FREE BOUNDARY CONVERGENCE IN THE HOMOGENIZATION OF THE ONE PHASE STEFAN PROBLEM

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ABSTRACT. We consider the one phase Stefan problem in a "granular" medium, i.e., with nonconstant thermal diffusity, and we study the asymptotic behaviour of the free boundary with respect to homogenization. We prove the convergence of the coincidence set in measure and in the Hausdorff metric. We apply this result to the free boundary and we obtain the convergence in mean for the star-shaped case and the uniform convergence for the one-dimensional case, respectively. This gives an answer to a problem posed by J. L. Lions in [L].

1. Introduction. We consider the exterior nonhomogeneous one phase Stefan problem, that is, the melting of a frozen granular medium, maintained at zero degrees in contact with a liquified region. The unknown are the temperature and the free boundary which consists of the solid-liquid interface.

Consider a given bounded domain $\Omega_0 \subset \mathbf{R}^n$ (the initial liquid region) whose boundary consists of two connected manifolds Γ' and Γ_0 of class C^1, with Γ' lying in the interior of Γ_0 and bounding a domain ω.

Let $\mathcal{B}$ be a large ball with center 0 containing Ω_0 and set $\mathcal{O} = \mathcal{B} \setminus \bar{\omega}$. Consider the family of operators

$$(1.1) \qquad A^\varepsilon v = -\frac{\partial}{\partial x_i}\left(a_{ij}\left(\frac{x}{\varepsilon}\right)\frac{\partial v}{\partial x_j}\right), \quad \text{for } \varepsilon > 0,$$

where the a_{ij} are given functions in $\mathbf{R}^n$, representing the coefficients of the thermal diffusity of the medium, with period 1 in all variables, $a_{ij} = a_{ji}$ and verifying, for a.a. $y \in]0, 1[^n$,

$$(1.2) \qquad \alpha|\xi|^2 \leq a_{ij}(y)\xi_i\xi_j \leq M|\xi|^2, \quad \forall \xi \in \mathbf{R}^n \ (\alpha, M > 0).$$

We are looking for a function $\theta^\varepsilon(x, t) \geq 0$ (the temperature) which satisfies

$$(1.3) \qquad \frac{\partial\theta^\varepsilon}{\partial t} + A^\varepsilon\theta^\varepsilon = 0 \quad \text{when } \theta^\varepsilon > 0$$

where x belongs to a moving domain $\Omega^\varepsilon(t)$ bounded by Γ', where a prescribed temperature is given by

$$(1.4) \qquad \theta^\varepsilon(x, t) = g(x, t), \qquad x \in \Gamma', t > 0,$$

and by free boundary $\Gamma^\varepsilon(t)$ contained in $\mathcal{B}$ for $0 \leq t \leq T$, where one imposes

$$(1.5) \qquad\qquad\qquad\qquad \theta^\varepsilon = 0,$$

$$(1.6) \qquad\qquad\qquad a_{ij}\left(\frac{x}{\varepsilon}\right)\frac{\partial\theta^\varepsilon}{\partial x_j}n_i^\varepsilon = -k(V^\varepsilon \cdot n^\varepsilon),$$

being $n^\varepsilon = \{n_i^\varepsilon\}$ the normal to $\Gamma^\varepsilon(t)$, V^ε the velocity of propagation of $\Gamma^\varepsilon(t)$ and $k > 0$ a given constant representing the heat of fusion. The initial temperature is given in the known domain Ω_0:

$$(1.7) \qquad\qquad\qquad \theta^\varepsilon(x,0) = h(x), \qquad x \in \overline{\Omega}_0.$$

We assume that g and h are positive functions and Hölder continuous in their variables with $g(x,0) = h(x)$ if $x \in \Gamma'$, and $h(x) = 0$ if $x \in \Gamma_0$.

