UNIQUE MINIMALITY OF FOURIER PROJECTIONS[1]

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ABSTRACT. The question of when the Fourier projection is the only one of least norm from a space of continuous functions on the circle onto spaces spanned by trigonometric polynomials is studied in two settings. In the first the domain space is the disc algebra and the range is finite-dimensional. In the second the domain space consists of all real continuous functions and the range has finite codimension.

Introduction.

General background. Let $\mathcal{F}_n$ be the Fourier projection of the continuous (real or complex) periodic functions, $C_{2\pi}$, onto π_n the trigonometric polynomials of degree less than or equal to n. One of the extremal properties of $\mathcal{F}_n$ is that it has the minimal operator norm along all the projections (i.e. idempotent operators) of $C_{2\pi}$ onto π_n (Lozinski [8]). The numbers $\|\mathcal{F}_n\|$ are known to be unbounded. A consequence of this, for example, is that for any sequence of projections $\{L_n\}$ onto π_n–such as interpolation at $n + 1$ points–there is an f in $C_{2\pi}$ and a point x such that $(L_n f)(x)$ is unbounded (Cheney [2]).

The minimal norm property of $\mathcal{F}_n$ follows from an averaging process due to Berman [1] which shows that for any projection P,

$$\mathcal{F}_n f = \frac{1}{2\pi} \int_{-\pi}^{\pi} T_{-\lambda} \circ P \circ T_\lambda f \, d\lambda,$$

where $T_\lambda f(x) = f(x + \lambda)$. This equation is also attributed to Marcinkiewicz [9].

The Berman formula immediately shows more than we stated. Let Γ be any set of integers and π_Γ the closed linear span of $\{e^{ikx}: k \in \Gamma\}$. Let E be a subspace of $C_{2\pi}$ which contains π_Γ and which is the closed linear span of those functions e^{inx} contained in it. The Fourier projection is then the (obviously unique if it exists) bounded projection $\mathcal{F}_\Gamma$ of E onto π_Γ such that

$$(\mathcal{F}_\Gamma p)(t) = \sum_{j \in \Gamma} \hat{p}(j) e^{ijt}$$

for all trigonometric polynomials p in E. In this setting Berman's formula obtains (since its right-hand side satisfies the defining properties for $\mathcal{F}_\Gamma$) and so if P is any projection of E onto π_Γ then $\|\mathcal{F}_\Gamma\| \leq \|P\|$.

More recently it was found that, for the classical case, $\mathcal{F}_n$ is in fact the only projection of minimal norm [3]. It is the extension of this uniqueness property to

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other basic settings that is our interest in this study. It is not always true that $\mathscr{F}_\Gamma$ is the unique minimal norm projection [6]. Specifically if $\Gamma = \{0, \pm 3, \pm 6, \pm 9, \pm 12\}$ and $Pf = \mathscr{F}_\Gamma([1 + \cos x]f(x))$ then $\|P\| = \|\mathscr{F}_\Gamma\|$. For that matter it is not always true that there are any bounded projections of $C_{2\pi}$ onto π_Γ. From Berman's formula, the Riesz representation theorem, and P. J. Cohen's theorem on idempotent measures (actually Helson's earlier result for our special case) it follows that π_Γ is complemented if and only if Γ is in the Boolean algebra generated by groups of integers and their cosets.

Summary of results. This study separates into two distinct settings. One considers projections of the complex analytic function on the disc onto a subspace π_Γ where Γ is a finite set of positive integers. It is shown that the Fourier projection is the unique minimal norm projection onto the span of $\{e^{ikx} : 0 \leqslant k \leqslant n\}$.

The other setting involves only real functions, and has a complete characterization for uniqueness of minimal norm projections onto translation invariant subspaces of finite codimension. That is, in this setting, Γ contains all but a finite number of integers and their negatives. For example, similar to the classical settings, uniqueness is obtained when $\mathbf{Z} - \Gamma = \{0, \pm 1, \pm 2, \ldots, \pm n\}$. But contrary to the finite analog we also have uniqueness when $\mathbf{Z} - \Gamma = \{0, \pm 3, \ldots, \pm 3n\}$. The minimal norm projection is not unique, for example, when $\mathbf{Z} - \Gamma = \{0, \pm 2, \pm 3, \pm 4\}$. This seems to be the first counterexample that is not of the type described in the beginning paragraphs above.

Part I. Minimal projections of analytic functions

Let Γ be a finite set of nonnegative integers and let π_Γ be the linear span of $\{e^{ijx} : j \in \Gamma\}$. Let A be the space of functions f which are continuous on the closed unit disc $\Delta = \{z : |z| \leqslant 1\}$ and analytic on the interior of Δ. The restriction of A to the unit circle T is a closed subspace of $C(T)$ which we again denote by A; A is the uniform closure of $\{e^{ikx} : k \geqslant 0\}$. The Fourier projection $\mathscr{F}_\Gamma$ maps A onto π_Γ by the rule

$$(\mathscr{F}_\Gamma f)(e^{ix}) = \sum_{j \in \Gamma} \hat{f}(j)e^{ijx}, \tag{1.1}$$

where

$$\hat{f}(j) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{ix})e^{-ijx}\,dx = f^{(j)}(0)/j!, \qquad j = 0, 1, \ldots. \tag{1.2}$$

We shall show in this part of the paper that when $\Gamma = \{0, \ldots, n\}$, $\mathscr{F}_\Gamma$ is the unique minimal norm projection of A onto π_Γ. Unless otherwise indicated all sums in this part of the paper are over the set Γ. §1 of this part closely follows the early development in [3].

