)$ is (the term for) $\varnothing$, the least element of A_β, for all but $\leqslant \aleph_0$ β's$\}$, ordered by $f \leqslant g$ iff for all $\beta < \alpha$, $f \restriction \beta \leqslant g \restriction \beta$.

Each $\mathcal{P}_\alpha$ is countably closed. We are done as in Solovay-Tennenbaum [7] if the A_β's can be chosen so that each $\mathcal{P}_\alpha$ has the κcc, and therefore that every $\aleph_2$ $(= \kappa)$-Souslin tree which crops up gets killed by some A_β.

If T is a tree then $(T)_\lambda$ is the λth level of T, $(T)_{<\lambda} = \bigcup_{\mu<\lambda} T_\mu$. Regarding the previous problem, it is a theorem of Mitchell that if CH and $\diamond\{\alpha < \omega_2\colon cf(\alpha) = \aleph_1\}$ hold, then there are countably closed $\aleph_2$-Souslin trees T_n, $n < \omega$, such that for

Received by the editors October 18, 1978.

1980 *Mathematics Subject Classification*. Primary 02K35, 04A20.

[1]Supported by NSF grant MCS-76-06942.

[2]Supported by NSF grant MCS-76-08479.

each $m < \omega$, $\bigotimes_{n<m} T_n$ has the $\aleph_2 cc$, but $\bigotimes_{n<\omega} T_n$ does not have the $\aleph_2 cc$. We give for interest his proof modulo the usual Jensen methods. At stage $\mu < \omega_2$ construct each $(T_n)_\mu$ normally above $(T_n)_{<\mu}$. If $\mu = \nu + 1$ let each $x \in (T_n)_\nu$ have at least two successors in $(T_n)_\mu$. If $cf(\mu) = \omega$ let all branches in $(T_n)_{<\mu}$ go through. If $cf(\mu) = \omega_1$ make sure that the antichain given by the $\diamond$-sequence for $\bigotimes_{n<m_\mu} T_n$ is taken care of, and choose $\langle c_{\mu n}: n < \omega \rangle \in \bigotimes_{n<\omega} (T_n)_\mu$ so that if $\mu' < \mu$, $cf(\mu') = \omega_1$, then $\langle c_{\mu' n}: n < \omega \rangle \nleq \langle c_{\mu n}: n < \omega \rangle$. We also carry along the following induction hypothesis: if $\nu < \mu$, $\langle x_n: n < \omega \rangle \in \bigotimes_{n<\omega} (T_n)_\nu$, $m < \omega$, $\langle y_n: n < m \rangle \in \bigotimes_{n<m} (T_n)_\mu$, $x_n < y_n$ $(n < m)$ and $\langle x_n: n < \omega \rangle \ngeq \langle c_{\lambda n}: n < \omega \rangle$, for all $\lambda \leqslant \nu$ with $cf(\lambda) = \omega_1$, then there are $y_n \in (T_n)_\mu$ $(m \leqslant n < \omega)$ with $x_n < y_n$, such that $\langle y_n: n < \omega \rangle \ngeq \langle c_{\lambda n}: n < \omega \rangle$, for all $\lambda \leqslant \mu$ with $cf(\lambda) = \omega_1$.

If δ is inaccessible, then forcing with $L_{\aleph_1 \delta}$ (whence $2^{\aleph_0} = \aleph_1$, $2^{\aleph_1} = \aleph_2 = \delta$, and $\diamond\{\alpha < \omega_2: cf(\alpha) = \omega_1\}$ hold) followed by forcing with the $\bigotimes_{n<\omega} T_n$ constructed previously, gives a countably closed length ω iteration of countably closed, δcc partial orderings which does not have δcc.