Extending θ^ε by zero outside the union of the domains $\Omega^\varepsilon(t)$, for $0 < t < T$, Duvaut [D] introduced the new function defined in the fixed domain $Q_T = \mathcal{O} \times {]}0, T{[}$

$$(1.8) \qquad\qquad\qquad u^\varepsilon(x,t) = \int_0^t \theta^\varepsilon(x,\tau)\, d\tau,$$

which is characterized as the unique solution of the following parabolic variational inequality (see [D, FK]):

$$(1.9) \qquad\qquad u^\varepsilon \in L^2(0, T; H^1(\mathcal{O})), \qquad \frac{\partial u^\varepsilon}{\partial t} \in L^2(Q_T),$$

$$(1.10) \qquad u^\varepsilon(t) \in K(t) = \left\{ v \in H^1(\mathcal{O}) : v = \int_0^t g(\tau)\, d\tau \text{ on } \Gamma', \right.$$
$$\left. v = 0 \text{ on } \partial\mathcal{B}, \text{ and } v \geq 0 \text{ a.e. in } \mathcal{O} \right\},$$

$$(1.11) \qquad \left(\frac{\partial u^\varepsilon}{\partial t}, v - u^\varepsilon\right) + a^\varepsilon(u^\varepsilon, v - u^\varepsilon) \geq (f, v - u^\varepsilon), \qquad \forall v \in K(t),$$

$$(1.12) \qquad\qquad\qquad\qquad u^\varepsilon(x,0) = 0,$$

where $(\ ,\ )$ denotes the scalar product in $L^2(\mathcal{O})$,

$$a^\varepsilon(u, v) = \int_\mathcal{O} a_{ij}\left(\frac{x}{\varepsilon}\right)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}\, dx$$

is the bilinear form associated with A^ε and

$$(1.13) \qquad\qquad\qquad f(x) = \begin{cases} h(x) & \text{if } x \in \Omega_0, \\ -k & \text{if } x \in \mathcal{O} \setminus \Omega_0. \end{cases}$$

With minor adaptations from [FK] one can state

PROPOSITION 1. *The unique solution u^ε to (1.9)–(1.12) is such that*

$$(1.14) \qquad\qquad\qquad 0 \leq \frac{\partial u^\varepsilon}{\partial t} \leq C \quad a.e. \text{ in } Q_T = \mathcal{O} \times {]}0, T{[}$$

where $C > 0$ is a constant independent of ε;

$$(1.15) \qquad\qquad \Omega_0 \subset \Omega^\varepsilon(t) \subset \Omega^\varepsilon(t') \quad \text{for } 0 < t < t' \leq T,$$

where $\Omega^\varepsilon(t) = \{x \in \mathcal{O}: u^\varepsilon(x,t) > 0\}$, for $t > 0$.

PROOF. The unique part of the proposition which does not follow from [FK] is the inclusion $\Omega_0 \subset \Omega^\varepsilon(t)$, $t > 0$. To prove this we must show that $u^\varepsilon(x, t) > 0$ for all $(x, t) \in \Omega_T = \Omega_0 \times (0, T)$. Since $u^\varepsilon \geq 0$, this is an immediate consequence of the strong minimum principle for super-solutions to parabolic equations with discontinuous coefficients (cf. [Ch]). Indeed, in Ω_T, u^ε verifies

$$\frac{\partial u^\varepsilon}{\partial t} - \frac{\partial}{\partial x_i}\left(a_{ij}\left(\frac{x}{\varepsilon}\right)\frac{\partial u^\varepsilon}{\partial x_j}\right) \geq h(x) > 0$$

(cf. (1.11) and (1.13)) and therefore we cannot have $u^\varepsilon(x, t) = 0$ in any point (x, t) of Ω_T. □

REMARK 1. The stronger inclusion $\Omega^\varepsilon(t) \supset \Omega_0 \cup \Gamma_0$ for $t > 0$ has been proved in [CF] for the homogeneous problem with $a_{ij} \equiv \delta_{ij}$ and assuming that Γ_0 satisfies the uniform inside ball property. However we were unable to extend this result, which is based on a comparison argument with a radially symmetric classical solution, to general discontinuous coefficients a_{ij}.