1. Necessary conditions for minimality. In this section we make no special assumptions about Γ. Let P be a projection from A onto π_Γ with minimal norm. Since P is a projection it has the representation

$$(Pf)(e^{ix}) = \sum e^{ijx}\int_T f(e^{it})\,d\mu_j(t), \tag{1.3}$$

where μ_j is a finite regular Borel measure on the unit circle T. We note that μ_j is unique up to adding an element of H_0^1, the annihilator of A in the space of measures; see [4] for a discussion of this and related facts. Now write $d\mu_j = g_j \, dt + d\rho_j$ where $g_j \in L^1$ and ρ_j is singular with respect to Lebesgue measure. Berman's equation is

$$\mathscr{F}_\Gamma = \frac{1}{2\pi} \int_T T_{-s} P T_s \, ds, \tag{1.4}$$

where T_s is the rotation operator: $(T_s f)(e^{ix}) = f(e^{i(x+s)})$. Let F be a function in H^∞ which solves the extremal problem

$$\alpha = \max\left\{ \left| \sum f^{(j)}(0)/j! \right| : f \in H^\infty, \|f\|_\infty \leqslant 1 \right\}. \tag{1.5}$$

F is unique if we specify that $\sum F^{(j)}(0)/j!$ be positive. It is known that F is a finite Blaschke product with n or fewer zeros in $\{|z| < 1\}$; see [4, 8.4]. Then F (or a rotation of F) is the extremal function for $\mathscr{F}_\Gamma$:

$$\|\mathscr{F}_\Gamma\| = \|\mathscr{F}_\Gamma(F)\|_\infty = \mathscr{F}_\Gamma(F)(1)$$

since $\mathscr{F}(F)(1) = \sum F^{(j)}(0)/j! = \sum \hat{F}(j)$. We rewrite Berman's equation in the form

$$\mathscr{F}_\Gamma = \frac{\alpha + \pi}{2\pi} \left[\frac{1}{\alpha + \pi} \int_{-\pi}^{\alpha} T_{-s} P T_s \, ds \right] + \frac{\pi - \alpha}{2\pi} \left[\frac{1}{\pi - \alpha} \int_{\alpha}^{\pi} T_{-s} P T_s \, ds \right]$$

$$= \theta P_1 + (1 - \theta) P_2,$$

where $0 < \theta < 1$ and P_1, P_2 are minimal projections onto π_Γ. We now let P stand for either P_1 or P_2. For simpler notation we let $\hat{x}$ be the linear functional defined by $\hat{x}f = f(e^{ix})$, $-\pi \leqslant x \leqslant \pi$. Hence we have

$$\|\mathscr{F}_\Gamma\| = \hat{x}\mathscr{F}_\Gamma T_{-x}(F) = \theta \hat{x} P_1 T_{-x}(F) + (1 - \theta)\hat{x} P_2 T_{-x}(F)$$

$$\leqslant \theta \|P_1\| + (1 - \theta)\|P_2\| = \|\mathscr{F}_\Gamma\|.$$

Thus, $\hat{x} P T_{-x}(F) = \|\mathscr{F}_\Gamma\|$ for each x. Let $H(x, e^{it})$ be the best H_0^1 approximation to $\sum g_j(t) e^{ijx}$ for each x in $[-\pi, \pi]$. Then

$\|\hat{x}P\|$ = distance from $\sum e^{ijx} \, d\mu_j(t)$ to H_0^1 in space of measures

$\qquad = L^1$ distance from $\sum g_j(t)e^{ijx}$ to H_0^1 plus measure norm of $\sum e^{ijx} \, d\rho_j(t)$

$\qquad = \hat{x} P T_{-x}(F)$

$\qquad = \int F(e^{i(t-x)}) \left[\sum g_j(t) e^{ijx} + H(x, e^{it}) \right] dt$

$\qquad\quad + \int F(e^{i(t-x)}) \left[\sum e^{ijx} \, d\rho_j(t) \right]$

$\qquad \leqslant \left\| \sum g_j(t) e^{ijx} + H(x, e^{it}) \right\|_{L^1(dt)} + \left\| \sum e^{ijx} \, d\rho_j(t) \right\|$

