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EMBEDDING PROCESSES IN BROWNIAN MOTION IN $\mathbf{R}^n$

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ABSTRACT. We give a potential-theoretic characterization of the right-continuous processes which can be embedded in Brownian motion in $\mathbf{R}^n$ by means of an increasing family of standard stopping times. In general it is necessary to use a Brownian motion process whose filtration is richer than the natural one.

1. Introduction. In an article entitled *Potential processes*, Chacon [1] showed how potential theory may be applied to the study of discrete-time martingales, with particular emphasis on embedding theorems of Skorohod type. In this article we show how potential theory may be used to characterize those processes which can be embedded in Brownian motion in $\mathbf{R}^n$ by means of an increasing family of stopping times which are not "too big". The stopping times which are not too big are the ones we call standard, following Chacon [1]; see 2.7 and 2.8. If $n \geqslant 3$ then every stopping time is standard but if $n = 1$ or 2 there are stopping times which are not standard. Doob (see Meyer [1]) noticed that absolutely any discrete-time process on the line can be embedded in Brownian motion in $\mathbf{R}$ by means of an increasing sequence of stopping times, by a trivial construction. This illustrates the interest of considering a restricted class of stopping times. (For continuous-time processes on the line there is not so much freedom and the construction required is much more difficult–see Monroe [2]. The question of what processes can be embedded in Brownian motion in $\mathbf{R}^2$ by means of an increasing family of unrestricted stopping times has not been investigated.)

In §2 we define potential processes and standard stopping times and show that a process which can be embedded in Brownian motion in $\mathbf{R}^n$ by means of an increasing family of standard stopping times is a potential process. In §3 we show that a discrete-time potential process can be embedded in Brownian motion in $\mathbf{R}^n$ by means of an increasing sequence of standard stopping times. If $n \geqslant 2$ it is necessary to use a Brownian motion process whose filtration is richer than the natural filtration of the Brownian motion in the sense that its time 0 σ-field admits a continuously distributed random variable which is independent of the Brownian motion. For embedding continuous-time potential processes it seems to be necessary to use a more complex enlargement of the filtration of the Brownian motion. This kind of enlargement is discussed in §4 and turns out to be precisely the kind of enlargement that preserves the Markov property–see 4.8 and 4.9.

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In §5 we show that a continuous-time potential process with right-continuous sample paths can be embedded in Brownian motion in $\mathbf{R}^n$ by means of a right-continuous increasing family of standard stopping times. In this case the use of a Brownian motion process with an enriched filtration is necessary even if $n = 1$ and the process being embedded has continuous sample paths.

2. Elementary properties of potential processes. In this section we shall recall the definition of the potential of a measure on $\mathbf{R}^n$, we shall give the definition and some characterizations of n-dimensional potential processes, we shall recall the notion of standard stopping times of a Brownian motion process, and we shall show that if $(\Omega, \mathcal{B}, \mathcal{B}_t, B_t, P)$ is a Brownian motion process and if $(T(i))_{i \in I}$ is an increasing family of standard stopping times indexed by a set $I \subseteq [-\infty, \infty]$ then the process $(\Omega, \mathcal{B}, \mathcal{B}_{T(i)}, B_{T(i)}, P)$ is an n-dimensional potential process.

2.1. *Notation.* Define $\Phi\colon \mathbf{R}^n \to (-\infty, \infty]$ by

$$\Phi(x) = \begin{cases} -\frac{1}{2}|x| & \text{if } n = 1, \\ -(1/2\pi)\log\|x\| & \text{if } n = 2, x \neq 0, \\ \dfrac{1}{(n-2)\sigma_n \|x\|^{n-2}} & \text{if } n \geqslant 3, x \neq 0, \\ +\infty & \text{if } n \geqslant 2, x = 0 \end{cases}$$

where σ_n denotes the $(n-1)$-dimensional Lebesgue measure of $\{x \in \mathbf{R}^n\colon \|x\| = 1\}$. The function Φ is called the Newtonian potential kernel. It is invariant under rotations and its Laplacian is equal to $-\delta$, where δ denotes the Dirac measure. Any other locally Lebesgue integrable function on $\mathbf{R}^n$ with these two properties differs from Φ only by a constant, except on a set of Lebesgue measure zero.

2.2. DEFINITION. If μ is a measure on $\mathbf{R}^n$ then we define μU_+ and $\mu U_-\colon \mathbf{R}^n \to [0, \infty]$ by

$$\mu U_+(x) = \int \Phi^+(x - y)\, d\mu(y), \qquad \mu U_-(x) = \int \Phi^-(x - y)\, d\mu(y)$$

and we define μU, on the subset of $\mathbf{R}^n$ where μU_+ and μU_- are not both infinite, by $\mu U = \mu U_+ - \mu U_-$. We call μU *the potential of* μ. We say μ is *special* if μU is defined on all of $\mathbf{R}^n$ and is superharmonic. One can show that this happens iff μ is finite on compact sets and

$$\int |1 \wedge \Phi(x)|\, d\mu(x) < \infty.$$

In particular, if $n = 1$ then μ is special iff μ is finite and $\int |x|\, d\mu(x) < \infty$, while if $n = 2$ then μ is special iff μ is finite and $\int \log^+ \|x\|\, d\mu(x) < \infty$. If $n \geqslant 3$ then every finite measure μ on $\mathbf{R}^n$ is special (and so are many infinite ones).

If μ is a special measure on $\mathbf{R}^n$ then μ can be recovered from μU; indeed μ is minus the Laplacian of μU in the sense of Schwartz distributions.

2.3. *Notation.* The symbol ∂ will denote the point at infinity. $\mathbf{R}^n_\partial$ will denote the topological space $\mathbf{R}^n \cup \{\partial\}$ where ∂ is adjoined as an isolated point. If μ is a measure on $\mathbf{R}^n_\partial$ then μU will denote the potential of the restriction of μ to $\mathbf{R}^n$ and we shall say μ is special iff its restriction to $\mathbf{R}^n$ is special.

2.4. DEFINITION. Let I be a subset of $[-\infty, \infty]$ and let n be a positive integer. An *n-dimensional potential process with time set I* is a system $(\Omega, \mathcal{F}, \mathcal{F}_i, X_i, P)$ where:

(a) $(\Omega, \mathcal{F}, P)$ is a probability space;

(b) $(\mathcal{F}_i)$ is an increasing family of sub-σ-fields of $\mathcal{F}$ indexed by I;

(c) (X_i) is a family of $\mathbf{R}^n_\partial$-valued random variables over $(\Omega, \mathcal{F}, P)$ indexed by I;

(d) For each $i \in I$,

(i) X_i is $\mathcal{F}_i$-measurable,

(ii) law(X_i) is a special measure on $\mathbf{R}^n_\partial$,

(iii) if $n \leqslant 2$, $P(X_i = \partial) = 0$;

(e) whenever S and T are $(\mathcal{F}_i)$-stopping times taking on only finitely many values and satisfying $S \leqslant T$ we have law$(X_S)U \geqslant$ law$(X_T)U$.

With regard to (e), we remark that an $(\mathcal{F}_i)$-stopping time is assumed to take values in I and that it follows from (ii) of (d) that law(X_S) and law(X_T) are special measures. The condition (e) is of course the main one of this definition and it is the reason for calling these processes potential processes. The definition of potential processes that we have given here is the restriction to the setting of classical potential theory of a more general definition envisaged by R. V. Chacon.

2.5. PROPOSITION. *Let I be a subset of* $[-\infty, \infty]$, *let n be a positive integer, and let* $(\Omega, \mathcal{F}, \mathcal{F}_i, X_i, P)$ *be a system satisfying* (a)–(d) *of* 2.4. *Then* (a) *and* (b) *below are equivalent*:

(a) $(\Omega, \mathcal{F}, \mathcal{F}_i, X_i, P)$ *is an n-dimensional potential process.*

(b) *For each y in* $\mathbf{R}^n$, $(\Phi(X_i - y)1_{\{X_i \neq \partial\}})$ *is a supermartingale in the extended sense over* $(\Omega, \mathcal{F}, \mathcal{F}_i, X_i, P)$.

Moreover, if $n = 1$ then a third equivalent condition is

(c) (X_i) *is a martingale over* $(\Omega, \mathcal{F}, \mathcal{F}_i, P)$.

(The equivalence of (a) and (c) when $n = 1$ was established by Chacon [**1**].)

PROOF. (a) $\Rightarrow$ (b). Let $Z_i^y = \Phi(X_i - y)1_{\{X_i \neq \partial\}}$ for i in I and y in $\mathbf{R}^n$. Let i and j be in I with $i \leqslant j$ and let F be in $\mathcal{F}_i$. We wish to show that

$$\int_F Z_i^y \, dP \geqslant \int_F Z_j^y \, dP$$

for each y in $\mathbf{R}^n$. Let $S = i$ on F and j on $\Omega \setminus F$, and let $T = j$ on all of Ω. Then S and T are $(\mathcal{F}_i)$-stopping times taking on only finitely many values and $S \leqslant T$ so law$(X_S)U \geqslant$ law$(X_T)U$. But law$(X_S)U(y) = \int Z_S^y \, dP$ and similarly for T so we see that $\int_F Z_i^y \, dP \geqslant \int_F Z_j^y \, dP$ for all y in $\mathbf{R}^n$ such that law$(X_S)U(y) < \infty$. Thus if we let μ and ν be the special measures on $\mathbf{R}^n$ defined by $\mu(dx) = P(\{X_i \in dx\} \cap F)$ and $\nu(dx) = P(\{X_j \in dx\} \cap F)$ then $\mu U \geqslant \nu U$ except possibly on the polar set $\{\text{law}(X_S)U = \infty\}$. Hence $\mu U \geqslant \nu U$ on all of $\mathbf{R}^n$. That is, $\int_F Z_i^y \, dP \geqslant \int_F Z_j^y \, dP$ for all $y \in \mathbf{R}^n$, as desired.

(b) $\Rightarrow$ (a). This follows from the optional sampling theorem, on noting that if S is any $(\mathcal{F}_i)$-stopping time taking on only finitely many values and if y is any point in $\mathbf{R}^n$ then law(X_S) is special iff

$$E(\Phi^-(X_S - y)1_{\{X_S \neq \partial\}}) < \infty$$

and that in this case,

$$\operatorname{law}(X_S)U(y) = E\big(\Phi(X_S - y)1_{\{X_S \neq \partial\}}\big).$$

(c) $\Rightarrow$ (b), if $n = 1$. $P(X_i = \partial) = 0$ and $\Phi(X_i - y) = -\frac{1}{2}|X_i - y|$ so the process $(\Phi(X_i - y)1_{\{X_i \neq \partial\}})$ is a supermartingale over $(\Omega, \mathfrak{F}, \mathfrak{F}_i, P)$ by Jensen's inequality for conditional expectations.

(b) $\Rightarrow$ (c), if $n = 1$. Let i and j be in I with $i \leqslant j$ and let $F \in \mathfrak{F}_i$. Consider the measures μ and ν on $\mathbf{R}$ defined by $\mu(dx) = P(\{X_i \in dx\} \cap F)$ and $\nu(dx) = P(\{X_j \in dx\} \cap F)$. Then μ and ν have the same total mass, namely $P(F)$, and

$$\mu U(y) = \int_F \Phi(X_i - y)\, dP \geqslant \int_F \Phi(X_j - y)\, dP = \nu U(y)$$

for all y in $\mathbf{R}$. Hence $\int_{\mathbf{R}} x\, d\mu(x) = \int_{\mathbf{R}} x\, d\nu(x)$ by Lemma 2.3 of Chacon [**1**]. That is,

$$\int_F X_i\, dP = \int_F X_j\, dP.$$

Thus (X_i) is a martingale over $(\Omega, \mathfrak{F}, \mathfrak{F}_i, P)$. □

2.6. *Terminology*. By a Brownian motion process in $\mathbf{R}^n$ we mean a continuous Markov process $(\Omega, \mathfrak{B}, \mathfrak{B}_t, B_t, P)$ with state space $\mathbf{R}^n$ (augmented by the cemetery point ∂) and having the transition function of Brownian motion. To make the formula relating Brownian motion to potential theory "cleaner", we adopt the somewhat unorthodox normalization $E(\|B_t - B_0\|^2) = 2nt$. (Usually one has $E(\|B_t - B_0\|^2) = nt$.) We do not assume that the process starts from the origin or that the filtration $(\mathfrak{B}_t)$ is the natural filtration of (B_t).

2.7. DEFINITION. Let $(\Omega, \mathfrak{B}, \mathfrak{B}_t, B_t, P)$ be a Brownian motion process in $\mathbf{R}^n$. A stopping time T for this process is said to be *standard* iff whenever R and S are stopping times and $R \leqslant S \leqslant T$ then $\operatorname{law}(B_R)$ and $\operatorname{law}(B_S)$ are special and $\operatorname{law}(B_R)U \geqslant \operatorname{law}(B_S)U$.

2.8. *Discussion of standard stopping times*. If $n \geqslant 3$ then every stopping time is standard. This follows from the fact that when $n \geqslant 3$ and A is any Borel subset of $\mathbf{R}^n$ then

$$\int_A \operatorname{law}(B_T)U(x)\, dx = E\left[\int_T^\infty 1_A(B_t)\, dt\right]$$

for any stopping time T. If $n = 1$ and $P(B_0 = \partial) = 0$ then T is standard iff $(B_{T\wedge t})_{0 \leqslant t < \infty}$ is uniformly integrable iff $P(T = \infty) = 0$ and whenever S is a stopping time and $S \leqslant T$ then $E(|B_S|) < \infty$ and $E(B_S) = E(B_0)$. If $n = 2$ and $P(B_0 = \partial) = 0$ then T is standard iff $(\log^+\|B_{T\wedge t}\|)_{0 \leqslant t < \infty}$ is uniformly integrable. In general T is standard iff $\operatorname{law}(B_T)$ is special and $\operatorname{law}(B_{T\wedge t})U \geqslant \operatorname{law}(B_T)U$ for $0 \leqslant t < \infty$ iff for every compact subset K of $\mathbf{R}^n$ we have

$$\int_K \left[\operatorname{law}(B_0)U(x) - \operatorname{law}(B_T)U(x)\right] dx = E\left[\int_0^T 1_K(B_t)\, dt\right] < \infty. \qquad (*)$$

The reader is referred to Falkner [**1**] for a detailed discussion of standard stopping times. Intuitively the standard stopping times are the ones which are not too big. Traditionally (see Skorohod [**1**, p. 163], Dubins [**1**], Root [**1**], Meyer [**1**], Baxter and Chacon [**1**]) the stopping times which have been considered not too big, at least in the one-dimensional case, have been those with finite expectation. Let T be a stopping time. Assume $P(B_0 = \partial) = 0$ and law$(B_0; P)$ is special. If $E(T) < \infty$ then $(\|B_{T\wedge t} - B_0\|^2)_{0 \leq t < \infty}$ is uniformly integrable whence T is standard. If, on the other hand, T is standard then first, if $n = 1$ or 2 then automatically $T < \infty$ a.s., and second, for any n, if $T < \infty$ a.s. then from Lemma 5 of Baxter and Chacon [1] together with (*) above, $E(\|B_T\|^2) = E(\|B_0\|^2) + 2nE(T)$ so, in this case, $E(\|B_T\|^2) < \infty$ iff both $E(\|B_0\|^2) < \infty$ and $E(T) < \infty$.

