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ON EXTENDING FREE GROUP ACTIONS ON SPHERES AND A CONJECTURE OF IWASAWA

BY

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ABSTRACT. A transfer map for Reidemeister torsion is defined and used to determine whether free actions of $\mathbf{Z}/k$ on S^{2n+1}, $n > 1$, extend to free actions of $\mathbf{Z}/hk$. It is shown that for k odd, every free $\mathbf{Z}/k$ action on S^{2n+1}, $n > 1$, extends to a free $\mathbf{Z}/2k$ action. For prime p, extension of an arbitrary free $\mathbf{Z}/p$ action to a free $\mathbf{Z}/p^2$ action is reduced to a long-standing conjecture of Iwasawa.

0. Introduction and statement of results. In this paper we address the following basic question: Let π be a finite group, ρ a subgroup. When does a free action of ρ on S^{2n+1} extend to a free action of π?

Some of our results apply in general, but we concentrate on the case where π is a cyclic group. In this case we answer the above question completely, reducing the issue to algebraic number theory. If $\rho = 1$, our results can be viewed as an addendum to [**1**] which solved this question for $\rho = 1$ and π an odd order cyclic group. We will now briefly summarize our results.

We write an action of π as a map $\mu: \pi \times S^{2n+1} \to S^{2n+1}$ and we say μ_1, μ_2 are equivalent if there is a homeomorphism $f: S^{2n+1} \to S^{2n+1}$ such that $f\mu_1(T, x) = \mu_2(T, f(x))$ for all $(T, x) \in \pi \times S^{2n+1}$. For π cyclic of order h, the prototype is of course the linear action of $\mathbf{Z}/h$ on $S^{2n+1} \subset \mathbf{C}^{n+1}$ given by $\mu_0(T,(z_0, z_1,\ldots,z_n)) = \exp(2\pi i/h)(z_0, z_1,\ldots,z_n)$. We write $\Delta(\mu)$ for the Reidemeister torsion of the action $\mu: \pi \times S^{2n+1} \to S^{2n+1}$, following Milnor [**4**]. $\Delta(\mu)$ is a unit in the ring $\mathbf{Q}R_\pi$, defined by $\mathbf{Q}\pi/(\Sigma)$, where Σ denotes the sum of the elements of π. It is a basic invariant of the action. For the standard linear action mentioned above, $\Delta(\mu_0) = (T-1)^{n+1}$ (see [**4**, p. 406]). Actually it is more convenient to keep tabs on the quotient

$$\Delta(\mu)/\Delta(\mu_0) = \Delta(\mu) \cdot (T-1)^{-n-1} \in \mathbf{Q}R_\pi^\times/(\pm\pi).$$

In fact, let $R_\pi = \mathbf{Z}\pi/(\Sigma)$ with $j: (R_\pi)^\times \to \mathbf{Q}R_\pi^\times$ the natural inclusion map. Then Wall has shown [**9**, Theorem 14.E.3] that

0.1. LEMMA. *$\Delta(\mu)/\Delta(\mu_0) = j(u)$ for some unique unit $u \in (R_\pi)^\times/(\pm\pi)$. Moreover, given a unit $u \in (R_\pi)^\times$ there is a cellular free action μ on a finite complex homotopically equivalent to S^{2n+1} such that $\Delta(\mu)/\Delta(\mu_0) = j(u) \bmod(\pm\pi)$.*

For this reason, this unit $u \in (R_\pi)^\times/(\pm\pi)$ is called the *associated unit* of μ.

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After a surgery calculation—carried out in Part I—we define, for each finite group π and subgroup ρ, a transfer map,

$$\mathrm{tr}\colon K_1(R_\pi)/(\pm\pi) \to K_1(R_\rho)/(\pm\rho) \quad \text{and}$$
$$\mathrm{tr}\colon K_1(\mathbf{Q}R_\pi)/(\pm\pi) \to K_1(\mathbf{Q}R_\rho)/(\pm\rho)$$

and we prove that $\mathrm{tr}\,\Delta(\mu) = \Delta(\bar{\mu})$ whenever $\mu\colon \pi \times X \to X$ is a free action for which the torsion is defined, and $\bar{\mu}$ is its restriction to ρ. This is Theorem 1.7, and we feel it should be of independent interest.

