TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 269, Number 2, February 1982

THE AUTOMORPHISM GROUP OF A COMPOSITION OF QUADRATIC FORMS[1]

BY

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ABSTRACT. Let $U \times X \to X$ be a (bilinear) composition $(u, x) \mapsto ux$ of two quadratic spaces U and X over a field F of characteristic $\neq 2$ and assume there is a vector in U which induces the identity map on X via this composition. Define G to be the subgroup of $O(U) \times O(X)$ consisting of those pairs (ϕ, ψ) satisfying $\phi(u)\psi(x) = \psi(ux)$ identically and define G_X to be the projection of G on $O(X)$. The group G is investigated and in particular it is shown that its connected component, as an algebraic group, is isogenous to a product of two or three classical groups and so is reductive. Necessary and sufficient conditions are given for G_X to be transitive on the unit sphere of X when U and X are Euclidean spaces.

Let U and X be nondegenerate quadratic spaces over a field F of characteristic $\neq 2$. We denote both of the quadratic forms by q, $q\colon U \to F$, $q\colon X \to F$. A *composition* of the two forms is a bilinear map $U \times X \to X$, $(u, x) \mapsto ux$ satisfying the identity $q(ux) = q(u)q(x)$.

We shall assume that there is a vector u_0 in U satisfying $u_0 x = x$ for all x.

Define an automorphism group

$$G = \{(\phi, \psi) \in O(U) \times O(X)\colon \psi(ux) = \phi(u)\psi(x) \text{ for all } u \text{ and } x\}.$$

O is the orthogonal group. Let G_X be the projection of G in $O(X)$. A. Kaplan[2] has asked the following questions in the case when U and X are Euclidean spaces (i.e. $F = \mathbf{R}$, the real numbers, and both quadratic forms are positive definite).

(1) What kind of (algebraic) group is G?

(2) Is G_X transitive on the unit sphere $X^* = \{x \in X\colon q(x) = 1\}$?

We shall answer (1) for any F by showing, inter alia, that the connected component of G is a group isogenous to the product of the spin group of $(V, -q)$ and of one or two classical groups (see Corollary to Theorem 4, and Theorems 1, 5–7). We also show (Theorems 8–11) that G_X is transitive on X^* (in the Euclidean case)

always	when $\dim U = 1, 2$ or 3,
iff $\lambda_{v_1}\lambda_{v_2}\lambda_{v_3}$ is a scalar transformation	when $\dim U = 4$,
iff $\dim X = 8$	when $\dim U = 6, 7$ or 8,
never	when $\dim U = 5$ or is $\geqslant 9$.

Received by the editors August 23, 1979 and, in revised form, September 12, 1980.

1980 *Mathematics Subject Classification*. Primary 10C05; Secondary 10C04.

Key words and phrases. Quadratic forms, composition of quadratic forms, algebraic groups, nilmanifolds.

[1] Research supported by N.S.E.R.C. (Canada) grant A8778.

[2] The group G and question (2) are of interest for a certain class of nilmanifolds (see [6] and [7]).

Here λ_u denotes the linear transformations $x \mapsto ux$ of X for $u \in U$, V is the orthogonal complement $(Fu_0)^\perp$ of u_0 in U, and v_1, v_2, v_3 is any orthogonal basis of V (when $\dim U = 4$).

A. Kaplan has also derived these theorems (unpublished) on transitivity when $\dim U = 2$ or $\dim U = \dim X$. Also, D. B. Shapiro has independently given proofs (of a somewhat different nature) of some of these results, especially in the Euclidean case.

I am grateful to W. Rossmann for his generous help on some critical points and to D. B. Shapiro for several helpful conversations.

1. The basic result. If $(\phi, \psi) \in G$, then $\phi u_0 = u_0$ so we may assume that $G \subseteq O(V) \times O(X)$. We define

$$G^+ = G \cap (O^+(V) \times O(X)).$$

It is clearly $= G$ or a subgroup of index 2.

Consider the Clifford algebra $C(V, -q)$ belonging to the quadratic form $-q$ on V. We also denote it by C or $C(m)$. By the universal property of Clifford algebras, the map $v \mapsto -v$ of V can be extended, first of all, to an involution of J of C and, secondly, to an involutory automorphism K of C. Now define a "trace" $\tau\colon C \to F$ by letting $\tau(\alpha)$ be the trace of the linear map $\beta \mapsto 2^{-m}\alpha\beta$ of C into itself so $\tau(1) = 1$ and $\tau(\alpha\beta) = \tau(\beta\alpha)$. If $v_1, \ldots, v_m$ is an orthogonal basis of V, the 2^m products $v_{i_1} \cdots v_{i_k}$, $1 \leqslant i_1 < \cdots < i_k \leqslant m$, form a basis ("standard") of C, each member of which, other than 1, has trace 0. Thus $\tau(\alpha^J) = \tau(\alpha)$. Define

$$\gamma = v_1 v_2 \ldots v_m. \tag{1}$$

We now show that

(1) *The action $x \mapsto vx$ of V on X can be extended to an action of C on X under which X is a C-module and such that (if f is the bilinearization of q on X)*

$$f(\alpha x, y) = f(x, \alpha^J y) \tag{2}$$

for all x, y in X and α in C.

(2) *There is a unique nondegenerate hermitian form $h\colon X \times X \to C$ making*

$$\begin{array}{ccc} & \overset{h}{\nearrow} & C \\ & & \downarrow \tau \\ X \times X & \xrightarrow{\quad f \quad} & F \end{array}$$

commutative.