2.9. PROPOSITION. *Let $(\Omega, \mathfrak{B}, \mathfrak{B}_t, B_t, P)$ be a Brownian motion process in $\mathbf{R}^n$. If $n \leq 2$ assume $P(B_0 = \partial) = 0$. Let I be a subset of $[-\infty, \infty]$ and let $(T(i))$ be an increasing family of standard stopping times indexed by I. Then $(\Omega, \mathfrak{B}, \mathfrak{B}_{T(i)}, B_{T(i)}, P)$ is an n-dimensional potential process with time set I.*

PROOF. Clearly this process satisfies (a), (b), (c), (d)(i), and (d)(ii) of Definition 2.4. It also satisfies (d)(iii) since if $n \leq 2$ and $P(B_0 = \partial) = 0$ then a standard stopping time is P-a.s. finite. To finish the proof it suffices to show that it satisfies (b) of Proposition 2.5. Let i and j be in I with $i \leq j$ and let F be in $\mathfrak{B}_{T(i)}$.

Let $S = T_i$ on F and T_j on $\Omega \setminus F$, and let $T = T_j$.

Note that S is a stopping time. As T is standard, law$(B_S)U \geq$ law$(B_T)U$.

From this we can conclude that for y in $\{\text{law}(B_S)U < \infty\}$ we have

$$E\left[\Phi(B_{T(i)} - y)1_{\{B_{T(i)} \neq \partial\}}1_F\right] \geq E\left[\Phi(B_{T(j)} - y)1_{\{B_{T(j)} \neq \partial\}}1_F\right].$$

Now by the method that was used in concluding the proof of (a) $\Rightarrow$ (b) in Proposition 2.5, one can show that this inequality actually holds for all y in $\mathbf{R}^n$. □

3. Embedding discrete-time potential processes in Brownian motion. Let $(W, \mathfrak{W}, \mathfrak{W}_t, \beta_t, \theta_t, \Pi^x)$ be the canonical realization of Brownian motion in $\mathbf{R}^n$. If $n = 1$ let $(\Omega, \mathfrak{B}, \mathfrak{B}_t, B_t, \theta_t, P^x) = (W, \mathfrak{W}, \mathfrak{W}_t, \beta_t, \theta_t, \Pi^x)$. If $n \geq 2$ then let

$\Omega = W \times (0, 1)$,

$\mathfrak{B} = \mathfrak{W} \otimes \text{Borel}(0, 1)$,

$\mathfrak{B}_t = \bigcap_{s > t}[\mathfrak{W}_s \otimes \text{Borel}(0, 1)]$,

ψ = projection of Ω on W,

$B_t = \beta_t \circ \psi$,

$\theta_t = \theta_t \circ \psi$ (thus we use θ_t in 2 different senses, but this will not cause confusion),

$P^x = \Pi^x \otimes$ (Lebesgue measure on Borel(0, 1)).

The notation $(\Omega, \mathfrak{B}, \mathfrak{B}_t, B_t, \theta_t, P^x)$ will carry the same meaning, dependent on n, throughout this section and the appendix which follows. We remark that, for any n and any probability measure μ on $\mathbf{R}^n_\partial$, $(\Omega, \mathfrak{B}, \mathfrak{B}_t, B_t, P^\mu)$ is a Brownian motion process with right-continuous filtration. The purpose of this section is to prove the following theorem.

3.1. THEOREM. *Let $I = \{0, 1, 2, \ldots\}$ and let $(\Lambda, \mathfrak{F}, \mathfrak{F}_i, X_i, Q)$ be an n-dimensional potential process with time set I. Let μ = law(X_0). Then there is an increasing family $(T(i))_{i \in I}$ of standard stopping times for the Brownian motion process $(\Omega, \mathfrak{B}, \mathfrak{B}_t, B_t, P^\mu)$ such that the joint law of (X_i) with respect to Q is the same as the joint law of $(B_{T(i)})$ with respect to P^μ.*

Discussion of the theorem. As we have seen, in the one-dimensional case the potential processes are precisely the martingales and the standard stopping times are precisely the stopping times T such that $(B_{T\wedge t})_{0<t<\infty}$ is uniformly integrable. Thus when $n = 1$ the theorem says that any discrete-time martingale $(X_0, X_1, X_2, \ldots)$ can be "embedded" in Brownian motion by means of an increasing sequence $T(0) \leqslant T(1) \leqslant T(2) \leqslant \ldots$ of stopping times such that for each i the process $(B_{T(i)\wedge t})$ is uniformly integrable. This was proved by Monroe [**1**, Lemma 8]. Earlier Dubins [**1**] had shown that if $(X_0, X_1, X_2, \ldots)$ is a martingale then stopping times $T(0) \leqslant T(1) \leqslant T(2) \leqslant \ldots$ can be constructed which embed $(X_0, X_1, X_2, \ldots)$ in (B_t) and which satisfy $E(T(i)) < \infty$ if $E(X_i^2) < \infty$. What Monroe showed is that if the stopping times $T(i)$ are constructed as in the article of Dubins then, for each i, $(B_{T(i)\wedge t})$ is uniformly integrable; he proved this without any assumption on $E(X_i^2)$ of course.

Our approach to the proof of Theorem 3.1 is to obtain it as a corollary of an embedding theorem for measures ("ETM") which runs as follows.

ETM. "Let μ be a special probability measure on $\mathbf{R}^n$. Then a measure ν on $\mathbf{R}^n_\partial$ is of the form law$(B_T; P^\mu)$ where T is a standard stopping time for the Brownian motion process $(\Omega, \mathfrak{B}, \mathfrak{B}_t, B_t, P^\mu)$ iff the following two conditions are satisfied:

(a) ν is a special probability measure;

(b) $\mu U \geqslant \nu U$."

For $n \geqslant 3$, ETM is a particular case of a result of Rost [**1**]. For $n = 2$, ETM is established in Falkner [**2**]. The method used in this article differs from Rost's and can also be used to establish ETM for $n \geqslant 3$. (We should say that to see that ETM for $n = 2$ actually does follow from Falkner [**2**] it is necessary to know that the conditions (a) and (b) of ETM entail that $\nu(\mathbf{R}^n) = 1$ in this case. This follows from the fact that if α and β are special measures on $\mathbf{R}^n$, where $n = 1$ or 2, and if $\alpha U \geqslant \beta U$ then $\alpha(\mathbf{R}^n) \leqslant \beta(\mathbf{R}^n)$; see Lemma 3.10 of Falkner [**1**].) When $n = 1$ and $\mu =$ the unit point mass at the origin then a probability measure ν on $\mathbf{R}_\partial$ is special and satisfies $\nu U \leqslant \mu U$ iff $\nu(\{\partial\}) = 0$, $\int |x|\, d\nu(x) < \infty$, and $\int x\, d\nu(x) = 0$. Thus when $n = 1$ and $\mu = \delta_0$, ETM follows from Proposition 1 of Dubins [**1**] in the special case that $\int x^2\, d\nu(x) < \infty$, and is a particular instance of Lemma 8 of Monroe [**1**] when ν is not thus restricted. For an alternative treatment of the one-dimensional case (and with μ not restricted to be a point mass) the reader may consult Falkner [**1**, §4].

Let us make a few remarks about the method of proving Theorem 3.1. For simplicity, in these remarks we shall consider a potential process (X_0, X_1) with just the two points 0 and 1 in its time set. We wish to find standard stopping times $T(0) \leqslant T(1)$ such that

$$\text{law}(B_{T(0)}, B_{T(1)}) = \text{law}(X_0, X_1).$$

We start (B_t) with law$(B_0) = \mu \equiv$ law(X_0) so we can just take $T(0) = 0$. Now as (X_0, X_1) is a potential process, law$(X_1 | X_0 = x)U \leqslant \delta_x U$ for law(X_0)-a.a. x in $\mathbf{R}^n$ so by ETM we can choose P^x-standard stopping times $S(x)$ such that law$(B_{S(x)}; P^x)$ = law$(X_1 | X_0 = x)$. (The case $x = \partial$ may be dealt with easily also.) One sets $T(1) = S(x)$ on $\{B_0 = x\}$. Then law$(B_{T(0)}, B_{T(1)}; P^\mu) =$ law(X_0, X_1) and $T(1)$ is

P^μ-standard. Of course one must show that the stopping times $S(x)$ can be chosen to depend on x in a suitably measurable fashion. For Dubins who was working in the case $n = 1$, this was no problem as he had an algorithm for producing a stopping time to embed a measure. Azéma and Yor **[1]** have found a nice formula for such a stopping time, in the case $n = 1$. As soon as $n \geqslant 2$ though, no such algorithm is known so another approach is needed. We overcome this difficulty by showing that the possibility of choosing the stopping times $S(x)$ in a suitably measurable fashion follows from the general theory of measurable selection. In the proof of Theorem 3.1 which we give below we say what kind of measurability is needed, assume that it can be obtained and fill in the details of the sketch of the proof given above. Then in the appendix to this section we show how the abstract measurable selection theorem given on p. 251 of Dellacherie and Meyer **[1]** can be applied to obtain the needed measurability. Before proceeding to the proof of Theorem 3.1 we state and prove a simple lemma which will be used several times in the remainder of this paper.

3.2. LEMMA. *Let $\mathcal{V}$ be the collection of open balls V in $\mathbf{R}^n$ such that the centre of V has rational coordinates and the radius of V is rational. Let f and g be superharmonic functions on $\mathbf{R}^n$ such that*

$$\int_V f(y)\,dy \leqslant \int_V g(y)\,dy$$

for all V in $\mathcal{V}$. Then $f(x) \leqslant g(x)$ for all x in $\mathbf{R}^n$.

PROOF. Let $V(x, \varepsilon)$ denote the open ball in $\mathbf{R}^n$ of centre x and radius ε. Then $V(x, \varepsilon)$ can be approximated arbitrarily well with respect to Lebesgue measure by elements of $\mathcal{V}$. Also f and g are locally Lebesgue integrable. Thus we find that

$$\int_{V(x,\varepsilon)} f(y)\,dy \leqslant \int_{V(x,\varepsilon)} g(y)\,dy.$$

Since f and g are superharmonic, we obtain $f(x) \leqslant g(x)$ upon dividing by the Lebesgue measure of $V(x, \varepsilon)$ and letting ε decrease to 0. □

PROOF OF THEOREM 3.1. Let $T(0) = 0$. Then $T(0)$ is P^μ-standard and $\operatorname{law}(B_{T(0)}; P^\mu) = \mu = \operatorname{law}(X_0; Q)$. Suppose $i \in I$ and P^μ-standard stopping times $T(0) \leqslant \cdots \leqslant T(j)$ have been chosen such that $\operatorname{law}(Y_0, \ldots, Y_j; P^\mu) = \operatorname{law}(X_0, \ldots, X_j; Q)$ where $Y_i = B_{T(i)}$ for $i = 0, \ldots, j$. We shall show that one can choose a P^μ-standard stopping time $T(j+1) \geqslant T(j)$ such that

$$\operatorname{law}(Y_0, \ldots, Y_j, B_{T(j+1)}; P^\mu) = \operatorname{law}(X_0, \ldots, X_j, X_{j+1}; Q).$$

Let D denote $\mathbf{R}^n$ and let E denote $\mathbf{R}^n_\partial$. Let $\alpha = \operatorname{law}(X_0, \ldots, X_j; Q)$ and let $\beta = \operatorname{law}(X_0, \ldots, X_j, X_{j+1}; Q)$. Then α is a probability measure on E^{j+1}, β is a probability measure on E^{j+2}, and the map $(x_0, \ldots, x_j, x_{j+1}) \mapsto (x_0, \ldots, x_j)$ carries β onto α. Hence by the measure disintegration theorem there is an α-essentially unique Borel measurable family $(\gamma(x)\colon x \in E^{j+1})$ of probability measures on E such that for all nonnegative Borel functions f on E^{j+2} we have

$$\int_{E^{j+2}} f\,d\beta = \int_{E^{j+1}} \alpha(dx) \int \gamma(x)(dx_{j+1}) f(x, x_{j+1}).$$

Now

$$\int_{E^{j+1}} \alpha(dx) \int_D \gamma(x)(dx_{j+1}) \Phi^-(x_{j+1}) = \int_{E^{j+2}} \Phi^-(x_{j+1}) 1_D(x_{j+1})\, d\beta(x_0, \ldots, x_j, x_{j+1})$$

$$= \int_D \Phi^-(y) \text{law}(X_{j+1})\,(dy).$$

Since (X_i) is a potential process, law(X_{j+1}) is special so the latter integral is finite. Hence for α-a.a. x in E^{j+1} we have

$$\int_D \gamma(x)(dx_{j+1}) \Phi^-(x_{j+1}) < \infty.$$

That is, $\gamma(x)$ is special for α-a.a. $x \in E^{j+1}$. Let $\mathcal{V}$ be the set of open balls V in D such that the centre of V has rational coordinates and the radius of V is rational. Since $X_0, \ldots, X_j, X_{j+1}$ is a potential process,

$$\int_{F \times E} \Phi(x_j - y) 1_D(x_j)\, d\beta(x_0, \ldots, x_{j+1})$$

$$\geqslant \int_{F \times E} \Phi(x_{j+1} - y) 1_D(x_{j+1})\, d\beta(x_0, \ldots, x_{j+1})$$

for all F in Borel(E^{j+1}) and all y in D. This may be deduced from (a) $\Rightarrow$ (b) of 2.5 by change of variables. Now the first integral in this inequality may also be written as $\int_F \Phi(x_j - y) 1_D(x_j)\, d\alpha(x_0, \ldots, x_j)$ while the second may be written as $\int_F \gamma(x_0, \ldots, x_j) U(y)\, d\alpha(x_0, \ldots, x_j)$. Now if A is any bounded Borel subset of D then

$$\int_A \int_F \Phi^-(x_j - y) 1_D(x_j)\, d\alpha(x_0, \ldots, x_j)\, dy \leqslant \int_A \text{law}(X_j) U_-(y)\, dy < \infty$$

and

$$\int_A \int_F [\gamma(x_0, \ldots, x_j) U(y)]^-\, d\alpha(x_0, \ldots, x_j)\, dy \leqslant \int_A \text{law}(X_{j+1}) U_-(y)\, dy < \infty.$$

Thus if V is any element of $\mathcal{V}$ then we may integrate over V with respect to y and interchange orders of integration to obtain

$$\int_F \int_V \Phi(x_j - y) 1_D(x_j)\, dy\, d\alpha(x_0, \ldots, x_j)$$

$$\geqslant \int_F \int_V \gamma(x_0, \ldots, x_j) U(y)\, dy\, d\alpha(x_0, \ldots, x_j).$$

it follows that for α-a.a. $(x_0, \ldots, x_j)$ in E^{j+1} we have, for all V in $\mathcal{V}$,

$$\int_V \Phi(x_j - y) 1_D(x_j)\, dy \geqslant \int_V \gamma(x_0, \ldots, x_j) U(y)\, dy.$$

Applying Lemma 3.2 we see that for α-a.a. $(x_0, \ldots, x_j)$ in E^{j+1} we have, for all y in D, $\delta_{x_j} U(y) \geqslant \gamma(x_0, \ldots, x_j) U(y)$. Thus by ETM, for α-a.a. $(x_0, \ldots, x_j)$ in $E^j \times D$ there exists a P^{x_j}-standard stopping time $S(x_0, \ldots, x_j)$ such that

$$\text{law}(B_{S(x_0, \ldots, x_j)}; P^{x_j}) = \gamma(x_0, \ldots, x_j). \qquad (*)$$

If $n = 1$ or 2 then $Q(X_j = \partial) = 0$ so this takes care of α-a.a. x in E^{j+1}. If $n \geqslant 3$ and $(x_0, \ldots, x_j)$ is in $E^j \times \{\partial\}$ and $\delta_{x_j} U \geqslant \gamma(x_0, \ldots, x_j) U$ then $\gamma(x_0, \ldots, x_j) U \leqslant 0$ which implies that the probability measure $\gamma(x_0, \ldots, x_j)$ can only be δ_∂. (This implication is not valid for $n = 1$ or 2.) Thus in such a case $S(x_0, \ldots, x_j) = 0$ is a stopping time such that (*) is satisfied. Thus we see that one can find a family $(S(x): x \in E^{j+1})$ of stopping times such that, for α-a.a. $(x_0, \ldots, x_j)$ in E^{j+1}, $S(x_0, \ldots, x_j)$ is P^{x_j}-standard and (*) is satisfied. Now as we shall see in the appendix to this section, this family of stopping times may be chosen so that for each t in $(0, \infty)$ the map $(x, \omega) \mapsto t \wedge S(x)(\omega)$ is (Borel $E^{j+1}) \otimes \mathfrak{B}_t$-measurable. Let Y denote the map $\omega \mapsto (Y_0(\omega), \ldots, Y_j(\omega))$ of Ω into E^{j+1}. Now Y is $(\mathfrak{B}_{T(j)}$, Borel $E^{j+1})$-measurable and for each t in $(0, \infty)$ the translation operator $\theta_{T(j)}$ is $(\mathfrak{B}_{T(j)+t}, \mathfrak{B}_t)$-measurable. Thus if we define $R: \Omega \to [0, \infty]$ by

$$R(\omega) = S(Y(\omega))(\theta_{T(j)}\omega)$$

then $t \wedge R$ is $\mathfrak{B}_{T(j)+t}$-measurable for each t in $(0, \infty)$. As the filtration $(\mathfrak{B}_{T(j)+t})$ is right-continuous, it follows that R is a $(\mathfrak{B}_{T(j)+t})$-stopping time.

