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WEAK P-POINTS IN COMPACT CCC F-SPACES

BY

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ABSTRACT. Using a technique due to van Mill we show that each compact ccc F-space of weight greater than 2^{ω} contains a weak P-point, i.e. a point $x \in X$ such that $x \notin \bar{F}$ for each countable $F \subset X - \{x\}$. We show that, assuming $BF(c)$, each nowhere separable compact F-space has a weak P-point. We show the existence of points which are not limit points of any countable nowhere dense set in compact F-spaces of weight $\aleph_1$. We also discuss remote points and points not the limit point of any countable discrete set.

Introduction. All spaces considered are completely regular and X^* denotes $\beta X - X$. A space X is an *F-space* if each cozero set is C^*-embedded. A ccc F-space is easily seen to be extremally disconnected; the closure of each open set is open. A point $x \in X$ is a *weak P-point* if $x \notin \bar{F}$ for each countable $F \subset X - \{x\}$. Kunen **[K]** has shown that ω^* has a dense set of weak P-points. Jan van Mill then showed that each compact infinite F-space of weight 2^{ω} in which nonempty G_δ's have nonempty interior has weak P-points **[vM]**. He also showed that if there is a ccc nonseparable growth of ω then he could remove the weight restriction. Subsequently Murray Bell **[B]** constructed such a growth of ω. Then in **[DvM]** the author and van Mill extended van Mill's result to "each compact nowhere ccc F-space has weak P-points ". It is easy to see that a separable space cannot have weak P-points. In §5 we give an example of a nonseparable F-space in which there are no weak P-points. We see, therefore, that we need to assume nowhere separable rather than nonseparable. However in the case of ccc spaces we can consider all nonseparable spaces because such a space contains a nowhere separable open set. We address the open question "do all compact nowhere separable F-spaces have weak P-points?".

We are able to show that for compact ccc F-spaces of weight greater than c the answer is yes and that assuming $BF(c)$ it is also true for nonseparable compact ccc F-spaces of weight $\leqslant c$. We are also able to show that for spaces of weight $\aleph_1$ we do not need to assume $BF(c)$, that is, there are points not the limit point of any countable nowhere dense set.

The point $x \in X^*$ is called a *remote point* of X if $x \notin \mathrm{cl}_{\beta X} A$ for each nowhere dense subset of A of X. It is known that if X is a nonpseudocompact space with countable Π-weight then X has a remote point **[vD**, 1.5], **[CS]**. In **[vDvM]** the authors show that not every nonpseudocompact space has remote points but ask if $\omega \times 2^{\omega_2}$ has remote points if CH fails. Our methods enable us to show that under $BF(c)$ each nonpseudocompact ccc space of weight $\leqslant c$ has a remote point.

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Finally we also show that under some further assumptions on the Π-character of a ccc F-space there are points which are not the limit point of any countable discrete set.

1. Nice filters. Let X be a normal space. A *filter base* on X is a collection of closed subsets of X which is closed under finite intersections and which does not contain the empty set. The *filter* generated by the filterbase $\mathfrak{F}$ is the collection of $\{A \subset X: A = \overline{A}$ and $\exists F \in \mathfrak{F}$ with $F \subset A\}$. Let X be the topological sum of countably many nonempty compact spaces, say X_n $(n < \omega)$. In **[vM]** van Mill defines a filter $\mathfrak{F}$ to be *nice* provided that for each $F \in \mathfrak{F}$, the set $\{n < \omega: F \cap X_n = \varnothing\}$ is finite while, in addition, $\bigcap \mathfrak{F} = \varnothing$. We will say that a nice filter $\mathfrak{F}$ on X *avoids* countable sets if for each countable $D \subset X$ there is an $F \in \mathfrak{F}$ with $\overline{D} \cap F = \varnothing$. In **[vM]**, Jan van Mill has done most of the work of constructing weak P-points.