$\qquad = \|\hat{x}P\|.$

Thus, we conclude that

$$0 \leqslant F(e^{i(t-x)}) \left[\sum g_j(t) e^{ijx} + H(x, e^{it}) \right] \quad \text{a.e. } dt \tag{1.6}$$

and

$$F(e^{i(t-x)}) \sum e^{ijx} \, d\rho_j(t) \text{ is a nonnegative measure.} \qquad (1.7)$$

Equations (1.6) and (1.7) hold for every $x \in [-\pi, \pi]$. We shall use equations (1.6) and (1.7) to show that $g_j(t) = e^{-ijt}$ and $\rho_j = 0$ for all j. We make mention here of the fact that since P is a projection

$$\int_T e^{ikt} \, d\mu_j(t) = \delta_{jk} \quad \text{for } j, k \in \Gamma. \qquad (1.8)$$

2. Analysis of the minimal kernel. Associated with the extremal problem (1.5) for bounded holomorphic functions is an extremal kernel K. In the case $\Gamma = \{0, \ldots, n\}$ we have both of the following formulas for K:

$$\begin{cases} \text{(a)} \quad K(e^{it}) = \sum_{j=0}^{n} e^{-ijt} + J(e^{it}), \\[2mm] \text{(b)} \quad K(z) = z^{-n} \prod_{j=1}^{n} (1 - \alpha_j z)^2, \quad z \in \mathbf{C}. \end{cases} \qquad (1.9)$$

Here J is the best H_0^1 approximation to $\sum e^{-ijt}$ and $0 < |\alpha_j| < 1$, $1 \le j \le n$ (the latter is shown in §3; see equation (11), p. 138 and Example (i), p. 139, of [4]). We also know from the basic theory of extremal problems (again see [4, Chapter 8]) that

$$0 \le F(e^{it})K(e^{it}) \quad \text{for all } t$$

so that in our case

$$0 \le e^{-in\theta}F(e^{i\theta}) \prod_{1}^{n} (1 - \alpha_j e^{i\theta})^2 \quad \text{a.e. } d\theta. \qquad (1.10)$$

We now exploit equations (1.6) and (1.10) to reach the desired conclusions. We first note that

$$0 \le e^{int}e^{-inx}G(e^{it}e^{-ix}) \Big[\sum g_j(t)e^{ijx} + H(x, e^{it}) \Big], \qquad (1.11)$$

where we have written

$$G(z) = 1 / \prod_{1}^{n} (1 - \alpha_j z)^2. \qquad (1.12)$$

Multiply equation (1.11) by $e^{irt} + e^{-irt}$ for $0 \le r \le n$ and then integrate on t. The result is a real-valued function of the variable e^{ix}. The integral

$$\int_0^{2\pi} e^{int}(e^{irt} + e^{-irt})G(e^{it}e^{-ix})H(x, e^{it}) \, dt$$

vanishes identically in x since H lies in H_0^1, since G is analytic for $|z| < 1$, and since $0 \le r \le n$. Replacing x by $-x$ in (1.11) we see that the function

$$e^{inx} \int_0^{2\pi} e^{int}(e^{irt} + e^{-irt})G(e^{it}e^{ix}) \Big[\sum g_j(t)e^{-ijx} \Big] \, dt,$$

which is real-valued, has only *nonnegative* powers of e^{ix} and hence must be identically constant. The same holds if we multiply (1.11) by $e^{irt} - e^{-irt}$ for

$0 \leqslant r \leqslant n$ (except the resulting function is purely imaginary). Hence, we find that

$$w_r(e^{ix}) = \int_0^{2\pi} e^{int} e^{irt} G(e^{it}e^{ix}) \left[\sum g_j(t) e^{i(n-j)x} \right] dt \qquad (1.13)$$

is identically constant for $r = 0, \pm 1, \ldots, \pm n$. Now

$$G(e^{i\theta}) = \sum_{\nu=0}^{\infty} C_\nu e^{i\nu\theta},$$

where the series is absolutely convergent since $G(z)$ is holomorphic on $|z| < 1$. Substituting this into the expression (1.13) for w_r and remembering that w_r is a constant, we have for $m = 1, 2, \ldots$ and $r = 0, \pm 1, \ldots, \pm n$,

$$0 = \sum_{\nu+n-j=m} C_\nu \int_T g_j(t) e^{int} e^{irt} e^{i\nu t} \, dt. \qquad (1.14)$$

Set $C_\nu = 0$ if $\nu < 0$; then (1.14) may be rewritten as

$$0 = \int_T \left(\sum C_{m+j-n} g_j(t) e^{ijt} \right) e^{imt} e^{irt} \, dt, \qquad m = 1, 2, \ldots, \ r = 0, \pm 1, \ldots, \pm n.$$
$$(1.15)$$

To make the notation even simpler, set $g_j \equiv 0$ if $j \notin \Gamma$; then

$$0 = \int_T \left(\sum_{j=0}^{2n-1} C_{m-n+j} g_j(t) e^{ijt} \right) e^{imt} e^{irt} \, dt. \qquad (1.16)$$