To prove (1), we note that X is a C-module satisfying $f(vx, y) = f(x, -vy)$ for all x, y in X and v in V by Theorem 5.5 and Remark 5.7, Chapter 5 [**8**]. Then (1) follows by the definition of J. Statement (2) follows from Theorem 22 [**12**] or 7.1, [**5**].

Note that the unitary group $U(h)$ satisfies

$$U(h) = O(X) \cap \mathrm{End}_C X.$$

The inclusion $\subseteq$ is clear; if $\phi \in O(X)$, then $\tau h(\phi x, \phi y) = f(\phi x, \phi y) = f(x, y)$ so $h(\phi x, \phi y) = h(x, y)$ by the uniqueness of h, whence $\supseteq$.

The algebra C has dimension 2^m and the subalgebra C^+ generated by the products vv' has dimension 2^{m-1}. Furthermore as a vector space $C = C^+ \oplus C^-$ where C^- is the subspace spanned by all odd products $v_1 \cdots v_{2k+1}$ of vectors. Define

$$\Gamma = \{\alpha \in C^+ \cup C^- : \alpha^J\alpha = 1 \text{ and } \alpha V \alpha^{-1} = V\}, \quad \Gamma^+ = \Gamma \cap C^+.$$

Γ^+ is the spin group of $(V, -q)$. We occasionally denote Γ by $\Gamma(m)$, Γ^+ by $\Gamma^+(m)$. The canonical homomorphism $\rho: \Gamma \to O(V, q)$, $\rho(\alpha)v = \alpha v \alpha^{-1}$, satisfies

$$\rho\Gamma^+ = O_0^+(V, q) = O_0^+(V, -q)$$

where $O_0^+(V, q)$ is the "reduced orthogonal group" or "spinorial kernel" and is a subgroup of $O^+(V)$ (cf. n°5, 9 **[2]**). Note that

$$[\Gamma : \Gamma^+] = 1 \text{ or } 2.$$

The groups Γ and Γ^+ are (the F-rational points of) algebraic groups. Over the algebraic closure of F, $\rho\Gamma^+ = O^+$ and $\rho\Gamma$ is O or O^+ depending on the parity of m; the same is true when U and X are Euclidean. The kernel of ρ (on Γ^+ or Γ) has order 2 or 4.

THEOREM 1. *The groups* Γ, Γ^+, $U(h)$ *and* G *are the* F*-rational points of* F*-closed algebraic groups* (*hence defined over* F *if* F *is perfect*). *There is a homomorphism, rational over* F,

$$\theta: \Gamma \times U(h) \to G$$

given by

$$\theta(\alpha, \psi) = (\rho(\alpha), \alpha\psi)$$

with central kernel of order 2 *or* 4 *and satisfying*

$$[G : \operatorname{im}\theta] \leqslant [O(V) : \rho\Gamma], \tag{3}$$

$$[G^+ : \theta(\Gamma^+ \times U(h))] \leqslant [O^+(V) : \rho\Gamma^+]. \tag{4}$$

REMARK. By the theorem, the projection G_V of G on the first factor in $O(V) \times O(X)$ contains the spinorial kernel $O_0^+(V)$. When U (and X) are Euclidean $O_0^+(V) = O^+(V)$ which is transitive on the unit sphere in V if $m \geqslant 2$; by a later result (Theorem 7), $G_V = \{\pm 1\} = O(V)$ when $m = 1$ so G_V is transitive on the unit sphere here as well. These facts were also proved by A. Kaplan (unpublished). The transitivity, more generally, of G_V can be investigated, in particular cases, by using knowledge of the orbits in V of $O_0^+(V)$–in this regard see **[11]**.

PROOF. If $\alpha \in \Gamma$ then $(x \mapsto \alpha x) \in O(X)$ by (2) since $\alpha^J\alpha = 1$. Thus $\operatorname{im}\theta \subseteq O(V) \times O(X)$.

Suppose $(\phi, \psi) \in G \cap (\rho\Gamma \times O(X))$, say $\phi = \rho\alpha$. Then

$$\alpha v \alpha^{-1}\psi(x) = \psi(vx)$$

whence $\alpha^{-1}\psi \in (\operatorname{End}_C X) \cap O(X) = U(h)$, $(\phi, \psi) = \theta(\alpha, \alpha^{-1}\psi)$, and (3) follows easily (note that $U(h)$ consists of C-endomorphisms of X so Γ and $U(h)$ commute in $\operatorname{End}_F X$ and θ is a homomorphism). (4) follows similarly and the other statements are easily checked.

2. Determination of $U(h)$. Assume F algebraically closed, let R be the matrix algebra F_n and let J be an involution on R of the first kind, i.e. J is the identity on F. Then it is known that there is a symmetric or skew-symmetric matrix S such that $A^J = S^{-1}({}^tA)S$ for all A in R where ${}^tA$ is the transpose of A. Since S is uniquely determined up to a scalar, we can say that J is of symmetric, resp. skew-symmetric, type.