LEMMA 1.1. *Let K be a compact ccc F-space and let $\{Z_n: n \in \omega\}$ be disjoint nonempty clopen subsets of K. Then if $\mathfrak{F}$ is a nice filter on $X = \bigcup Z_n$, there is a point $x \in \bigcap\{\mathrm{cl}_K F: F \in \mathfrak{F}\}$ which is not in the closure of any countable subset of $K \setminus (X \cup \{x\})$. In particular, if $\mathfrak{F}$ avoids countable subsets of X then x is a weak P-point of K.*

Due to the length and complexity we will not include this proof. The reader is referred to the proof of Theorem 0.3 in **[vM]**. One need only observe that the restriction of $\mathfrak{F}$ to any infinite subcollection of the Z_n's is again a nice filter on this union and that Bell **[B]** has shown the existence of a ccc nowhere separable growth of ω. We then follow the proof of Theorem 0.3 in **[vM]** verbatim. □

Our investigation therefore turns to constructing nice filters on topological sums of compact ccc F-spaces which avoid countable sets.

2. Large F-spaces. We will first investigate compact ccc F-spaces with weight greater than c. We begin by stating some results which we will require.

THEOREM 2.1 **[BF]**. *Every infinite complete boolean algebra B contains a free subalgebra A with $|A| = |B|$.*

The clopen subsets of a compact ccc F-space form a complete boolean algebra so by the above result and Stone's duality theorem **[W]** we obtain

THEOREM 2.2. *Each compact ccc F-space of weight greater than c can be mapped onto 2^{c^+}.*

Using Theorem 2.2 we can now prove

THEOREM 2.3. *Each compact ccc F-space X of weight greater than c contains weak P-points.*

PROOF. It follows easily from 2.2 that we can choose countably many disjoint clopen subsets $\{Z_n: n \in \omega\}$ of X each of weight greater than c. From Lemma 1.1 we need only construct a nice filter $\mathfrak{F}$ on $\bigcup Z_n$ which avoids countable subsets of $\bigcup Z_n$. Our technique is very similar to that used in **[vM]**. By Theorem 2.2, we can let g_n be a continuous surjection from Z_n to 2^{c^+}. Note that 2^{c^+} is ccc nowhere separable. For each countable $D \subset \bigcup Z_n$, let $D' = \bigcup_n g_n[D \cap Z_n]$. Choose a cellular family of clopen subsets $\{A_k^D\colon k \in \omega\}$ of 2^{c^+} whose union is dense and $A_k^D \cap D' = \varnothing$ for each k. This can be done because D' is nowhere dense in 2^{c^+}. We will let $\mathcal{F}$ be the filter generated by $\{\bigcup_{n\in\omega} \bigcup_{k\leqslant n} g_n^{\leftarrow}[A_k^D]\colon D \in [\bigcup Z_n]^\omega\}$.

It is a simple matter to check that $\mathcal{F}$ is a nice filter on $\bigcup Z_n$ and obviously $\mathcal{F}$ avoids countable sets. Let $\{D_j\colon 1 \leqslant j \leqslant m\}$ be countable subsets of $\bigcup Z_n$. Then for each of the cellular families $\{A_k^{D_j}\colon k \in \omega\}$, $\bigcup_{k\in\omega} A_k^{D_j}$ is dense in 2^{c^+}. Hence we can recursively choose k_j, $1 \leqslant j \leqslant m$, so that $\bigcap_{j=1}^m A_{k_j}^{D_j} \neq \varnothing$. Let $N = \max\{k_j\colon 1 \leqslant j \leqslant m\}$ and let $n \geqslant N$. Hence

$$Z_n \cap \Big(\bigcup_{k\leqslant n} g_n^{\leftarrow}[A_k^{D_1}]\Big) \cap \cdots \cap \Big(\bigcup_{k\leqslant n} g_n^{\leftarrow}[A_k^{D_m}]\Big)$$

contains $g_n^{\leftarrow}[\bigcap_{j=1}^m A_{k_j}^{D_j}]$ and is therefore not empty. This proves that $\mathcal{F}$ is a nice filter and completes the proof of the theorem. □

REMARK. It is worth noting that any ccc nowhere separable space Y could have taken the place of 2^{c^+} in the above proof so long as one has surjections from each Z_n to Y. This fact can be used to conclude that many well-known F-spaces have weak P-points, for instance any ccc nowhere separable space which has countably many disjoint clopen sets which are pairwise homeomorphic.