Fix $M \geqslant n + 1$; set $m = M + s$ and $r = -s$ for $s = 0, \pm 1, \ldots, \pm n$. Thus

$$0 = \int_T \left(\sum_{j=0}^{2n-1} C_{M-n+s+j} g_j(t) e^{ijt} \right) e^{iMt} \, dt, \qquad s = 0, \pm 1, \ldots, \pm n.$$

We now refer to Proposition 2 in §3 to conclude that the Hankel matrix $\| C_{p+i+k} \|_{i,k=0}^{2n-1}$, $p = -n + 1, -n + 2, \ldots$, has a nonzero determinant. Thus, if $J \in \Gamma$ is fixed, there are scalars $A_{-n}, \ldots, A_{n-1}$ with

$$\sum_{s=-n}^{n-1} A_s C_{M-n+s+j} = \begin{cases} 1, & j = J, \\ 0, & j \neq J. \end{cases}$$

Hence,

$$0 = \int_T \left(g_J(t) e^{iJt} \right) e^{iMt} \, dt, \qquad J \in \Gamma, \qquad (1.17)$$

for each $M \geqslant n + 1$. This argument may be repeated with $\sum e^{ijx} \, d\rho_j(t)$ in place of $\sum e^{ijx} g_j(t) + H(x, e^{it})$; the conclusion reached is that

$$0 = \int_T e^{iJt} e^{iMt} \, d\rho_J(t), \qquad M \geqslant n + 1, J \in \Gamma.$$

Hence, the measure $e^{iJt} e^{int} \, d\rho_J(t)$ is absolutely continuous with respect to Lebesgue measure by the F. and M. Riesz theorem [4, p. 41]. Since this measure is also singular with respect to Lebesgue measure it must vanish and hence $\rho_J = 0$ for $J \in \Gamma$.

We now know that for each $j \in \Gamma$, $g_j(t) = R_j(e^{it}) + h_j(e^{it})$ where R_j is in the linear span of $\{ e^{-ikt} : k = 0, \ldots, 2n \}$ and h_j lies in H_0^1. Let us absorb the term

$\sum h_j(e^{it})e^{ijx}$ into $H(x, e^{it})$. We then know that

$$0 \leqslant \frac{\sum R_j(e^{it})e^{ijx} + H(x, e^{it})}{e^{it}e^{-ix}K(e^{it}e^{-ix})} \qquad \text{a.e. } dt \text{ for each } x, \qquad (1.18)$$

where K is given in (1.9). Let x be held fixed. Multiply top and bottom of (1.18) by e^{2int} and make use of (1.9) to obtain

$$0 \leqslant \frac{S(e^{it}) + H_1(e^{it})}{e^{int}Q(e^{it})} \qquad \text{a.e. } dt, \qquad (1.18)'$$

where

$$S(e^{it}) = \sum_j e^{2int}R_j(e^{it})e^{ijx},$$

$$H_1(e^{it}) = e^{2int}H(x, e^{it}),$$

$$Q(e^{it}) = e^{inx}\prod_1^n \left(1 - \alpha_j e^{it}e^{-ix}\right)^2.$$

Now each of S and Q is a polynomial in e^{it} of degree $2n$ or less and H_1 is in H_0^1 so that the Schwarz reflection principle immediately implies that $(S(z) + H_1(z))/z^nQ(z)$ is a rational function of z and thus the same holds for $H_1(z)$. Consequently, $S + H_1$ is rational on the sphere and holomorphic on $|z| < 1$ while Q is a polynomial of degree $2n$ with zeros of order 2 at the points $e^{ix}/\alpha_1, \ldots, e^{ix}/\alpha_n$, $0 < |\alpha_j| < 1$. Consider the rational function

$$z^n \frac{\bar{S}(1/\bar{z}) + \bar{H}_1(1/\bar{z})}{\bar{Q}(1/\bar{z})},$$

where the bar denotes complex conjugation. When $|z| = 1$ this rational function equals the complex conjugate of $(S(z) + H_1(z))/z^nQ(z)$. But the latter is real-valued by (1.18)'. Hence these two rational functions coincide. Thus,

$$z^{2n}\left[\bar{S}(1/\bar{z}) + \overline{H}_1(1/\bar{z})\right]/\left[S(z) + H_1(z)\right] = \bar{Q}(1/\bar{z})/Q(z) \qquad (1.19)$$

for all z. The right-hand side of (1.19) has a pole of order two at e^{ix}/α_j for $1 \leqslant j \leqslant n$. Since $0 < |\alpha_j| < 1$, we see that $S + H_1$ must have a zero of order two at e^{ix}/α_j. Hence, $S + H_1 = MQ$, where M is rational, holomorphic on $|z| < 1$ (and depends on x). Thus, from (1.19) we obtain

$$M(z) = z^{2n}\overline{M}(1/\bar{z}) \qquad (1.20)$$

so that M is a polynomial of degree at most $2n$ of the form

$$M(z) = M(z, x) = c_n(x)z^n + \sum_{k=0}^{n-1} \left\{c_k(x)z^k + \bar{c}_k(x)z^{2n-k}\right\},$$

where c_n is real. We shall show that $c_n = 1$ and $c_k = 0$ for $k = 0, \ldots, n - 1$.