REMARK 2. For the homogeneous Stefan problem, that is, when the functions a_{ij} are constants, it has been proved that the formulation in variational inequality of the one phase Stefan problem is equivalent to the classical formulation in terms of the temperature $\theta = u_t$, which is a continuous function (see [C] and [CF]). Moreover in that case it is known that the free boundary is strictly contained in $\mathcal{B}$, for finite times T and for $\mathcal{B}$ large enough (see [FK]) and is very smooth, at least differentiable in both variables (see [C]), analytic in the space variables and analytic in all variables during any interval during which the heat supply is analytic (see [KN]).

2. The homogenization result. For ε "small" the operator A^ε has highly oscillating coefficients with period ε, and it is quite natural to "replace" A^ε by a much simpler operator $\mathcal{C}$, called homogenized operator, given by (see [BLP]):

$$(2.1) \qquad\qquad \mathcal{C}v = -q_{ij}\frac{\partial^2 v}{\partial x_i \partial x_j}$$

to which we associate

$$\mathcal{C}(u, v) = \int_{\mathcal{O}} q_{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}\,dx,$$

where the q_{ij} are constants verifying (1.2) and uniquely defined by

$$q_{ij} = a(\chi^i - y_i, \chi^j - y_j).$$

Here we have set

$$a(\phi, \psi) = \int_Y a_{ij}(y)\frac{\partial \phi}{\partial y_i}\frac{\partial \psi}{\partial y_j}\,dy,$$

$Y = {]}0, 1{[}^n$, and χ^i are the solutions of the following problems

$$\begin{cases} \chi^i \in W = \{\phi \in H^1(Y) : \phi \text{ equal on opposite sides of } \partial Y\}, \\ a(\chi^i - Y_i, \psi) = 0 \quad \forall \psi \in W. \end{cases}$$

Introduce the following evolution variational inequality

$$(2.2) \qquad u \in L^2(0, T; H^1(\mathbb{O})), \qquad \frac{\partial u}{\partial t} \in L^2(Q_T),$$

$$(2.3) \qquad u(t) \in K(t), \qquad u(0) = 0,$$

$$(2.4) \qquad \left(\frac{\partial u}{\partial t}, v - u \right) + \mathcal{C}(u, v - u) \geqslant (f, v - u), \quad \forall v \in K(t),$$

corresponding to the homogenized Stefan problem.

Using the general theory of [BLP] for homogenization of parabolic variational inequalities one can state the following theorem:

THEOREM 1. *Under the hypothesis of the introduction, if u^ε (resp. u) denotes the solution of (1.9)–(1.12) (resp. (2.2)–(2.4)) and if we take $\varepsilon \to 0$ then we have*

$$(2.5) \qquad u^\varepsilon \rightharpoonup u \text{ in } L^2(0, T; H^1(\mathbb{O})) \text{-weakly,}$$

$$(2.6) \qquad \frac{\partial u^\varepsilon}{\partial t} \rightharpoonup \frac{\partial u}{\partial t} \text{ in } L^\infty(Q_T) \text{-weakly*,}$$

$$(2.7) \qquad A^\varepsilon u^\varepsilon \rightharpoonup \mathcal{C}u \text{ in } L^\infty(Q_T) \text{-weakly*,}$$

$$(2.8) \qquad u^\varepsilon(x, t) \to u(x, t) \text{ uniformly in } (x, t) \in \overline{Q}_T.$$

PROOF. The first part is classical (see [BLP, p. 288]) and it has been already stated in [L]. Then (2.6) is an immediate consequence of (1.14).