Let $T(j + 1) = T(j) + R$. Clearly $T(j + 1) \geqslant T(j)$. Using the right-continuity of the filtration $(\mathfrak{B}_t)$ one easily verifies that $T(j + 1)$ is a $(\mathfrak{B}_t)$-stopping time. Let f and g be nonnegative Borel functions on E^{j+1} and E respectively. Then

$$\begin{aligned}
&\int f(X_0, \ldots, X_j) g(X_{j+1})\, dQ \\
&\quad = \int f(x_0, \ldots, x_j) g(x_{j+1})\, d\beta(x_0, \ldots, x_j, x_{j+1}) \\
&\quad = \int f(x_0, \ldots, x_j)\left[\int g(x_{j+1}) \gamma(x_0, \ldots, x_j)\,(dx_{j+1})\right] d\alpha(x_0, \ldots, x_j) \\
&\quad = \int f(x_0, \ldots, x_j) E^{x_j}\left[g\left(B_{S(x_0, \ldots, x_j)}\right)\right] d\alpha(x_0, \ldots, x_j) \\
&\quad = \int f(Y(\omega)) E^{Y_j(\omega)}\left[g\left(B_{S(Y(\omega))}\right)\right] dP^\mu(\omega) \\
&\quad = \int f(Y(\omega)) g\left(B_{S(Y(\omega))}(\theta_{T(j)}\omega)\right) dP^\mu(\omega) \\
&\quad = \int f(Y(\omega)) g\left(B_{T(j+1)}(\omega)\right) dP^\mu(\omega)
\end{aligned}$$

where the second-to-last step follows from the fact that $f(Y)$ is $\mathfrak{B}_{T(j)}$-measurable together with a version of the strong Markov property for which the reader may consult Meyer [**2**, p. 45, T21]. This shows that $\text{law}(Y_0, \ldots, Y_j, B_{T(j+1)}; P^\mu) = \text{law}(X_0, \ldots, X_j, X_{j+1})$ as desired. If $n \geqslant 3$ this is all we have to do. Otherwise it remains to show that $T(j + 1)$ is P^μ-standard. We shall prove this by showing that, for any compact subset K of D,

$$E^\mu\left[\int_0^{T(j+1)} 1_K(B_t)\, dt\right] = \int_K \left[\mu U(x) - \text{law}(B_{T(j+1)}; P^\mu) U(x)\right] dx < \infty.$$

Observe that we already know that the right-hand integral is finite since μ and $\text{law}(B_{T(j+1)}; P^\mu)$ are equal to $\text{law}(X_0)$ and $\text{law}(X_{j+1})$ respectively and so are both special. Thus it remains only to show the equality of the two integrals. Now $T(j)$ is P^{μ}-standard so

$$E^{\mu}\left[\int_0^{T(j)} 1_K(B_t)\,dt\right] = \int_K \left[\mu U - \mathrm{law}(Y_j;\, P^{\mu})U\right].$$

Next,

$$
\begin{aligned}
E^{\mu}&\left[\int_{T(j)}^{T(j+1)} 1_K(B_t)\,dt\right] \\
&= \int \int_0^{S(Y(\omega))(\theta_{T(j)}\omega)} 1_K\big(B_t(\theta_{T(j)}\omega)\big)\,dt\,dP^{\mu}(\omega) \\
&= \int E^{Y_j(\omega)}\left[\int_0^{S(Y(\omega))} 1_K(B_t)\,dt\right] dP^{\mu}(\omega) \\
&= \int E^{x_j}\left[\int_0^{S(x_0,\ldots,x_j)} 1_K(B_t)\,dt\right] d\alpha(x_0,\ldots,x_j) \\
&= \int\int_K \left[\delta_{x_j}U(y) - \mathrm{law}(B_{S(x_0,\ldots,x_j)};\, P^{x_j})U(y)\right] dy\, d\alpha(x_0,\ldots,x_j) \\
&= \int\int_K \left[\delta_{x_j}U(y) - \gamma(x_0,\ldots,x_j)U(y)\right] dy\, d\alpha(x_0,\ldots,x_j) \\
&= \int_K\int \left[\delta_{x_j}U(y) - \gamma(x_0,\ldots,x_j)U(y)\right] d\alpha(x_0,\ldots,x_j)\, dy \\
&= \int_K\int \left[\Phi(x_j - y) - \int \Phi(x_{j+1} - y)\gamma(x_0,\ldots,x_j)\,(dx_{j+1})\right] d\alpha(x_0,\ldots,x_j)\, dy \\
&= \int_K\int \Phi(x_j - y)\, d\alpha(x_0,\ldots,x_j) - \int \Phi(x_{j+1} - y)\, d\beta(x_0,\ldots,x_{j+1})\, dy \\
&= \int_K \mathrm{law}(X_j)U(y) - \mathrm{law}(X_{j+1})U(y)\, dy \\
&= \int_K \mathrm{law}(Y_j;\, P^{\mu})U(y) - \mathrm{law}(B_{T(j+1)};\, P^{\mu})U(y)\, dy.
\end{aligned}
$$

Here the second step follows from the same version of the strong Markov property cited earlier in the proof and the fourth step follows from the fact that $S(x_0, \ldots, x_j)$ is P^{x_j}-standard for α-a.a. $(x_0, \ldots, x_j)$. Thus we see that

$$E^{\mu}\left[\int_0^{T(j+1)} 1_K(B_t)\,dt\right] = \int_K \mu U(y) - \mathrm{law}(B_{T(j+1)};\, P^{\mu})U(y)\, dy$$

as desired. This completes the proof. □

3A. Appendix to §3. In this appendix we establish the result on measurable selection of stopping times which was used in the proof of Theorem 3.1. We begin by fixing some notation which will be used throughout this appendix. Let $(\Lambda, \mathcal{F}, \mathcal{F}_t, X_t, P)$ be an adapted measurable stochastic process with state space $(E, \mathcal{E})$ and time set $[0, \infty]$. By this we mean that:

(a) $(\Lambda, \mathcal{F}, P)$ is a probability space;

(b) $(E, \mathcal{E})$ is a measurable space;

(c) $(\mathcal{F}_t)$ is an increasing family of sub-σ-fields of $\mathcal{F}$ indexed by $[0, \infty]$;
(d) (X_t) is a family of maps from Λ to E indexed by $[0, \infty]$;
(e) for each t, X_t is $(\mathcal{F}_t, \mathcal{E})$-measurable;
(f) the map $(t, \omega) \mapsto X_t(\omega)$ is $((\text{Borel}[0, \infty]) \otimes \mathcal{F}, \mathcal{E})$-measurable.

Note that if μ is any probability measure on $\mathbf{R}^n_\partial$ and we take $(\Lambda, \mathcal{F}, \mathcal{F}_t, X_t, P) = (\Omega, \mathcal{B}, \mathcal{B}_t, B_t, P^\mu)$ and $(E, \mathcal{E}) = (\mathbf{R}^n_\partial, \text{Borel } \mathbf{R}^n_\partial)$ then (a)–(f) are satisfied. (See the beginning of §3 for the definition of $(\Omega, \mathcal{B}, \mathcal{B}_t, B_t, P^\mu)$.) The assumption that this particular choice of $(\Lambda, \mathcal{F}, \mathcal{F}_t, X_t, P)$ and $(E, \mathcal{E})$ has been taken will be referred to as the hypothesis (BM^μ). If $\mathcal{F}$ is countably generated mod P (that is, if there exists a countably generated sub-σ-field $\mathcal{G}$ of $\mathcal{F}$ such that for every F in $\mathcal{F}$ there exists G in $\mathcal{G}$ such that $P(F \triangle G) = 0$, where $\triangle$ denotes symmetric difference) then we shall say that *the hypothesis* (CG) is satisfied. Note that (BM^μ) implies (CG). As is well known, (CG) is satisfied iff $L^1(\Lambda, \mathcal{F}, P)$ is separable.

For any $\mathcal{F}$-measurable map $f\colon \Lambda \to [0, \infty]$ let $[f] = \{g\colon \Lambda \to [0, \infty] \mid g$ is $\mathcal{F}$-measurable and $P(f \neq g) = 0\}$. Let $\mathfrak{R}\mathfrak{T}$ denote the space $\{[f] \mid f\colon \Lambda \to [0, \infty]$ is $\mathcal{F}$-measurable$\}$ equipped with the topology of convergence in measure. As is well known, $\mathfrak{R}\mathfrak{T}$ is completely metrizable; it is separable iff (CG) is satisfied. Let $\mathfrak{S}\mathfrak{T}$ be the subspace of $\mathfrak{R}\mathfrak{T}$ defined by

$$\mathfrak{S}\mathfrak{T} = \{[T]\colon T \text{ is an } (\mathcal{F}_t)\text{-stopping time}\}.$$

(Of course $\mathfrak{R}\mathfrak{T}$ stands for "random times" and $\mathfrak{S}\mathfrak{T}$ stands for "stopping times".)

3A.1. Lemma. *If $(\mathcal{F}_t)$ is right-continuous and if $f\colon \Lambda \to [0, \infty]$ has the property that for every t in $[0, \infty)$ there exists F in $\mathcal{F}_t$ such that $P(\{f < t\} \triangle F) = 0$ then there is an $(\mathcal{F}_t)$-stopping time T such that $P(T \neq f) = 0$.*

Proof. For each nonnegative rational r choose a set F_r in $\mathcal{F}_r$ such that $P(\{f < r\} \triangle F_r) = 0$. Define T by $T(\omega) = \sup\{r\colon \omega \notin F_r\}$, where $\sup \varnothing = 0$. Then $T(\omega) < t$ iff there exists a rational $r < t$ such that $\omega \in F_r$. Thus $\{T < t\} = \bigcup_{r<t} F_r$, which belongs to $\mathcal{F}_t$. As $(\mathcal{F}_t)$ is right-continuous this implies that T is a stopping time. Now

$$\{f < t\} = \bigcup_{r<t} \{f < r\} = \bigcup_{r<t} F_r = \{T < t\} \quad (\text{mod } P)$$

so $P(f < t \leqslant T) = 0$ and $P(T < t \leqslant f) = 0$ for all t. It follows that $P(f < T) = 0$ and $P(T < f) = 0$, Thus $P(T \neq f) = 0$. □

3A.2. Corollary. *If $(\mathcal{F}_t)$ is right-continuous then $\mathfrak{S}\mathfrak{T}$ is closed in $\mathfrak{R}\mathfrak{T}$.*

Proof. Let $([T_i])$ be a sequence in $\mathfrak{S}\mathfrak{T}$ converging in $\mathfrak{R}\mathfrak{T}$ to $[f]$. (It is enough to consider sequences as $\mathfrak{R}\mathfrak{T}$ is metrizable.) Passing to a subsequence, we may suppose that $T_i \to f$ P-a.s. For t in $[0, \infty]$ let $F_t = \bigcup_i \bigcap_{j>i} \{T_j < -2^{-i}\}$. Then $F_t \in \mathcal{F}_t$ and $P(\{f < t\} \triangle F_t) = 0$. Thus by 3A.1 there is an $(\mathcal{F}_t)$-stopping time T such that $P(f \neq T) = 0$. That is, $[f] \in \mathfrak{S}\mathfrak{T}$. □

3A.3. Proposition. *Suppose* (CG) *is satisfied. Then there is a map $\tau \mapsto \hat{\tau}$ defined on $\mathfrak{S}\mathfrak{T}$ such that:*

(a) *for every τ in $\mathfrak{S}\mathfrak{T}$, $\hat{\tau} \in \tau$;*

(b) *for every* t *in* $[0, \infty]$, *the map* $(\tau, \omega) \mapsto t \wedge \hat{\tau}(\omega)$ *is* (*Borel* $\mathbb{S}\mathfrak{T}$) $\otimes$ $\mathfrak{F}_t$-*measurable.*

(*Observe that if* ($\mathfrak{F}_t$) *is right-continuous then* (b) *implies that* $\hat{\tau}$ *is a stopping time for every* τ *in* $\mathbb{S}\mathfrak{T}$.)

PROOF. For each nonnegative rational r and each τ in $\mathbb{S}\mathfrak{T}$ let λ_τ^r be the measure on $\mathfrak{F}_r$ defined by $\lambda_\tau^r(F) = P(F \cap \{\tau < r\})$. Note that $\tau \mapsto \lambda_\tau^r(F)$ is Borel($\mathbb{S}\mathfrak{T}$)-measurable (in fact it is continuous on $\mathbb{S}\mathfrak{T}$) for any $F \in \mathfrak{F}_r$. Now $\mathfrak{F}_r$ is countably generated mod P as the larger σ-field $\mathfrak{F}$ is. Thus by the principal of measurable selection of .epresentatives (see, for example, Meyer [3, pp. 194–195]) we can choose, for each τ, a version f_τ^r of the Radon-Nikodým derivative of λ_τ^r with respect to $P|\mathfrak{F}_r$ in such a way that the map $(\tau, \omega) \mapsto f_\tau^r(\omega)$ is (Borel $\mathbb{S}\mathfrak{T}$) $\otimes$ $\mathfrak{F}_r$-measurable. Since $\{\tau < r\}$ is P-almost equal to a set in $\mathfrak{F}_r$, we have that $f_\tau^r = 1_{\{\tau < r\}}$ P-a.s. for each τ in $\mathbb{S}\mathfrak{T}$. Let $F_\tau^r = \{f_\tau^r = 1\}$. Then we have $P(\{\tau < r\} \triangle F_\tau^r) = 0$ for each τ in $\mathbb{S}\mathfrak{T}$. Now for τ in $\mathbb{S}\mathfrak{T}$ define $\hat{\tau}$: $\Lambda \to [0, \infty]$ by

$$\hat{\tau}(\omega) = \sup\{r: \omega \notin F_\tau^r\}$$

where as usual we take $\sup \varnothing = 0$.