THEOREM 2. *Let* $g\colon W \times W \to R$ *be a nondegenerate hermitian form on the* R*-module* W *of finite length* t. *Then the unitary group* $U(g)$ *is isomorphic, as an algebraic group, to the orthogonal group* $O(t, F)$, *resp. symplectic group* $\mathrm{Sp}(t, F)$, *if* J *is of symmetric, resp. skew-symmetric, type.*

PROOF. This is a special case of Morita theory–see Theorems 8.2 and 8.1 in [5]–although some additional checking is needed to see that the isomorphism is rational. The form h (in the notation of [5]) is a symmetric, resp. skew-symmetric, form on F^t whose adjoint is J. A direct and elementary proof can be given as follows: Write W as a direct sum of $Rw_1, \ldots, Rw_t$ where $w_1, \ldots, w_t$ have annihilator equal to the ideal of matrices with zero first column. This enables one to identify $\mathrm{End}_R W$ with F_t. It follows that $e_{ij}g(w_k, w_l) = 0 = g(w_k, w_l)e_{ij}^J$ when $j \geqslant 2$ where e_{ij} is the usual matrix unit, and then that $g(w_k, w_l) = c_{kl}e_{11}$, resp. $c_{kl}e_{12}$, where $C = (c_{kl})$ is symmetric, resp. skew-symmetric. It is easy to check that $B \in U(g) \subset F_t$ iff ${}^tBCB = C$, whence the theorem.

The next theorem is known and has an elementary proof which we do not include. The context is again a finite-dimensional algebra R over an algebraically closed field F but this time R is a direct sum of two matrix algebras which the involution J interchanges.

THEOREM 3. *Let* $g\colon W \times W \to R$ *be a nondegenerate hermitian form on the* R*-module* W *of finite length* t. *Then* t *is even and the unitary group* $U(g)$ *is isomorphic, as an algebraic group, to* $\mathrm{GL}(\frac{1}{2}t, F)$.

Now we return to the main problem.

THEOREM 4. *Let* F *be algebraically closed,* $h\colon X \times X \to C$ *the hermitian form in Theorem* 1. *Let* $\dim_F X = n$. *As an algebraic group,* $U(h)$ *is isomorphic to*

$$\begin{array}{ll} O(n2^{-m/2}, F), & m \equiv 0, 6 \pmod 8, \\ \mathrm{GL}(n2^{-(m-1)/2}, F), & m \equiv 1, 5 \pmod 8, \\ \mathrm{Sp}(n2^{-m/2}, F), & m \equiv 2, 4 \pmod 8, \\ \mathrm{Sp}(n_1 2^{-(m-1)/2}, F) \times \mathrm{Sp}(n_{-1}2^{-(m-1)/2}, F), & m \equiv 3 \pmod 8, \\ O(n_1 2^{-(m-1)/2}, F) \times O(n_{-1}2^{-(m-1)/2}, F), & m \equiv 7 \pmod 8. \end{array}$$

When $m \equiv 3$ *or* 7 $\pmod 8$, n_1 *and* $n_{-1} = n - n_1$ *are the dimensions of the eigensubspaces of* X *with respect to the linear transformation* γ *defined in* (1).

It is understood that the fractions, such as $n2^{-m/2}$, are all integers–this will follow from the fact that X is a C-module.

PROOF. If $u_1, \ldots, u_k$ are mutually orthogonal vectors in V, $(u_1 \cdots u_k)^J = (-1)^{[(k+1)/2]} u_1 \ldots u_k$ where [ ] is the greatest integer function, whence (with $\binom{m}{-1} = 0 = \binom{m}{r}$ if $r > m$ and $C^{\{J\}} = \{\alpha \in C: \alpha^J = \alpha\}$)

$$\dim C^{\{J\}} = \sum_{k=0}^{\infty} \left(\binom{m}{4k+1} + \binom{m}{4k} \right) = \sum_{k=0}^{\infty} \binom{m+1}{4k}.$$

By expanding the 4 binomial powers $(1 \pm 1)^{m+1}$ and $(1 \pm i)^{m+1}$ and adding, the last sum can be evaluated, giving

$$\dim C^{\{J\}} = \tfrac{1}{4}\left(2^{m+1} + 2(\sqrt{2}\,)^{m+1}\cos((m+1)/4)\pi\right).$$

But if J is any involution of the first kind on F_l, $\dim F_l^{\{J\}}$ is $\frac{1}{2}l(l+1)$, resp. $\frac{1}{2}l(l-1)$, if J is symmetric, resp. skew-symmetric. Thus if $m = 2p$, the theorem follows from Theorem 2 since $C \cong F_{2^p}$ and $\dim_F X = 2^p \operatorname{len}_C X$.

Suppose $m = 2p + 1$ so $C \cong F_{2^p} + F_{2^p}$ with idempotents of the form $\varepsilon_1 = \frac{1}{2}(1 + a\gamma)$, $\varepsilon_{-1} = \frac{1}{2}(1 - a\gamma)$ for some scalar a. The components of C are interchanged by J iff $m \equiv 1 \pmod 4$ and then the theorem follows from Theorem 3. If $m \equiv 3 \pmod 4$, $X = \varepsilon_1 X \perp \varepsilon_{-1} X$ and the restrictions h_1 and h_{-1} of h to these two components are hermitian forms over F_{2^p}. The rest of Theorem 4 now follows from Theorem 2 as in the case $m = 2p$, taking into account that $\varepsilon_1 X$ and $\varepsilon_{-1} X$ are the eigenspaces of γ.

COROLLARY. *The connected component of the group G is a reductive algebraic group, isogenous via θ to the connected component of $\Gamma \times U(h)$, which is a product of two or three classical groups.*

PROOF. Over the algebraic closure of F, ρ: $\Gamma^+ \to O^+(V)$ is surjective and hence θ: $\Gamma^+ \times U(h) \to G^+$ is surjective by (4), so θ is an isogeny. The connected component of Γ is the spin group Γ^+ so the Corollary follows from Theorem 4.