The previous theorem does not rule out that an iteration of $\aleph_2$-Souslin trees can give CH and $SH_{\aleph_2}$; in this paper, though, the $\aleph_2$-Souslin trees are killed by a different method. Let T be an $\aleph_2$-Souslin tree (we may assume without loss of generality that T is normal and $\mathrm{Card}(T)_1 = \aleph_1$). The antichain partial order A_T is defined to be $(\{x \subseteq T: x$ a countable antichain, root $T \notin x\}, \subseteq)$. Now A_T need not have the $\aleph_2 cc$, as shown by the following result of the first author: $\mathrm{Con}(ZFC)$ implies $\mathrm{Con}(ZFC +$ "there is an $\aleph_2$-Souslin tree T and a sequence $\langle d_{\alpha n}: n < \omega \rangle$ from $(T)_\alpha$, for each $\alpha < \omega_2$, such that if $\alpha < \beta$, there is an $m < \omega$ with $d_{\alpha n} < d_{\beta n}$, for all $n > m$".) Namely, start with a model of CH. Determine in advance that, say, $(T)_\alpha = [\omega_1 \alpha, \omega_1(\alpha + 1))$ and that $d_{\alpha n} = \omega_1 \alpha + n$. Conditions are countable subtrees S of T such that if $S \cap (T)_\alpha \neq \varnothing$ then $\{d_{\alpha n}: n < \omega\} \subseteq S$, which meet the requirements on the $d_{\beta n}$'s.

Devlin **[2]** has shown that such a tree exists in L.

We show now that if each A_β is an A_T, T an $\aleph_2$-Souslin tree, then each $\mathcal{P}_\alpha$ has the κcc, which will prove the theorem (we actually just use that Card $T <$ the cardinal designated as the new $2^{\aleph_1}$ and T has no ω_2-paths; see remarks at the end). This theorem was originally proved by the first author when κ is measurable; that the assumption can be weakened to weak compactness of κ is due to the second author.

We consider now only the case $\alpha \leqslant \kappa^+$ (which will suffice, assuming $2^\kappa = \kappa^+$ in $\mathfrak{M}$, for $CH + SH_{\aleph_2} + 2^{\aleph_1} = \aleph_3$); α arbitrary will be dealt with at the end.

Fix α for the rest of the proof. We assume by induction that

(1) For each $\beta < \alpha$, $\mathcal{P}_\beta$ has the κcc.

(One more induction hypothesis is listed later.)

For $\beta < \alpha$, let T_β be the βth $\aleph_2$-Souslin tree, so $\mathcal{P}_{\beta+1} = \mathcal{P}_\beta \tilde{\otimes} A_\beta$, where $A_\beta = A_{T_\beta}$. Assume without loss of generality that for each $\lambda < \kappa$,

$$(T_\beta)_\lambda \subseteq [\omega_1 \lambda, \omega_1(\lambda + 1)).$$

An $f \in \mathcal{P}_\beta$, $\beta \leqslant \alpha$, is said to be determined if there is in $\mathfrak{M}$ a sequence $\langle z_\gamma: \gamma \in \mathrm{dom} f - \{0\} \rangle$ of countable sets of ordinals such that for all $\gamma \in \mathrm{dom} f - \{0\}$,

$f \restriction \gamma \Vdash_{\mathcal{P}_\gamma} f(\gamma) = z_\gamma$. If $\langle f_n : n < \omega \rangle$ is a sequence of determined members of $\mathcal{P}_\beta$, with $f_n \leqslant f_{n+1}$, then the coordinatewise union f_ω of the f_n's is seen to be a determined member of $\mathcal{P}_\beta$ extending each f_n. From this it may be seen, by induction on $\beta \leqslant \alpha$, that the set of determined members of $\mathcal{P}_\beta$ is cofinal in $\mathcal{P}_\beta$. Redefine each $\mathcal{P}_\beta$ then to consist just of the determined conditions. Clearly Card $\mathcal{P}_\beta \leqslant \kappa$, for all $\beta \leqslant \alpha$.

For $f, g \in \mathcal{P}_\beta, f \sim g$ means that f and g are compatible.

Fix for the rest of the proof a one-one enumeration $\alpha = \{\alpha_\mu : \mu \in S\}$, for some $S \subseteq \kappa$ (this induces a similar enumeration of each $\beta < \alpha$, the induction hypothesis (2) for β below, is with respect to this induced enumeration). For notational simplicity we now assume that S is some $\kappa' \leqslant \kappa$.

If $\lambda < \kappa$, $\beta \leqslant \alpha, f \in \mathcal{P}_\beta$, define $f|\lambda$ to be the function h with domain β such that $h(\gamma) = \varnothing$ unless $\gamma \in \{\alpha_\mu : \mu < \lambda\} \cap \beta$, in which case,

$$\gamma = 0 \Rightarrow h(\gamma) = f(\gamma) \restriction (\omega_1 \times \lambda), \qquad \gamma > 0 \Rightarrow h(\gamma) = f(\gamma) \cap \lambda.$$

The function $f|\lambda$ need not be a condition, but for $g \in \mathcal{P}_\beta$, we will still write $f|\lambda \leqslant g$ to mean that $f|\lambda$ is coordinatewise a subset of g. Let $\mathcal{P}_\beta|\lambda = \{f \in \mathcal{P}_\beta : f|\lambda = f\}$.