Then $\{(\tau, \omega): \hat{\tau}(\omega) < t\} = \bigcup_{r<t}\{(\tau, \omega): f_\tau^r(\omega) = 1\}$ which clearly belongs to (Borel $\mathbb{S}\mathfrak{T}$) $\otimes$ $\mathfrak{F}_t$. This implies (b). Also for each τ,

$$\{\hat{\tau} < t\} = \bigcup_{r<t} F_\tau^r = \bigcup_{r<t} \{\tau < r\} = \{\tau < t\} \quad (\text{mod } P)$$

whence $\hat{\tau} \in \tau$. $\square$

3A.4. LEMMA. *Let* M *be the set of probability measures on* $(E, \mathcal{E})$ *and let* $\mathfrak{M}$ *be the smallest* σ-*field of subsets of* M *which makes the functions of the form* $m \in M \mapsto m(A)$ $(A \in \mathcal{E})$ *measurable. Then the map* $[f] \mapsto \text{law}(X_f)$ *from* $\mathfrak{R}\mathfrak{T}$ *to* M *is* (*Borel* $\mathfrak{R}\mathfrak{T}$, $\mathfrak{M}$)-*measurable.*

PROOF. Let $\mathfrak{Y}$ be the set of bounded measurable real-valued processes Y over $(\Lambda, \mathfrak{F})$ with time set $[0, \infty]$ such that $[f] \mapsto E(Y_f)$ is a Borel function on $\mathfrak{R}\mathfrak{T}$. If Y has continuous sample paths then this function is actually continuous on $\mathfrak{R}\mathfrak{T}$. By a monotone class argument it then follows that $\mathfrak{Y}$ contains all bounded measurable real-valued processes over $(\Lambda, \mathfrak{F})$ with time set $[0, \infty]$. Let A be in $\mathcal{E}$ and let $Y_t = 1_A(X_t)$ $(0 \leqslant t \leqslant \infty)$. By what we have just seen, $[f] \mapsto E(Y_f)$ is a Borel function on $\mathfrak{R}\mathfrak{T}$. But $E(Y_f) = P(X_f \in A) = \text{law}(X_f)(A)$. $\square$

3A.5. LEMMA. *Let* μ *be a special probability measure on* $\mathbf{R}^n$ *and suppose the hypothesis* (BM$^\mu$) *is satisfied. Let*

$$\mathbb{S}\mathbb{S}\mathfrak{T} = \{[T]: T \textit{ is a } P^\mu\textit{-standard stopping time}\}.$$

Then $\mathbb{S}\mathbb{S}\mathfrak{T}$ *is a Borel subset of* $\mathbb{S}\mathfrak{T}$.

PROOF. By 3A.4, the map $[T] \mapsto \text{law}(B_T)(A)$ is Borel($\mathbb{S}\mathfrak{T}$)-measurable for every Borel subset A of $\mathbf{R}_0^n$. Similarly, so are the maps $[T] \mapsto \text{law}(B_{T\wedge t})(A)$, where t is in $[0, \infty)$, since $[T] \to [T \wedge t]$ is a continuous map of $\mathbb{S}\mathfrak{T}$ into itself for each such t. Now T is P^μ-standard iff $\text{law}(B_T)$ is special and $\text{law}(B_{T\wedge k})U \succ \text{law}(B_T)U$ for each positive integer k. Combining these observations with Lemma 3.2 we see that $\mathbb{S}\mathbb{S}\mathfrak{T}$ is indeed a Borel set in $\mathbb{S}\mathfrak{T}$. □

3A.6. PROPOSITION. *Assume* $(\mathfrak{F}_t)$ *is right-continuous and* (CG) *is satisfied. Suppose also that* $(E, \mathcal{E})$ *is countably generated. Let* $\mathfrak{T}$ *be a Borel subset of* $\mathbb{S}\mathfrak{T}$, *let* $(Z, \mathfrak{D}, \alpha)$ *be a* σ*-finite measure space, let* $(M, \mathfrak{M})$ *be the measurable space of probability measures on* $(E, \mathcal{E})$ *as described in* 3A.4, *and let* $z \mapsto \nu(z)$ *be a* $(\mathfrak{D}, \mathfrak{M})$*-measurable map of* Z *into* M. *Let*

$$Z_1 = \{z \in Z: \nu(z) = \operatorname{law}(X_T) \text{ for some } [T] \in \mathfrak{T}\}.$$

Then:

(a) Z_1 *is measurable with respect to the completion* $\bar{\alpha}$ *of* α;

(b) *if* $\bar{\alpha}(Z \setminus Z_1) = 0$ *then there is a family* $(T(z))_{z\in Z}$ *of stopping times such that*:

(i) *for each* t *in* $(0, \infty)$ *the map* $(z, \omega) \mapsto t \wedge T(z)(\omega)$ *is* $\mathfrak{D} \otimes \mathfrak{F}_t$*-measurable*;

(ii) *for* α*-a.a.* $z \in Z$, $[T(z)] \in \mathfrak{T}$ *and* $\nu(z) = \operatorname{law}(X_{T(z)})$.

PROOF. As (CG) is satisfied, $\mathfrak{R}\mathfrak{T}$ is a Polish space. As $(\mathfrak{F}_t)$ is right-continuous, $\mathbb{S}\mathfrak{T}$ is closed in $\mathfrak{R}\mathfrak{T}$, by 3A.2. Thus ($\mathbb{S}\mathfrak{T}$, Borel $\mathbb{S}\mathfrak{T}$) is a Lusinian measurable space in the sense of Meyer and Dellacherie [**1**]. Now $(M, \mathfrak{M})$ is separated and countably generated since $(E, \mathcal{E})$ is countably generated. Also, $[T] \mapsto \operatorname{law}(X_T)$ is (Borel $\mathbb{S}\mathfrak{T}$, $\mathfrak{M}$)-measurable by 3A.4. It follows that

$$M_1 \equiv \{\operatorname{law}(X_T): [T] \in \mathfrak{T}\}$$

belongs to Souslin($\mathfrak{M}$). Therefore $Z_1 = \{z \in Z: \nu(z) \in M_1\}$ is an element of Souslin($\mathfrak{D}$) and hence Z_1 is $\bar{\alpha}$-measurable. This proves (a).

Now assume $\bar{\alpha}(Z \setminus Z_1) = 0$. By the measurable selection theorem on p. 251 of Dellacherie and Meyer [**1**], there is a function Ψ: $M_1 \to \mathfrak{T}$ such that Ψ is (σ(Souslin($\mathfrak{M}|M_1$)), Borel($\mathbb{S}\mathfrak{T}$))-measurable and law("$X_{\Psi(\mu)}$") $= \mu$ for all μ in M_1. Let

$$\sigma(z) = \begin{cases} \Psi(\nu(z)) & \text{if } z \in Z_1, \\ [0] & \text{if } z \in Z \setminus Z_1. \end{cases}$$

Then the map $z \mapsto \sigma(z)$ is (σ(Souslin($\mathfrak{D}$)), Borel($\mathbb{S}\mathfrak{T}$))-measurable and for $\bar{\alpha}$-a.a. z in Z we have that $\sigma(z)$ is in $\mathfrak{T}$ and law("$X_{\sigma(z)}$") $= \nu(z)$. Now in particular, $z \mapsto \sigma(z)$ is ($\bar{\mathfrak{D}}$, Borel($\mathbb{S}\mathfrak{T}$))-measurable, where $\bar{\mathfrak{D}}$ is the completion of $\mathfrak{D}$ with respect to α. As Borel($\mathbb{S}\mathfrak{T}$) is countably generated, it follows that for some Z_0 in $\mathfrak{D}$ satisfying $\alpha(Z \setminus Z_0) = 0$ we have that $z \mapsto \sigma(z)$ is ($\mathfrak{D}$, Borel($\mathbb{S}\mathfrak{T}$))-measurable on Z_0. Now let

$$\tau(z) = \begin{cases} \sigma(z) & \text{if } z \in Z_0, \\ [0] & \text{if } z \in Z \setminus Z_0. \end{cases}$$

Then $z \mapsto \tau(z)$ is ($\mathfrak{D}$, Borel($\mathbb{S}\mathfrak{T}$))-measurable and for α-a.a. z in Z we have that $\tau(z)$ is in $\mathfrak{T}$ and law("$X_{\tau(z)}$") $= \nu(z)$. Finally, let $\tau \mapsto \hat{\tau}$ be as in 3A.3 and let $T(z) = (\tau(z))\hat{}$ for all z in Z. Clearly the family $(T(z))_{z\in Z}$ has the desired properties. □

Now here is the result on measurable selection of stopping times we used in the proof of Theorem 3.1.

3A.7. COROLLARY. *Let* $(Z, \mathfrak{D}, \alpha)$ *be a* σ*-finite measure space. Let* $D = \mathbf{R}^n$ *and* $E = \mathbf{R}^n_\partial$. *Suppose* π *is a measurable map of* Z *into* D *and* γ *is a measurable map from* Z *into the space of probability measures on* E *such that, for* α*-a.a.* z *in* Z, $\gamma(z)$ *is special and* $\delta_{\pi(z)}U \geqslant \gamma(z)U$. *Then there is a family* $(S(z))_{z\in Z}$ *of* $(\mathfrak{B}_t)$*-stopping times such that*:

(a) *for each* t *in* $(0, \infty)$ *the map* $(z, \omega) \mapsto t \wedge S(z)(\omega)$ *is* $\mathfrak{D} \otimes \mathfrak{B}_t$*-measurable*;

(b) *for* α*-a.a.* z *in* Z, $S(z)$ *is* $P^{\pi(z)}$*-standard and* $\text{law}(B_{S(z)}; P^{\pi(z)}) = \gamma(z)$.

PROOF. Let δ be the unit point mass at the origin in $\mathbf{R}^n$ and let $(\Lambda, \mathfrak{F}, \mathfrak{F}_t, X_t, P) = (\Omega, \mathfrak{B}, \mathfrak{B}_t, B_t, P^\delta)$. Then (CG) is satisfied and $(\mathfrak{F}_t)$ is right-continuous. Note also that $(E, \text{Borel } E)$ is countably generated. Let $\mathfrak{T}$ be the set of P^δ-standard stopping times. Then $\mathfrak{T}$ is a Borel set in $\mathfrak{S}\,\mathfrak{T}$ by 3A.5. For each z in Z let $\nu(z)$ be the translate of $\gamma(z)$ by $-\pi(z)$, so that $\delta U \geqslant \nu(z)U$ for α-a.a. z in Z. By ETM, for α-a.a. z in Z there exists T in $\mathfrak{T}$ such that $\text{law}(B_T; P^\delta) = \nu(z)$. Thus by 3A.6 there is a family $(T(z))_{z\in Z}$ of stopping times such that:

(a′) for each t in $(0, \infty)$ the map $(z, \omega) \mapsto t \wedge T(z)(\omega)$ is $\mathfrak{D} \otimes \mathfrak{B}_t$-measurable;

(b′) for α-a.a. z in Z, $[T(z)]$ is in $\mathfrak{T}$ and $\text{law}(B_{T(z)}; P^\delta) = \nu(z)$.

Now for each z in Z let $S(z)$ be "the translate of $T(z)$ by $+\pi(z)$". Clearly the family $(S(z))_{z\in Z}$ has the required properties. □

4. Enlargements of probability spaces. In §3 we considered an enlargement of the canonical realization of Brownian motion for the case $n \geqslant 2$. This enlargement was of a particularly simple type; essentially we just took the product of the canonical realization of Brownian motion with $(L, \mathfrak{L}, \lambda)$ where $L = (0, 1)$, $\mathfrak{L} = \text{Borel } L$, and $\lambda =$ Lebesgue measure on L. That such an enlargement is needed for the embedding of a discrete-time potential process in Brownian motion follows from the fact that it is already necessary for ETM. For example if $n \geqslant 2$ and if μ is the unit point mass at 0 in $\mathbf{R}^n$ while ν is the probability measure on $\mathbf{R}^n$ which has half its mass at 0 and the other half uniformly distributed on $\{x \in \mathbf{R}^n\colon \|x\| = 1\}$ then the hypotheses of ETM are satisfied but there is no $(\mathfrak{W}_t)$-stopping time T such that $\text{law}(\beta_T; \Pi^\mu) = \nu$. This follows from the zero-one law (i.e., the fact that $\mathfrak{W}_0$ is Π^μ-trivial) and the fact that since $n \geqslant 2$, $\Pi^\mu(B_t = 0 \text{ for some } t > 0) = 0$. (Here we are using the notations from the beginning of §3.)

It is an open question whether a product enlargement such as we considered in the discrete-time case suffices for the embedding of continuous-time potential processes in Brownian motion. However we shall see in the next section that a right-continuous potential process with initial law μ can be embedded in an "optional" enlargement of the canonical realization of Brownian motion with initial law μ by means of a right-continuous increasing family of standard stopping times. In this section we shall discuss enlargements of probability spaces and especially optional enlargements of filtered probability spaces. We remark that what we call optional enlargements are essentially the distributional enlargements of Baxter and Chacon [2].

4.1. DEFINITION. Let $(\Omega, \mathscr{F}, P)$ be a probability space. An *enlargement* of $(\Omega, \mathscr{F}, P)$ is a probability space $(\bar{\Omega}, \bar{\mathscr{F}}, \bar{P})$ together with a map $\psi: \bar{\Omega} \to \Omega$ such that ψ is $(\bar{\mathscr{F}}, \mathscr{F})$-measurable and $\psi(\bar{P}) = P$. (By $\psi(\bar{P})$ we mean the measure Q on $\mathscr{F}$ defined by $Q(F) = \bar{P}(\psi^{-1}[F])$.)

4.2. DEFINITION. Let $(\Omega, \mathscr{F})$ and $(X, \mathscr{Q})$ be measurable spaces. By a *randomized random variable* in $(X, \mathscr{Q})$ (over $(\Omega, \mathscr{F})$) we mean a map χ defined on Ω taking values in the set of probability measures on $\mathscr{Q}$, such that the function $\omega \mapsto \chi(\omega)(A)$ is $\mathscr{F}$-measurable for each A in $\mathscr{Q}$.

If X is a topological space then by a randomized random variable in X we mean a randomized random variable in $(X, \text{Borel } X)$, where Borel X is of course the σ-field generated by the open subsets of X.

4.3. *Abbreviations.*

rv = random variable;

rrv = randomized random variable;

rt = random time = rv in $[0, \infty]$;

rrt = randomized random time = rrv in $[0, \infty]$.