Returning to the definitions of S, H_1, and Q we find that

$$e^{int}\left\{\sum_{j=0}^n R_j(e^{it})e^{ijx} + H(x, e^{it})\right\} = e^{-int}e^{inx}M(e^{it}, x)\prod_1^n \left(1 - \alpha_j e^{it}e^{-ix}\right)^2.$$

$$(1.21)$$

Let r be an integer, $0 \leqslant r \leqslant n$, and refer to (1.13). We then compute

$$w_r(e^{ix}) = \int_0^{2\pi} e^{int} e^{irt} G(e^{it} e^{ix}) \left\{ \sum_{j=0}^{n} R_j(e^{it}) e^{i(n-j)x} \right\} dt$$

$$= \int_0^{2\pi} e^{irt} e^{-int} M(e^{it}, -x) \, dt$$

$$= c_{n-r}(-x), \qquad 0 \leqslant r \leqslant n.$$

But w_r is constant. Thus, c_k is constant for $k = 0, \ldots, n$. Now

$$w_r = \frac{1}{2\pi} \int_0^{2\pi} w_r(e^{ix}) \, dx = \int_0^{2\pi} e^{int} e^{irt} R_n(e^{it}) G(0) \, dt$$

$$= \begin{cases} 0, & 0 < r \leqslant n, \\ 1, & r = 0, \end{cases}$$

by (1.8). Thus, $M(e^{it}, x) = e^{int}$. Next refer to (1.21) and conclude that

$$\sum_{j=0}^{n} R_j(e^{it}) e^{ijx} + H(x, e^{it}) = e^{-int} e^{inx} \prod_{j=1}^{n} \left(1 - \alpha_j e^{it} e^{-ix}\right)^2. \qquad (1.22)$$

However, the right-hand side of (1.22) is precisely $K(e^{it} e^{-ix})$ by (1.9)(b). Thus, using (1.9)(a) we have

$$\sum_{j=0}^{n} R_j(e^{it}) e^{ijx} + H(x, e^{it}) = \sum_{j=0}^{n} e^{-ijt} e^{ijx} + J(e^{it} e^{-ix}).$$

Since both $H(x, e^{it})$ and $J(e^{it} e^{-ix})$ lie in H_0^1 for each x as functions of e^{it}, we see that

$$\sum_{j=0}^{n} R_j(e^{it}) e^{ijx} = \sum_{j=0}^{n} e^{-ijt} e^{ijx} \quad \text{for all } t, x.$$

Finally, a comparison of Fourier coefficients of e^{ijx} yields $R_j(e^{it}) = e^{-ijt}$, the desired conclusion.

3. Details.

PROPOSITION 1. *Let* $(\sum_{k=0}^{n} z^k)^{1/2} = \sum_0^{\infty} \lambda_j z^j$ *for* $|z|$ *small. Then* $0 < \lambda_n < \cdots < \lambda_0$.

PROOF. $\sum_0^n z^k = (1 - z^{n+1})(1 - z)^{-1}$ so that $(\sum_0^n z^k)^{1/2} = (1 - z)^{-1/2} + o(z^n)$ for $|z|$ small. However, $(1 - z)^{-1/2} = \sum_{j=0}^{\infty} A_j z^j$ where

$$A_\nu = \frac{1}{\nu!} \frac{(2\nu - 1) \ldots (3)(1)}{2^\nu}, \qquad \nu = 1, 2, 3 \ldots,$$

$$A_0 = 1$$

and $\lambda_\nu = A_\nu$ for $0 \leqslant \nu \leqslant n$.

COROLLARY. *If* $\Gamma = \{0, 1, 2, \ldots, n\}$, *then* K *has the form* (1.9) *and hence* $\mathscr{F}_\Gamma$ *is the unique projection onto* π_Γ *of minimal norm.*

PROOF. According to the discussion on p. 140 of [4] it is sufficient to show that if

$$\left(\sum_{j \in \Gamma} z^{n-j}\right)^{1/2} = \sum_0^{\infty} \lambda_j z^j \quad \text{for } |z| \text{ small}$$

then $P_n(z) = \lambda_0 + \lambda_1 z + \cdots + \lambda_n z^n$ does not vanish in $|z| \leq 1$. However, $\Sigma_\Gamma z^{n-j}$ $= \Sigma_\Gamma z^j$ and so $P_n(z) = \Sigma_{k=0}^n \lambda_k z^k$, where $\lambda_0, \ldots, \lambda_n$ are determined by

$$\left(\sum_{k=0}^n z^k \right)^{1/2} = \lambda_0 + \lambda_1 z + \cdots + \lambda_n z^n + \ldots .$$

According to Proposition 1, $1 = \lambda_0 > \cdots > \lambda_n > 0$ and so P_n has no zeros in $|z| \leq 1$ by the well-known condition of Eneström-Kakeya (see [10, p. 136]).