3. Small *F*-spaces. In the case of small F-spaces, that is, spaces of weight less than or equal to c, we cannot use the above method because 2^κ for $\kappa \leqslant c$ is separable. The method we use is an attempt to capture within the space the essential idea behind the above method. We only managed to succeed with the aid of the set-theoretic principle $BF(c)$. Let F be the set of all functions from ω into ω. If f and g belong to F, define $g \leqslant f$ provided $\{n \in \omega\colon g(n) > f(n)\}$ is finite. A subset G of F is *bounded* if there is an $f \in F$ such that for each $g \in G$, $g \leqslant f$. $BF(c)$ is equivalent to the statement: each subset of F of cardinality less than c is bounded. $BF(c)$ is known to be consistent with the usual axioms of set theory and follows from MA or even $P(c)$ [**R**, pp. 82, 88].

To prove our result for small nowhere separable compact ccc F-spaces we will first prove a lemma in greater generality than is needed for weak P-points.

LEMMA 3.1. *Assume $BF(c)$. Let $\{Z_n\colon n \in \omega\}$ be compact ccc spaces of weight less than or equal to c and let X be the topological sum of $\{Z_n\colon n \in \omega\}$. There is a nice filter $\mathcal{F}$ on X which avoids all nowhere dense subsets of X.* (*Such a filter has been called a remote filter* [**vM**].)

PROOF. Since X is ccc and of weight less than or equal to c there are only c maximal cellular families of regular closed sets. Let $\{\{A_{n,m}^\alpha\colon n, m \in \omega\}\colon \alpha < c\}$ list all maximal cellular families of regular closed sets such that for each $\alpha < c$ and $n \in \omega$, $A_{n,m}^\alpha \subset Z_n$ for all $m \in \omega$. Let $\mathcal{D}$ be the set of nowhere dense subsets of X. Notice that for each $D \in \mathcal{D}$ there is an $\alpha < c$ such that $\overline{D} \cap (\bigcup_{n,m} A_{n,m}^\alpha) = \varnothing$.

Our plan is to select, for each $\alpha < c$, a function h_α from ω into ω. We will define our filter $\mathcal{F}$ to be generated by the set of closed sets $\{\bigcup_{n\in\omega} \bigcup_{j\leqslant h_\alpha(n)} A_{n,j}^\alpha\colon \alpha < c\}$.

So the idea is to select the h_α's to ensure that this filter is nice. This procedure is actually a simple recursion using $BF(c)$.

Let $h_0(n) = n$ for each $n \in \omega$. Suppose we have defined h_γ for $\gamma < \alpha < c$ such that for any finite sequence $\gamma_1 < \gamma_2 < \cdots < \gamma_k < \alpha$ there is an $N \in \omega$ such that for $n \geqslant N$,

$$Z_n \cap \left[\bigcup_{j \leqslant h_{\gamma_1}(n)} A_{n,j}^{\gamma_1} \right] \cap \cdots \cap \left[\bigcup_{j \leqslant h_{\gamma_k}(n)} A_{n,j}^{\gamma_k} \right] \neq \varnothing .$$

(The Z_n is here only for emphasis.) This is the condition we require to ensure we get a nice filter.

Let us select h_α. For each $E \in [\alpha]^{<\omega}$ we define a function g_E as follows. Let E be the sequence $\gamma_1 < \gamma_2 < \cdots < \gamma_k$. Let $g_E(n) = 0$ if $\bigcap_{i=1}^{k} [\bigcup_{j \leqslant h_{\gamma_i}(n)} A_{n,j}^{\gamma_i}] = \varnothing$. Otherwise let $g_E(n)$ be the smallest integer p such that

$$A_{n,p}^{\alpha} \cap \bigcap_{i=1}^{k} \left[\bigcup_{j \leqslant h_{\gamma_i}(n)} A_{n,j}^{\gamma_i} \right] \neq \varnothing .$$

Since $| \{ g_E : E \in [\alpha]^{<\omega} \} | \leqslant | [\alpha]^{<\omega} | < c$, the set $\{ g_E : E \in [\alpha]^{<\omega} \}$ is a bounded family if we assume $BF(c)$. Hence we can choose a function h_α so that for each $E \in [\alpha]^{<\omega}$ the set $\{ n : g_E(n) > h_\alpha(n) \}$ is finite.