Suppose $0 < \beta \leqslant \alpha$, $\lambda < \kappa$. Define

$$\#_\lambda^\beta(f, g, h) \Leftrightarrow f, g \in \mathcal{P}_\beta, f|\lambda = g|\lambda = h,$$

$$*_\lambda^\beta(f, h) \Leftrightarrow f \in \mathcal{P}_\beta, h \in \mathcal{P}_\beta|\lambda \text{ and for every } h' \geqslant h \text{ with } h' \in \mathcal{P}_\beta|\lambda,\ h' \sim f,$$

$$*_\lambda^\beta(f, g, h) \Leftrightarrow *_\lambda^\beta(f, h) \text{ and } *_\lambda^\beta(g, h).$$

For $P \subseteq Q$, Q a partial ordering, $P \subseteq_{\text{reg}} Q$ means that P is a regular subordering of Q, that is, any two members of P compatible in Q are compatible in P, and every maximal antichain of P is a maximal antichain of Q. If $\mathcal{P}_\beta|\lambda \subseteq_{\text{reg}} \mathcal{P}_\beta$, then $*_\lambda^\beta(f, h)$ states that $h \Vdash_{\mathcal{P}_\beta|\lambda} [\![f]\!] \neq 0$.

Recall that the sets of the form $\{\lambda < \kappa : (R_\lambda, \in, A \cap R_\lambda) \models \Phi\}$, where $A \subseteq \kappa$, Φ is π_1^1, and $(R_\kappa, \in, A) \models \Phi$, belong to a normal uniform filter $\mathcal{F}_{wc}$, the weakly compact filter on κ (see **[9]**, **[0]**). The second thing we assume by induction is

(2) for all $\beta < \alpha$, for $\mathcal{F}_{wc}$-almost all $\lambda < \kappa$, for all f, g, h, $\#_\lambda^\beta(f, g, h)$ implies that for some $h' \geqslant h$, $*_\lambda^\beta(f, g, h')$.

If $\beta < \alpha$, $\lambda < \kappa$, say that $(T_\beta)_{<\lambda}$ is determined by $\mathcal{P}_\beta|\lambda$ if for each θ, τ in $(T_\beta)_{<\lambda}$ there is a $\mathcal{P}_\beta$-maximal antichain R of conditions deciding the ordering between θ and τ in T_β, such that $R \subseteq \mathcal{P}_\beta|\lambda$.

LEMMA 1. *There is a closed unbounded set of $\lambda < \kappa$ such that for all $\mu < \lambda$, $(T_{\alpha_\mu})_{<\lambda}$ is determined by $\mathcal{P}_{\alpha_\mu}|\lambda$.*

PROOF. This is a consequence of the strong inaccessibility of κ and the assumption that each $\mathcal{P}_\beta$, $\beta < \alpha$, has κcc.

LEMMA 2. *For $\mathcal{F}_{wc}$-almost all $\lambda < \kappa$,*

(a) λ *is strongly inaccessible.*

(b) *For all* $\mu < \lambda$, $\mathcal{P}_{\alpha_\mu}|\lambda$ *has the* λcc.

(c) *For all* $\mu < \lambda$, $\mathcal{P}_{\alpha_\mu}|\lambda \subseteq_{\mathrm{reg}} \mathcal{P}_{\alpha_\mu}$.

(d) *For all* $\mu < \lambda, \Vdash_{\mathcal{P}_{\alpha_\mu}|\lambda} \lambda = \aleph_2$.

(e) *For all* $\mu < \lambda, \Vdash_{\mathcal{P}_{\alpha_\mu}|\lambda} (T_{\alpha_\mu})_{<\lambda}$ *is an* $\aleph_2$*-Souslin tree.*

PROOF. By π_1^1 reflection and the normality of $\mathcal{F}_{wc}$.

LEMMA 3. *Let* $\beta \leqslant \alpha, \lambda < \kappa, \mathcal{P}_\beta|\lambda \subseteq_{\mathrm{reg}} \mathcal{P}_\beta$.