Now it is known that

LEMMA. *$K_1(R_\pi) = R_\pi^\times$ if π is a finite cyclic group.*

(For a proof, see the Remark following 1.6.B.)

Hence, if π is a cyclic group our definition of the transfer provides a homomorphism

$$\mathrm{tr}\colon (R_\pi)^\times/(\pm\pi) \to (R_\rho)^\times/(\pm\rho).$$

We then prove

THEOREM 1. *Let $\bar{\mu}$ be a free action of $\rho = \mathbf{Z}/k$ on S^{2n+1}, $n \geqslant 2$. Let $\bar{u} \in (R_\rho)^\times/(\pm\rho)$ be its associated unit. Then $\bar{\mu}$ extends to a free action of $\mathbf{Z}/hk = \pi$ if and only if $\bar{u}$ is in the image of* $\mathrm{tr}\colon (R_\pi)^\times/(\pm\pi) \to (R_\rho)^\times/(\pm\rho)$.

If p is a prime number it then turns out that the problem of extending a $\mathbf{Z}/p^r$ action to a $\mathbf{Z}/p^{r+1}$ action is closely related to an old conjecture of Iwasawa in [**2**]. This conjecture states that the norm map $N\colon \mathbf{Z}(\zeta_{r+1})^\times \to \mathbf{Z}(\zeta_r)^\times$ is always onto, where ζ_r denotes a primative p^rth root of unity. Iwasawa's conjecture would follow at once from Vandiver's conjecture (which says that p does not divide the second factor of the ideal class group of $\mathbf{Z}(\zeta_1)$). Vandiver's conjecture is known to be true for all primes $< 125{,}000$. (Cf. [**2**, p. 556 and **12**].)

The relationship of this to group actions is as follows: If $\pi = \mathbf{Z}/p^{r+1}$, $\rho = \mathbf{Z}/p^r$, define a map $\varepsilon\colon R_\pi \to \mathbf{Z}(\zeta_r)$ by sending T to ζ_r. ε induces $\varepsilon_*\colon (R_\rho)^\times \to \mathbf{Z}(\zeta_r)^\times$ sending $(\pm\rho)$ to C, the group of roots of unity.

THEOREM 2. *Let $\bar{\mu}$ be a free action of $\mathbf{Z}/p^r$ on S^{2n+1}, $n \geqslant 2$. Let $\bar{u}$ be its associated unit.*

(A) *If $\varepsilon_*(\bar{u})$ is not in the image of $N\colon \mathbf{Z}(\zeta_{r+1})^\times \to \mathbf{Z}(\zeta_r)^\times/C$, then $\bar{\mu}$ does not extend to a $\mathbf{Z}/p^{r+1}$ action.*

(B) *Suppose $r = 1$, and suppose Iwasawa's conjecture holds for p. Then $\bar{\mu}$ extends to a free $\mathbf{Z}/p^2$ action on S^{2n+1}.*

REMARK. Despite the explicit nature of Theorem 1, we can provide no example of a free $\mathbf{Z}/k$ action which fails to extend to a free $\mathbf{Z}/hk$ action. John Ewing has shown [**11**], using our results, that for each p, there is a $\mathbf{Z}/p^r$ action which does not extend to a $\mathbf{Z}/p^{r+1}$ action.

THEOREM 3. *Let k be an odd integer. Every free $\mathbf{Z}/k$ action on S^{2n+1}, $n > 1$, extends to a free $\mathbf{Z}/2k$ action.*

Here is an outline of the rest of the paper.

Part I of §1 is a surgery theoretic calculation showing that the natural transfer map of the two surgery exact sequences involved is surjective. Part II constructs the transfer map needed to study the Reidemeister torsion; we then use this to prove Theorem 1. §2 gives the proof of Theorems 2 and 3.