We now generalize this to an arbitrary F. Let D be the division algebra in the Brauer class of C if C is simple, otherwise in the Brauer class of C^+. This Brauer class is easily determined from the form f on V, especially when F is a local or global field (see [**8**, Proposition 3.20 in Chapter 5, Theorem 3.6 in Chapter 4]). The l's in Theorem 5 are easily determined from those given in Theorem 4 (by extending the scalars to an algebraic closure of F). The nature of J' and of h' (insofar as being hermitian, symmetric of skew-symmetric) follows from Theorem 4 and a knowledge of D. The precise nature of h' is difficult to determine, but in the Euclidean case it is very easy (see Theorem 6).

THEOREM 5. *$U(h)$ is isomorphic as an algebraic group to*

$\mathrm{GL}(l, D)$	*if* $m \equiv 1 \pmod 4$ *and* $\gamma^2 \in F^2$,
$U(l_1, h_1') \times U(l_{-1}, h_{-1}')$	*if* $m \equiv 3 \pmod 4$ *and* $\gamma^2 \in F^2$,
$U(l, h')$	*otherwise,*

where the h''s are hermitian, symmetric, or skew-symmetric forms.

This is proved in substantially the same way as Theorem 4, using Morita theory (cf. 2.4 in [9] and 8.2, 8.1 in [5]). Although this process may yield a skew-hermitian form h', it can be replaced by an hermitian form by scaling.

Let $\mathbf{C}$ denote the complex numbers, $\mathbf{H}$ the classical real quaternions.

THEOREM 6. *Suppose U and X are Euclidean and let $h\colon X \times X \to C$ be the hermitian form in Theorem* 1. *Let* $\dim_{\mathbf{R}} X = n$. *As an algebraic group $U(h)$ is isomorphic to*

$$\begin{array}{ll} O(n2^{-m/2}, \mathbf{R}), & m \equiv 0, 6 \pmod 8, \\ U(n2^{-(m+1)/2}, \mathbf{C}), & m \equiv 1, 5 \pmod 8, \\ U(n2^{-(m/2)-2}, \mathbf{H}), & m \equiv 2, 4 \pmod 8, \\ U(n_1 2^{-(m+3)/2}, \mathbf{H}) \times U(n_{-1} 2^{-(m+3)/2}, \mathbf{H}), & m \equiv 3 \pmod 8, \\ O(n_1 2^{-(m-1)/2}, \mathbf{R}) \times O(n_{-1} 2^{-(m-1)/2}, \mathbf{R}), & m \equiv 7 \pmod 8, \end{array}$$

where each symmetric and each hermitian form involved is positive definite, the involutions on $\mathbf{C}$ *and* $\mathbf{H}$ *are the standard ones, and n_1 and n_{-1} are as defined in Theorem* 4.

PROOF. The group $U(h)$ is a real form of the group $U(h_{\mathbf{C}})$ where $h_{\mathbf{C}}$ is the hermitian form arising from the complexification $U_{\mathbf{C}} \times X_{\mathbf{C}} \to X_{\mathbf{C}}$ of the given real composition. Thus we can apply Theorem 4. Since $U(h)$ is compact (as a closed subgroup of $O(X)$) and the groups listed in Theorem 4 have unique compact real forms, $U(h)$ is as stated in Theorem 6 (cf. [13], e.g.). The dimensions can be derived from those in Theorem 4 by dividing by 1, 2 or 4 according as the division ring of coefficients is $\mathbf{R}$, $\mathbf{C}$ or $\mathbf{H}$.

3. F algebraically closed, or U and X Euclidean. In this section we determine the index $[G : G^+]$ in these two cases. Table 1 is derived from [8, p. 131]. Put $m = 2p$ when m is even, $m = 2p + 1$ when m is odd.

m (mod 8)	0	1	2	3	4	5	6	7
C	$\mathbf{R}_{2^p}$	$\mathbf{C}_{2^p}$	$\mathbf{H}_{2^{p-1}}$	$\mathbf{H}_{2^{p-1}} \oplus \mathbf{H}_{2^{p-1}}$	$\mathbf{H}_{2^{p-1}}$	$\mathbf{C}_{2^p}$	$\mathbf{R}_{2^p}$	$\mathbf{R}_{2^p} \oplus \mathbf{R}_{2^p}$
$\dim_{\mathbf{R}}$ (simple module)	2^p	2^{p+1}	2^{p+1}	2^{p+1}	2^{p+1}	2^{p+1}	2^p	2^p

TABLE 1. Euclidean case

LEMMA 1. *Suppose that X is a simple C-module and that either F is algebraically closed or U and X are Euclidean. Then every symmetric bilinear form f' on X satisfying*

$$f'(\alpha x, y) = f'(x, \alpha^J y)$$

for all α in C, x and y in X, is of the form $f' = af$ for some $a \in F$.

PROOF. We prove only the Euclidean case; the proof when F is algebraically closed is similar and easier.

The map $y \mapsto h(\cdot, y)$, $X \to \mathrm{Hom}_C(X, C)$, is bijective since h is nondegenerate. Let h' be the hermitian form satisfying $\tau h' = f'$, and let $\sigma \in \mathrm{End}_C X$ be the composite of $y \mapsto h'(\cdot, y)$ followed by the inverse of $y \mapsto h(\cdot, y)$. Then

$$h'(x, y) = h(x, \sigma y) \text{ for all } x \text{ and } y \text{ so also } f'(x, y) = f(x, \sigma y).$$

If $C \cong \mathbf{C}_l$, then $\sigma \in \mathrm{cen}\, C = \mathbf{C}$ since X is simple. Now $\mathrm{cen}\, C = \mathbf{R} + \mathbf{R}\gamma$ and it is easy to see that J is complex conjugation on it; since we may assume f' is not identically 0, $h'(x, x) \neq 0$ for some x and since it and $h(x, x)$ are J-symmetric, $h'(x, x) = h(x, x)\bar{\sigma}$ implies $\sigma \in \mathbf{R}$. Thus $f' = \sigma f$ as desired.