PROPOSITION 2. *Let* $0 < |a_k| < 1$ *for* $k = 1, \ldots, n$ *and set*

$$G(z) = \prod_1^n (1 - a_k z)^{-2} = \sum_{\nu=0}^\infty C_\nu z^\nu, \qquad |z| < 1.$$

Let $C_\nu = 0$ *if* $\nu < 0$. *Then for* $p = -2n + 1, -2n + 2, \ldots$ *the* $(2n) \times (2n)$ *matrix* $\|C_{p+j+k}\|_{j,k=0}^{2n-1}$ *has rank* $2n$.

PROOF. Replace z by $1/z$ in the expansion of G; then

$$\frac{z^{2n-1}}{\prod_1^n (z - a_k)^2} = \sum_0^\infty C_\nu / z^{\nu+1}, \qquad |z| > 1.$$

If $p = 0, 1, 2, \ldots$, then

$$\sum_{\nu=0}^\infty \frac{C_{p+\nu}}{z^{\nu+1}} = z^p \frac{z^{2n-1}}{\prod_1^n (z - a_k)^2} - \sum_{j=0}^{p-1} C_j z^{p-1-j}. \tag{1.23}$$

The right-hand side of (1.23) is a rational function with precisely $2n$ poles. If $p = -1, -2, \ldots, -2n + 1$, then we have

$$\frac{C_0}{z^{1-p}} + \frac{C_1}{z^{2-p}} + \ldots = \frac{z^{2n-1+p}}{\prod_1^n (z - \alpha_k)^2}. \tag{1.24}$$

The right-hand side of (1.24) is also a rational function with precisely $2n$ poles. In both cases, then, we may conclude from [5, Theorem 8, p. 207] that the infinite Hankel matrix $\|C_{p+j+k}\|_{j,k \geq 0}$ has rank $2n$ and then from [5, Corollary, p. 206] that

$$\det \|C_{p+j+k}\|_{j,k=0}^{2n-1} \neq 0.$$

(We wish to thank Professor R. G. Douglas for suggesting [5] as a source of information on the rank of Hankel matrices.)

EXAMPLE 1. NONUNIQUENESS. The uniqueness of $\mathcal{F}_\Gamma$ as a projection of minimal norm can fail.

Let Γ be the multiples of $r > 1$: $\Gamma = \{0, r, 2r, \ldots, mr\}$. Let H_0 be the best H_0^1 approximation to $D(e^{i\theta}) = \Sigma_{k=0}^m e^{-irk\theta}$; note H_0 is a function of $r\theta$. Let $0 < l < r$ and set

$$(\tilde{P}f)(e^{ix}) = \int_\Gamma (1 + \cos l\theta) \big[D(\theta - x) + H_0(\theta - x) \big] f(e^{i\theta}) \, d\theta.$$

Since $(\cos l\theta) e^{ir\theta} f(e^{i\theta})$ lies in H^∞, it follows that P has range in π_Γ. It also follows that $\tilde{P}$ is a projection not equal to $\mathcal{F}_\Gamma$. Finally,

$$\|\tilde{P}\| \leq \max_x \int |1 + \cos l\theta| \, |D(\theta - x) + H_0(\theta - x)| \, d\theta = \|\mathcal{F}_\Gamma\|$$

since $1 + \cos l\theta \geq 0$ and $\cos l\theta$ convolved with a function of $r\theta$ is identically zero.

4. The major result. Let Γ be a finite set of integers symmetric about 0 and let $\Lambda = \mathbf{Z} \setminus \Gamma$. Define π_Γ and π_Λ to be the closed real linear span of $\{\cos jt, \sin jt: j \in \Gamma\}$ and $\{\cos jt, \sin jt: j \in \Lambda\}$, respectively. In this part of the paper we provide a characterization of these (symmetric) sets Λ for which the Fourier projection $\mathcal{F}_\Lambda$ is the unique minimal norm projection of $C(T)$ onto π_Λ. In this part all functions considered are real-valued.

DEFINITION. An element $g \in \pi_\Gamma$ is *determined by its zeros* if, whenever h is an element of π_Γ and h/g is continuous on T, then h is a constant multiple of g. That is, if $\lambda_1, \ldots, \lambda_m$ is a listing of the zeros of g, counting multiplicities, then the only elements of π_Γ which vanish at $\lambda_1, \ldots, \lambda_m$ are the constant multiples of g. We remark also that g being determined by its zeros is equivalent to: whenever h is in π_Γ with $\operatorname{sgn}(g(s) + \delta h(s)) = \operatorname{sgn} g(s)$ a.e.$-ds$, for $|\delta| \leqslant 1$, then h is a constant multiple of g.

Set

$$D_\Gamma(t) = \sum_{j \in \Gamma} e^{ijt}. \tag{2.1}$$

The major result of this part is the following.