We would like to give a word of thanks here to John Ewing and to John Cruthirds for a number of very useful conversations.

1. The basic condition for extending actions.

PART I. Our goal in this part of §1 is to establish

PROPOSITION 1.1. *Let L^{2n+1} be a finite Poincaré complex with $\pi_1(L) = \mathbf{Z}/hk$ and universal cover $\simeq S^{2n+1}$. Let $\bar{L}$ denote its h-fold cover. Then the transfer map,*

$$\pi^*\colon S^s_{\mathrm{Top}}(L) \to S^s_{\mathrm{Top}}(\bar{L}),$$

is onto.

(As in Wall [9], $S^s_{\mathrm{Top}}(L)$ denotes the Top-structures on L.)

This will be done analyzing the following map of surgery exact sequences:

$$(1.2)\qquad \begin{array}{ccccccc} \tilde{L}^s_{2n+2}(\mathbf{Z}/k) & \to & S^s_{\mathrm{Top}}(\bar{L}) & \to & NM(\bar{L}) & \xrightarrow{\bar{\sigma}} & L^s_{2n+1}(\mathbf{Z}/k) \\ \uparrow \mathrm{tr} & & \uparrow \pi^* & & \uparrow \pi^* & & \uparrow \mathrm{tr} \\ \tilde{L}^s_{2n+2}(\mathbf{Z}/hk) & \to & S^s_{\mathrm{Top}}(L) & \to & NM(L) & \xrightarrow{\sigma} & L^s_{2n+1}(\mathbf{Z}/hk) \end{array}$$

Here $\tilde{L}^s_n(G)$ = the cokernel of $L^s_n(e) \to L^s_n(G)$, and NM denotes the set of bordism classes of Top-normal maps.

It is well known that such an L is homotopically equivalent to some standard lens space $L(hk, \theta_0, \theta, \ldots, \theta_n)$ where θ_i are integers prime to hk. Choose such a homotopy equivalence: it yields an element of $NM(L)$ and its cover is an element of $NM(\bar{L})$. This choice then allows us to identify $NM(L)$ with $[L, G/\mathrm{Top}]$ (see [6 or 9]) and π^* with the group homomorphism, $\pi^*\colon [L, G/\mathrm{Top}] \to [\bar{L}, G/\mathrm{Top}]$. Our first step is

LEMMA 1.3. $\pi^*\colon [L, G/\mathrm{Top}] \to [\bar{L}, G/\mathrm{Top}]$ *is an epimorphism. Its kernel is a finite group of order $h^{[n/2]}$ unless k is odd and h is even. In this case its order is* $2^{[(n+1)/2]}h^{[n/2]}$.

PROOF. $[L, G/\mathrm{Top}] = [L, G/\mathrm{Top}_{(2)}] \oplus [L, G/\mathrm{Top}_{(\mathrm{odd})}]$ (since both localizations are finite) and this is isomorpic to $\tilde{H}^{4*}(L; \mathbf{Z}_{(2)}) \oplus H^{4*+2}(L; \mathbf{Z}/2) \oplus \widetilde{KO}^{\circ}(L)_{(\mathrm{odd})}$. Similarly for $\bar{L}$. So the lemma is an immediate consequence of the following three statements. We first write h as $h = h_2 \cdot h_{\mathrm{odd}}$ where h_2 is a power of 2 and h_{odd} is odd.

(i) $\pi^*\colon \tilde{H}^{4*}(L; \mathbf{Z}_{(2)}) \to \tilde{H}^{4*}(\bar{L}; \mathbf{Z}_{(2)})$ is surjective; its kernel has order $h_2^{[n/2]}$.

(ii) $\pi^*\colon H^{4*+2}(L; \mathbf{Z}/2) \to H^{4*+2}(\bar{L}; \mathbf{Z}/2)$ is surjective; its kernel has order $2^{[n+1/2]}$ if k is odd and h is even, and otherwise it is injective.