To see that we have preserved our induction assumption, let $E = \{ \gamma_i : 1 \leqslant i \leqslant k, \gamma_i < \alpha \}$. By assumption, there is an integer N such that for $n \geqslant N$,

$$Z_n \cap \bigcap_{i=1}^{k} \left[\bigcup_{j \leqslant h_{\gamma_i}(n)} A_{n,j}^{\gamma_i} \right] \neq \varnothing .$$

Therefore, for $n \geqslant N$,

$$A_{n,g_E(n)}^{\alpha} \cap \bigcap_{i=1}^{k} \left[\bigcup_{j \leqslant h_{\gamma_i}(n)} A_{n,j}^{\gamma_i} \right] \neq \varnothing .$$

By the definition of h_α, there is an integer N_1, so that for $n \geqslant N_1$, $h_\alpha(n) \geqslant g_E(n)$. Therefore, for $n \geqslant \max(N, N_1)$,

$$Z_n \cap \bigcap_{i=1}^{k} \left[\bigcup_{j \leqslant h_{\gamma_i}(n)} A_{n,j}^{\gamma_i} \right] \cap \left[\bigcup_{j \leqslant h_\alpha(n)} A_{n,j}^{\alpha} \right] \neq \varnothing .$$

This completes the induction.

Let $\mathfrak{F}$ be the filter generated by the set $\{ \bigcup_{n<\omega} \bigcup_{j \leqslant h_\alpha(n)} A_{n,j}^{\alpha} : \alpha < c \}$. Since each $A_{n,j}^{\alpha}$ is closed in Z_n and we are only taking a finite union in each Z_n, $\mathfrak{F}$ is indeed a filter of closed sets. It follows easily from the induction that $\mathfrak{F}$ is nice. Finally, for each $D \in \mathfrak{D}$, as pointed out above, there is an $\alpha < c$ such that $\overline{D} \cap (\bigcup_{n,m} A_{n,m}^{\alpha}) = \varnothing$. This completes the proof. □

We can now state the main result of this section.

THEOREM 3.2. *Assume $BF(c)$. Each compact ccc nowhere separable F-space of weight less than or equal to c contains weak P-points.*

PROOF. Let K be such an F-space and let $\{Z_n : n \in \omega\}$ be nonempty disjoint clopen subsets of K (recall that K is extremally disconnected). By Lemma 1.1, we need only construct a nice filter $\mathfrak{F}$ on $X = \bigcup_n Z_n$ which avoids countable sets. From Lemma 3.1, we have a nice filter $\mathfrak{F}$ on X which avoids all nowhere dense sets. Since X is nowhere separable we are done. □

Our theorem concerning remote points also follows from Lemma 3.1.

THEOREM 3.3. *Assume $BF(c)$. Each nonpseudocompact ccc space of weight less than or equal to c has remote points.*

PROOF. Let X be such a space. Since X is not pseudocompact there is a nonempty zero set Z of βX contained in X^*. Let $Y = \beta X \setminus Z$. Since X is ccc and of weight less than or equal to c, so is Y. Let $\{Z_n : n \in \omega\}$ be a locally finite collection of disjoint nonempty regular closed subsets of Y. Let $\mathfrak{F}$ be a remote filter on $\bigcup Z_n$ as constructed in Lemma 3.1. Then any $p \in \bigcap_{F \in \mathfrak{F}} \mathrm{cl}_{\beta X} F$ is a remote point of Y and hence of X. To see this, first observe that since each Z_n is a regular closed subset of Y, there is an $F \in \mathfrak{F}$ such that $F \subset \mathrm{int}_Y(\bigcup Z_n)$. Since Y is normal, $\mathrm{cl}_{\beta X} F \cap \mathrm{cl}_{\beta X}(Y \setminus \bigcup Z_n) = \varnothing$. Therefore the only nowhere dense subsets of X we have to worry about are those contained in $\bigcup Z_n$, but $\mathfrak{F}$ avoids all such sets. □

4. Another technique. In this section we develop another technique of constructing nice filters, which is an adaptation of Eric van Douwen's construction of remote points [**vD**]; the construction in [**CS**] is also similar. The Π-*weight*, ΠwX, of a space X is the least cardinality of a Π-base. A Π-*base* is a collection of nonempty open sets such that every nonempty open set of the space contains one from the collection. We construct nice filters in compact ccc F-spaces of Π-weight $\aleph_1$ which avoid all countable nowhere dense sets. We cannot hope to avoid all countable sets because it is consistent that all such spaces are separable [**T**]. We use the same method to construct nice filters which avoid countable discrete sets in compact ccc F-spaces with further Π-weight assumptions.