(a) *If* $f \in \mathcal{P}_\beta, j \in \mathcal{P}_\beta|\lambda$, *and* $f \sim j$, *then there is an* $h \geqslant j$ *with* $*_\lambda^\beta(f, h)$.

(b) *If* $*_\lambda^\beta(f, g, h)$ *and D, E are cofinal subsets of* $\mathcal{P}_\beta$, *then there exists* $\langle f', g', h'\rangle \geqslant \langle f, g, h\rangle$ *with* $*_\lambda^\beta(f', g', h'), f' \in D, g' \in E, h \leqslant f', g'$.

PROOF. These are standard facts about forcing.

The following is T. Carlson's version of the lemma we originally used here.

LEMMA 4. *Suppose* λ *satisfies Lemma* 1 *and Lemma* 2(c), $\mu < \lambda$, *and* $*_\lambda^{\alpha_\mu}(f, h)$. *Then* $f|\lambda \leqslant h$.

PROOF. Otherwise there is a $\nu < \lambda$ with $\alpha_\nu < \alpha_\mu$, and a $\theta \in f(\alpha_\nu) \cap \lambda$ such that $\theta \notin h(\alpha_\nu)$. We have that $h \restriction \alpha_\nu \Vdash_{\mathcal{P}_{\alpha_\nu}|\lambda} \theta$ is T_{α_ν}-incomparable with each member of $h(\alpha_\nu)$; otherwise $*_\lambda^{\alpha_\mu}(f, h)$ would be contradicted. Pick an $h' \in \mathcal{P}_{\alpha_\nu}|\lambda$, $h' \geqslant h \restriction \alpha_\nu$, and a $\theta' < \lambda$ such that $h' \Vdash_{\mathcal{P}_{\alpha_\nu}|\lambda} \theta <_{T_{\alpha_\nu}} \theta'$. Let $\bar{h}$ be $h' {}^\frown \langle h(\alpha_\nu) \cup \{\theta'\}\rangle {}^\frown h \restriction [\alpha_\nu + 1, \alpha_\mu)$. Then $\bar{h} \in \mathcal{P}_{\alpha_\mu}|\lambda$, $h \leqslant \bar{h}$, and $\bar{h} \nsim f$, a contradiction.

LEMMA 5. *Suppose* λ *satisfies Lemma* 1 *and Lemma* 2(c), $\mu < \lambda$, *and* $*_\lambda^{\alpha_\mu}(f, g, h)$. *Then there is an* $(f', g', h') \geqslant (f, g, h)$ *with* $\#_\lambda^{\alpha_\mu}(f', g', h')$.

PROOF. Choose $(f, g, h) = (f_0, g_0, h_0) \leqslant \cdots \leqslant (f_n, g_n, h_n) \leqslant \cdots$ so that $*_\lambda^{\alpha_\mu}(f_n, g_n, h_n)$, $h_n \leqslant f_{n+1}$, $h_n \leqslant g_{n+1}$. This is done by repeated applications of Lemma 3(a), (b). Then Lemma 4 implies that the coordinatewise union (f', g', h') of the (f_n, g_n, h_n)'s is as desired.

DEFINITION. Suppose $\lambda < \kappa$, $\mu < \lambda$, $f, g \in \mathcal{P}_{\alpha_\mu}$, and suppose θ, τ are nodes of $(T_{\alpha_\mu})_{\geqslant\lambda}$ ($\theta = \tau$ allowed). Then $\langle f, g\rangle$ is said to λ-separate $\langle \theta, \tau\rangle$ if there is a $\gamma < \lambda$ and $\theta', \tau' \in (T_{\alpha_\mu})_\gamma$, with $\theta' \neq \tau'$, such that

$$f \Vdash_{\mathcal{P}_{\alpha_\mu}} \theta' <_{T_{\alpha_\mu}} \theta, \qquad g \Vdash_{\mathcal{P}_{\alpha_\mu}} \tau' <_{T_{\alpha_\mu}} \tau.$$

LEMMA 6. *Suppose* λ *satisfies Lemmas* 1 *and* 2, $\mu < \lambda$, $*_\lambda^{\alpha_\mu}(f, g, h)$, $\{\theta, \tau\} \subseteq (T_{\alpha_\mu})_{\geqslant\lambda}$, *with* $\theta = \tau$ *allowed. Then there is an* $\langle f', g', h'\rangle \geqslant \langle f, g, h\rangle$ *such that* $*_\lambda^{\alpha_\mu}(f', g', h')$ *and* $\langle f', g'\rangle$ λ*-separates* $\langle \theta, \tau\rangle$.