From (2.5) and (1.14) one finds that $A^\varepsilon u^\varepsilon$ belongs to a bounded set of $L^2(0, T; H^{-1}(\mathbb{O}))$ independently of ε, and then we can extract a subsequence such that

$$A^\varepsilon u^\varepsilon \rightharpoonup \xi \text{ in } L^2(0, T; H^{-1}(\mathbb{O})) \text{-weakly.}$$

But, for a.a. $t \in \,]0, T[$ one has

$$A^\varepsilon u^\varepsilon(t) + \frac{\partial u^\varepsilon}{\partial t}(t) - f \geqslant 0 \text{ in } H^{-1}(\mathbb{O}),$$

$$u^\varepsilon(t) \rightharpoonup u(t) \text{ weakly in } H^1(\mathbb{O}),$$

$$\frac{\partial u^\varepsilon}{\partial t}(t) \rightharpoonup \frac{\partial u}{\partial t}(t) \text{ strongly in } H^{-1}(\mathbb{O}) \text{ (by compactness).}$$

Then from Theorem 2 of [M1] one finds

$$\mathcal{C}u(t) = \xi(t) \quad \text{a.a. } t \in \,]0, T[,$$

and (2.7) will be proved if we show that

$$\|A^\varepsilon u^\varepsilon\|_{L^\infty(Q_T)} \leqslant C \qquad (C \text{ independent of } \varepsilon > 0).$$

But this is a consequence of the following inequalities due to Charrier and Troianiello (cf. [CT]):

$$(2.9) \qquad f^- \leqslant \frac{\partial u^\varepsilon}{\partial t} + A^\varepsilon u^\varepsilon \leqslant f^+ .$$

Recall (1.14) and that f is bounded by hypothesis (see (1.13)).

From (2.9) and the Hölder estimates for linear parabolic equations (cf. [**LSU**, p. 204]) there exist $\alpha > 0$ and $C > 0$, independent of ε, such that

$$\| u^\varepsilon \|_{C^\alpha(\bar{Q}_T)} \leqslant C,$$

and (2.8) follows by the Ascoli-Arzelá theorem. $\square$

3. The convergence of the coincidence set. Let us denote by

$$I^\varepsilon = \left\{ (x, t) \in \bar{Q}_T \colon u^\varepsilon(x, t) = 0 \right\} \quad \text{and} \quad I = \left\{ (x, t) \in \bar{Q}_T \colon u(x, t) = 0 \right\},$$

the coincidence sets of the variational inequalities (1.9)–(1.12) and (2.2)–(2.4) and by χ^ε and χ their characteristic functions respectively.

Since u^ε and u are continuous functions, the sets I^ε and I are compact subsets of $\bar{Q}_T$. The I^ε may have a very complicated structure but we shall show that they approach the homogenized coincidence set I, for which some regularity is known.

Consider all compact subsets of $\bar{Q}_T$, and introduce the Hausdorff distance

$$\delta(I, J) = \max\left[\sup_{X \in I} d(X, J), \sup_{Y \in J} d(Y, I) \right]$$

where $d(X, J) = \inf_{Y \in J} | X - Y |$.

THEOREM 2. *Under the conditions of Theorem 1 one has*

(3.1) $$I^\varepsilon \to I \text{ in the Hausdorff distance}.$$

PROOF. Since the family of all compact subsets of $\bar{Q}_T$ is a compact metric space for the Hausdorff distance (see [**De**, p. 42], for instance), one can extract a subsequence $I^{\varepsilon'}$ such that

$$I^{\varepsilon'} \to I^* \text{ (Hausdorff)},$$

where I^* is a compact of $\bar{Q}_T$. Then for any $(x, t) \in I^*$ there exists $(x^{\varepsilon'}, t^{\varepsilon'}) \in I^{\varepsilon'}$, such that $(x^{\varepsilon'}, t^{\varepsilon'}) \to (x, t)$. Using the triangle inequality, the continuity of u and (2.8) one easily deduces that $u(x, t) = 0$, and therefore $I^* \subset I$.

