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A CLASS OF L^1-CONVERGENCE

BY

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ABSTRACT. It is proved that if the Fourier coefficients $\{a_n\}$ of $f \in L^1(0, \pi)$ satisfy $(*)$ $n^{-1}\sum_{k=n}^{2n} k^p|\Delta a_n|^p = o(1)$, for some $1 < p \leq 2$, then $\|s_n - f\| = o(1)$, if and only if $a_n \lg n = o(1)$. For cosine trigonometric series with coefficients of bounded variation and satisfying $(*)$ it is proved that a necessary and sufficient condition for the series to be a Fourier series is $\{a_n\} \in \mathcal{C}$, where $\mathcal{C}$ is the Garrett-Stanojević [4] class.

1. Introduction. Let f be a 2π-periodic and even function in $L^1(0, \pi)$, and let $\{a_k\}$ be the sequence of its Fourier coefficients. Denote by $\mathcal{F}$ the class of sequences of Fourier coefficients of all such functions. It is well known that, in general, it does not follow from $\{a_k\} \in \mathcal{F}$ that

$$s_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx$$

converges to f in the $L^1(0, \pi)$-norm, i.e. it does not follow that $\|s_n - f\| = o(1)$, $n \to \infty$, where $\|\cdot\|$ is the $L^1(0, \pi)$-norm.

However, there are examples of subclasses of $\mathcal{F}$ for which $a_n \lg n = o(1)$, $n \to \infty$, is a necessary and sufficient condition for $\|s_n - f\| = o(1)$, $n \to \infty$.

DEFINITION 1.1. A subclass $\mathcal{K}$ of $\mathcal{F}$ is called a class of L^1-convergence if

(1.1) $\quad \|s_n - f\| = o(1)$, $n \to \infty$, is equivalent with $a_n \lg n = o(1)$, $n \to \infty$.

There are three classical examples of classes of L^1-convergence. The first one is due to Young [1] and $\mathcal{K}$ is defined to be the class of all convex sequences $\{a_k\}$ $(\Delta^2 a_k \geq 0)$. The second one is the class of all sequences $\{a_k\}$ such that $\sum_{k=1}^{\infty} k|\Delta^2 a_k| < \infty$, introduced by Kolmogorov [2]. The latter one is an obvious extension of the first one, and both are subclasses of $\mathcal{BV}$, the class of all null-sequences of bounded variation.

The third example is the Telyakovskii [3] class $\mathcal{S}$. A sequence $\{a_k\}$ belongs to $\mathcal{S}$ if there exists a monotone sequence $\{A_k\}$ such that $\sum_{k=1}^{\infty} A_k < \infty$ and $|\Delta a_k| \leq A_k$, for all k. Clearly $\mathcal{S} \subset \mathcal{BV}$.

Garrett and Stanojević [4] introduced the following class $\mathcal{C}$. A null-sequence $\{a_k\}$ belongs to the class $\mathcal{C}$ if for every $\varepsilon > 0$ there exists $\delta > 0$, independent of n, and such that

$$C_n(\delta) = \frac{1}{\pi}\int_0^{\delta}\left|\sum_{k=n}^{\infty} \Delta a_k D_k(x)\right| dx < \varepsilon,$$

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for all n, where D_k is the Dirichlet kernel. As a corollary to their main result in [4], they proved that $\mathcal{C} \cap \mathcal{BV}$ is a class of L^1-convergence. Later, Garrett, Rees and Stanojević [5] proved that $\mathcal{S} \subset \mathcal{C} \cap \mathcal{BV}$.

Recently Fomin [6] extended the Telyakovskii class $\mathcal{S}$ in the following way. A sequence $\{a_k\}$ belongs to the class $\mathcal{F}_p$, if for some $1 < p \leqslant 2$,

$$\sum_{n=1}^{\infty} \left(\frac{\sum_{k=n}^{\infty} |\Delta a_k|^p}{n} \right)^{1/p} < \infty.$$

Fomin [6] proved that $\mathcal{F}_p$ is a class of L^1-convergence and that $\mathcal{S} \subset \mathcal{F}_p$.