4.4. DEFINITION. Let $(\Omega, \mathscr{F}, P)$ be a probability space and let $(\bar{\Omega}, \bar{\mathscr{F}}, \bar{P}, \psi)$ be an enlargement of $(\Omega, \mathscr{F}, P)$. Let $(X, \mathscr{Q})$ be a measurable space and let f be an rv in $(X, \mathscr{Q})$ over $(\bar{\Omega}, \bar{\mathscr{F}})$. An rrv χ in $(X, \mathscr{Q})$ over $(\Omega, \mathscr{F})$ will be said to *correspond* to f iff we have

$$\bar{P}\big(f^{-1}[A] \cap \psi^{-1}[F]\big) = \int_F \chi(\omega)(A)\, dP(\omega)$$

for all A in $\mathscr{Q}$ and F in $\mathscr{F}$. In other words, χ corresponds to f iff χ is a regular conditional distribution of f with respect to $\bar{P}$ given $(\psi, \mathscr{F})$.

If χ_1 and χ_2 are two rrv's in $(X, \mathscr{Q})$ over $(\Omega, \mathscr{F})$ corresponding to f then for each A in $\mathscr{Q}$ we have $P(\{\omega: \chi_1(\omega)(A) \neq \chi_2(\omega)(A)\}) = 0$. If $\mathscr{Q}$ is countably generated then by a monotone class argument, $\{\chi_1 \neq \chi_2\}$ belongs to $\mathscr{F}$ and $P(\chi_1 \neq \chi_2) = 0$. Thus when $\mathscr{Q}$ is countably generated, an rrv in $(X, \mathscr{Q})$ over $(\Omega, \mathscr{F})$ corresponding to f is P-essentially unique if it exists. If $\mathscr{Q}$ is countably generated and the measure law(f) is inner regular with respect to a semicompact class then an rrv in $(X, \mathscr{Q})$ over $(\Omega, \mathscr{F})$ corresponding to f exists; see Sazonov **[1**, Theorem 7]. (Let us remark in passing that if one weakens slightly the definition of a regular conditional distribution of f with respect to $\bar{P}$ given $(\psi, \mathscr{F})$ then one can prove that such a thing exists assuming that law(f) is inner regular with respect to a semicompact class. That is, the requirement that $\mathscr{Q}$ be countably generated can be dropped. This is due to Pachl **[1]**.) If the measurable space $(X, \mathscr{Q})$ is "nice" then the requirement that law(f) be inner regular with respect to a semicompact class is satisfied automatically. More precisely, let $(X', \mathscr{Q}')$ be the quotient of $(X, \mathscr{Q})$ obtained by identifying points of X which cannot be separated by an element of $\mathscr{Q}$. Then (a) through (d) below are equivalent:

(a) $(X, \mathscr{Q})$ is countably generated and every finite measure on $\mathscr{Q}$ is inner regular with respect to a semicompact class (depending on the measure).

(b) $(X', \mathscr{Q}')$ is Borel isomorphic to a universally measurable subspace of $\mathbf{R}$.

(c) $(X', \mathcal{Q}')$ is Borel isomorphic to a universally measurable subspace of a Polish space.

(d) $(X, \mathcal{Q})$ is countably generated and $(X', \mathcal{Q}')$ is universally measurable in every countably separated measurable space in which it is Borel embedded.

If we are given an rrv χ in $(X, \mathcal{Q})$ over $(\Omega, \mathcal{F}, P)$ then we can always find an enlargement $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}, \psi)$ of $(\Omega, \mathcal{F}, P)$ such that χ corresponds to an rv f in $(X, \mathcal{Q})$ over $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})$; namely let

$\bar{\Omega} = \Omega \times X$,

$\bar{\mathcal{F}} = \mathcal{F} \otimes \mathcal{Q}$,

$\bar{P}(F) = \int_\Omega \chi(\omega)(F(\omega))\, dP(\omega)$ for F in $\bar{\mathcal{F}}$ (here $F(\omega) = \{x \in X: (\omega, x) \in F\}) =$ section of F over ω,

$\psi =$ projection of $\bar{\Omega}$ on Ω,

$f =$ projection of $\bar{\Omega}$ on X.

If we are given rv's f_1 and f_2 over an enlargement of a probability space, and corresponding rrv's χ_1 and χ_2 over the original probability space, there is no way to recover the joint distribution of f_1 and f_2 from χ_1, χ_2, and the original probability space alone. This is why one must consider enlargements and not just rrv's.

4.5. *Terminology*. By a filtered probability space we mean a system $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ where $(\Omega, \mathcal{F}, P)$ is a probability space and $(\mathcal{F}_t)$ is an increasing family of sub-σ-fields of $\mathcal{F}$ indexed by $[0, \infty]$.

4.6. DEFINITION. An *enlargement* of a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ is a filtered probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathcal{F}}_t, \bar{P})$ together with a map $\psi: \bar{\Omega} \to \Omega$ such that $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}, \psi)$ is an enlargement of $(\Omega, \mathcal{F}, P)$ and ψ is $(\bar{\mathcal{F}}_t, \mathcal{F}_t)$-measurable for all t in $[0, \infty]$.

Now any old enlargement of $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ may be rather bad in the sense that its filtration may be so large as to bear no useful relationship to the original filtration $(\mathcal{F}_t)$. For example, we could have $\bar{\mathcal{F}}_t = \bar{\mathcal{F}}$ for all t. In the next definition we single out the "good" enlargements. In results which follow, it will become clear why these enlargements are good.

4.7. DEFINITION. An enlargement $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathcal{F}}_t, \bar{P}, \psi)$ of a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ will be called *optional* iff for each t in $[0, \infty)$ and each F in $\bar{\mathcal{F}}_t$ there exists an $\mathcal{F}_t$-measurable version of $\bar{P}(F|\psi, \mathcal{F})$. (We remark that by a version of $\bar{P}(F|\psi, \mathcal{F})$ we mean an $\mathcal{F}$-measurable function f on Ω such that $\bar{P}(F \cap \psi^{-1}[G]) = \int_G f\, dP$ for all G in $\mathcal{F}$.)

Equivalently, the enlargement is optional iff for each t in $[0, \infty)$ and each F in $\bar{\mathcal{F}}_t$ there exists a $\psi^{-1}(\mathcal{F}_t)$-measurable version of $\bar{P}(F|\psi^{-1}(\mathcal{F}))$. In other words, the enlargement is optional iff, for $0 \leqslant t < \infty$, $\bar{\mathcal{F}}_t$ and $\psi^{-1}(\mathcal{F})$ are conditionally independent given $\psi^{-1}(\mathcal{F}_t)$; see Meyer [**3**, p. 52, T51]. The notion of optional enlargement is a slight generalization of the notion of distributional enlargement defined in Baxter and Chacon [**2**]. Recall that stopping times are also referred to as optional times. The equivalence of (a) and (b) in the following proposition is the reason for our choice of the term "optional" to describe these enlargements.

4.8. PROPOSITION. *Let* $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathcal{F}}_t, \bar{P})$ *be an enlargement of a filtered probability space* $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$. *For any process* (X_t) *with sample space* Ω *let* $(\bar{X}_t)$ *denote the* *process* $(X_t \circ \psi)$, *which has sample space* $\bar{\Omega}$. *Consider the following statements*:

(a) *The enlargement is optional.*

(b) *Whenever* T *is an* $(\bar{\mathcal{F}}_t)$*-stopping time and* τ *is an rrv in* $[0, \infty]$ *over* $(\Omega, \mathcal{F})$ *corresponding to* T *we have that for each* t *in* $[0, \infty)$ *there exists an* $\mathcal{F}_t$*-measurable function* h *such that*

$$P\big(\{\omega \in \Omega : \tau(\omega)([0, t]) \neq h(\omega)\}\big) = 0.$$

(c) *Whenever* $(X_t)_{0 \leq t \leq \infty}$ *is a martingale over* $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ *then* $(\bar{X}_t)_{0 \leq t \leq \infty}$ *is a martingale over* $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathcal{F}}_t, \bar{P})$.

(d) *Whenever* $(E, \mathcal{E})$ *is a measurable space and* $(X_t)_{0 \leq t \leq \infty}$ *is a Markov process in* $(E, \mathcal{E})$ *over* $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ *then* $(\bar{X}_t)_{0 \leq t \leq \infty}$ *is a Markov process over* $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathcal{F}}_t, \bar{P})$.

(e) *Whenever* $(E, \mathcal{E})$ *is a measurable space,* $(P_{s,t})_{0 \leq s \leq t \leq \infty}$ *is a Markov transition function on* $(E, \mathcal{E})$ *and* $(X_t)_{0 \leq t \leq \infty}$ *is a Markov process in* $(E, \mathcal{E})$ *over* $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ *with transition function* $(P_{s,t})$ *then* $(\bar{X}_t)$ *is a Markov process over* $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathcal{F}}_t, \bar{P})$ *with transition function* $(P_{s,t})$.

Then (a) *is equivalent to* (b), *and* (a) *implies* (c), (d), *and* (e). *If for every* F *in* $\mathcal{F}$ *there exists* G *in* $\mathcal{F}_\infty$ *such that* $P(F \triangle G) = 0$ *then* (a), (b), (c), *and* (d) *are equivalent.*

PROOF. (a) $\Rightarrow$ (b). Fix t in $[0, \infty)$. Now $H \equiv \{T \leq t\}$ belongs to $\bar{\mathcal{F}}_t$ so there exists an $\mathcal{F}_t$-measurable version h of $\bar{P}(H|\psi, \mathcal{F})$. But since τ corresponds to T, $\tau(\cdot)([0, t])$ is a version of $\bar{P}(H|\psi, \mathcal{F})$. Hence $P(\tau(\cdot)([0, t]) \neq h) = 0$.

(b) $\Rightarrow$ (a). Choose t in $[0, \infty)$ and H in $\bar{\mathcal{F}}_t$. Let

$$T = \begin{cases} t & \text{on } H, \\ \infty & \text{on } \bar{\Omega} \setminus H. \end{cases}$$

Then T is an $(\bar{\mathcal{F}}_t)$-stopping time and $\{T \leq t\} = H$. By the discussion following Definition 4.4, there exists an rrv τ in $[0, \infty]$ over $(\Omega, \mathcal{F})$ corresponding to T.

By assumption there exists an $\mathcal{F}_t$-measurable function h such that

$$P\big(\tau(\cdot)([0, t]) \neq h\big) = 0.$$

Then h is an $\mathcal{F}_t$-measurable version of $\bar{P}(H|\psi, \mathcal{F})$. Thus the enlargement is optional.

Now let us make an observation. Let $\mathcal{G}$ and $\mathcal{H}$ be sub-σ-fields of $\mathcal{F}$ with $\mathcal{G} \subseteq \mathcal{H}$ and let $\bar{\mathcal{G}}$ be a sub-σ-field of $\bar{\mathcal{F}}$ such that ψ is $(\bar{\mathcal{G}}, \mathcal{G})$-measurable. Then (*) and (**) below are equivalent:

(*) For every G in $\bar{\mathcal{G}}$, there exists a $\mathcal{G}$-measurable version of $\bar{P}(G|\psi, \mathcal{H})$.

(**) For every h in $L^1(\Omega, \mathcal{H}, P)$ we have $\bar{E}(h \circ \psi|\bar{\mathcal{G}}) = E(h|\mathcal{G}) \circ \psi$ $\bar{P}$-a.s.

To see that (*) implies (**), choose h in $L^1(\Omega, \mathcal{H}, P)$ and G in $\bar{\mathcal{G}}$ and let g be a $\mathcal{G}$-measurable version of $\bar{P}(G|\psi, \mathcal{H})$. Then

$$\int_G h \circ \psi \, d\bar{P} = E\big[\bar{P}(G|\psi, \mathcal{H})h\big] = E[gh]$$

$$= E[gE(h|\mathcal{G})] = \int_G E(h|\mathcal{G}) \circ \psi \, d\bar{P}$$

and $E(h|\mathcal{G}) \circ \psi$ is $\bar{\mathcal{G}}$-measurable. Hence $E(h|\mathcal{G}) \circ \psi = \bar{E}(h \circ \psi|\bar{\mathcal{G}})$ $\bar{P}$-a.s. To see that (**) implies (*), choose G in $\bar{\mathcal{G}}$. Then for any h in $L^1(\Omega, \mathcal{H}, P)$,

$$\begin{aligned} E\{E[\bar{P}(G|\psi, \mathcal{H})|\mathcal{G}]h\} &= E\{\bar{P}(G|\psi, \mathcal{H})E(h|\mathcal{G})\} \\ &= \int_G E(h|\mathcal{G}) \circ \psi \, d\bar{P} = \int_G \bar{E}(h \circ \psi|\bar{\mathcal{G}}) \, d\bar{P} = \int_G h \circ \psi \, d\bar{P} \\ &= E\{\bar{P}(G|\psi, \mathcal{H})h\}. \end{aligned}$$

Therefore $E[\bar{P}(G|\psi, \mathcal{H})|\mathcal{G}] = \bar{P}(G|\psi, \mathcal{H})$ P-a.s.

(a) $\Rightarrow$ (c). This follows immediately from (*) $\Rightarrow$ (**).

(a) $\Rightarrow$ (d). The assumption is that (X_t) is adapted to $(\mathcal{F}_t)$ and $E(f(X_t)|\mathcal{F}_s) = E(f(X_t)|\sigma(X_s))$ P-a.s. whenever $0 \leqslant s < t \leqslant \infty$ and f is a bounded $\mathcal{E}$-measurable function on E. Let s, t, and f be so chosen. Then using (*) $\Rightarrow$ (**) we see that

$$\begin{aligned} \bar{E}(f(\bar{X}_t)|\bar{\mathcal{F}}_s) &= \bar{E}(f(X_t) \circ \psi|\bar{\mathcal{F}}_s) = E(f(X_t)|\mathcal{F}_s) \circ \psi \\ &= E(f(X_t)|\sigma(X_s)) \circ \psi = \bar{E}[f(X_t) \circ \psi|\psi^{-1}(\sigma(X_s))] \\ &= \bar{E}(f(\bar{X}_t)|\sigma(\bar{X}_s)) \quad \bar{P}\text{-a.s.} \end{aligned}$$

Also, it is clear that $(\bar{X}_t)$ is adapted to $(\bar{\mathcal{F}}_t)$.

(a) $\Rightarrow$ (e). Here the assumption is that (X_t) is $(\mathcal{F}_t)$-adapted and $E(f(X_t)|\mathcal{F}_s) = P_{s,t}(X_s, f)$ P-a.s. whenever $0 \leqslant s < t \leqslant \infty$ and f is a bounded $\mathcal{E}$-measurable function on E. The proof that this is also true of the enlarged process is similar to the proof of (a) $\Rightarrow$ (d).

For the remainder of the proof we suppose that for every F in $\mathcal{F}$ there exists G in $\mathcal{F}_\infty$ such that $P(F \triangle G) = 0$.

(c) $\Rightarrow$ (a). Let h be in $L^1(\Omega, \mathcal{F}, P)$ and let $X_t = E(h|\mathcal{F}_t)$ for $0 \leqslant t \leqslant \infty$. Then (X_t) is a martingale over $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$. Hence $(\bar{X}_t)$ is a martingale over $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathcal{F}}_t, \bar{P})$. Thus for any s in $[0, \infty)$, $\bar{E}(\bar{X}_\infty|\bar{\mathcal{F}}_s) = \bar{X}_s$ $\bar{P}$-a.s. But $X_\infty = h$ P-a.s. and $\bar{X}_s = E(h|\mathcal{F}_s) \circ \psi$. Thus $\bar{E}(h \circ \psi|\bar{\mathcal{F}}_s) = E(h|\mathcal{F}_s) \circ \psi$ $\bar{P}$-a.s. By (**) $\Rightarrow$ (*), it follows that the enlargement is optional.

