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HARDY SPACES AND JENSEN MEASURES[1]

BY

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ABSTRACT. Suppose A is a subalgebra of $L^\infty(m)$ on which m is multiplicative. In this paper, we show that if m is a Jensen measure and $A + \bar{A}$ is dense in $L^2(m)$, then $A + \bar{A}$ is weak-* dense in $L^\infty(m)$, that is, A is a weak-* Dirichlet algebra. As a consequence of it, it follows that if $A + \bar{A}$ is dense in $L^4(m)$, then A is a weak-* Dirichlet algebra. (It is known that even if $A + \bar{A}$ is dense in $L^3(m)$, A is not a weak-* Dirichlet algebra.) As another consequence, it follows that if B is a subalgebra of the classical Hardy space H^∞ containing the constants and dense in H^2, then B is weak-* dense in H^∞.

1. Introduction. Let $(X, \mathcal{Q}, m)$ be a nontrivial probability measure space and A a subalgebra of $L^\infty = L^\infty(m)$ containing the constants. Suppose m is multiplicative on A, that is, for $f, g \in A$ we have $\int_X fg\,dm = \int_X f\,dm \int_X g\,dm$. The abstract Hardy space $H^p = H^p(m)$, $0 < p \leq \infty$, associated with A is defined as follows. For $0 < p < \infty$, H^p is the $L^p = L^p(m)$-closure of A, while H^∞ is defined to be the weak-* closure of A. If $A + \bar{A}$ is weak-* dense in L^∞, A is called a weak-* Dirichlet algebra, which was introduced by T. P. Srinivasan and J. K. Wang [8]. The theory of weak-* Dirichlet algebras has emerged as the correct setting for many of the central results of abstract analytic function theory.

K. Hoffman and H. Rossi [5] gave an example such that even if $A + \bar{A}$ is dense in L^3, A is not a weak-* Dirichlet algebra. While G. Lumer [6] showed that if $A + \bar{A}$ is dense in L^p for all finite p, $H^p \cap L^\infty$ is a weak-* Dirichlet algebra. Recently, the author [7] proved that if $A + \bar{A}$ is dense in L^4, then $H^4 \cap L^\infty$ is a weak-* Dirichlet algebra. Hence if $f \in A$, then [8]

$$\int_X \log|f|\,dm \geq \log\left|\int_X f\,dm\right|.$$

We say m a Jensen measure when functions in A satisfy the inequality above.

In this paper, we show that if m is a Jensen measure and $A + \bar{A}$ is dense in $L^2(m)$, then A is a weak-* Dirichlet algebra. If $A + \bar{A}$ is dense in L^4, then m is a Jensen measure by the remark above and so A is a weak-* Dirichlet algebra. Hence we answer affirmatively the question left open by Hoffman-Rossi [5], Lumer [6] and the author [7]. Moreover the result is applied to give another proof of a theorem of S. D. Fisher for backward shift invariant subalgebras [1].

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2. Jensen measure. Let A be a subalgebra of L^∞ containing the constants and m a multiplicative measure on it. Set $A_0 = \{f \in A: \int f\,dm = 0\}$.

LEMMA 1. *Suppose $A + \bar{A}$ is dense in L^2. If $w \in L^\infty$ is a nonnegative function with $w^{-1} \in L^\infty$, then*

$$\inf_{f \in A_0} \int |1 - f|^2 w\,dm = \left(\inf_{g \in A_0} \int |1 - g|^2 w^{-1}\,dm \right)^{-1}.$$

PROOF. If $f, g \in A_0$, by Schwarz's inequality,

$$\int |1 - f|^2 w\,dm \geqslant \left(\int |1 - g|^2 w^{-1}\,dm \right)^{-1}.$$

There exists a unique f_0 in the closure of A_0 in $L^2(w\,dm)$ such that

$$\inf_{f \in A_0} \int |1 - f|^2 w\,dm = \int |1 - f_0|^2 w\,dm.$$

Then, by the minimum property of $\int |1 - f_0|^2 w\,dm$, $1 - f_0$ is orthogonal to A_0 in $L^2(w\,dm)$. Set $h_0 = (1 - f_0)w$. Since the infimum is positive, then

$$\int |1 - f_0|^2 w\,dm = \int (1 - \bar{f}_0)(1 - f_0) w\,dm$$

$$= \int (1 - f_0) w\,dm = \int h_0\,dm > 0.$$

By the hypothesis, $\bar{H}^2$ is the orthogonal complement of A_0 in L^2 and so $h_0 \in \bar{H}^2$. Hence $g_0 = \bar{h}_0 / \int \bar{h}_0\,dm$ belongs to H^2 and

$$\left(\int h_0\,dm \right)^2 \int |1 - (1 - g_0)|^2 w^{-1}\,dm = \int |1 - f_0|^2 w\,dm = \int h_0\,dm.$$

Since $w, w^{-1} \in L^\infty$, $1 - g_0$ belongs to the closure of A_0 in $L^2(w^{-1}dm)$ and

$$\int h_0\,dm = \left(\int |1 - (1 - g_0)|^2 w^{-1}\,dm \right)^{-1}.$$

This implies the lemma.