(iii) $\pi^*\colon \widetilde{KO}(L) \to \widetilde{KO}(\bar{L})$ is onto; its kernel has order $(h_{\mathrm{odd}})^{[n/2]}$.

PROOF OF (i) AND (ii). It is an easy exercise (using the fibration $S^{2n+1} \to \bar{L} \to K(\mathbf{Z}/k, 1)$) to prove, for $A = \mathbf{Z}_{(2)}$ or $\mathbf{Z}/2$, that $H^{2*}(\bar{L}; A) = (A/kA)[\bar{x}]/(\bar{x})^{n+1}$ where $\deg \bar{x} = 2$, and similarly for L. The surjectivity of π^*: $H^2(L; A) \to H^2(\bar{L}; A)$ is trivial from the universal coefficient theorem and the behavior of π_* on the fundamental groups. From this it is clear that π^* is onto with kernel as specified in (i) and (ii).

To prove (iii) let C be the mapping cone of π. Proceeding as above, one easily computes: $H^i(C; \mathbf{Z}_{(\mathrm{odd})}) = \mathbf{Z}/h_{\mathrm{odd}}$ if i is even and $0 < i \leqslant 2n + 2$, and $H^i(C; \mathbf{Z}_{(\mathrm{odd})}) = 0$ otherwise. Hence the Atiyah-Hirzebruch spectral sequence for $KO^*(C)_{(\mathrm{odd})}$ collapses ($E_2^{pq} = 0$ if p or q is odd) and we get $\tilde{K}O^1(C)_{(\mathrm{odd})} = 0$, $\tilde{K}O^0(C)_{(\mathrm{odd})} =$ a group of order $(h_{\mathrm{odd}})^{[(n+1)/2]}$ ($E_2^{4i,-4i} = \mathbf{Z}/(h_{\mathrm{odd}})$, $0 < i \leqslant [(n+1)/2]$). so we get an exact sequence

$$0 \to \tilde{K}O^{-1}(L)_{(\mathrm{odd})} \xrightarrow{\pi^*} \tilde{K}O^{-1}(\bar{L})_{(\mathrm{odd})} \to \tilde{K}O^0(C)_{(\mathrm{odd})}$$

$$\to \tilde{K}O^0(L)_{(\mathrm{odd})} \xrightarrow{\pi^*} \tilde{K}O^0(\bar{L})_{(\mathrm{odd})} \to 0.$$

However, $\tilde{K}O^{-1}(\bar{L})_{(\mathrm{odd})} = 0$ if n is even (use the Atiyah-Hirzebruch spectral sequence), and for n odd $\tilde{K}O^{-1}(\bar{L}) = \mathbf{Z}_{(\mathrm{odd})}$ with $\mathrm{Im}(\pi^*)$ of index (h_{odd}). So for any n, π^* on the right is an epimorphism with kernel of order $(h_{\mathrm{odd}})^{[n/2]}$. This proves Lemma 1.3.

Now we return to diagram (1.2) and glue together some ideas of Petrie and Wall to observe:

LEMMA 1.4. *The map*

$$\mathrm{tr}: \tilde{L}^s_{2n+2}(\mathbf{Z}/hk) \to \tilde{L}^s_{2n+2}(\mathbf{Z}/k)$$

is surjective. Also $\tilde{L}^s_{2n+2}(\mathbf{Z}/k)$ acts freely on $S^s_{\mathrm{Top}}(\bar{L})$ and similarly for $\mathbf{Z}/hk$ and L.