We first need a result of Efimov.

THEOREM 4.1 [**E**, p. 260]. *A compact ccc F-space X with no isolated points can be represented uniquely as $X = \overline{\bigcup_{n \in N} U_{\kappa_n}}$, where the U_{κ_n} are disjoint open and closed sets of homogeneous Π-weight: that is, $\Pi wV = \Pi wU_{\kappa_n}$ for any open $V \subset U_{\kappa_n}$, and if $n \neq m$ then $\kappa_n \neq \kappa_m$.*

From the above theorem it follows that when we are considering compact ccc F-spaces we may assume they have homogeneous Π-weight. For the remainder of this section all hypothesized spaces will be compact ccc F-spaces with homogeneous Π-weight. The Π-*character* of a point x of X, denoted $\Pi\chi(x, X)$, is the least cardinal of a local Π-base at x.

LEMMA 4.2. *Let X have Π-weight $\aleph_1$. Then*

(i) $\Pi\chi(x, X) = \aleph_1$ for all $x \in X$, and

(ii) $\Pi\chi(D, X) = \aleph_1$ for any countable nowhere dense set D, i.e. given countably many open subsets of X there is a neighborhood of D containing none of them.

PROOF. Recall that we are assuming that the Π-weight of every open set is $\aleph_1$.

(i) Let $x \in X$ and let $\{A_n: n \in \omega\}$ be nonempty clopen subsets of X. It suffices to find a neighborhood of x which does not contain any of the A_n's. Since $\Pi w(A_n) = \aleph_1$ for each $n \in \omega$, we can begin by finding a clopen set B_0 such that $B_0 \subset A_0$ and $A_n \setminus B_0 \neq \varnothing$ for all $n \in \omega$. Similarly the set $\{A_n \setminus B_0: n \in \omega\}$ is not a Π-base for any open set, in particular, not for $A_0 \setminus B_0$. We can therefore choose a clopen set B_1 such that $B_1 \subset A_0 \setminus B_0$ and $(A_n \setminus B_0) \setminus B_1 \neq \varnothing$ for all $n \in \omega$. Similarly, recursively select clopen sets B_{2n} and B_{2n+1} such that

(1) $B_i \cap B_j = \varnothing$ for $i < j \leq 2n+1$,

(2) $B_{2n} \cup B_{2n+1} \subset A_n$, and

(3) $A_k \setminus [\bigcup_{j=1}^{2n+1} B_j] \neq \varnothing$ for all $k \in \omega$.

Therefore both $\overline{\bigcup_{j\in\omega} B_{2j}}$ and $V = \overline{\bigcup_{j\in\omega} B_{2j+1}}$ intersect each A_n by (2) and $U \cap V = \varnothing$ because X is extremally disconnected. Hence either U or $X \setminus U$ is the required neighborhood of x because neither set contains any of the A_n's.

(ii) Let $D = \{d_n: n \in \omega\}$ be nowhere dense in X and let $\{A_n: n \in \omega\}$ be clopen subsets of X. We again wish to find a neighborhood of D which does not contain any A_n. One recursively selects nonempty clopen sets B_k and C_k for $k \in \omega$ such that:

(1) $d_k \in B_k$, $B_k \cap [\bigcup_{j<k} C_j] = \varnothing$ and $A_n \setminus [B_k \cup \bigcup_{j<k}(B_j \cup C_j)] \neq \varnothing$ for $n \in \omega$; and

(2) $C_k \cap \overline{D} = \varnothing$, $C_k \subset A_k \setminus [\bigcup_{j\leq k} B_j]$ and $A_n \setminus [\bigcup_{j\leq k}(B_j \cup C_j)] \neq \varnothing$ for $n \in \omega$.