THEOREM. *Let Γ be a finite set of integers, symmetric about 0. The Fourier projection $\mathcal{F}_\Lambda$ is the unique minimal norm projection of $C(T)$ onto π_Λ if and only if D_Γ is determined by its zeros.*

PROOF. Let us assume first that D_Γ is *not* determined by its zeros. We will show that it is possible to construct a projection P onto π_Λ with $\|P\| = \|\mathcal{F}_\Lambda\|$ but yet $P \neq \mathcal{F}_\Lambda$. The assumption that D_Γ is not determined by its zeros implies that the subspace of π_Γ consisting of functions which vanish at the zeros of D_Γ, including multiplicities, is at least two dimensional. Hence, there is a function $h \in \pi_\Gamma$ with h/D_Γ continuous on T and

$$\int_T h \operatorname{sgn} D_\Gamma = 0. \tag{2.2}$$

Consequently,

$$\operatorname{sgn}(D_\Gamma + \varepsilon h) = \operatorname{sgn} D_\Gamma \tag{2.3}$$

if $|\varepsilon| < \|h/D_\Gamma\|_\infty^{-1}$. Thus, for the same range of ε,

$$\int |D_\Gamma + \varepsilon h| = \int (D_\Gamma + \varepsilon h)\operatorname{sgn}(D_\Gamma + \varepsilon h) = \int |D_\Gamma| \tag{2.4}$$

by (2.2) and (2.3). Hence, $\|D_\Gamma + \varepsilon h\|_1 = \|\mathcal{F}_\Gamma\|$ for all small ε. Now choose a nonzero function $\varphi \in C(T)$ with

$$\int \varphi(s)e^{i(n-m)s}\, ds = 0, \quad \text{for all } n, m \in \Gamma. \tag{2.5}$$

There is no loss in assuming $\|\varphi\|_\infty < \|h/D_\Gamma\|_\infty^{-1}$. It follows from (2.5) that

$$\int \varphi(s)e^{-iks}h(s-t)\, ds = 0 \tag{2.6}$$

for all t and all $k \in \Gamma$. Now define an operator P by

$$(Pf)(s) = f(s) - \int \left[D_\Gamma(s - t) + \varphi(s)h(s - t) \right] f(t) \, dt$$

$$= (\mathcal{F}_\Lambda f)(s) - \varphi(s) \int h(s - t)f(t) \, dt.$$

We must show that P is a projection onto π_Λ and $\|P\| = \|\mathcal{F}_\Lambda\|$. First suppose $k \in \Gamma$; then

$$(Pf)\hat{\ }(k) = \int (Pf)(s)e^{-iks} \, ds$$

$$= -\int f(t) \int \varphi(s)h(s - t)e^{-iks} \, ds \, dt$$

$$= 0$$

by (2.6). Thus, the range of P lies in π_Λ. Next, let $f = e^{ims}$ for some $m \in \Lambda$. Then

$$Pf = \mathcal{F}_\Lambda f - \varphi(s) \int h(s - t)e^{imt} \, dt = f$$

since $h \in \pi_\Gamma$. Consequently, P is a projection onto π_Λ. Finally, we compute the norm of P:

$$\|P\| \leqslant 1 + \sup_{s \in [0, 2\pi]} \int |D_\Gamma(s - t) + \varphi(s)h(s - t)| \, dt$$

$$= 1 + \|\mathcal{F}_\Gamma\| = \|\mathcal{F}_\Lambda\|.$$

The first equality results from (2.4) and the second from the observation that if S is any finite rank operator on a space $C(X)$, with X perfect, then

$$\|I + S\| = 1 + \|S\|. \tag{2.7}$$

This, in turn, follows readily from the fact that S has a representation $Sf = \Sigma \varphi_i(f)f_i$, for some φ_i's in $C(X)^*$, f_i's in $C(X)$. We shall have further use of (2.7) in what follows. This concludes the proof that if $\mathcal{F}_\Lambda$ is the unique minimal norm projection onto π_Λ then D_Γ is determined by its zeros.

Now for the proof that if D_Γ is determined by its zeros then $\mathcal{F}_\Lambda$ is unique. Using Berman's formula (see the Introduction for an explanation of the validity of Berman's formula in this setting) as in §1 we obtain that if $\mathcal{F}$ is not uniquely minimal then it is not an extreme point of the convex set of projections of least norm. We show that this contradicts the assumption that D_Γ is determined by its zeros.