PROOF. Following Wall [8] we write χ_k: $L^s_{2n+2}(\mathbf{Z}/k) \to \mathbf{C}(\mathbf{Z}/k)$ for the signature map, where $\mathbf{C}(\pi)$ denotes the representation ring of π (over $\mathbf{C}$). According to [8], $\mathrm{Im}\, \chi_k = \{4(x + (-1)^{n+1}\bar{x}) \mid x \in \mathbf{C}(\mathbf{Z}/k)\}$. Similarly for χ_{hk}. The diagram

$$\begin{array}{ccc} L^s_{2n+2}(\mathbf{Z}/k) & \xrightarrow{\chi_k} & \mathbf{C}(\mathbf{Z}/k) \\ \uparrow \mathrm{tr} & & \uparrow p \\ L^s_{2n+2}(\mathbf{Z}/hk) & \xrightarrow{\chi_{hk}} & \mathbf{C}(\mathbf{Z}/hk) \end{array}$$

commutes where p denotes restriction. Since each representation of $\mathbf{Z}/k$ is the restriction of a $\mathbf{Z}/hk$ representation it is clear that $p\, \mathrm{Im}\, \chi_{hk} = \mathrm{Im}\, \chi_k$.

But according to Wall [8], χ is injective for n odd and $\mathrm{Ker}\, \chi = L_{2n+2}(e)$ for n even for these groups. It follows at once that tr is a surjection on $\tilde{L}^s_{2n+2}(\mathbf{Z}/k)$.

Now $\tilde{L}^s_{2n+2}(\mathbf{Z}/k)$ is a free abelian, and Petrie [5, 2.3 and 2.10] shows that a subgroup of $\tilde{L}^s_{2n+2}(\mathbf{Z}/k)$ acts freely on $S^s_{\mathrm{Top}}(\bar{L})$. But the rank of this subroup equals the rank of $\tilde{L}^s_{2n+2}(\mathbf{Z}/k)$ (as computed by Wall [7]). It follows then that $\tilde{L}^s_{2n+2}(\mathbf{Z}/k)$ itself is acting freely. This completes the proof.

The final step in this section concerns the right side of (1.2), where we now look at the kernel of σ: $NM(L) \to L^s_{2n+1}(\pi_1(L))$. Recall $NM(L)$ is a group via its identification with $[L, G/\mathrm{Top}]$.

LEMMA 1.5. *The map σ is a homomorphism and the restriction of π^* to π^*: $\operatorname{Ker}\sigma \to \operatorname{Ker}\bar{\sigma}$ is an epimorphism.*

PROOF. $L^s_{2n+1}(\mathbf{Z}/hk) = 0$ unless n is odd and hk even, and in this case $L^s_{2n+1}(\mathbf{Z}/hk) = \mathbf{Z}/2$, with the surgery obstruction given by

$$\sigma(f) = \langle a_1(1 + a_2^2)^r f^*k, [L]\rangle$$

(see [**9**, p. 210]). Here $f \in [L, G/\mathrm{Top}]$, $2r = n + 1$, $a_i \in H^i(L; \mathbf{Z}/2)$ is the nonzero element, and $k \in H^{4*+2}(G/\mathrm{Top}; \mathbf{Z}/2)$ is a certain primitive class. The primitivity of k yields immediately that σ is a homomorphism. To prove $\pi^*\operatorname{Ker}\sigma = \operatorname{Ker}\bar{\sigma}$ we can assume, using 1.3, that n is odd and hk is even.

Case 1. k even. Then if $\bar{f} \in \operatorname{Ker}\bar{\sigma}$ and f is an element of $[L, G/\mathrm{Top}]$ such that $\pi^*f = \bar{f}$ we see

$$\begin{aligned}
\sigma(f) &= \langle a_1(1 + a_2^2)^r f^*k, [L]\rangle \\
&= \langle (\operatorname{tr}\bar{a}_1)(1 + a_2^2)^r f^*k, [L]\rangle \quad \text{since } \operatorname{tr}\bar{a}_1 = a_1 \\
&= \langle \operatorname{tr}\{\bar{a}_1 \cdot \pi^*((1 + a_2^2)^r f^*k)\}, [L]\rangle \quad \text{since } (\operatorname{tr} x)y = \operatorname{tr}(x \cdot \pi^* y) \\
&= \langle \operatorname{tr}\{\bar{a}_1 \cdot (1 + \bar{a}_2^2)^r \bar{f}^*k\}, [L]\rangle \quad \text{since } \pi^* a_2 = \bar{a}_2 \\
&= \langle \bar{a}_1(1 + \bar{a}_2^2)^r \bar{f}^*k, [\bar{L}]\rangle \quad \text{since } \operatorname{tr}[L] = [\bar{L}] \\
&= \sigma(\bar{f}) = 0.
\end{aligned}$$

Hence $f \in \operatorname{Ker}\sigma$ and π^*: $\operatorname{Ker}\sigma \to \operatorname{Ker}\bar{\sigma}$ is onto in this case.