LEMMA 2 (SZEGÖ'S THEOREM). *Suppose $A + \bar{A}$ is dense in L^2 and m is a Jensen measure. If $w \in L^1$ is a nonnegative function, then*

$$\inf_{f \in A_0} \int |1 - f|^2 w\,dm = \exp \int \log w\,dm.$$

PROOF. By the inequality of arithmetic and geometric means and Jensen's inequality, for any $f, g \in A_0$,

$$\int |1 - f|^2 w\,dm \geqslant \exp \int \log w\,dm$$

and

$$\int |1 - g|^2 w^{-1} dm \geqslant \exp \int \log w^{-1} m$$

if m is a Jensen measure. If $w, w^{-1} \in L^\infty$, by Lemma 1

$$\inf_{f \in A_0} \int |1 - f|^2 w \, dm = \exp \int \log w \, dm.$$

If $w^{-1} \notin L^\infty$ with $w \in L^\infty$, for any $\varepsilon > 0$,

$$\exp \int \log(w + \varepsilon) \, dm = \inf \int |1 - f|^2 (w + \varepsilon) \, dm$$

$$\geqslant \inf \int |1 - f|^2 w \, dm \geqslant \exp \int \log w \, dm$$

and so letting ε tend to zero, the lemma follows. For any $w \in L^1$, let $w_n = \min\{w, n\}$, then

$$\exp \int \log w \, dm \geqslant \left(\inf \int |1 - g|^2 w^{-1} \, dm \right)^{-1}$$

$$\geqslant \left(\inf \int |1 - g|^2 w_n^{-1} \, dm \right)^{-1} = \exp \int \log w_n \, dm$$

and so letting n tend to infinity, the lemma follows.

THEOREM 1. *$A + \bar{A}$ is dense in L^2 and m is a Jensen measure if and only if A is a weak-* Dirichlet algebra.*

PROOF. If A is a weak-* Dirichlet algebra, then $A + \bar{A}$ is dense in L^2 clearly and it is known **[8]** that m is a Jensen measure. If $A + \bar{A}$ is dense in L^2 and m is a Jensen measure, then Szegö's theorem is valid by Lemma 2. Srinivasan and Wang **[8]** imply that Szegö's theorem is equivalent to that $A + \bar{A}$ is weak-* dense in L^∞.

THEOREM 2. *Let A be a subalgebra of L^∞ containing the constants and m a multiplicative measure on it. If $A + \bar{A}$ is dense in L^2, then the following* (1) ~ (6) *are equivalent.*

(1) *$A + \bar{A}$ is weak-* dense in L^∞, that is, A is a weak-* Dirichlet algebra.*

(2) *$A + \bar{A}$ is dense in L^4.*

(3) *m is a Jensen measure.*

(4) *If $f \in H^1$ is a real function, then f is a constant.*

(5) *If $f \in H^{1/2}$ is a nonnegative function, then f is a constant.*

(6) *There is a constant γ_p, defined for $0 < p < 1$, such that*

$$\| f \|_p \leqslant \gamma_p \| f + \bar{g} \|_1, \qquad f \in A, g \in A_0.$$

PROOF. (1) $\Leftrightarrow$ (3) is equivalent to Theorem 1. (1) $\Leftrightarrow$ (2) is clear by the remark in Introduction and (1) $\Leftrightarrow$ (3). (1) $\Rightarrow$ (6) is known (cf. **[2**, p. 107]). (6) $\Rightarrow$ (4) is clear. (4) $\Rightarrow$ (3) Suppose $w \in L^\infty$ is a nonnegative function with $w^{-1} \in L^\infty$. Let f_0 (resp. g_0) be the orthogonal projection of 1 into the closure of A_0 in $L^2(w \, dm)$ (resp. $L^2(w^{-1} \, dm)$). Then by Lemma 1 and Schwarz's lemma, $|1 - f_0|^2 w = |1 - g_0|^2 w^{-1}$ and $(1 - f_0)(1 - g_0) = k \geqslant 0$. If $f = 1 - g_0$, then $kw = |f|^2$ and $f \in H^2$ and $k \in H^1$ because $w, w^{-1} \in L^\infty$. By the hypothesis, k is a constant 1. Thus $w = |f|$ for $f, f^{-1} \in H^2 \cap L^\infty$. This implies that $H^2 \cap L^\infty$ is a logmodular algebra and so m is a Jensen measure **[4]**. (1) $\Rightarrow$ (5) is known **[9]**. (5) $\Rightarrow$ (4) is clear.