PROOF.

Claim. There are $f_0, f_1 \geqslant f$, $\bar{h} \geqslant h$, with $*_\lambda^{\alpha_\mu}(f_0, \bar{h})$, $*_\lambda^{\alpha_\mu}(f_1, \bar{h})$, such that $\langle f_0, f_1\rangle$ λ-separates $\langle \theta, \theta\rangle$ via a $\langle \theta_1, \theta_2\rangle \in (T_{\alpha_\mu})_\gamma$, for some $\gamma < \lambda$.

PROOF. Consider the result of taking a generic set $G_{\alpha_\mu}|\lambda$ over $\mathcal{P}_{\alpha_\mu}|\lambda$ which contains h. In $\mathfrak{M}[G_{\alpha_\mu}|\lambda]$, $(T_{\alpha_\mu})_{<\lambda}$ is a λ ($= \aleph_2$)-Souslin tree. In the further extension $\mathfrak{M}[G_{\alpha_\mu}]$, θ determines a λ-path through $(T_{\alpha_\mu})_{<\lambda}$. Since this path is not in $\mathfrak{M}[G_{\alpha_\mu}|\lambda]$, there must be $\bar{h} \in G_{\alpha_\mu}|\lambda$, $\bar{h} \geqslant h$, $f_0, f_1 \geqslant f$, $\gamma < \lambda$, $\theta_0, \theta_1 \in (T_{\alpha_\mu})_\gamma$, $\theta_0 \neq \theta_1$, with

$\bar{h}\Vdash f_0\Vdash \theta_0 <_{T_{\alpha_\mu}} \theta$, $\bar{h}\Vdash f_1\Vdash \theta_1 <_{T_{\alpha_\mu}} \theta$, such that $*^{\alpha_\mu}_\lambda(f_0, \bar{h})$ and $*^{\alpha_\mu}_\lambda(f_1, \bar{h})$. This gives the claim.

Now, by Lemma 3, choose $(g', h') \geqslant (g, h)$ and a $\tau' \in (T_{\alpha_\mu})_\gamma$ so that $*^{\alpha_\mu}_\lambda(g', h')$ and $g'\Vdash \tau' <_{T_{\alpha_\mu}} \tau$. Pick $i \in \{0, 1\}$ with $\tau' \neq \theta_i$. Let $f' = f_i$, $\theta' = \theta_i$. Then (f', g', h') are as desired. This proves the lemma.

We claim that the induction hypotheses (1) and (2) automatically pass up to α if $cf(\alpha) > \omega$. Namely, (1) holds at α by a Δ-system argument. For (2), suppose that for an $\mathcal{F}_{wc}$-positive set W of λ's there is a counterexample $\langle f_\lambda, g_\lambda, h_\lambda\rangle$. Let $N_\lambda = (\text{support}\, f_\lambda \cup \text{support}\, g_\lambda)$. If $cf(\alpha) \neq \kappa$ then for some $\beta < \alpha$ and $\mathcal{F}_{wc}$-positive $V \subseteq W$, $\lambda \in V$ implies $N_\lambda \subseteq \beta$, and we are done. If $cf(\alpha) = \kappa$, pick a closed unbounded set $C \subseteq \kappa$ such that $\langle \sup\{\alpha_\nu: \nu < \lambda\}: \lambda \in C\rangle$ is increasing, continuous and cofinal in α and an $\mathcal{F}_{wc}$-positive $V \subseteq W \cap C$ such that for some $\beta < \alpha$ and all $\lambda \in V$, $N_\lambda \cap \sup\{\alpha_\nu: \nu < \lambda\} \subseteq \beta$, then apply (2) at β.

Thus, we may assume for the rest of the proof that α is a successor ordinal or $cf(\alpha) = \omega$. Fix $\langle \mu_n: n < \omega\rangle$ such that if $\alpha = \beta + 1$ then each μ_n is the μ with $\alpha_\mu = \beta$, and if $cf(\alpha) = \omega$ then $\langle \alpha_{\mu_n}: n < \omega\rangle$ is an increasing sequence converging to α.