Since $\mathcal{F}_\Lambda$ is not extreme there exists a nonzero operator $Q: C_{2\pi} \to C_{2\pi}$ such that $P_\delta = \mathcal{F}_\Lambda - \delta Q$ is a projection of least norm onto π_Λ for all δ in $[-1, 1]$. For each such δ, define

$$L_\delta = I - P_\delta = \mathcal{F}_\Gamma + \delta Q.$$

Now observe that Q is an operator with range in π_Λ such that

    (i) Q annihilates π_Λ;

    (ii) Q has a kernel representation constructed as follows. For $j \in \Gamma$, let $h_j = Q(e^{ijx})$. By (i), if $f \in C_{2\pi}$,

$$Qf(t) = Q(\mathscr{F}_\Gamma f)(t) = Q\left(\sum_\Gamma \hat{f}(j)e^{ijx}\right)(t)$$

$$= \int f(s)\sum_\Gamma e^{-ijs}\, h_j(t)ds.$$

Then $H(s, t) = \sum_\Gamma e^{-ijs}h_j(t)$ is the desired kernel. It follows now that if we define $K_\delta(s, t) = D_\Gamma(t - s) + \delta H(s, t)$, $|\delta| \leqslant 1$, we have $L_\delta f(t) = \int f(s)K_\delta(s, t)ds$. Now L_δ has finite rank so, by (2.7), $\|L_\delta\| = \|P_\delta\| - 1 = \|\mathscr{F}_\Lambda\| - 1 = \|\mathscr{F}_\Gamma\|$. Also we obviously have $\|L_\delta\| = \sup_t \int|K_\delta(s, t)|ds$. Now define $\sigma(s) = \operatorname{sgn} D_\Gamma(s)$. Then we have

$$\|\mathscr{F}_\Gamma\| = \int |D_\Gamma(t - s)|ds = \int \sigma(t - s)D_\Gamma(t - s)ds,$$

for all t. Note that

$$\int H(s, t)\sigma(t - s)ds = 0, \quad \text{for all } t.$$

It follows that, for all t,

$$\|\mathscr{F}_\Gamma\| \geqslant \int |K_\delta(s, t)|ds \geqslant \int K_\delta(s, t)\sigma(t - s)ds = \|\mathscr{F}_\Gamma\|,$$

so that

$$\|\mathscr{F}_\Gamma\| = \int |K_\delta(s, t)|ds = \int \sigma(t - s)K_\delta(s, t)ds.$$

Hence, for all t, $\operatorname{sgn} K_\delta(s, t) = \sigma(t - s)$ a.e.–ds. By the remark made following the definition of "determined by its zeros", it follows that, for each t, there is a constant c_t such that

$$H(s, t) = c_t D_\Gamma(t - s), \quad \text{for all } s.$$

But then

$$|1 + \delta c_t|\int |D_\Gamma(t - s)|ds = \int |K_\delta(s, t)|ds$$

$$= \int |D_\Gamma(t - s)|ds,$$

implying that $c_t = 0$ for all t. This means that $Q = 0$, which is a contradiction. The proof is now complete.

It is interesting that the condition on D_Γ of being determined by its zeros has been shown to be sufficient for the unique minimality of the Fourier projection from $L^1[-\pi, \pi]$ onto π_Γ [7].

COROLLARY. *Let m be the number of zeros of D_Γ, counting multiplicities. If $m < \dim \pi_\Gamma$, then $\mathscr{F}_\Lambda$ is not the unique minimal norm projection onto π_Λ.*

5. Some examples.

EXAMPLE 2. NONUNIQUENESS. Let $\Gamma = \{0, \pm 2, \pm 3, \pm 4\}$; we will show that D_Γ is not determined by its zeros by showing that D_Γ has only 4 zeros, counting multiplicities. Since π_Γ is 7 dimensional there will be many elements of π_Γ which have the same zeros as D_Γ, including multiplicities, but which are not constant

multiples of D_Γ. Now with $x = \cos t$

$$D_\Gamma(t) = 1 + 2T_2(x) + 2T_3(x) + 2T_4(x) = p(x),$$

where T_j is the jth Chebyshev polynomial of the first kind. Hence,

$$\frac{1}{2}p(x) = 8x^4 + 4x^3 - 6x^2 - 3x + \frac{1}{2}.$$

Some computation shows that p has only 2 roots in $[-1, 1]$ and that p' is not zero at these roots. Thus, D_Γ has just 4 zeros in $[0, 2\pi)$.

EXAMPLE 3. UNIQUENESS. Here are some sets Γ for which D_Γ is determined by its zeros.

(a) $\Gamma = \{0, \pm 1, \ldots, \pm n\}$; here $D_\Gamma(t) = D_n(t) = \sin((n + 1/2)t)/\sin(t/2)$ has $2n$ distinct zeros. Since no nonzero element of π_Γ can have more than this many zeros, D_Γ is determined by its zeros.

(b) Fix an integer $r \geqslant 2$ and let $\Gamma = \{0, \pm r, \ldots, \pm mr\}$ then $D_\Gamma(x) = D_m(rx)$. If $h \in \pi_\Gamma$, then $h(x) = Q(rx)$ for some $Q \in \pi_{\Gamma'}$ where

$$\Gamma' = \{0, \pm 1, \ldots, \pm m\}.$$

Hence, if h/D_Γ is continuous then so is Q/D_m and so $Q = \lambda D_m$, λ = scalar, by (a). Thus, $h = \lambda D_\Gamma$, and D_Γ is determined by its zeros.

(c) $\Gamma = \{-m, m\}$; here $D_\Gamma(x) = 2 \cos mx$ and so D_Γ has $2m$ distinct zeros which is the most that any element of π_Γ can have.

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