If $C \cong \mathbf{R}_l$ or $\mathbf{R}_l \oplus \mathbf{R}_l$, again $\sigma \in \mathrm{cen}\, C = \mathbf{R}$ or $\mathbf{R} \oplus \mathbf{R}$ and it is easy to see that we may assume $\sigma \in \mathbf{R}$ since X is simple.

Now suppose $C \cong \mathbf{H}_l$ or $\mathbf{H}_l \oplus \mathbf{H}_l$. In the latter case, one of the components, say the first, acts nontrivially on X and the other acts as 0. Choose an orthonormal basis $v_1, \ldots, v_m$ of V and consider the subalgebra $C(2) \cong \mathbf{H}$ of C generated by v_1 and v_2. The restriction of J to $C(2)$ is the usual involution $\alpha \to \bar{\alpha}$ of $\mathbf{H}$. If C is simple let $C_1 = C$, otherwise let C_1 be the first simple component. Let D be the projection of $C(2)$ on C_1. Since the centralizer D' of D in C_1 is $\cong \mathbf{R}_l$ and $D \otimes_{\mathbf{R}} D' \cong C_1$ [3, Corollary of Theorem 2, 10], there is a system of matrix units in C_1 so that $C_1 = D_l$. Thus we may suppose that $X = D^l$ (column vectors), that $C_1 = D_l$ acts on X by left multiplication, and that $\mathrm{End}_C X = \mathrm{End}_{C_1} X$ consists of right multiplication by D; suppose in particular that $\sigma(x) = x\alpha$ for all x in D^l, where $\alpha \in D$. We now define $\tau\beta = \frac{1}{2}(\beta + \bar{\beta})$ and get hermitian forms h_1 and h_1' satisfying $\tau h_1 = f$, $\tau h_1' = f'$. Since $f'(x, y) = f(x, y\alpha)$, $h_1'(x, y) = h_1(x, y\alpha)$ by the uniqueness of h_1'. Now we may assume $f' \neq 0$; since D^l has a basis lying in $\mathbf{R}^l$, there exists x in $\mathbf{R}^l$ with $h'(x, x) \neq 0$; since then $x\alpha = \alpha x$, $h_2'(x, x) = h_2(x, x)\bar{\alpha}$ implies $\alpha \in \mathbf{R}$.

Suppose F is algebraically closed or U and X are Euclidean. We know that $[G : G^+] = 2$ when m is even.

Now suppose m is odd, so $\rho\Gamma = \rho\Gamma^+ = O^+(V)$. If $G^+ \neq G$, there exists $(\phi, \psi) \in G$ with $\phi \notin O^+(V)$ and by Theorem 1 we may assume $\phi = -1$. Thus

$$\psi(vx) = -v\psi(x) \tag{5}$$

for all v in V and x in X. Thus ψ is a K-linear map $\in O(X)$. Conversely if $O(X)$ contains a K-linear map ψ, then $(-1, \psi) \in G$ and so $G^+ \neq G$.

If C is not simple, it is the direct sum of two simple algebras $C_1 \oplus C_2$ and so any C-module X is $= X_1 \oplus X_2$ where $X_1 = C_1X$ and $X_2 = C_2X$ are the isotypic components of X.

THEOREM 7. *Suppose F is algebraically closed or that U and X are Euclidean.*

(a) *If m is even,*

$$\theta(\Gamma^+ \times U(h)) = G^+, \qquad \theta(\Gamma \times U(h)) = G, \qquad [G : G^+] = 2.$$

(b) *If m is odd,*

$$\theta(\Gamma^+ \times U(h)) = \theta(\Gamma \times U(h)) = G^+.$$

Moreover $[G : G^+] = 2$ except that $G = G^+$ in the following case: C is not simple and the isotypic components of X have different lengths.

PROOF. The canonical homomorphism $\rho\colon \Gamma^+ \to O^+(V)$ is surjective (under the hypotheses), as are $\rho\colon \Gamma \to O(V)$ when m is even and $\rho\colon \Gamma \to O^+(V)$ when m is odd; the statements concerning the images of θ now follow from Theorem 1, as does $[G : G^+] = 2$ when m is even. So assume m is odd.

Take $\gamma = v_1 \cdots v_m$ where $v_1, \ldots, v_m$ is an orthonormal basis of (V, q) so $\gamma^2 = \pm 1$. If C is not simple, K interchanges the simple components of C and so a K-linear automorphism of X maps each isotypic component of X bijectively on the other one; thus $G^+ \neq G$ implies that the isotypic components have the same length.

To prove the necessity of the last condition of the theorem, we first assume that C is simple so we are in the Euclidean case and $C \cong \mathbf{C}_{2^p}$. If Y is any C-submodule, its orthogonal complement with respect to f is also a C-module by (2) and so it suffices to show the existence of a K-linear map in $O(X)$ in the case that X is simple. We may therefore suppose $X = \mathbf{C}^{2^p}$ and we let $x \mapsto \bar{x}$ be complex conjugation on the coordinates. Since K is complex conjugation on cen $C = \mathbf{C}$, the map $\overline{A \mapsto A^K}$ is a C-automorphism of $\underline{\mathbf{C}_{2^p}}$, hence an inner automorphism by the Skolem-Noether theorem, so $A^K = \overline{BAB^{-1}}$ for some $B \in \mathbf{C}_{2^p}$. Thus $\phi(x) = \overline{Bx}$ is K-linear:

$$\phi(Ax) = \overline{BAx} = \overline{BAB^{-1}}\ \overline{Bx} = A^K\phi(x).$$

Define $f'(x, y) = f(\phi x, \phi y)$. It is easy to see that J and K commute (since they do so on V) and it follows easily that $f'(\alpha x, y) = f'(x, \alpha^J y)$ for all x and y, all $\alpha \in C$. By Lemma 1, $f' = af$ for some $a \in \mathbf{R}$, and since f' is also positive definite, $a > 0$. Therefore $a^{-1/2}\phi \in O(X)$ and is K-linear.