On the other hand Stanojević [7] proved that $\mathcal{F}_p \subset \mathcal{C} \cap \mathcal{BV}$, and obtained several new classes of L^1-convergence. A sequence $\{a_k\}$ of Fourier coefficients belongs to the class $\mathcal{C}_p$ if, for some $1 < p \leqslant 2$, $\sum_{k=1}^{\infty} |\Delta a_k|^p < \infty$, and

$$n \left(\frac{\sum_{k=n}^{\infty} |\Delta a_k|^p}{n} \right)^{1/p} = o(1), \qquad n \to \infty. \tag{1.2}$$

Clearly $\{a_k\} \in \mathcal{F}_p$ implies that $\{a_k\} \in \mathcal{C}_p$. Hence the result of Stanojević [7] is stronger than the one of Fomin [6].

In [7], the class $\mathcal{BV}$ is extended in the following manner.

DEFINITION 1.2. A null-sequence $\{a_k\}$ belongs to the class $\mathcal{P}$ if

$$\frac{1}{n} \sum_{k=1}^{n} k |\Delta a_k| = o(1), \qquad n \to \infty.$$

It is proved in [7] that if $\{a_k\} \in \mathcal{P} \cap \mathcal{F}$ and if $n\Delta a_n = O(1)$, $n \to \infty$, then (1.1) holds. The proof is based on the estimate

$$\| g_n - \sigma_n \| = O\left(\frac{1}{n} \sum_{k=1}^{n} k |\Delta a_k| \right), \qquad n \to \infty, \tag{1.3}$$

where $g_n(x) = s_n(x) - a_{n+1} D_n(x)$, and σ_n is the Fejér's sum of s_n.

We have a threefold objective in this paper. First, we shall, within a subclass of $\mathcal{P}$, remove the condition $n\Delta a_n = O(1)$, $n \to \infty$. The subclass $\mathcal{V}_p$ of $\mathcal{P}$ is defined as follows.

DEFINITION 1.3. A sequence $\{a_k\}$ belongs to the class $\mathcal{V}_p$ if, for some $1 < p \leqslant 2$,

$$\frac{1}{n} \sum_{k=1}^{n} k^p |\Delta a_k|^p = o(1), \qquad n \to \infty. \tag{1.4}$$

Secondly we shall estimate $| \, \|s_n - f\| - |a_n| \, \|D_n\| \, |$ directly without using the estimate (1.3).

Finally we shall obtain a necessary and sufficient condition for $a_0/2 + \sum_{k=1}^{\infty} a_k \cos kx$, $\{a_n\} \in \mathcal{V}_p \cap \mathcal{BV}$, to be a Fourier series.

2. Lemmas. In the proof of our main theorem, instead of the condition (1.4) we shall use an equivalent one

$$\frac{1}{n} \sum_{k=n}^{2n-1} k^p |\Delta a_k|^p = o(1), \qquad n \to \infty. \tag{2.1}$$

The following lemma establishes that equivalence.

LEMMA 2.1. *Let* $\{c_k\}$ *be a sequence of nonnegative numbers. Then*

$$\alpha_n = \frac{1}{n} \sum_{k=1}^{n} c_k = o(1), \qquad n \to \infty,$$

if and only if

$$\beta_n = \frac{1}{n} \sum_{k=n}^{2n} c_k = o(1), \qquad n \to \infty.$$

PROOF. The "only if" part is obvious. For the "if" part notice that

$$2^n \beta_{2^n} = 2^{n+1} \alpha_{2^{n+1}} - 2^n \alpha_{2^n}$$

or

$$\alpha_{2^m} = \frac{\alpha_1}{2^m} + \frac{1}{2^m} \sum_{k=0}^{m-1} 2^k \beta_{2^k}. \tag{2.2}$$

For $2^m \leqslant n < 2^{m+1}$, we have

$$\alpha_n \leqslant \frac{2^{m+1}}{n} \alpha_{2^{m+1}} \leqslant 2\alpha_{2^{m+1}}. \tag{2.3}$$

Hence from (2.2) and (2.3) it follows that $\beta_n \to 0$, $n \to \infty$, implies that $\alpha_n \to 0$, $n \to \infty$.