LEMMA 7. *For $\mathcal{F}_{wc}$-almost all λ, the following holds: if $f, g \in \mathcal{P}_\alpha$, $h \in \mathcal{P}_\alpha|\lambda$ and $\#^\alpha_\lambda(f, g, h)$ then there exists $\langle f', g', h'\rangle \geqslant \langle f, g, h\rangle$ such that $\#^\alpha_\lambda(f', g', h')$ and such that for each $\mu < \lambda$ with $\alpha_\mu \neq 0$, and each $\theta \in f'(\alpha_\mu) - \lambda$, each $\tau \in g'(\alpha_\mu) - \lambda$, $\langle f' \restriction \alpha_\mu, g' \restriction \alpha_\mu\rangle$ λ-separates $\langle \theta, \tau\rangle$.*

PROOF. We prove the lemma for λ, assuming that λ satisfies Lemmas 1 and 2, $\lambda > \mu_n$ $(n < \omega)$ and for each $n < \omega$, λ is in the $\mathcal{F}_{wc}$ set given by induction hypothesis (2) for α_{μ_n}. Construct $\langle f_n, g_n, h_n\rangle$, $n < \omega$, so that

(a) $f_n, g_n \in \mathcal{P}_{\alpha_{\mu_n}}$, $\#^{\alpha_{\mu_n}}_\lambda(f_n, g_n, h_n)$,

(b) $\langle f \restriction \alpha_{\mu_n}; g \restriction \alpha_{\mu_n}, h \restriction \alpha_{\mu_n}\rangle \leqslant \langle f_n, g_n, h_n\rangle$,

(c) $\langle f_n, g_n, h_n\rangle \leqslant \langle f_{n+1}, g_{n+1}, h_{n+1}\rangle$,

(d) if, at stage $n > 1$, $\langle \theta_n, \tau_n\rangle$ is the nth pair (in the appropriate bookkeeping list for exhausting them) with $\theta_n \in f_n(\alpha_{\nu_n}) - \lambda$, $\tau_n \in g_n(\alpha_{\nu_n}) - \lambda$, $\nu_n < \lambda$, $\alpha_{\nu_n} \leqslant \alpha_{\mu_n}$, then

$$\langle f_n \restriction \alpha_{\nu_n}, g_n \restriction \alpha_{\nu_n}\rangle \text{ λ-separates } \langle \theta, \tau\rangle.$$

Let $f_0 = f \restriction \alpha_{\mu_0}$, $g_0 = g \restriction \alpha_{\mu_0}$, $h_0 = h \restriction \alpha_{\mu_0}$. Suppose $n > 1$ and f_{n-1}, g_{n-1}, h_{n-1} have been constructed. Let

$$f'_n = f_{n-1}{}^\frown f \restriction [\alpha_{\mu_{n-1}}, \alpha_{\mu_n}), \qquad g'_n = g_{n-1}{}^\frown g \restriction [\alpha_{\mu_{n-1}}, \alpha_{\mu_n}),$$
$$h'_n = h_{n-1}{}^\frown h \restriction [\alpha_{\mu_{n-1}}, \alpha_{\mu_n}).$$

Then $\#^{\alpha_{\mu_n}}_\lambda(f'_n, g'_n, h'_n)$. By induction hypothesis (2), there is an $\bar{h}_n \geqslant h'_n$ such that $*^{\alpha_\mu}_\lambda(f'_n, g'_n, \bar{h}_n)$. By Lemma 6, there is $\langle f''_n, g''_n, h''_n\rangle \geqslant \langle f'_n, g'_n, \bar{h}_n\rangle$ such that $*^{\alpha_{\mu_n}}_\lambda(f''_n, g''_n, h''_n)$ and

$$\langle f''_n \restriction \alpha_{\nu_n}, g''_n \restriction \alpha_{\nu_n}\rangle \text{ separates } \langle \theta_n, \tau_n\rangle.$$

Finally, by Lemma 5 we may choose $\langle f_n, g_n, h_n\rangle \geqslant \langle f''_n, g''_n, h''_n\rangle$ so that $\#^{\alpha_{\mu_n}}_\lambda(f_n, g_n, h_n)$.

Taking f', g', h' to be the coordinatewise unions of the f_n's, g_n's, h_n's gives the lemma.

We now verify the two induction hypotheses.

(1) $\mathcal{P}_\alpha$ has the κcc.