(d) $\Rightarrow$ (a). Choose $(E, \mathcal{E})$ and (X_t) so that $\mathcal{F}_t = X_t^{-1}(\mathcal{E})$ for $0 \leqslant t \leqslant \infty$. (This is possible. For example, we can take $E = [0, \infty] \times \Omega$, $\mathcal{E}$ = the σ-field of $(\mathcal{F}_t)$-progressively measurable sets, and $X_t(\omega) = (t, \omega)$.) It is trivial that (X_t) is Markov over $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$. Hence by assumption, $(\bar{X}_t)$ is Markov over $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathcal{F}}_t, \bar{P})$. Let h be a bounded $\mathcal{F}$-measurable function on Ω. Then h is equal P-a.s. to some bounded $\mathcal{F}_\infty$-measurable function g. As $\mathcal{F}_\infty = X_\infty^{-1}(\mathcal{E})$, there is a bounded $\mathcal{E}$-measurable function f on E such that $g = f(X_\infty)$. Then for any s in $[0, \infty)$ we have

$$\begin{aligned} \bar{E}(f(\bar{X}_\infty)|\bar{\mathcal{F}}_s) &= \bar{E}(f(\bar{X}_\infty)|\sigma(\bar{X}_s)) \\ &= \bar{E}[f(X_\infty) \circ \psi|\psi^{-1}(\sigma(X_s))] = E(f(X_\infty)|\sigma(X_s)) \circ \psi \\ &= E(f(X_\infty)|\mathcal{F}_s) \circ \psi \quad \bar{P}\text{-a.s.} \end{aligned}$$

As $h = f(X_\infty)$ P-a.s., it follows that $\bar{E}(h \circ \psi|\bar{\mathcal{F}}_s) = E(h|\mathcal{F}_s) \circ \psi$ $\bar{P}$-a.s. By approximation, this extends to any h in $L^1(\Omega, \mathcal{H}, P)$. Then by (**) $\Rightarrow$ (*), it follows that the enlargement is optional.

REMARKS. (1) *A randomized* $(\mathcal{F}_t)$*-stopping time* (abbreviated $(\mathcal{F}_t)$-rst) is an rrt τ over $(\Omega, \mathcal{F})$ such that the function $\omega \mapsto \tau(\omega)([0, t])$ is $\mathcal{F}_t$-measurable for each t in $[0, \infty]$. If $(\mathcal{F}_t)$ is right-continuous and χ is an rrt over $(\Omega, \mathcal{F})$ such that for every t in $[0, \infty)$ there exists an $\mathcal{F}_t$-measurable function f such that $P(\chi(\cdot)([0, t]) \neq f) = 0$ then one can show there exists an $(\mathcal{F}_t)$-rst τ such that $P(\tau \neq \chi) = 0$. (This is analogous to Lemma 3A.1.) Thus when $(\mathcal{F}_t)$ is right-continuous, (a) is also equivalent to:

(b′) To every $(\overline{\mathcal{F}}_t)$-stopping time there corresponds an $(\mathcal{F}_t)$-rst.

(2) The statements (c), (d), and (e) have been phrased in terms of processes with time set $[0, \infty]$. One can also consider the analogous statements (c′), (d′), and (e′) for processes with time set $[0, \infty)$. Then one can show that (a) implies (c′), (d′), and (e′), and that if for every F in $\mathcal{F}$ there exists G in $\sigma(\bigcup_{0 \leq t < \infty} \mathcal{F}_t)$ such that $P(F \triangle G) = 0$ then (a), (c′), and (d′) are equivalent.

4.9. PROPOSITION. *Let* $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathcal{F}}_t, \overline{P}, \psi)$ *be an enlargement of a filtered probability space* $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$. *Let* $(E, \mathcal{E})$ *be a measurable space and let* $(X_t)_{0 \leq t \leq \infty}$ *be a Markov process in* $(E, \mathcal{E})$ *over* $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$. *Let* $\overline{X}_t = X_t \circ \psi$ *for* $0 \leq t \leq \infty$. *Suppose*

(a) *for every* F *in* $\mathcal{F}$ *there exists* G *in* $\sigma(X_t\colon 0 \leq t \leq \infty)$ *such that* $P(F \triangle G) = 0$;

(b) $(\overline{X}_t)_{0 \leq t \leq \infty}$ *is Markov over* $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathcal{F}}_t, \overline{P})$.

Then $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathcal{F}}_t, \overline{P}, \psi)$ *is an optional enlargement of* $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$.

PROOF. By $(**) \Rightarrow (*)$ of the proof of 4.8, it is sufficient (and necessary) to show that for each f in $L^1(\Omega, \mathcal{F}, P)$ and each t in $[0, \infty)$ we have

$$\overline{E}\big(f \circ \psi | \overline{\mathcal{F}}_t\big) = E(f | \mathcal{F}_t) \circ \psi \quad \overline{P}\text{-a.s.}$$

In view of (a) it is enough to show this for f of the form $g h_1(X_{t_1}) \cdots h_k(X_{t_k})$ where $t < t_1 < \cdots < t_k \leq \infty$, g is a bounded $\mathcal{F}_t$-measurable function on Ω, and $h_1, \ldots, h_k$ are bounded $\mathcal{E}$-measurable functions on E. Then

$$\begin{aligned}
\overline{E}\big(f \circ \psi | \overline{\mathcal{F}}_t\big) &= (g \circ \psi)\overline{E}\big(h_1\big(\overline{X}_{t_1}\big) \cdots h_k\big(\overline{X}_{t_k}\big) | \overline{\mathcal{F}}_t\big) \\
&= (g \circ \psi)\overline{E}\big(h_1\big(\overline{X}_{t_1}\big) \cdots h_k\big(\overline{X}_{t_k}\big) | \sigma\big(\overline{X}_t\big)\big) \\
&= (g \circ \psi)\overline{E}\big\{\big[h_1(X_{t_1}) \cdots h_k(X_{t_k})\big] \circ \psi | \psi^{-1}(\sigma(X_t))\big\} \\
&= (g \circ \psi)\big(E\big\{h_1(X_{t_1}) \cdots h_k(X_{t_k}) | \sigma(X_t)\big\} \circ \psi\big) \\
&= (g \circ \psi)\big(E\big\{h_1(X_{t_1}) \cdots h_k(X_{t_k}) | \mathcal{F}_t\big\} \circ \psi\big) \\
&= \big[gE\big(h_1(X_{t_1}) \cdots h_k(X_{t_k}) | \mathcal{F}_t\big)\big] \circ \psi \\
&= E(f | \mathcal{F}_t) \circ \psi \quad \overline{P}\text{-a.s.}
\end{aligned}$$

In the second and fifth steps here we have used (iii) $\Rightarrow$ (ii) of (a) of Theorem 1.3 on p. 12 of Blumenthal and Getoor [**1**]. □

REMARKS. (1) I thank Dr. Martin Barlow of the University of Liverpool for showing me how to prove the above proposition.

(2) The analogous proposition for the case when (X_t) has time set $[0, \infty)$ can be obtained as a corollary by setting X_∞ equal to a constant function.

(3) Let μ be a probability measure on $\mathbf{R}^n$ and let $(\Omega, \mathfrak{B}^0, \mathfrak{B}_t^0, B_t, P^\mu)$ be the canonical realization of Brownian motion in $\mathbf{R}^n$ having initial law μ and with uncompleted σ-fields. To be a bit more explicit, Ω is the set of maps ω: $[0, \infty] \to \mathbf{R}_\partial^n$ such that ω is continuous on $[0, \infty)$ (recall ∂ is an isolated point of $\mathbf{R}_\partial^n$) and $\omega(\infty) = \partial$, $B_t(\omega) = \omega(t)$ for ω in Ω and $0 \leqslant t \leqslant \infty$, $\mathfrak{B}_t^0 = \sigma(B_s\colon 0 \leqslant s \leqslant t)$ for $0 \leqslant t \leqslant \infty$, and $\mathfrak{B}^0 = \mathfrak{B}_\infty^0$. If $(\overline{\Omega}, \overline{\mathfrak{B}}, \overline{\mathfrak{B}}_t, \overline{P}, \psi)$ is an optional enlargement of $(\Omega, \mathfrak{B}^0, \mathfrak{B}_t^0, P^\mu)$ and $\overline{B}_t = B_t \circ \psi$ for $0 \leqslant t \leqslant \infty$ then $(\overline{\Omega}, \overline{\mathfrak{B}}, \overline{\mathfrak{B}}_t, \overline{B}_t, \overline{P})$ is a Brownian motion process in $\mathbf{R}^n$ with initial law μ, by (a) $\Rightarrow$ (e) of 4.8. Also for each $\overline{\omega}$ in $\overline{\Omega}$, $\psi(\overline{\omega})$ is the sample path $(\overline{B}_t(\overline{\omega}))_{0 \leqslant t \leqslant \infty}$.

On the other hand, suppose we are given a Brownian motion process $(\overline{\Omega}, \overline{\mathfrak{B}}, \overline{\mathfrak{B}}_t, \overline{B}_t, \overline{P})$ in $\mathbf{R}^n$ with initial law μ. For each $\overline{\omega}$ in $\overline{\Omega}$ let $\psi(\overline{\omega})$ be the sample path $(\overline{B}_t(\overline{\omega}))_{0 \leqslant t \leqslant \infty}$. Then $B_t \circ \psi = \overline{B}_t$ for $0 \leqslant t \leqslant \infty$ and one verifies easily that $(\overline{\Omega}, \overline{\mathfrak{B}}, \overline{\mathfrak{B}}_t, \overline{P}, \psi)$ is an enlargement of $(\Omega, \mathfrak{B}^0, \mathfrak{B}_t^0, P^\mu)$; it follows from Proposition 4.9 that it is an *optional* enlargement. Thus Brownian motion processes in $\mathbf{R}^n$ with initial law μ are essentially the same thing as optional enlargements of $(\Omega, \mathfrak{B}^0, \mathfrak{B}_t^0, B_t, P^\mu)$.

4.10. *A topology on the randomized random variables.* Let H be a compact metrizable space. Then $C(H)$, the space of continuous real-valued functions on H, equipped with the supremum norm, is a separable Banach space. Let $(\Omega, \mathfrak{F})$ be a measurable space. A Carathéodory function (over $(\Omega, \mathfrak{F}; H)$) is a real-valued function ϕ on $\Omega \times H$ such that $\phi(\omega, \cdot)$ is continuous on H for every ω in Ω and $\phi(\cdot, h)$ is $\mathfrak{F}$-measurable for every h in H. It is a simple matter to show that every Carathéodory function is $\mathfrak{F} \otimes$ (Borel H)-measurable and that a map $f\colon \Omega \to C(H)$ is $\mathfrak{F}$-measurable iff the function $(\omega, h) \mapsto f(\omega)(h)$ is Carathéodory. Now let P be a probability measure on $\mathfrak{F}$. Then $\mathfrak{L}^1(\Omega, \mathfrak{F}, P; C(H))$ will denote the space of $\mathfrak{F}$-measurable functions $f\colon \Omega \to C(H)$ such that

$$\|f\|_1 \equiv \int_\Omega \|f(\omega)\|_{\sup}\, dP(\omega)$$

is finite. $\mathfrak{L}^1(\Omega, \mathfrak{F}, P; C(H))$ is a complete seminormed linear space when equipped with the seminorm $\| \ \|_1$. If χ is an rrv in H over $(\Omega, \mathfrak{F})$ then χ gives rise to a continuous linear functional $\langle \cdot , \chi \rangle_P$ on $\mathfrak{L}^1(\Omega, \mathfrak{F}, P; C(h))$ defined by

$$\langle f, \chi \rangle_P = \int_\Omega \int_H f(\omega)(h)\, \chi(\omega)(dh)\, P(d\omega).$$

Using a couple of Riesz representation theorems, one verifies easily that a continuous linear functional Λ on $\mathfrak{L}^1(\Omega, \mathfrak{F}, P; C(H))$ is of the form $\langle \cdot , \chi \rangle_P$ for some rrv χ in H over $(\Omega, \mathfrak{F})$ iff $f \geqslant 0$ implies $\Lambda(f) \geqslant 0$, and $\Lambda(1) = 1$. Note that in this case one necessarily has $\|\Lambda\| = 1$. Thus the set of linear functionals of the form $\langle \cdot , \chi \rangle_P$, where χ is an rrv in H over $(\Omega, \mathfrak{F})$, is a bounded weak*-closed subset of the dual of $\mathfrak{L}^1(\Omega, \mathfrak{F}, P; C(H))$. It follows that if we equip the space of randomized random variables in H over $(\Omega, \mathfrak{F})$ with the topology generated by the maps of the form $\chi \mapsto \langle f, \chi \rangle_P$ where f is in $\mathfrak{L}^1(\Omega, \mathfrak{F}, P; C(H))$, then it is a compact regular (almost never Hausdorff) space which we shall denote by $\mathfrak{RRV}(\Omega, \mathfrak{F}, P; H)$. Note that if $\mathfrak{F}$ is countably generated mod P then $\mathfrak{L}^1(\Omega, \mathfrak{F}, P; C(H))$ is separable and $\mathfrak{RRV}(\Omega, \mathfrak{F}, P; H)$ is pseudometrizable. If $\mathfrak{G}$ is a sub-σ-field of $\mathfrak{F}$ containing all the P-null sets of $\mathfrak{F}$ then $\mathfrak{RRV}(\Omega, \mathfrak{G}, P; H)$ is closed in $\mathfrak{RRV}(\Omega, \mathfrak{F}, P; H)$. From this it follows that the closure of every countable subset of $\mathfrak{RRV}(\Omega, \mathfrak{F}, P; H)$ is pseudometrizable and that $\mathfrak{RRV}(\Omega, \mathfrak{F}, P; H)$ is sequentially compact, even when $\mathfrak{F}$ is not countably generated mod P.

4.11. *The space of right-continuous increasing functions.* Let H denote the set of increasing right-continuous maps h: $[0, \infty) \to [0, \infty]$. (When we say h is increasing we just mean that $a \leq b$ implies $h(a) \leq h(b)$.) Since $[0, \infty]$ may be identified with $[0, 1]$ by means of an order-preserving homeomorphism, H may be identified in a natural way with the space of distribution functions of positive measures μ on $[0, \infty)$ satisfying $\mu([0, \infty)) \leq 1$. The space of such measures is compact and metrizable in its vague topology, so H becomes a compact metrizable space by means of this identification. Clearly the topology so induced on H does not depend on the particular order-preserving homeomorphism between $[0, \infty]$ and $[0, 1]$ which we used. As is well known, a sequence (h_n) in H converges to h in H iff $h_n(a) \to h(a)$ for each continuity point a of h. Now each h in H is continuous at all but countably many points of $[0, \infty)$. Thus if ϕ is any continuous function on $[0, \infty]$ and f is any Lebesgue integrable function on $[0, \infty)$ then the function

$$h \mapsto \int_{[0,\infty)} f(a)\phi(h(a))\, da$$

is continuous on H. Using the right-continuity of the h's we can deduce from this that, for each a in $[0, \infty)$, the evaluation map $h \mapsto h(a)$ is a Borel function on $\mathfrak{H}$. Now countably many of these evaluation maps suffice to separate the points of H. Let $\mathfrak{H}$ = Borel H. As $(H, \mathfrak{H})$ is a Souslinian (indeed Lusinian) measurable space in the sense of Dellacherie and Meyer [**1**], this implies that $\mathfrak{H}$ is the smallest σ-field on H making all the evaluation maps measurable. For each t in $[0, \infty]$ let $\mathfrak{H}_t$ be the σ-field on H generated by the sets of the form $\{h \in H: h(a) \in E\}$ where a is in $[0, \infty)$ and E is in Borel$[0, t]$. Clearly $(\mathfrak{H}_t)$ is an increasing family of countably generated sub-σ-fields of $\mathfrak{H}$ and $\mathfrak{H}_\infty = \mathfrak{H}$.