The next lemma provides an identity that we need in order to avoid using (1.3) explicitly in our proof.

LEMMA 2.2. *Let*

$$s_n(x) = \frac{1}{2} a_0 + \sum_{k=1}^{n} a_k \cos kx$$

and let $\{\sigma_n(x)\}$ *be the sequence of Fejér sums of the sequence* $\{s_n(x)\}$. *Then*

$$\begin{aligned} s_n(x) - f(x) &= 2(\sigma_{2n-1}(x) - f(x)) - (\sigma_{n-1}(x) - f(x)) \\ &\quad - \frac{1}{n} \sum_{k=n+1}^{2n} (2n-k) a_k \cos kx. \end{aligned} \tag{2.4}$$

The core of the proof is supplied by the following lemma.

LEMMA 2.3. *Let* $\{c_k\}$ *be a sequence of real numbers. Then for any* $1 < p \leqslant 2$ *and* $n \geqslant 1$

$$\frac{1}{n} \int_0^{\pi} \left| \sum_{k=n}^{2n-1} c_k D_k(x) \right| dx \leqslant A_p \left(\frac{1}{n} \sum_{k=n}^{2n-1} |c_k|^p \right)^{1/p},$$

where A_p *is an absolute constant.*

PROOF. We write

$$\frac{1}{n}\int_0^{\pi}\left|\sum_{k=n}^{2n-1} c_k D_k(x)\right| dx = \frac{1}{n}\int_0^{\pi/n}\left|\sum_{k=n}^{2n-1} c_k D_k(x)\right| dx + \frac{1}{n}\int_{\pi/n}^{\pi}\left|\sum_{k=n}^{2n-1} c_k D_k(x)\right| dx = I_1 + I_2. \tag{2.5}$$

Since $|D_k| \leqslant k + 1$, for the first integral in (2.5) we have $I_1 \leqslant n^{-1}\sum_{k=n}^{2n-1}|c_k|$, and by Hölder's inequality

$$I_1 \leqslant \left(\frac{1}{n}\sum_{k=n}^{2n-1}|c_k|^p\right)^{1/p}.$$

Let $1/p + 1/q = 1, p > 1$. Then by applying the Hölder inequality to the second integral in (2.5) we get

$$I_2 = \frac{1}{n}\int_{\pi/n}^{\pi}\frac{1}{2\sin(x/2)}\left|\sum_{k=n}^{2n-1} c_k \sin\left(k+\tfrac{1}{2}\right)x\right| dx$$
$$\leqslant \frac{1}{2n}\left(\int_{\pi/n}^{\pi}\frac{dx}{(2\sin(x/2))^p}\right)^{1/p}\left(\int_{\pi/n}^{\pi}\left|\sum_{k=n}^{2n-1} c_k \sin\left(k+\tfrac{1}{2}\right)x\right|^q dx\right)^{1/q}.$$

Since

$$\int_{\pi/n}^{\pi}\frac{dx}{(\sin(x/2))^p} \leqslant \pi^p\int_{\pi/n}^{\pi}\frac{dx}{x^p} \leqslant \frac{\pi}{p-1}n^{p-1},$$

it follows that

$$I_2 \leqslant \frac{1}{2}\left(\frac{\pi}{p-1}\right)^{1/p} n^{-1/p}\left(\int_0^{\pi}\left|\sum_{k=n}^{2n-1} c_k \sin\left(k+\tfrac{1}{2}\right)x\right|^q dx\right)^{1/q}.$$