This recursion can be carried out because $\Pi\chi(d_k, X) = \aleph_1$ for each $k \in \omega$, $\Pi w(A_n \setminus [\bigcup_{j\leq k}(B_j \cup C_j)]) = \aleph_1$ and because $\overline{D}$ is nowhere dense in X. This completes the proof. $\square$

THEOREM 4.3. *Let X have Π-weight $\aleph_1$. There are points in X which are not in the closure of any countable nowhere dense sets.*

PROOF. Let $\{Z_n: n \in \omega\}$ be disjoint nonempty clopen subsets of X. By Lemma 1.1 we must construct a nice filter on $\bigcup Z_n$ which avoids all countable nowhere dense subsets of $\bigcup Z_n$. For each $n \in \omega$, let $\{B_{n,\alpha}: \alpha < \omega_1\}$ be nonempty clopen subsets of Z_n which form a Π-base for Z_n. Let $\mathfrak{D} = \{D \subset \bigcup Z_n: D$ is countable and nowhere dense$\}$. We will construct a family $\{F_{D,n}: D \in \mathfrak{D}, n \in \omega\}$ of clopen sets satisfying:

(1) $F_{D,n} \cap \overline{D} = \varnothing$;

(2) for any $\mathcal{E} \subset \mathfrak{D}$ with $1 \leq |\mathcal{E}| \leq n$, $\bigcap_{D\in\mathcal{E}} F_{D,n} \neq \varnothing$; and

(3) $F_{D,n} \subseteq Z_n$; and then define $F_D = \bigcup_n F_{D,n}$.

Our nice filter will be $\mathfrak{F}$ which has $\{F_D: D \in \mathfrak{D}\}$ as a filter base.

For each $D \in \mathfrak{D}$ and $n \in \omega$ we define $\alpha(D,n,0) = \min\{\alpha: B_{n,\alpha} \cap \overline{D} = \varnothing\}$. Note that $\alpha(D,n,0)$ exists because $\overline{D}$ is nowhere dense. By Lemma 4.2(ii) there is a clopen neighborhood $U(D,n,1)$ of D such that $B_{n,\gamma} \setminus U(D,n,1) \neq \varnothing$ for each $\gamma < \alpha(D,n,0)$. Let $\alpha(D,n,1) = \min\{\alpha$: for each $\gamma \leq \alpha(D,n,0)$ there is a $\xi < \alpha$ with $B_{n,\xi} \subset B_{n,\gamma} \setminus U(D,n,1)\}$.

Then let $K(D,n,1) = \{\xi < \alpha(D,n,1): B_{n,\xi} \cap U(D,n,1) = \varnothing\}$. Note that $\overline{\bigcup \{B_{n,\xi}: \xi \in K(D,n,1)\}} \cap \overline{D} = \varnothing$ and that, for each $\gamma < \alpha(D,n,0)$, there is a $\xi \in K(D,n,1)$ with $B_{n,\xi} \subset B_{n,\gamma}$. Recursively construct for $m < n$, $U(D,n,m+1)$, clopen neighborhoods of D such that for each $\gamma < \alpha(D,n,m)$,

$$B_{n,\gamma} \setminus U(D,n,m+1) \neq \varnothing,$$

ordinals $\alpha(D, n, m+1) = \min\{\alpha$: for each $\gamma \leqslant \alpha(D, n, m)$ there is a $\xi < \alpha$ with $B_{n,\xi} \subset B_{n,\gamma} \setminus U(D, n, m+1)\}$, and sets $K(D, n, m) = \{\xi < \alpha(D, n, m+1)$: $B_{n,\xi} \cap U(D, n, m+1) = \varnothing\}$. Define

$$F_{D,n} = B_{n,\alpha(D,n,0)} \cup \bigcup_{m=1}^{n} \left[\overline{\cup \{B_{n,\xi}; \xi \in K(D, n, m)\}} \right].$$

Let us check that this satisfies (2). Let $\mathfrak{E}$ be a subfamily of $\mathfrak{D}$ with $1 \leqslant |\mathfrak{E}| \leqslant n$. Let $|\mathfrak{E}| = e$. With recursion on j pick $E_j \in \mathfrak{E} - \{E_i\colon 0 \leqslant i$ and $i < j\}$, for $0 \leqslant j < e$ in such a way that

(4) $\alpha(E_j, n, j) \leqslant \alpha(E, n, j)$ for all $E \in \mathfrak{E} - \{E_i\colon 0 \leqslant i$ and $i < j\}$.