4.12. PROPOSITION. *Let* $(\Omega, \mathfrak{F}, \mathfrak{F}_t, P)$ *be a filtered probability space. Assume* $(\mathfrak{F}_t)$ *is right-continuous. Let* $H, \mathfrak{H}, \mathfrak{H}_t$ *be as in* 4.11. *Let* χ *be an rrv in H over* $(\Omega, \mathfrak{F})$. *Also let*

$\overline{\Omega} = \Omega \times H$,
$\overline{\mathfrak{F}} = \mathfrak{F} \otimes \mathfrak{H}$,
$\overline{\mathfrak{F}}_t = \bigcap_{s>t} \mathfrak{F}_s \otimes \mathfrak{H}_s$ *for* $0 \leq t < \infty$,
$\overline{\mathfrak{F}}_\infty = \mathfrak{F}_\infty \otimes \mathfrak{H}_\infty$,
$\overline{P}(F) = \int_\Omega \chi(\omega)(F(\omega))\, dP(\omega)$ *for F in* $\overline{\mathfrak{F}}$,
ψ = *projection of* $\overline{\Omega}$ *on* Ω.

Then the following are equivalent:

(a) $(\overline{\Omega}, \overline{\mathfrak{F}}, \overline{\mathfrak{F}}_t, \overline{P}, \psi)$ *is an optional enlargement of* $(\Omega, \mathfrak{F}, \mathfrak{F}_t, P)$.

(b) *For any t in $[0, \infty)$ and any A in $\mathcal{H}_t$ there is an $\mathcal{F}_t$-measurable function ξ such that*

$$P(\{\omega \in \Omega: \chi(\omega)(A) \neq \xi(\omega)\}) = 0.$$

(c) *Whenever $t \in [0, \infty)$, $k \in \mathbf{N}$, $a_0, \ldots, a_k \in [0, \infty)$, $\varepsilon \in (0, \infty)$, $\phi_0, \ldots, \phi_k$ are continuous functions on $[0, \infty]$ which vanish on $[t, \infty]$, g is the function on H defined by*

$$g(h) = \prod_{i=0}^{k} \left[\frac{1}{\varepsilon} \int_{a_i}^{a_i+\varepsilon} \phi_i(h(a))\, da \right]$$

and $f \in L^1(\Omega, \mathcal{F}, P)$ such that $E(f|\mathcal{F}_t) = 0$ then $\langle f \otimes g, \chi \rangle_P = 0$; that is,

$$\int_\Omega \int_H f(\omega)\, g(h)\, \chi(\omega)(dh)\, P(d\omega) = 0.$$

PROOF. First note that $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathcal{F}}_t, \overline{P}, \psi)$ is always an enlargement of $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$.

(a) $\Rightarrow$ (b). If A is in $\mathcal{H}_t$ then $\Omega \times A$ is in $\overline{\mathcal{F}}_t$ and $\overline{P}(\Omega \times A|\psi, \mathcal{F}) = \chi(\cdot)(A)$ P-a.s.

(b) $\Rightarrow$ (c). For each i, $\phi_i = 0$ on $[t, \infty]$ so $h \mapsto \phi_i(h(a))$ is $\mathcal{H}_{t-}$-measurable for each a in $[0, \infty]$. It follows that g is $\mathcal{H}_{t-}$-measurable. Thus $\chi(\cdot)(g)$ is equal P-a.s. to an $\mathcal{F}_t$-measurable function. Hence $E(f\chi(\cdot)(g)) = 0$; i.e., $\langle f \otimes g, \chi \rangle_P = 0$.

(c) $\Rightarrow$ (b). First, for any g of the sort described, we can conclude that $\chi(\cdot)(g)$ is equal P-a.s. to an $\mathcal{F}_t$-measurable function. Then (letting $\varepsilon \to 0$) we find that whenever $t \in [0, \infty)$, $k \in \mathbf{N}$, $a_0, \ldots, a_k \in [0, \infty)$, and $\phi_0, \ldots, \phi_k$ are continuous functions on $[0, \infty]$ which vanish on $[t, \infty]$ then the function $\int_H \chi(\cdot)\,(dh)\prod_{i=0}^k \phi_i(h(a_i))$ is equal P-a.s. to an $\mathcal{F}_t$-measurable function. From this (by a monotone class argument) we can conclude that, for any t in $[0, \infty)$ and any A in $\mathcal{H}_{t-}$, $\chi(\cdot)(A)$ is equal P-a.s. to an $\mathcal{F}_t$-measurable function. Then using the right-continuity of $(\mathcal{F}_t)$ we obtain the desired conclusion.

(b) $\Rightarrow$ (a). By a monotone class argument, one finds that if t is in $[0, \infty)$ and F is in $\mathcal{F}_t \otimes \mathcal{H}_t$ then $\omega \mapsto \chi(\omega)(B(\omega))$ is equal P-a.s. to an $\mathcal{F}_t$-measurable function. Using the right-continuity of $(\mathcal{F}_t)$, one finds that this conclusion is still valid if F is only in $\overline{\mathcal{F}}_t$. But for any F in $\overline{\mathcal{F}}$, $\chi(\cdot)(F(\cdot))$ is a version of $\overline{P}(F|\psi, \mathcal{F})$. □

REMARK. A right-continuous $(\mathcal{F}_t)$-time change is a right-continuous increasing family $(T_a)_{0 \leqslant a < \infty}$ of $(\mathcal{F}_t)$-stopping times. Another way of looking at such an object is to say that it is a map $T: \Omega \to H$ such that for t in $[0, \infty)$ and A in $\mathcal{H}_t$ we have $T^{-1}[A]$ in $\mathcal{F}_t$. (The correspondence is $(T_a) = (T(\cdot)(a))$.) From this point of view, condition (b) of the proposition just says that χ is what would logically be called a randomized right-continuous $(\mathcal{F}_t)$-time change (at least if $\mathcal{F}_0$ contains all the P-null sets of $\mathcal{F}$).

4.13. THEOREM. *Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ be a filtered probability space. Assume $(\mathcal{F}_t)$ is right-continuous. Let H, $\mathcal{H}$, $\mathcal{H}_t$ be as in* 4.11 *and let $\overline{\Omega}$, $\overline{\mathcal{F}}$, $\overline{\mathcal{F}}_t$, ψ be as in* 4.12. *Let $\mathcal{O}$ be the set of probability measures $\overline{P}$ on $\overline{\mathcal{F}}$ such that $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathcal{F}}_t, \overline{P}, \psi)$ is an optional enlargement of $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$. Let $\mathcal{O}$ be given the topology generated by the maps of the form*

$$\overline{P} \mapsto \int_{\Omega \times H} f(\omega)(h)\, \overline{P}(d\omega, dh)$$

where f ranges over $\mathfrak{L}^1(\Omega, \mathfrak{F}, P; C(H))$. *Then*:

(a) $\mathfrak{O}$ *is compact*;

(b) *if* $\mathfrak{F}$ *is countably generated* mod P *then* $\mathfrak{O}$ *is metrizable*;

(c) *the closure of any countable subset of* $\mathfrak{O}$ *is metrizable*;

(d) $\mathfrak{O}$ *is sequentially compact*.

PROOF. Let $\mathfrak{P}$ be the set of all probability measures $\bar{P}$ on $\bar{\mathfrak{F}}$ such that $\psi(\bar{P}) = P$, and let $\mathfrak{P}$ be topologized analogously to $\mathfrak{O}$. Let $\mathfrak{R} = \mathfrak{R}\mathfrak{R}\mathfrak{V}(\Omega, \mathfrak{F}, P; H)$ topologized as in 4.10. Define ϕ: $\mathfrak{R} \to \mathfrak{P}$ by

$$\phi(\chi)(F) = \int_\Omega \chi(\omega)(F(\omega))\, dP(\omega).$$

By the measure disintegration theorem, ϕ is onto $\mathfrak{P}$. From the definitions of the topologies of $\mathfrak{P}$ and $\mathfrak{R}$ it is evident that a set V is open in $\mathfrak{R}$ iff it is of the form $\phi^{-1}[W]$ for some W open in $\mathfrak{P}$. It follows that ϕ is continuous, open, and closed and that the assertions (a) through (d) hold when $\mathfrak{O}$ is replaced by $\mathfrak{P}$. Thus to complete the proof, it suffices to show that $\phi^{-1}[\mathfrak{O}]$ is closed in $\mathfrak{R}$. But this follows from 4.12 since the functions g of the form considered in 4.12(c) are continuous on H, as pointed out in 4.11. □

REMARKS. (1) Subsections 4.10 through 4.13 above are an adaptation of a portion of the article of Baxter and Chacon [**2**].

(2) In the next section we consider the problem of embedding a continuous-time potential process in a Brownian motion process. The approach we use is to approximate the continuous-time potential process by discrete-time potential processes, embed these by means of Theorem 3.1, and then pass to the limit with the aid of Theorem 4.13.

5. Embedding continuous-time potential processes in Brownian motion.

5.1. THEOREM. *Let* $(\Lambda, \mathfrak{F}, \mathfrak{F}_a, X_a, Q)$ *be an n-dimensional potential process with time set* $[0, \infty)$ *and with right-continuous sample paths. Then there is a Brownian motion process* $(\Omega, \mathfrak{B}, \mathfrak{B}_t, B_t, P)$ *in* $\mathbf{R}^n$ *and a right-continuous increasing family* $(T(a))_{0 \leq a < \infty}$ *of P-standard* $(\mathfrak{B}_t)$*-stopping times such that the processes* $(X_a)_{0 \leq a < \infty}$ *and* $(B_{T(a)})_{0 \leq a < \infty}$ *have the same finite-dimensional joint distributions.*

PROOF. Let $\mu = \text{law}(X_0)$ and let $(W, \mathfrak{W}, \mathfrak{W}_t, \beta_t, \Pi)$ be the canonical realization of Brownian motion in $\mathbf{R}^n$ with initial law μ. Let

$\bar{W} = W \times (0, 1)$,

$\bar{\mathfrak{W}} = \mathfrak{W} \otimes \text{Borel}(0, 1)$,

$\bar{\mathfrak{W}}_t = \bigcap_{s>t} [\mathfrak{W}_s \otimes \text{Borel}(0, 1)]$ for $0 \leq t < \infty$,

$\bar{\mathfrak{W}}_\infty = \mathfrak{W}_\infty \otimes \text{Borel}(0, 1)$ $(= \bar{\mathfrak{W}})$,

ϕ = projection of $\bar{W}$ on W,

$\bar{\beta}_t = \beta_t \circ \phi$ for $0 \leq t \leq \infty$,

$\bar{\Pi} = \Pi \otimes$ (Lebesgue measure on Borel(0, 1)).

For each natural number k, let $I(k) = \{j2^{-k}: j = 0, 1, 2, \ldots\}$ and (appealing to Theorem 3.1) let $(S(k, a))_{a \in I(k)}$ be an increasing family of $\overline{\Pi}$-standard $(\overline{\mathcal{W}}_t)$-stopping times such that the discrete-time processes $(X_a)_{a \in I(k)}$ and $(\bar{\beta}_{S(k,a)})_{a \in I(k)}$ have the same joint distribution.

Now define X_a^k and T_a^k for all a in $[0, \infty)$ by

$$\left.\begin{aligned} X_a^k &= X_{j2^{-k}} \\ T_a^k &= S(k, j2^{-k}) \end{aligned}\right\} \quad \text{for } (j-1)2^{-k} \leqslant a < j2^{-k}, \ j = 0, 1, 2, \ldots.$$

Then the continuous-time processes $(X_a^k)_{0 \leqslant a < \infty}$ and $(\bar{\beta}_{T_a^k})_{0 \leqslant a < \infty}$ have the same finite-dimensional joint distributions and $X_a^k \to X_a$ pointwise on Λ as $k \to \infty$.

Now let H, $\mathcal{H}$, and $(\mathcal{H}_t)_{0 \leqslant t < \infty}$ be as in 4.11, and let

$\Omega = W \times H,$

$\mathcal{B} = \mathcal{W} \otimes \mathcal{H},$

$\mathcal{B}_t = \bigcap_{s > t} \mathcal{W}_s \otimes \mathcal{H}_s$ for $0 \leqslant t < \infty,$

$\mathcal{B}_\infty = \mathcal{W}_\infty \otimes \mathcal{H}_\infty \ (= \mathcal{B}),$

ψ = projection of Ω on W,

$B_t = \beta_t \circ \psi$ for $0 \leqslant t \leqslant \infty$.

For each k let Γ_k be the map of $\overline{W}$ into Ω defined by

$$\Gamma_k(\bar{w}) = \left(\phi(\bar{w}), \left(T_a^k(\bar{w})\right)_{0 \leqslant a < \infty}\right).$$

Then Γ_k is $(\overline{\mathcal{W}}, \mathcal{B})$-measurable and also $(\overline{\mathcal{W}}_t, \mathcal{B}_t)$-measurable for each t in $[0, \infty]$. ($\bar{w} \mapsto (T_a^k(\bar{w}))_{0 \leqslant a < \infty}$ is $(\overline{\mathcal{W}}_t, \mathcal{H}_t)$-measurable for each t and $(\overline{\mathcal{W}}_t)$ is right-continuous.) Next, for each k let P_k be the measure on $\mathcal{B}$ defined by $P_k(A) = \overline{\Pi}(\Gamma_k^{-1}[A])$. If t is in $[0, \infty)$ and F is in $\mathcal{B}_t$ then one verifies easily that $P_k(F|\psi, \mathcal{W}) = \overline{\Pi}(\Gamma_k^{-1}[F]|\phi, \mathcal{W})$ Π-a.s.; as there is a $\mathcal{W}_t$-measurable version of the latter conditional probability, we see that $(\Omega, \mathcal{B}, \mathcal{B}_t, P_k, \psi)$ is an optional enlargement of $(W, \mathcal{W}, \mathcal{W}_t, \Pi)$. For each a in $[0, \infty)$ define $T(a)$ on Ω by

$$T(a)(w, h) = h(a).$$

Then $T(a) \circ \Gamma_k = T_a^k$, $B_t \circ \Gamma_k = \bar{\beta}_t$ and $(T(a))_{0 \leqslant a < \infty}$ is a right-continuous increasing family of P_k-standard $(\mathcal{B}_t)$-stopping times such that the process $(B_{T(a)})_{0 \leqslant a < \infty}$ has the same finite-dimensional joint distributions relative to P_k as $(X_a^k)_{0 \leqslant a < \infty}$ has (relative to Q).