Let $1 < p \leqslant 2$. Then using the Hausdorff-Young inequality we get

$$\left(\frac{1}{\pi}\int_0^{\pi}\left|\sum_{k=n}^{2n-1} c_k \sin\left(k+\tfrac{1}{2}\right)x\right|^q dx\right)^{1/q} \leqslant \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\sum_{k=n}^{2n-1} c_k e^{ikx}\right|^q dx\right)^{1/q} \leqslant \left(\sum_{k=n}^{2n-1}|c_k|^p\right)^{1/p}.$$

Thus $I_2 \leqslant (\pi/2)(p-1)^{-1/p}(n^{-1}\sum_{k=n}^{2n-1}|c_k|^p)^{1/p}$. Combining all estimates we get

$$I_1 + I_2 \leqslant \pi\left(2 + \tfrac{1}{2}(p-1)^{-1/p}\right)\left(\frac{1}{n}\sum_{k=n}^{2n-1}|c_k|^p\right)^{1/p},$$

and the proof of Lemma 2.3 is completed.

3. Main result. In our main theorem we shall prove that for some $1 < p \leqslant 2$, the class $\mathcal{V}_p \cap \mathcal{F}$ is an L^1-convergence class. To make use of our lemmas we shall reformulate that result in the following manner.

THEOREM 3.1. *Let $f \in L^1(0, \pi)$ be an even and 2π-periodic function and for some $1 < p \leqslant 2$ let the sequence $\{a_k\}$ of its Fourier coefficients satisfy*

$$\frac{1}{n}\sum_{k=n}^{2n} k^p|\Delta a_k|^p = o(1), \qquad n \to \infty. \tag{2.1}$$

Then (1.1) *holds.*

PROOF. Since

$$\sum_{k=n+1}^{2n-1}(2n-k)a_k \cos kx = \sum_{k=n+1}^{2n-1}(2n-k)a_k(D_k(x) - D_{k-1}(x))$$

$$= -na_nD_n(x) + \sum_{k=n}^{2n-1}((2n-k)a_k - (2n-k-1)a_{k+1})D_k(x),$$

from (2.4) we get

$$\begin{aligned} s_n(x) - f(x) = a_nD_n(x) &+ 2(\sigma_{2n-1}(x) - f(x)) \\ &- (\sigma_{n-1}(x) - f(x)) - \frac{1}{n}\sum_{k=n}^{2n-1} a_{k+1}D_k(x) \\ &+ \frac{1}{n}\sum_{k=n}^{2n-1}(2n-k-1)\Delta a_k D_k(x); \end{aligned} \tag{3.1}$$

hence

$$\begin{aligned} |\,\|s_n - f\| - |a_n|\,\|D_n\|\,| \leqslant 2\|\sigma_{2n-1} - f\| &+ \|\sigma_{n-1} - f\| \\ &+ \frac{1}{n}\int_0^\pi \left|\sum_{k=n}^{2n-1} a_{k+1}D_k(x)\right| dx \\ &+ \frac{1}{n}\int_0^\pi \left|\sum_{k=n}^{2n-1}(2n-k-1)\Delta a_k D_k(x)\right| dx. \end{aligned}$$

Applying Lemma 2.3 to the last two integrals we get

$$\frac{1}{n}\int_0^\pi \left|\sum_{k=n}^{2n-1} a_k D_k(x)\right| dx \leqslant A_p\left(\frac{1}{n}\sum_{k=n}^{2n-1}|a_k|^p\right)^{1/p}$$

and

$$\frac{1}{n}\int_0^\pi \left|\sum_{k=n}^{2n-1}(2n-k-1)\Delta a_k D_k(x)\right| dx \leqslant A_p\left(\frac{1}{n}\sum_{k=n}^{2n-1}k^p|\Delta a_k|^p\right)^{1/p}.$$

Since $\|D_n\| = (4/\pi^2)\lg n + O(1)$, $n \to \infty$, and since $f \in L^1(0, \pi)$ implies that $\|\sigma_n - f\| = o(1)$, $n \to \infty$, we finally get

$$|\,\|s_n - f\| - |a_n|\lg n\,| = O\left(\left(\frac{1}{n}\sum_{k=n}^{2n}k^p|\Delta a_k|^p\right)^{1/p}\right), \qquad n \to \infty.$$

In view of (2.1) this completes the proof of Theorem 3.1. We have now a corollary to this theorem.