Next define $s(j) \in \omega_1$, for $0 \leqslant j < e$ by $s(0) = \alpha(E_0, n, 0)$;

$$s(j+1) = \min\{\xi \in K(E_{j+1}, n, j+1)\colon B_{n,\xi} \subseteq B_{n,s(j)}\}.$$

This is possible by the definition of $K(E_j, n, j+1)$ and the fact that $s(j) < \alpha(E_j, n, j+1)$ for each $0 \leqslant j < e$ by (4). Since each $B_{n,s(j)} \subset F_{E_j,n}$ for $0 \leqslant j < e$, it follows that $\bigcap_{j<e} F_{E_j,n} \supseteq B_{n,s(e)} \neq \varnothing$. This completes the proof. □

COROLLARY 4.4. *Let X have Π-weight $\aleph_1$ and be nowhere separable. Then X contains weak P-points.*

REMARK. An easy adaptation of the proof of Theorem 4.3 can be used to show the following. Let X be a nonpseudocompact space of Π-weight at most $\aleph_1$. Then there is a point $x \in X^*$ such that $x \notin \overline{F}$ for any countable nowhere dense $F \subset X$ of Π-character $\aleph_1$.

THEOREM 4.5. *Let $\Pi w(X) = \kappa$ be a regular cardinal and suppose that $\Pi\chi(x, X) = \kappa$ for each $x \in X$. Then X contains points which are not the limit point of any countable discrete set.*

PROOF. If $\Pi w(X) = \omega$, this is shown in [**vM**]. For larger cardinals κ, it is simple to show that $\Pi\chi(D, X) = \kappa$ for each countable discrete set D in X. Then simply replace $\aleph_1$ by κ in the proof of 4.3.

We do not know if it is necessarily true that $\Pi\chi(x, X) = \Pi w(X)$ for each $x \in X$ in a compact ccc F-space of homogeneous Π-weight. We showed that it was true for Π-weight $\aleph_1$ in 4.2 and in the following theorem we show it for separable compact F-spaces assuming Martin's axiom for σ-centred posets, denoted MAS.

THEOREM 4.6 (MAS). *Let X be a compact separable F-space. Then $\Pi\chi(x, X) = \Pi w(X)$ for each $x \in X$.*

PROOF. Let $\Pi w(X) = \kappa$ and note that $\kappa \leqslant c$ since X is separable. Let S be a countable dense subset of X and suppose there is an $x \in X$ with $\Pi\chi(x, X) < \kappa$. Let us first prove the following fact.

FACT 1. Suppose that x is the limit point of a discrete set D and that $\Pi\chi(d, X) > \lambda$ for each $d \in D$. Then $\Pi\chi(x, X) > \lambda$. Indeed, suppose $\{A_\alpha\colon \alpha < \lambda\}$ is a collection of clopen subsets of X. To show that it is not a Π-base at x we will find a clopen neighborhood of x not containing any A_α. Since D is discrete, let $D = \{d_n\colon n \in \omega\}$ and $\{V_n\colon n \in \omega\}$ be disjoint neighborhoods of the d_n's. Since $\Pi\chi(d_n, X) > \lambda$ for each n, we can choose a neighborhood U_n of d_n such that $U_n \subset V_n$ and, if $A_\alpha \cap V_n \neq \varnothing$, then $(A_\alpha \cap \overline{V_n}) \setminus U_n \neq \varnothing$ for $\alpha < \lambda$. Since X is extremally disconnected and $x \in U = \overline{\bigcup U_n}$, U is a neighborhood of x. Let $\alpha < \lambda$ and suppose that $A_\alpha \subset U$. Then there is an $n < \omega$ such that $A_\alpha \cap U_n \neq \varnothing$ and therefore $A_\alpha \cap V_n \neq \varnothing$. This is a contradiction because $(A_\alpha \cap V_n) \setminus U_n \neq \varnothing$ by assumption and $(U \setminus U_n) \cap (A_\alpha \cap V_n) = \varnothing$. Therefore Fact 1 is true.

Now suppose that $x \in X$ and $\Pi\chi(x, X) = \lambda < \kappa$. Note that $\{s \in S: \Pi\chi(s, X) \leqslant \lambda\}$ is nowhere dense in X because the union of local Π-bases of a dense set is a Π-base for X. Therefore we assume $\Pi\chi(s, X) > \lambda$ for each $s \in S$. Our plan now is to find a discrete subset of S which has x as a limit point, which will complete the proof.