PROOF. Given $f_\lambda \in \mathcal{P}_\alpha$, $\lambda < \kappa$. For each λ which satisfies Lemmas 1, 2 and 7, with $\lambda > \mu_n$ $(n < \omega)$, apply Lemma 7 to the triple $\langle f_\lambda, f_\lambda, f_\lambda|\lambda\rangle$, obtaining a triple $\langle f^*_\lambda, f^{**}_\lambda, j_\lambda\rangle$ (so $f_\lambda \leqslant f^*_\lambda, f^{**}_\lambda, j_\lambda = f^*_\lambda|\lambda = f^{**}_\lambda|\lambda$).

Let

$$B_\lambda = (\text{support}\, f^*_\lambda \cup \text{support}\, f^{**}_\lambda) \cap \{\alpha_\mu : \mu < \lambda\}.$$

If $0 \neq \alpha_\mu \in B_\lambda$, write

$$f^*_\lambda(\alpha_\mu) - \lambda = \{\theta_{\mu\lambda n}: n < r_{\mu\lambda}\}, \qquad r_{\mu\lambda} \leqslant \omega,$$

$$f^{**}_\lambda(\alpha_\mu) - \lambda = \{\tau_{\mu\lambda m}: m < s_{\mu\lambda}\}, \qquad s_{\mu\lambda} \leqslant \omega.$$

To each pair $\langle \theta_{\mu\lambda n}, \tau_{\mu\lambda m}\rangle$, $n < r_{\mu\lambda}$, $m < s_{\mu\lambda}$, $\langle f^*_\lambda, f^{**}_\lambda\rangle$ assigns a separating pair $\langle \theta'_{\mu\lambda n}, \tau'_{\mu\lambda m}\rangle \in \lambda \times \lambda$.

Let $J_\lambda = (\text{dom}\, f^*_\lambda(0) \cup \text{dom}\, f^{**}_\lambda(0)) - (\omega_1 \times \lambda)$.

By the normality of $\mathcal{F}_{wc}$, there is an $\mathcal{F}_{wc}$-positive set U such that on U, the sets B_λ, $r_{\mu\lambda}$, $s_{\mu\lambda}$, $\theta'_{\mu\lambda n}$, $\tau'_{\mu\lambda m}$, J_λ are independent of λ, and such that if $\lambda, \lambda' \in U$, $\lambda < \lambda'$, then (support $f^*_\lambda \cup$ support f^{**}_λ) $\cap$ (support $f^*_{\lambda'} \cup$ support $f^{**}_{\lambda'}$) $= B_\lambda$, and $J_\lambda \cap J_{\lambda'} = \varnothing$.

By induction on $\gamma \leqslant \alpha$ it is seen that if $\lambda, \mu \in U$ and $\lambda < \mu$, then $f^*_\lambda \sim f^{**}_\mu$. Namely, there is no trouble with coordinates in the support of at most one of these functions; coordinates in both supports, being in B_λ, are taken care of by the construction. Since $f_\lambda \leqslant f^*_\lambda$ and $f_\mu \leqslant f^{**}_\mu$, we are done.

The following strengthening of κcc for P_α has thus been proved: if for an $\mathcal{F}_{wc}$-positive set W of λ's, $\#^\alpha_\lambda(f_\lambda, g_\lambda, h_\lambda)$, then there is an $\mathcal{F}_{wc}$-positive $U \subseteq W$ and $(f'_\lambda, g'_\lambda, h'_\lambda)$, $\lambda \in U$, such that $\langle f_\lambda, g_\lambda, h_\lambda\rangle \leqslant \langle f'_\lambda, g'_\lambda, h'_\lambda\rangle$, $\#^\alpha_\lambda(f'_\lambda, g'_\lambda, h'_\lambda)$, and so that if $\lambda, \mu \in W$, $\lambda < \mu$, then $f'_\lambda \sim g'_\mu$ in the strong sense that the coordinatewise union of f'_λ and g'_μ is a condition extending both f'_λ and g'_μ.

Lastly, we prove the second induction hypothesis for α.

(2) For $\mathcal{F}_{wc}$-almost all $\lambda < \kappa$, for all f, g, h, $\#^\alpha_\lambda(f, g, h)$ implies that for some $h' \geqslant h$, $*^\alpha_\lambda(f, g, h')$.