Now suppose $C = C_1 \oplus C_2$. Since K interchanges the simple algebras C_1 and C_2, we may assume that C is the external direct sum $C = D \oplus D$ of a simple algebra D with itself and that $(d_1, d_2)^K = (d_2, d_1)$ for all $(d_1, d_2) \in D \oplus D$. Now $\gamma^J = (-1)^{(m+1)/2}\gamma$ so each of C_1 and C_2 is stable under J if $m \equiv 3 \pmod 4$, while if $m \equiv 1 \pmod 4$, J interchanges C_1 and C_2.

Assume first that C_1 and C_2 are stable under J (this is always so when U is Euclidean since then C is simple when $m \equiv 1 \pmod 4$). Since J and K commute, the involutions induced by J on D through the two isomorphisms $D \cong C_i$ are identical and are again denoted by J. The X_i are orthogonal with respect to both f and h: $X = X_1 \perp X_2$. Moreover $h(X_i, X_i) \subseteq C_i$ for $i = 1, 2$. Write each of X_1 and X_2 as an orthogonal direct sum of orthogonally indecomposable submodules. By [9, 2.4, 2.5(b)], each of these indecomposable submodules is a simple D-module, or each is a hyperbolic plane, i.e. is $\cong Y \oplus Y^*$ where Y is simple, Y^* is its dual made into a left D-module via $\alpha\eta = \eta\alpha^J$ ($\eta \in Y^*$, $\alpha \in D$) and the hermitian form on it is $h(y + \eta, y' + \eta') = \langle y, \eta' \rangle + \langle y', \eta \rangle^J$. We note that in the Euclidean case, only the first case can occur since the restriction of f to any simple submodule is nondegenerate, hence so also for h. Using Lemma 1 and len X_1 = len X_2, we can now easily construct a K-linear isometry in $O(X)$ by pairing off the indecomposables of X_1 with those of X_2 and choosing a D-isometry between the modules in each pair (in the case when indecomposables are simple, such isometries can be shown to exist using Lemma 1 in the same way as in the previous case of C simple).

Finally suppose C_1 and C_2 are not stable under J, so $C_1^J = C_2$, $C_2^J = C_1$, and F is algebraically closed. Since $h(C_ix, C_iy) = C_ih(x, y)C_i^J = 0$, it follows that $h(X_i, X_i) = 0 = f(X_i, X_i)$. One can make X_1 into a C_2-module rather than a C_1-module by defining $\alpha_2 \cdot x_1 = \alpha_2^K x_1$. It has the same length as a C_2-module as a C_1-module, hence has the same length as X_2. We can therefore find a K-linear isomorphism ϕ_1: $X_1 \to X_2$. Now the map $X_1 \to X_1^*$ given by $y_1 \mapsto h(\phi_1 \cdot, y_1)$ is JK-linear; since h is nondegenerate it is injective and hence bijective. Similarly $x_2 \mapsto h(\cdot, x_2)$ is a J-linear isomorphism $X_2 \to X_1^*$. Thus there is a unique $y_1 =: \phi_2(x_2)$ in X_1 so that

$$h(\phi_1 x_1, \phi_2 x_2)^K = h(x_1, x_2) \tag{6}$$

for all $x_1 \in X_1$, $x_2 \in X_2$. Uniqueness implies that ϕ_2: $X_2 \to X_1$ is K-linear, hence a K-isomorphism.

Define $\phi = \phi_1 \oplus \phi_2$: $X \to X$. Application of τ to (6) yields $f(\phi x_1, \phi x_2) = f(x_1, x_2)$ for all $x_1 \in X_1$ and $x_2 \in X_2$, and so ϕ is a K-linear isometry in $O(X)$.

4. *U* and *X* Euclidean. In this section, the transitivity of G_X on the unit sphere X^* is investigated; we assume throughout that U and X are Euclidean. Let G_X^+ be the projection of G^+ on $O(X)$. By Theorem 7

$$\begin{aligned} G_X^+ &= \Gamma^+ U(h) = \Gamma U(h) && \text{when } m \text{ is odd,} \\ G_X^+ &= \Gamma^+ U(h),\ G_X = \Gamma U(h) && \text{when } m \text{ is even.} \end{aligned} \tag{7}$$

The projection $G \to G_X$ is actually an isomorphism. For if $(\phi, 1) \in G$, then $vx = \phi(v)x$ so $(\phi(v) - v)x = 0$ for all v and x; since (V, f) is nondegenerate and $f(ux, u'x) = f(u, u')q(x)$ (see [**8**, (5.4), Chapter 5]), ϕ is the identity.

LEMMA 2. *G_X is transitive on X^* if and only if G_X^+ is transitive on X^*. Let G_X^+x be any orbit of G_X^+ on X^*. Then G_X is transitive on X^* if and only if the dimension of G_X^+x (as a real manifold) is equal to* $\dim X^* = \dim X - 1$.