Let $\{A_\alpha: \alpha < \lambda\}$ be a Π-base at x consisting of clopen subsets of X. Also let $\{B_\alpha: \alpha < \kappa\}$ be clopen subsets of X which form a Π-base for X. Recall that we are assuming that X has homogeneous Π-weight. We can inductively define B'_α so that $B'_\alpha \subset B_\alpha$ and for each $\xi < \lambda$ and $\gamma_1 < \cdots < \gamma_n < \alpha$, $B'_\alpha \not\supset A_\xi \setminus \bigcup_{i=1}^n B'_{\gamma_i}$. Therefore $\{B'_\alpha: \alpha < \kappa\}$ is a Π-base such that no finite union of its members contains any A_ξ. Let $(P, \leqslant)$ be the poset whose members are $\{(F, V): F \in [S]^{<\omega}$, V is a finite union of sets from the Π-base; and $F \subset V\}$. We will define $(G, W) \leqslant (F, V)$ if $F \subset G$, $V \subset W$ and $G \setminus F \cap V = \varnothing$. To see that $(P, \leqslant)$ is σ-centred, we simply let $P_F = \{(F, V): (F, V) \in P\}$ for each $F \in [S]^{<\omega}$. Let $E_\alpha = \{(F, V): F \cap A_\alpha \neq \varnothing\}$ for each $\alpha < \lambda$. It is easy to check that E_α is dense in $(P, \leqslant)$. Therefore MAS allows us to choose a generic filter $\mathcal{G}$ such that for each $\alpha < \lambda$ there is an $(F_\alpha, V_\alpha) \in \mathcal{G}$ with (F_α, V_α) also in E_α. Define $D = \bigcup\{F: \exists V \text{ with } (F, V) \in \mathcal{G}\}$.

First note that $D \cap A_\alpha \neq \varnothing$ for each $\alpha < \lambda$ and so $x \in \overline{D}$. All we have to show is that D is discrete. Let $d \in D$ and find $(F_D, V_D) \in \mathcal{G}$ with $d \in F_D$. By the definition of the partial ordering we see that $D \cap V_D = F_D$. It follows that D is discrete. □

We remark that MA implies that if X is compact, ccc and $\Pi w(X) < c$ then X is separable [**T**].

Corollary 4.7 (MAS). *Let X be a compact separable F-space with $\Pi w(X)$ regular; then X contains points not the limit point of any countable discrete set.*

Proof. Theorems 4.5 and 4.6.

5. Example and remarks.

Example. We now give the promised example of a compact nonseparable F-space which does not have any weak P-points. The author is grateful to the referee and also to Jan van Mill for suggesting this example. Let $\omega_1 + 1$ be the ordinals less than or equal to ω_1 endowed with the order topology. Let E be the projective cover of $\omega_1 + 1$ and let $k: E \to \omega_1 + 1$ be the canonical map. Let $Z = k^{\leftarrow}[\{\omega_1\}]$ and let $Y = \beta(\omega \times E)$. Observe that $Y - \mathrm{cl}_Y(\omega \times Z)$ is locally separable. Let Π: $\omega \times Z \to \omega$ be the projection map and let $\beta\Pi$: $\beta(\omega \times Z) \to \beta\omega$ be its Stone extension. Note that $\beta(\omega \times Z) = \mathrm{cl}_Y(\omega \times Z)$, as Y is an F-space and $\omega \times Z$ is σ-compact. We can form the adjunction space $M = Y \cup_{\beta\Pi} \beta\omega$ (see [**W**, Chapter 10]). Since $\mathrm{cl}_Y(\omega \times Z)$ is a P-set of Y it is easy to show that M is an F-space. Clearly M is not separable and has no weak P-points.

REMARKS. One would naturally conjecture that all compact nowhere separable F-spaces contain weak P-points. The remaining problem is to remove special set-theoretic assumptions for the case of compact ccc F-spaces of weight $\leqslant c$. Murray Bell has observed that if such a space has a σ-n-linked base for each n then it contains weak P-points. One would also like to show that all compact F-spaces contain points which are not the limit point of any countable discrete set. Is it true that all compact ccc F-spaces of homogeneous Π-weight also have homogeneous Π-character?

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