PROOF. Otherwise for an $\mathcal{F}_{wc}$-positive set W of λ's there exists a counterexample $\langle f_\lambda, g_\lambda, h_\lambda\rangle$. We may assume that for each $\lambda \in W$, $\lambda > \mu_n$ $(n < \omega)$ and λ satisfies Lemmas 1, 2 and 7. Furthermore, since we have already proved that $\mathcal{P}_\alpha$ has the κcc, we may assume that for each $\lambda \in W$, $\mathcal{P}_\alpha|\lambda \subseteq_{\text{reg}} \mathcal{P}_\alpha$ and $\mathcal{P}_\alpha|\lambda$ has the λcc. If f_λ or g_λ equals h_λ we are done, so assume, for each $\lambda \in W$, that $f_\lambda, g_\lambda \notin \mathcal{P}_\alpha|\lambda$.

Apply Lemma 7 to each $\langle f_\lambda, g_\lambda, h_\lambda\rangle$, $\lambda \in W$, getting $\langle f'_\lambda, g'_\lambda, h'_\lambda\rangle$. Now uniformize as in part (a) to get an $\mathcal{F}_{wc}$-positive $V \subseteq W$ such that if $\lambda, \mu \in V$ and $\lambda < \mu$ then $f'_\lambda \sim g'_\mu$. Since $\langle f_\lambda, g_\lambda, h_\lambda\rangle$ is a counterexample to (b), there is a maximal antichain H_λ of $\{h \in \mathcal{P}_\alpha|\lambda: h \geqslant h_\lambda\}$ such that for each $h \in H_\lambda$, $h \nsim f_\lambda$ or $h \nsim g_\lambda$. Then H_λ is a maximal antichain of $\{h \in \mathcal{P}_\alpha: h \geqslant h_\lambda\}$, and Card $H_\lambda < \lambda$. Pick an $\mathcal{F}_{wc}$-positive $U \subseteq V$ on which $H_\lambda = H$ is independent of λ and such that for each $h \in H$, the questions, whether or not $h \sim f_\lambda$, $h \sim g_\lambda$, are independent of λ. Pick

$\lambda, \mu \in U, \lambda < \mu$, and let $j \geqslant f'_\lambda, g'_\mu$. Now $j \geqslant h_\lambda$, and $j \notin \mathcal{P}_\alpha|\lambda$ (whence $j \notin H$). But for each $h \in H$, either $h \not\sim g_\lambda$ (whence $h \not\sim g_\mu$) or $h \not\sim f_\lambda$. In either case, $h \not\sim j$ since $j \geqslant f_\lambda, g_\mu$, so H is not maximal, a contradiction.

This completes the proof of the theorem.

Denote by an ω_2-tree a tree T of any cardinality with no paths of length ω_2. An ω_2-tree T is special if there is an $f\colon T \to \omega_1$ such that $x <_T y$ implies $f(x) \neq f(y)$. By the previous methods, using countable specializing functions instead of countable antichains, the consistency of "$2^{\aleph_0} = \aleph_1$, $2^{\aleph_1} > \aleph_2$, and every ω_2-tree of cardinality $< 2^{\aleph_1}$ is special" is obtained – the analogous theorem for the $\aleph_1$ case being Baumgartner-Malitz-Reinhardt [**1**]. We can also get this model to satisfy the "generalized Martin's axioms" (which are consistent relative to just ZFC but which do not imply $SH_{\aleph_2}$) that have been considered by the first author and by Baumgartner (see Tall [**8**]). Desirable, of course, would be the consistency of a generalized MA which is both simple and powerful.

The partial orderings appropriate for the prior methods can be iterated an arbitrary number of times, giving generalized MA models in which $2^{\aleph_1}$ is arbitrarily large. The ordering $\mathcal{R}_\alpha$ giving the first α steps of the iteration need not be of cardinality $\leqslant \kappa$, but, assuming each $\mathcal{R}_\beta$, $\beta < \alpha$, has κcc, any sequence $\langle p_\lambda\colon \lambda < \kappa\rangle$ from $\mathcal{R}_\alpha$ is a subset of a sufficiently closed model of power κ, in which the proof that two p_λ's are compatible can be carried out.

Regarding the analog of these results where $\aleph_2$ is replaced by γ^+–the relevant forcing is γ-directed closed, so by upward Easton forcing we may guarantee that, for example, γ remains supercompact if it was in the ground model.

For results involving consequences of $SH_{\aleph_2}$: with GCH, see Gregory [**3**], [**4**] (Con($SH_{\aleph_2}$ and GCH) is open); with just CH, see a forthcoming paper by Stanley and the second author.

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