Now by Theorem 4.13, there is a subsequence $(P_{k(l)})$ of (P_k) and a probability measure P on $\mathcal{B}$ such that $(\Omega, \mathcal{B}, \mathcal{B}_t, P, \psi)$ is an optional enlargement of $(W, \mathcal{W}, \mathcal{W}_t, \Pi)$ (so $(\Omega, \mathcal{B}, \mathcal{B}_t, B_t, P)$ is a Brownian motion process by 4.8) and $\int f(w)(h) P_{k(l)}(dw, dh) \to \int f(w)(h) P(dw, dh)$ for all f in $\mathcal{L}^1(W, \mathcal{W}, \Pi; C(H))$. We claim that the stopping times $T(a)$ are P-standard and that the process $(B_{T(a)})_{0 \leqslant a < \infty}$ has the same finite-dimensional joint distributions relative to P as $(X_a)_{0 \leqslant a < \infty}$ has (relative to Q). This is somewhat easier to prove when $n \geqslant 3$ so let us consider this case first. In this case, every stopping time is standard and we need only check that the joint distributions are right. Let $D = \mathbf{R}^n$, $E = \mathbf{R}_\partial^n$, let $0 \leqslant a_1 < \cdots < a_j < \infty$, and let g be a continuous function on E^j which vanishes outside a compact subset of D^j. Then

$$\int g\big(B_{T(a_1)}, \ldots, B_{T(a_j)}\big)\, dP$$

$$(*)\quad \begin{cases} = \lim_{\varepsilon\downarrow 0} \int \frac{1}{\varepsilon}\int_0^\varepsilon g\big(B_{T(a_1+s)}, \ldots, B_{T(a_j+s)}\big)\, ds\, dP \\ = \lim_{\varepsilon\downarrow 0} \lim_{l\to\infty} \int \frac{1}{\varepsilon}\int_0^\varepsilon g\big(B_{T(a_1+s)}, \ldots, B_{T(a_j+s)}\big)\, ds\, dP_{k(l)} \end{cases}$$

$$= \lim_{\varepsilon\downarrow 0} \lim_{l\to\infty} \int \frac{1}{\varepsilon}\int_0^\varepsilon g\Big(\bar\beta_{T^{k(l)}_{a_1+s}}, \ldots, \bar\beta_{T^{k(l)}_{a_j+s}}\Big)\, ds\, d\overline{\Pi}$$

$$= \lim_{\varepsilon\downarrow 0} \lim_{l\to\infty} \int \frac{1}{\varepsilon}\int_0^\varepsilon g\big(X^{k(l)}_{a_1+s}, \ldots, X^{k(l)}_{a_j+s}\big)\, ds\, dQ$$

$$= \lim_{\varepsilon\downarrow 0} \int \frac{1}{\varepsilon}\int_0^\varepsilon g\big(X_{a_1+s}, \ldots, X_{a_j+s}\big)\, ds\, dQ$$

$$= \int g\big(X_{a_1}, \ldots, X_{a_j}\big)\, dQ.$$

This suffices to show that $(B_{T(a)})$ has the right finite-dimensional joint distributions. However, the step $(*)$ requires further explanation. (We remark that the above calculation actually holds for any n, but the method of verification of the step $(*)$ depends on n and is simplest when $n \geqslant 3$.)

$$\int_0^\varepsilon g\big(B_{T(a_1+s)}(w, h), \ldots, B_{T(a_j+s)}(w, h)\big)\, ds$$
$$= \int_0^\varepsilon g\big(\beta_{h(a_1+s)}(w), \ldots, \beta_{h(a_j+s)}(w)\big)\, ds.$$

Now if $h_m \to h$ in H then $h_m(a_i + s) \to h(a_i + s)$ for $i = 1, \ldots, j$, for all but countably many s. Also, $\beta_t(w)$ depends continuously on t except at $t = \infty$; but g is supported by a compact subset of D^j and, as $n \geqslant 3$, $\|\beta_t\| \to \infty$ Π-a.s. as $t \to \infty$. Thus for Π-a.a. w in W the map

$$h \mapsto \int_0^\varepsilon g\big(B_{T(a_1+s)}(w, h), \ldots, B_{T(a_j+s)}(w, h)\big)\, ds$$

is continuous on H. In view of the sense in which $(P_{k(l)})$ converges to P, this justifies the step $(*)$ in the case $n \geqslant 3$.

Now suppose $n = 1$ or 2. Let D, E, $a_1, \ldots, a_j$, and g be as before. Fix ε in $(0, \infty)$. Define f on Ω by

$$f(w, h) = \int_0^\varepsilon g\big(\beta_{h(a_1+s)}(w), \ldots, \beta_{h(a_j+s)}(w)\big)\, ds$$

and for each t in $[0, \infty)$ define f_t on Ω by

$$f_t(w, h) = \int_0^\varepsilon g\big(\beta_{h(a_1+s)\wedge t}(w), \ldots, \beta_{h(a_j+s)\wedge t}(w)\big)\, ds.$$

Then f, f_t $(0 \leqslant t \leqslant \infty)$ are uniformly bounded, each f_t is continuous in h, and $f = f_t$ on $W \times H_t$, where $H_t \equiv \{h \in H\colon h \leqslant t \text{ on } [0, a_j + \varepsilon)\}$. Now

$$P_k(\Omega \setminus (W \times H_t)) \leqslant P_k\big(T(a_j + \varepsilon) > t\big) = \overline{\Pi}\big(T^k_{a_j+\varepsilon} > t\big),$$

$T^k_{a_j+\varepsilon}$ is $\overline{\Pi}$-standard, and $\text{law}(\bar{\beta}_{T^k_{a_j+\varepsilon}}, \overline{\Pi})U = \text{law}(X^k_{a_j+\varepsilon})U \succ \text{law}(X_{a_j+\varepsilon+1})U$. Combining these observations with Lemmas 4.3 and 4.4 of Falkner [1] we see that

$$\lim_{t\to\infty} \sup_k P_k(\Omega \setminus (W \times H_t)) = 0.$$

Now a moment's thought reveals that each H_t is closed in H. From this and the sense in which $(P_{k(l)})$ converges to P, it follows that $P(W \times H_t) \geqslant \limsup_{l\to\infty} P_{k(l)}(W \times H_t)$. Therefore $\lim_{t\to\infty} P(\Omega \setminus (W \times H_t)) = 0$. But

$$\int f\,dP - \int f\,dP_{k(l)} = \int f - f_t\,dP + \left[\int f_t\,dP - \int f_t\,dP_{k(l)}\right] + \int f_t - f\,dP_{k(l)}.$$

Now the first term on the right-hand side here can be made small by taking t large and the third term can be made small uniformly in l by taking t large, while the second goes to 0 for each t as $l \to \infty$. Thus $\int f\,dP_{k(l)} \to \int f\,dP$. This justifies the step (*) above in the case $n \leqslant 2$, so we now know that $(B_{T(a)})$ has the right finite-dimensional joint distributions relative to P in all cases.

It remains to show that the stopping times $T(a)$ are P-standard when $n = 1$ or 2. Fix a in $[0, \infty)$. Now

$$\text{law}(B_{T(a)}; P) = \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\int_0^\varepsilon \text{law}(B_{T(a+s)}; P)\,ds$$
$$= \lim_{\varepsilon\downarrow 0} \lim_{l\to\infty} \frac{1}{\varepsilon}\int_0^\varepsilon \text{law}(B_{T(a+s)}; P_{k(l)})\,ds$$

where the second step follows from the fact that $\text{law}(B_{T(a+s)}; P_{k(l)}) \to \text{law}(B_{T(a+s)}; P)$ for each s, as was shown above. (The limits are with respect to the vague topology.) Now consider any t in $[0, \infty)$. Then

$$\text{law}(B_{T(a)\wedge t}; P) = \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\int_0^\varepsilon \text{law}(B_{T(a+s)\wedge t}; P)\,ds$$
$$= \lim_{\varepsilon\downarrow 0} \lim_{l\to\infty} \frac{1}{\varepsilon}\int_0^\varepsilon \text{law}(B_{T(a+s)\wedge t}; P_{k(l)})\,ds$$

where in this case the second step follows from the fact that if f is a bounded continuous function on E then the map $(w, h) \mapsto \int_0^\varepsilon f(\beta_{h(a+s)\wedge t}(w))\,ds$ is continuous in h. Let $\alpha = \text{law}(X_{a+1})$. Then for $0 < s < \frac{1}{2}$ and $k(l) > 1$,

$$\text{law}(B_{T(a+s)\wedge t}; P_{k(l)})U \geqslant \text{law}(B_{T(a+s)}; P_{k(l)})U \succ \alpha U.$$

Averaging this inequality over s from 0 to ε and interchanging orders of integration we obtain

$$\rho_{t,\varepsilon,l}U \geqslant \nu_{\varepsilon,l}U \succ \alpha U$$

where $\rho_{t,\varepsilon,l} = \varepsilon^{-1}\int_0^\varepsilon \text{law}(B_{T(a+s)\wedge t}; P_{k(l)})\,ds$ and

$$\nu_{\varepsilon,l} = \frac{1}{\varepsilon}\int_0^\varepsilon \text{law}(B_{T(a+s)}; P_{k(l)})\,ds.$$

(That the interchange in order of integration is valid is obvious when $n = 1$ since in this case the potential kernel is of one sign. Consider the case $n = 2$. Let σ be the uniform unit distribution on $\{x \in \mathbf{R}^2: \|x\| = 1\}$. Then for any special measure λ on $\mathbf{R}^2$,

$$\lambda U_- = \Phi^- * \lambda = (-\Phi * \sigma) * \lambda = (-\Phi * \lambda) * \sigma = (-\lambda U) * \sigma$$

so that for $0 \leqslant s \leqslant \frac{1}{2}$ and $k(l) \geqslant 1$ we have

$$\text{law}(B_{T(a+s)\wedge t};\ P_{k(l)})U_- \leqslant \text{law}(B_{T(a+s)};\ P_{k(l)})U_- \leqslant \alpha U_-.$$

This justifies the interchange in the order of integration in the case $n = 2$.) Now first letting $l \to \infty$ and then letting $\varepsilon \downarrow 0$ and applying the convergence result 3.14 of Falkner [1] we find that $\text{law}(B_{T(a)\wedge t};\ P)U \geqslant \text{law}(B_{T(a)};\ P)U$. As this is true for arbitrarily large t, $T(a)$ is P-standard. □

Remark. When the dimension n is equal to 1, the above theorem reduces to Theorem 11 of Monroe [1] on embedding right-continuous martingales in Brownian motion. See also the discussion of Theorem 3.1.

Example. In the above theorem, the filtration $(\mathfrak{B}_t)$ is larger than the natural filtration of the process (B_t). When $n \geqslant 2$, the necessity of using an enlarged filtration is clear since enlargement of the filtration is necessary just to embed a measure, never mind a process, on account of difficulties with polar sets. When $n = 1$ however, we know that the embedding theorems for measures (ETM) and discrete-time processes (3.1) work for the natural filtration of (B_t). It is thus natural to ask whether a continuous-time martingale can be embedded in Brownian motion by means of an increasing family of *natural* standard stopping times. Now Dubins and Schwarz [1] have shown that if $(X_s)_{0 \leqslant s < \infty}$ is a continuous martingale with unbounded paths and without intervals of constancy then there is an increasing continuous family $(S(t))_{0 \leqslant t < \infty}$ of stopping times for the right-continuous modification of the natural filtration of (X_s) such that $(X_{S(t)})$ is Brownian motion. What we want though is an embedding in the other direction and, as was realized by R. V. Chacon, it is not generally possible to do this with natural stopping times, for the following reason: almost surely, Brownian motion (or more generally, any decent Markov process without holding points–see Chacon and Jamison [1]) traverses each of its trajectories at only one rate, whereas a martingale may traverse the same trajectory at a variety of rates. Hence to embed a martingale in Brownian motion, it is necessary to work with a Brownian motion process in which we have many copies of each trajectory and in which stopping times can depend on which copy of a trajectory we are on. Here is an explicit example showing the necessity of this:

Let $(W, \mathfrak{W}, \mathfrak{W}_s, \beta_s, \Pi)$ be the canonical version of Brownian motion in $\mathbf{R}$ starting from 0. Let

$\Lambda = W \times (0, 1)$,

$\mathfrak{F} = \mathfrak{W} \otimes \text{Borel}(0, 1)$,

$X_s(w, u) = \beta_{us}(w)$ for w in W and $0 < u < 1$,

$Q = \Pi \otimes$ (Lebesgue measure on Borel(0, 1)).

Then (X_s) is a continuous martingale over $(\Lambda, \mathfrak{F}, Q)$. Suppose $(\Omega, \mathfrak{B}, \mathfrak{B}_t, B_t, P)$ is a Brownian motion process in $\mathbf{R}$ and $(T(s))_{0 \leqslant s < \infty}$ is an increasing family of P-standard $(\mathfrak{B}_t)$-stopping times such that (X_s) and $(B_{T(s)})$ have the same finite-dimensional joint distributions. Now $(\Omega, \mathfrak{B}, \mathfrak{B}_{t+}, B_t, P)$ is still a Brownian motion process and $(T(s+))$ is an increasing *right-continuous* family of P-standard $(\mathcal{B}_{t+})$-stopping times such that (X_s) and $(B_{T(s+)})$ have the same finite-dimensional joint distributions. We have $B_{T(0+)} = 0$ P-a.s. since $X_0 = 0$ Q-a.s. As $T(0+)$ is P-standard, $E(|B_{T(0+)}|^2) = E(|B_0|^2) + 2E(T(0+))$. Hence $T(0+) = 0$ P-a.s. and $B_0 = 0$ P-a.s. Now by the law of the iterated logarithm,

$$\limsup_{s\downarrow 0} \frac{\beta_s}{2\sqrt{s \log|\log s|}} = 1 \quad \Pi\text{-a.s.}$$

See Lévy [**1**, p. 231]. (We have $2\sqrt{s \log|\log s|}$ rather than $\sqrt{2s \log|\log s|}$ in the denominator here because of the slightly unorthodox normalization of Brownian motion we use, as explained in 2.6.) Let

$$f = \limsup_{s\downarrow 0} \frac{X_s}{2\sqrt{s \log|\log s|}}, \qquad g = \limsup_{s\downarrow 0} \frac{B_{T(s+)}}{2\sqrt{s \log|\log s|}}.$$

As (X_s) is continuous and $(B_{T(s+)})$ is right-continuous, we actually have

$$f = \lim_{k\to\infty} \sup_{\substack{0<r<1/k \\ r \text{ rational}}} \frac{X_r}{2\sqrt{r \log|\log r|}}$$

and similarly for g. Thus f and g are measurable and are equal in law. But clearly $f(w, u) = \sqrt{u}$ for Q-a.a. (w, u) in Λ. Thus the law of g is diffuse. Now g is actually $\mathcal{B}_{T(0+)+}$-measurable and $\mathcal{B}_{T(0+)+} = \mathcal{B}_{0+}$ mod P since $P(T(0+) > 0) = 0$. Thus $\mathcal{B}_{0+}$ is P-nonatomic. But if $(\mathcal{B}_t)$ were the canonical filtration then we would have $\mathcal{B}_{0+} = \sigma(B_0)$ mod P by the right-continuity mod P of the canonical filtration of (B_t). Since $P(B_0 = 0) = 1$, $\sigma(B_0)$ is trivial mod P, so this is not the case.

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