(5) is Neuwirth-Newman's theorem and (6) is Kolmogoroff's theorem. Even if $A + \bar{A}$ is not dense in L^2, (3) implies (5) (cf. [**3**, pp. 135 ~ 136]).

3. Subalgebras of H^∞ on the unit circle. Let T be the unit circle in the complex plane and $d\theta/2\pi$ the normalized Lebesgue measure on T. In this section, we consider A, L^∞ in case $X = T$ and $m = d\theta/2\pi$. If A is the set of all analytic polynomials on T, then A is a weak-* Dirichlet algebra of $L^\infty(T) = L^\infty(d\theta/2\pi)$. Then, for $0 < p \leqslant \infty$, $H^p(T) = H^p(d\theta/2\pi)$ is the classical Hardy space.

THEOREM 4. *Let B be a subalgebra of $H^\infty(T)$ containing the constants. If B is L^2-dense in $H^2(T)$, then B is weak-* dense in $H^\infty(T)$.*

PROOF. Since $L^2(T) = H^2(T) + e^{-i\theta}\overline{H^2(T)}$ and $d\theta/2\pi$ is a Jensen measure, and B is L^2-dense in $H^2(T)$, so it is a weak-* Dirichlet algebra of $L^\infty(T)$ by Theorem 1, then $H^2(T) \cap L^\infty(T) = H^\infty(T)$ is a weak-* closure of B by Theorem 2.4.1 of [**8**].

COROLLARY (S. D. FISHER). *Let B be a nontrivial subalgebra of $H^\infty(T)$ which* (i) *contains the constants,* (ii) *is weak-* closed, and* (iii) *contains $e^{-i\theta}f$ whenever $f \in B$ and $\hat{f}(0) = \int_0^{2\pi} f\,d\theta/2\pi = 0$. Then $B = H^\infty(T)$.*

PROOF. Let $\mathfrak{M}$ be a closure of B in $L^2(T)$, then the orthogonal complement $\mathfrak{M}^\perp$ of $\mathfrak{M}$ in $H^2(T)$ is a shift invariant subspace, that is, $e^{i\theta}\mathfrak{M}^\perp \subset \mathfrak{M}^\perp$. By the well-known theorem of Beurling, $\mathfrak{M}^\perp = qH^2(T)$ for some inner function q with $\hat{q}(0) = 0$ if $\mathfrak{M}^\perp \neq \{0\}$. $e^{-i\theta}q$ is orthogonal to $qH^2(T)$ and so $e^{-i\theta}q \in \mathfrak{M} \cap L^\infty(T)$. Since $\mathfrak{M} \cap L^\infty$ is an algebra, $e^{-i2\theta}q^2$ is orthogonal to $qH^2(T)$ and so $\int qe^{-in\theta}d\theta/2\pi = 0$ for $n \geqslant 1$. Thus q is a zero constant and so this implies $\mathfrak{M}^\perp = \{0\}$ and $\mathfrak{M} = H^2(T)$. Theorem 4 implies $B = H^\infty(T)$.

REFERENCES

1. S. D. Fisher, *Algebras of bounded functions invariant under the restricted backward shift*, J. Funct. Anal. **12** (1973).

2. T. W. Gamelin, *Uniform algebras*, Prentice-Hall, Englewood Cliffs, N. J., 1969.

3. ______, *Uniform algebras and Jensen measures*, London Math. Soc. Lecture Note Series 32, Cambridge Univ. Press, London, 1978.

4. K. Hoffman, *Analytic functions and logmodular Banach algebras*, Acta Math. **108** (1962), 271–317.

5. K. Hoffman and H. Rossi, *Function theory from a multiplicative linear functional*, Trans. Amer. Math. Soc. **116** (1965), 536–543.

6. G. Lumer, *Herglotz transformation and H^p theory*, Bull. Amer. Math. Soc. **71** (1965), 725–730.

7. T. Nakazi, *A note on weak-* Dirichlet algebras*, Bull. Acad. Polon. Sci. Sér. Sci. Math. **27** (1979), 553–555.

8. T. P. Srinivasan and J. K. Wang, *Weak-* Dirichlet algebras*, Proc. Internat. Sympos. Function Algebras (Tuland Univ. 1965), Scott-Foresman, Chicago, Ill., 1966, pp. 216–249.

9. K. Yabuta, *A note on extremum problems in abstract Hardy spaces*, Arch. Math. **23** (1972), 54–57.

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