Case 2. k odd. Then $[\bar{L}, G/\mathrm{Top}] = \operatorname{Ker}\bar{\sigma}$ is an odd order group (as seen, for example, in the proof of 1.3) and the odd primary summand of $[L, G/\mathrm{Top}]$ must map onto it by Lemma 1.3. But this summand is in $\operatorname{Ker}\sigma$ since σ is a homomorphism. This completes the proof.

PROOF OF PROPOSITION 1.1. Using 1.3, 1.4 and 1.5 we see that diagram 1.2 reduces to:

$$\begin{array}{ccccccc}
0 & & & & 0 & & \\
\uparrow & & & & \uparrow & & \\
\tilde{L}^s_{2n+2}(\mathbf{Z}/k) & \to & S^s_{\mathrm{top}}(\bar{L}) & \to & \operatorname{Ker}\bar{\sigma} & \to & 0 \\
\uparrow \operatorname{tr} & & \uparrow \pi^* & & \uparrow \pi^* & & \\
\tilde{L}^s_{2n+2}(\mathbf{Z}/hk) & \to & S^s_{\mathrm{top}}(L) & \to & \operatorname{Ker}\sigma & \to & 0
\end{array}$$

Hence 1.1 becomes a simple diagram chase, and we are done.

PART II. We wish to make a statement concerning the behavior of the Reidmeister torsion under a transfer map so we will now define this transfer map.

1.6. *Construction*. Let A be a commutative ring (in practice $A = \mathbf{Q}$ or $\mathbf{Z}$). Set $AR_\pi = A(\pi)/(\Sigma)$, $AR_\rho = A(\rho)/(\bar{\Sigma})$, where ρ is a subgroup of the finite group π, and $\overline{\Sigma}$ is the sum of the elements of ρ. The multiplicative group $\pm\pi$ in $(A\pi)^{\times}$ determines a subgroup of $K_1(AR_\pi)$ denoted $(\pm\pi)$, the homomorphic image of $\pm\pi$. We construct here a homomorphism

$$\text{tr}: K_1(AR_\pi)/(\pm\pi) \to K_1(AR_\rho)/(\pm\rho).$$

$A(\pi)$ is a free left $A(\rho)$ module with base $g_1,\dots,g_h$, a complete set of right coset representatives of ρ in π. Let $T = \overline{\Sigma} \cdot A(\pi)$, a right ideal. $A(\pi)/T$ is a left $A(\rho)$ module with $\overline{\Sigma} \cdot (A(\pi)/t) = 0$ so that $A(\pi)/T$ becomes a left AR_ρ module. In fact, $A(\pi)/T$ is a free left AR_ρ module with base $g_1 \bmod T \cdots g_h \bmod T$. This base defines an isomorphism $A(\pi)/T \approx (AR_\rho)^h$ of left AR_ρ modules, and an induced isomorphism from $\text{End}(A(\pi)/T)$, the ring of AR_ρ endomorphisms of $A(\pi)/T$, to the matrix ring $M(h, AR_\rho)$. Denote this

$$\lambda: \text{End}(A(\pi)/T) \approx M(h, AR_\rho).$$

Now right multiplication by elements of $A(\pi)$ defines a map $A(\pi) \to \text{End}(A(\pi)/T)$ sending Σ to 0. This induces a ring homomorphism $r: AR_\pi \to \text{End}(A(\pi)/T)$.