PROOF. That G_X^+x and G_Xx are regular submanifolds of X^* follows from the compactness of G_X^+ and G_X (continuous images of G^+ and G which are closed subsets of the compact group $O(V) \times O(X)$). Clearly G_X is transitive on X^* if G_X^+ is. Conversely suppose $G_Xx = X^*$. Now G_X^+x is closed and is also open since $[G_X : G_X^+] < \infty$ (in fact $\leqslant 2$) implies that $\dim G_X^+x = \dim G_Xx = \dim X^*$. Thus $G_X^+x = X^*$ since X^* is connected.

Now if $\dim G_X^+x < \dim X^*$, G_X^+ is not transitive so neither is G_X. If $\dim G_X^+x = \dim X^*$, then the argument above using the connectivity of X^* yields $G_X^+x = X^*$, hence $G_Xx = X^*$.

THEOREM 8. *Suppose G_X is transitive on X^*. Then X is an isotypic C-module if $m = 3$, and is a simple C-module if $m \geqslant 4$.*

PROOF. $C(3) \cong \mathbf{H} \oplus \mathbf{H}$ and if $m \geqslant 4$, $C(4)$ is a matrix algebra which is not a division algebra or is the direct sum of two such algebras (see Table 1). Suppose X is not isotypic when $m = 3$ or is not simple when $m \geqslant 4$. Then it contains nonzero vectors (and so also unit vectors, by scaling) with annihilators in C of different dimensions. The elements of Γ, being units in C, preserve the dimension of annihilators as they operate on X. The same is true of transformations in $U(h)$ since they are C-linear automorphisms of X. Thus G_X^+ is not transitive on X^* by (7), so neither is G_X by Lemma 2.

THEOREM 9. *G_X is not transitive on X^* if $m = 4$ or $m \geqslant 8$.*

PROOF. By Theorem 8 we may assume X is simple so, by Table 1, $\dim X^* = 2^p - 1$ if $m \equiv 0, 6$ or $7 \pmod 8$, while $\dim X^* = 2^{p+1} - 1$ if $m \equiv 1, 2, 3, 4$, or $5 \pmod 8$. On the other hand, $\dim \Gamma^+ = \dim O^+(V) = \frac{1}{2}m(m-1)$ [**4**, Proposition 6, Chapter I, §II]. Since X is simple, $\mathrm{Aut}_C X = \mathbf{R}^\times, \mathbf{C}^\times$ or $\mathbf{H}^\times$ and so is a connected real algebraic group of dimension $\leqslant 4$. The subgroup $U(h)$ is a closed algebraic subgroup of $\mathrm{Aut}_C X$ and is proper since, e.g. $U(h) \cap \mathbf{R}^\times = \pm 1$. Thus $\dim U(h) \leqslant 3$ so $\dim G_X^+ = \dim \Gamma^+ U(h) \leqslant \frac{1}{2}m(m-1) + 3$. Thus any orbit of G_X^+ has dimension $\leqslant \frac{1}{2}m(m-1) + 3$ and it is straightforward to check that this in turn is $< \dim X^*$ when $m \geqslant 10$. Thus G_X is not transitive on X^* when $m \geqslant 10$ by Lemma 2.

Suppose that $m = 4$, 8 or 9. Now $C^+(m) \cong C(m-1)$ by [**8**, Corollary 2.10, Chapter 5], so Table 1 shows that a simple C^+-module has half the dimension of a simple C-module and so X is the direct sum of two simple C^+-modules. Each of the latter is stable under Γ^+ so Γ^+ has an orbit on X^* of dimension $\leqslant \frac{1}{2}\dim X - 1$, so $G_X^+ = \Gamma^+ U(h)$ has an orbit of dimension $\leqslant \frac{1}{2}\dim X + 2$. Since $\dim X \geqslant 8$, this is $< \dim X^*$ so G_X^+ and G_X are not transitive by Lemma 2.

THEOREM 10. *G_X is transitive on X^* if $m = 0$, 1 or 2, or if X is isotypic and $m = 3$.*

REMARK. When $m = 3$, $C \cong \mathbf{H} \oplus \mathbf{H}$. Let v_1, v_2, v_3 be an orthonormal basis of (V, f) and put $\gamma = v_1 v_2 v_3$ as usual. Then $\gamma^2 = 1$, and X is isotypic iff γ operates as 1 or -1 on all of X, hence iff $u_1 u_2 u_3$ is a scalar transformation on X, where u_1, u_2, u_3 is an arbitrary orthogonal basis of V.

PROOF. If $m = 0$, $G = \{1\} \times O(X)$ so $G_X = O(X)$ and is transitive on X^* by Witt's theorem. If $m = 1$, $C(1) \cong \mathbf{C}$ with $J =$ complex conjugation. Furthermore $\tau(a + bi) = a$ so it follows that h is a (positive definite) complex hermitian form on X with the same unit sphere as f. By Witt's theorem, $U(h)$ is transitive on X^*, hence so is the larger group G_X. A similar argument works when $m = 2$ for then $C(2) \cong \mathbf{H}$ with $J =$ the usual conjugation and $\tau(a + bi + cj + dk) = a$; thus h is a (positive definite) quaternionic hermitian form.