COROLLARY 3.1. *The class* $\mathcal{C}_p \cap \mathcal{F}$ *is a* L^1*-convergence class.*

PROOF. Since $n^{p-1}\sum_{k=n}^{2n-1}|\Delta a_k|^p = o(1)$, $n \to \infty$, implies

$$\frac{1}{n}\sum_{k=n}^{2n-1} k^p|\Delta a_k|^p = o(1), \qquad n \to \infty,$$

it follows that $\mathcal{C}_p \subset \mathcal{V}_p$.

4. Necessary and sufficient integrability conditions. Let $\{a_k\} \in \mathcal{B}\mathcal{V}$. Then

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx \tag{4.1}$$

converges pointwise to some f in $(0, \pi]$. In order to prove that (4.1) is a Fourier series it suffices to show that $f \in L^1(0, \pi)$.

THEOREM 4.1. *Let* $\{a_k\} \in \mathcal{V}_p \cap \mathcal{B}\mathcal{V}$. *Then* (4.1) *is a Fourier series if and only if* $\{a_k\} \in \mathcal{C}$.

PROOF. *The "if" part.* Garrett and Stanojević [4] proved that if $\{a_k\} \in \mathcal{B}\mathcal{V}$, then $\|g_n - f\| = o(1)$, $n \to \infty$, if and only if $\{a_k\} \in \mathcal{C}$. Hence $\{a_k\} \in \mathcal{C}$ is always a sufficient condition for $f \in L^1(0, \pi)$ whenever $\{a_k\} \in \mathcal{B}\mathcal{V}$.

The "only if" part. The identity (3.1) can be rewritten as

$$\begin{aligned} s_n(x) - a_{n+1}D_n(x) - f(x) &= g_n(x) - f(x) \\ &= 2(\sigma_{2n-1}(x) - f(x)) - (\sigma_{n-1}(x) - f(x)) \\ &\quad - \frac{1}{n}\sum_{k=n}^{2n-1} a_{k+1}D_k(x) - \frac{1}{n}\sum_{k=n-1}^{2n-1} \Delta a_k D_k(x) + \frac{\Delta a_n D_n(x)}{n}. \end{aligned}$$

Assuming that $f \in L^1(0, \pi)$ and applying Lemma 2.3 we get

$$\|g_n - f\| = O\left(\left(\frac{1}{n}\sum_{k=n}^{2n} k^p|\Delta a_k|^p\right)^{1/p}\right), \qquad n \to \infty.$$

Since $\{a_k\} \in \mathcal{V}_p \cap \mathcal{B}\mathcal{V}$ we have that $\|g_n - f\| = o(1)$, $n \to \infty$, and finally that $\{a_k\} \in \mathcal{C}$. This completes the proof of Theorem 4.1.

COROLLARY 4.1. *Let* $\{a_k\} \in \mathcal{C}_p \cap \mathcal{B}\mathcal{V}$. *Then* (4.1) *is a Fourier series if and only if* $\{a_k\} \in \mathcal{C}$.

PROOF. $\mathcal{C}_p \subset \mathcal{V}_p$.

Let $\mathcal{M}$ be a class of all monotone null-sequences $\{a_k\}$. As a consequence to Corollary 4.1 we obtain a partial answer to the classical outstanding question: Let $\{a_k\} \in \mathcal{M}$. What are necessary and sufficient conditions for (4.1) to be a Fourier series?

COROLLARY 4.2. *Let* $\{a_k\} \in \mathcal{V}_p \cap \mathcal{M}$. *Then* (4.1) *is a Fourier series if and only if* $\{a_k\} \in \mathcal{C}$.

Theorem 4.1, Corollary 4.1 and Corollary 4.2 extend the integrability classes found by Stanojević [7].

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