If we show the other inclusion the theorem will be proved. Since ∂I is smooth (see Remark 2), one has $I = \overline{\text{int } I}$ and the inclusion $I \subset I^*$ will follow if we prove

(3.2) $$\text{int } I = \text{int } I^*.$$

By contradiction, assume that there exists a nonempty open set $\Lambda \subset I \setminus I^*$. We claim that for every $\phi \in \mathcal{D}(\Lambda)$, if we consider ϕ extended by zero outside Λ, one has

(3.3) $$\int_{Q_T} \left(\frac{\partial u}{\partial t} + \mathcal{Q}u - f \right) \phi \, dx \, dt = \lim_{\varepsilon'} \int_{Q_T} \left(\frac{\partial u^{\varepsilon'}}{\partial t} + A^\varepsilon u^{\varepsilon'} - f \right) \phi \, dx \, dt = 0,$$

which is the desired contradiction, since $u \equiv 0$ in Λ and $f \neq 0$ a.e. Indeed (3.3) is a consequence of (2.6), (2.7) and the fact that supp $\phi \cap I^{\varepsilon'} = \varnothing$ for every ε' sufficiently small. This can easily be seen if we consider a limit point (x, t) of a sequence of points in supp $\phi \cap I^{\varepsilon'}$ which contradicts the facts $d((x, t), I^*) > 0$ and $I^{\varepsilon'} \to I^*$ (Hausdorff).

Once (3.2) is proved, one has $I^* = I$, and the whole sequence I^ε converges to I. The proof is complete. $\square$

REMARK 3. Similar results have been proved for elliptic variational inequalities by Codegone and Rodrigues (cf. [CR]). Theorem 3 may also be extended for smooth obstacles $\psi^\varepsilon(t) \not\equiv 0$ and for more general convergence on the operators A^ε, namely the G-convergence.

The following theorem extends the idea of Murat [M2] on the convergence in measure of the coincidence sets I^ε in a general case where no regularity hypotheses on ∂I^ε are asssumed.

THEOREM 3. *Under the hypothesis of Theorem* 1 *one has*

$$(3.4) \qquad\qquad \chi^\varepsilon \to \chi \text{ in } L^p(Q_T)\text{-strongly, for all } p < \infty.$$

In particular, we have the convergence in the Lebesgue measure of the coincidence sets I^ε to I.

Before proving this theorem in all generality we shall describe Murat's argument assuming the following regularity hypotheses:

$$(3.5) \qquad\qquad a_{ij} \text{ are } C^1\text{-functions; or}$$

$$(3.6) \qquad\qquad \text{mes}(\partial I^\varepsilon) = 0 \quad \text{for each } \varepsilon > 0.$$

If one of these two conditions is fulfilled, from the proof of Theorem 1 (see (2.9)) one easily finds that

$$\frac{\partial u^\varepsilon}{\partial t} + A^\varepsilon u^\varepsilon = f(1 - \chi^\varepsilon) \quad \text{a.e. in } Q_T.$$

Then if $\chi^\varepsilon \rightharpoonup q$ in $L^\infty(Q_T)$-weakly*, using (2.6) and (2.7), passing to the limit, one finds

$$\frac{\partial u}{\partial t} + \mathcal{C}u = f(1 - q).$$

But since the limit problem can also be written in the form

$$\frac{\partial u}{\partial t} + \mathcal{C}u = f(1 - \chi),$$

and remarking that $f \neq 0$ a.e. in Q_T, one has $\chi = q$.

We deduce $\chi^\varepsilon \to \chi$ in $L^p(Q_T)$ strongly, because if characteristic functions converge weakly to a characteristic function they also converge strongly for all $p < \infty$. $\square$

The proof of Theorem 3 without assumptions (3.5) or (3.6) needs more information on the sets I^ε. We shall give a proof which uses Theorem 2 and again the known regularity of the limit set I. For a necessary and sufficient condition to the equivalence between the convergence in measure and in the Hausdorff metric see [B].