Finally suppose $m = 3$. Let v_1, v_2, v_3 be an orthonormal basis of V, $\gamma = v_1 v_2 v_3$, and let $C(2)$ be the Clifford algebra of the subspace $\mathbf{R}v_1 + \mathbf{R}v_2$. Then $C(3) = C(2)\varepsilon_1 \oplus C(2)\varepsilon_{-1}$ with $\varepsilon_1 = \frac{1}{2}(1 + \gamma)$ and $\varepsilon_{-1} = \frac{1}{2}(1 - \gamma)$ the orthogonal idempotents. Now J fixes each ε_i and induces the standard involution on $C(2)$. Moreover h can be considered as an hermitian form over one of the factors $C(2)\varepsilon_i \cong \mathbf{H}$ and since the 2-sphere of h equals the unit sphere of f (as is easily checked), $U(h)$ is once more transitive on X^* by Witt's theorem, hence also G_X.

THEOREM 11. *Let $m = 5$, 6 or 7. Then G_X is transitive on X^* if and only if $\dim X = 8$.*

PROOF. The necessity follows from Theorem 8 and Table 1. By Theorem 1 the sufficiency will be proved once we show that $\Gamma^+(5)$, $\Gamma^+(6)$ and $\Gamma^+(7)$ are transitive on $X^* = S^7$. It is enough to do this for $\Gamma^+(5)$. Indeed we may suppose $C(5) \subset C(6) \subset C(7)$. Then $C(5)$ satisfies the adjointness condition (2) with respect to both $f(5)$ and $f(7)$ whence $f(7)$ is a scalar multiple of $f(5)$, so we can assume $f(7) = f(5)$ by Lemma 1. Similarly $f(6) = f(5)$ so the three unit spheres are identical. And $\Gamma^+(5) \subset \Gamma^+(6) \subset \Gamma^+(7)$.

What we must show then is that $\Gamma^+(5)$ ($= \mathrm{Spin}_5$) is transitive on S^7 under the spin representation. This is evidently a well-known fact (cf. [**10**, Theorem IV]) but I have not been able to find a reference (with a proof). We sketch briefly an elementary proof (cf. [**1**, §§3, 4, Chapter VIII]). In $\mathbf{C}_2$ define

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad I = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad K = iIJ = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and in $\bigotimes^3 \mathbf{C}_2 \cong \mathbf{C}_8$ define

$$V_1 = I \otimes E \otimes E, \quad V_2 = K \otimes I \otimes E, \quad V_3 = K \otimes K \otimes I,$$
$$V_4 = J \otimes E \otimes E, \quad V_5 = K \otimes J \otimes E.$$

The V_i anticommute and satisfy $V_i^2 = -1$ so we obtain a representation $C(5) \to \mathbf{C}_8$. Define

$$Y = \sum \mathbf{C}(e_1 \otimes e_2 \otimes e_3) \subset \bigotimes^3 \mathbf{C}^2 \cong \mathbf{C}^8$$

where the sum runs over all choices of e_1, e_2, e_3 from $\{\binom{1}{0}, \binom{0}{1}\}$ with exactly 0 or 2 of the three equal to $\binom{0}{1}$. It is easy to see that Y is a C-module, and so since $\dim_{\mathbf{R}} Y = 8$, we may take $X = Y$.

$\mathfrak{o}(5) = \sum_{j<k} \mathbf{R} V_j V_k$ is the image of the Lie algebra of Γ^+ in $\mathbf{C}_8$. A calculation shows that $\mathfrak{o}(5)\varepsilon$, where $\varepsilon = \binom{1}{0} \otimes \binom{1}{0} \otimes \binom{1}{0}$, has (real) dimension 7, so the annihilator of ε in $\mathfrak{o}(5)$ has dimension $10 - 7 = 3$, so the stabilizer of ε in Γ^+ has dimension 3 (as a real Lie group) so the orbit $\Gamma^+\varepsilon$ has dimension 7. Since some scalar multiple of ε is on $X^* = S^7$, the theorem follows by Lemma 2.

REFERENCES

1. H. Boerner, *Representations of groups*, North-Holland, Amsterdam, 1970.

2. N. Bourbaki, *Formes sesquilinéaires et formes quadratiques*, Actualités Sci. Indust., no. 1272, Hermann, Paris, 1959.

3. ______, *Modules et anneaux semi-simples*, Actualités Sci. Indust., no. 1261, Hermann, Paris, 1958.

4. C. Chevalley, *Theory of Lie groups*. I, Princeton Mathematical Series, no. 8, Princeton Univ. Press, Princeton, N. J., 1946.

5. A. Frohlich and A. M. McEvett, *Forms over rings with involution*, J. Algebra **12** (1969), 79–104.

6. A. Kaplan, *Fundamental solutions for a class of hypoelliptic PDE generated by composition of quadratic forms*, Trans. Amer. Math. Soc. **259** (1980), 147–153.

7. ______, *Riemann nilmanifolds attached to Clifford modules* (to appear).

8. T. Y. Lam, *The algebraic theory of quadratic forms*, Mathematical Lecture Note Series, Benjamin, Reading, Mass., 1973.

9. A. M. McEvett, *Forms over semisimple algebras with involution*, J. Algebra **12** (1969), 105–113.

10. D. Montgomery and H. Samelson, *Transformation groups on spheres*, Ann. of Math. **44** (1943), 454–470.

11. B. Pollak, *Transitivity in the spinorial kernel and the commutator subgroup of the orthogonal group*, Amer. J. Math. **85** (1963), 36–46.

12. C. Riehm and M. A. Shrader-Frechette, *The equivalence of sesquilinear forms*, J. Algebra **42** (1976), 495–530.

13. J. Tits, *Tabellen zu den einfachen Lie Gruppen und ihren Darstellungen*, Lecture Notes in Math., vol. 40, Springer-Verlag, Berlin, Heidelberg and New York, 1967.

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