PROOF OF THEOREM 3. From (2.9), for each $\varepsilon > 0$, there exists a function η^ε verifying

$$(3.7) \qquad\qquad 0 \leqslant 1 - \chi^\varepsilon \leqslant \eta^\varepsilon \leqslant 1, \quad \text{a.e. in } Q_T$$

and such that one has

$$(3.8) \qquad\qquad \frac{\partial u^\varepsilon}{\partial t} + A^\varepsilon u^\varepsilon = f\eta^\varepsilon \quad \text{a.e. in } Q_T.$$

Consider a subsequence $\eta^{\varepsilon'} \rightharpoonup \eta$ in $L^p(Q_T)$-weakly. Using Theorem 1 and passing to the limit in (3.8) one finds $f\eta = f(1 - \chi)$, since the limit problem is regular. Therefore, $\eta = 1 - \chi$, because $f \neq 0$ a.e.

Now consider a subsequence $\chi^{\varepsilon'} \rightharpoonup q$ in $L^p(Q_T)$-weakly. From (3.7) it follows that

$$0 \leqslant 1 - q \leqslant 1 - \chi \leqslant 1 \quad \text{a.e. in } Q_T,$$

which implies $q = 1$ a.e. in I.

To verify that $q = 0$ a.e. in $\Omega = Q_T \setminus I$, it is enough to prove that $q = 0$ in the sets $\Omega_\delta = \{(x, t) \in \Omega : d((x, t), \partial\Omega) > \delta\}$ for arbitrarily small $\delta > 0$. For any δ fixed, from (3.1) there exists $\varepsilon^* = \varepsilon^*(\delta) > 0$, such that

$$\Omega_\delta \cap I^\varepsilon = \varnothing, \quad \text{for all } 0 < \varepsilon \leqslant \varepsilon^*.$$

It follows that $\chi^\varepsilon = 0$ in Ω_δ for all $0 < \varepsilon \leqslant \varepsilon^*$, and consequently also $q = 0$ in Ω_δ.

Then $q = \chi$ and the convergence (of all the sequences) $\chi^\varepsilon \to \chi$ holds, first in $L^p(Q_T)$-weakly and afterwards also strongly. The proof is complete. $\square$

REMARK 4. We have assumed $\mathfrak{B}$ large enough to assure the existence of the free boundary for the homogenized problem. Therefore mes(I) > 0, and from (3.4), at least for ε sufficiently small one has mes(I^ε) > 0. This implies the existence of a free boundary for the nonhomogeneous Stefan problem, which seems a nontrivial problem when the coefficients a_{ij} are only assumed to be L^∞.

4. The convergence of the free boundaries.

In his report (see [L]) on the asymptotic behaviour of solutions of variational inequalities with highly oscillating coefficients J. L. Lions posed the question as to in what sense the free boundary

$$\Gamma^\varepsilon = \partial I^\varepsilon \cap Q_T = \bigcup_{0 < t < T} \Gamma^\varepsilon(t)$$

gives an "approximation" of the free boundary in the homogenized problem: $\Gamma = \partial I \cap Q_T$?

From the convergence of the coincidence sets and using some regularity of Γ^ε we can answer this question of J. L. Lions in two directions: from the convergence in measure of the coincidence sets we establish the convergence in mean of the star-shaped free boundaries and from the Hausdorff convergence we deduce the uniform convergence of the free boundary in the case of one-dimensional space variable.

4.1. *The star-shaped case.* Let us assume that for each $\varepsilon > 0$ the free boundary Γ^ε possesses a representation of the form

$$(4.1) \qquad \Gamma^\varepsilon(t) : \rho_\varepsilon = \rho_\varepsilon(\theta, t), \qquad \theta \in \Theta, 0 < t \leqslant T,$$

where $\Gamma^\varepsilon(\bar{t}) = \Gamma^\varepsilon \cap \{t = \bar{t}\}$ and ρ, θ are polar coordinates in $\mathbf{R}^n$, $n \geqslant 2$, $\rho = |x|$ and the angles $\theta = (\theta_1, \ldots, \theta_{n-1}) = \arg x$ range in the set

$$\Theta : -\pi \leqslant \theta_i \leqslant \pi, \quad i < n - 1, \ 0 \leqslant \theta_{n-1} \leqslant \pi \ (-\pi \leqslant \theta \leqslant \pi, \text{if } n = 2).$$

Friedman and Kinderlehrer have established sufficient conditions to (4.1) in their work [FK]. For instance if the a_{ij} are twice continuously differentiable functions, the fixed boundary Γ' is of class $C^{2+\alpha}$, representable by $\rho = r(\theta)$, $\theta \in \Theta$ (i.e., ω is star-shaped with reference to 0) and if Γ' is "quite flat" with respect to the growth of $g(x, t)$ (in particular, if g is constant and Γ' is a sphere this condition is fulfilled),

then for each fixed $\varepsilon > 0$ our assumption (4.1) is verified (see [**FK**, p. 1030]) ρ_ε being a continuous function of θ and t, Lipschitz in the angles θ and monotone increasing in t.

Consequently, we also assume that the homogenized free boundary Γ is also representable by

$$(4.2) \qquad \Gamma(t): \rho = \rho(\theta, t), \qquad \theta \in \Theta, 0 < t \leq T.$$

THEOREM 4. *If the free boundaries Γ^ε and Γ are star-shaped and representable in polar coordinates by (4.1) and (4.2), then as $\varepsilon \to 0$, we have*

$$(4.3) \qquad \rho_\varepsilon^n \to \rho^n \text{ in } L^1(\Theta \times]0, T[ )\text{-strongly.}$$

PROOF. The result is a corollary of Theorem 3. Indeed, passing to polar coordinates in $\mathbf{R}^n$ with $dx = \rho^{n-1} d\rho\, d\Theta$, one has

$$\int_{Q_T} |\chi^\varepsilon - \chi|dx\, dt = \frac{1}{n} \int_{\Theta \times]0, T[} |\rho_\varepsilon^n - \rho^n|d\Theta\, dt,$$

and the theorem is an immediate consequence of (3.4). $\square$

REMARK 5. As in Theorem 3 one can show that if $\varepsilon \to 0$, then $\chi^\varepsilon(t) \to \chi(t)$ in measure a.a. $t \in]0, T[$. From this one also deduces that $\rho_\varepsilon^n(t) \to \rho^n(t)$ in $L^1(\Theta)$-strongly a.a. t.

REMARK 6. Instead of the global result of Theorem 4, which needs strong geometrical hypotheses on the data, one could also give a local convergence in mean of the free boundary. For instance, if the free boundary Γ^ε admits the local representation

$$\Gamma^\varepsilon: x_n = \phi_\varepsilon(x', t), \quad \text{for } \varepsilon \geq 0, x' = (x_1, \ldots, x_{n-1})$$

from Theorem 3 one also obtains the convergence in L^1 of ϕ_ε to ϕ.

4.2. *The one-dimensional case.* In the one-dimensional homogenized problem the coefficient of $\mathcal{C}$ is a positive constant given by the inverse of the mean in $]0, 1[$ of $1/a(y)$, and Γ is a C^∞ curve.

If we only assume $a \in L^\infty(0, 1)$, one can still use the argument of Theorem 2.8 of [**KS**, p. 287] to show that the free boundary Γ^ε is a Lipschitz curve in the coordinates obtained from (x, t) by rotating through $\pi/4$, being the Lipschitz constant independent of ε. In particular, the sets I^ε verify an exterior cone property uniformly with respect to ε, which together with $I^\varepsilon \to I$ (Hausdorff) implies $\overline{\Gamma}^\varepsilon \to \overline{\Gamma}$ (Hausdorff), by Theorem 3.2 of [**R**].

Then we can state

THEOREM 5. *In the one-dimensional one phase Stefan problem the free boundary Γ^ε approaches uniformly the homogenized free boundary Γ.*

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