So the ring homomorphism $\lambda r: AR_\pi \to M(h, AR_\rho)$ yields a ring homomorphism $(\lambda r)_*: M(m, AR_\pi) \to M(mh, AR_\rho)$ given by $(\lambda r)_*(a_{ij}) = (\lambda r(a_{ij}))$ (a matrix of blocks). The induced homomorphism on the K_1 level is denoted

$$\text{tr}: K_1(AR_\pi) \to K_1(AR_\rho).$$

We claim that $\text{tr}(\pm\pi) \subset (\pm\rho)$. For, if $x \in \pi$, then for each coset representative g_i we have $g_i x = \bar{x}(i) \cdot g_{j(x,i)}$ for some element $\bar{x}(i) \in \rho$ and some integer $j(x, i)$ between 1 and h. It follows that $\lambda r(\pm x) = \pm P \cdot D(\bar{x}(1),\dots,\bar{x}(h))$ where P is a permutation matrix and D denotes a diagonal matrix. Hence, $\text{tr}(\pm x) = \pm\bar{x}(1) \cdot \bar{x}(2) \cdots \bar{x}(h) \in (\pm\rho)$.

This leaves us with an induced map

$$\text{tr}: K_1(AR_\pi)/(\pm\pi) \to K_1(AR_\rho)/(\pm\rho).$$

The elementary properties below are easily checked.

1.6.A. If $A \xrightarrow{f} A'$ is a ring homomorphism, the following diagram commutes:

$$\begin{array}{ccc} K_1(AR_\pi)/(\pm\pi) & \xrightarrow{\text{tr}} & K_1(AR_\rho)/(\pm\rho) \\ \downarrow f^* & & \downarrow f^* \\ K_1(A'R_\pi)/(\pm\pi) & \xrightarrow{\text{tr}} & K_1(A'R_\rho)/(\pm\rho) \end{array}$$

1.6.B. Suppose given a ring epimorphism $\varphi: A(\pi) \to \Lambda$ with $\overline{\Lambda} = \varphi(A(\rho))$ and with $\varphi(\overline{\Sigma}) = 0$. Suppose Λ is free over $\overline{\Lambda}$ with base $\varphi(g_1),\dots,\varphi(g_h)$. Then there is a commutative square:

$$\begin{array}{ccc} K_1(AR_\pi) & \xrightarrow{\varphi_*} & K_1(\Lambda) \\ \downarrow \text{tr} & & \downarrow N \\ K_1(AR_\rho) & \xrightarrow{\bar{\varphi}_*} & K_1(\overline{\Lambda}) \end{array}$$

Here $\bar{\varphi}\colon AR_\rho \to \bar{\Lambda}$ is the epimorphism induced by φ, and N is the "norm" map induced by the isomorphism of rings, $M(n, \Lambda) \to M(nh, \bar{\Lambda})$ which this basis of Λ over $\bar{\Lambda}$ defines.

REMARK. If π is a finite cyclic group, $SK_1(\mathbf{Z}\pi) = \{1\}$ (see [**10**, p. 623]). Hence, $K_1(\mathbf{Z}\pi) = \mathbf{Z}\pi^\times$ and one sees at once that $K_1(R_\pi) = R_\pi^\times$, by examining the exact sequence of the Cartesian square:

$$\begin{array}{ccc} \mathbf{Z}\pi & \xrightarrow{\varepsilon} & \mathbf{Z} \\ \downarrow & & \downarrow \\ R_\pi & \rightarrow & \mathbf{Z}/|\pi|\mathbf{Z} \end{array}$$

Also $K_1(\mathbf{Q}R_\pi) = (\mathbf{Q}R_\pi)^\times$ because $\mathbf{Q}R_\pi$ is a product of fields.

Now we show how the torsion of a complex behaves under this transfer map.

So let L denote a finite CW complex with fundamental group π, a finite group, universal cover $\tilde{L}$ and intermediate cover $\bar{L}$ corresponding to a subgroup ρ of π. We assume π acts trivially on $H_*(\tilde{L}; \mathbf{Q})$. In this case, Milnor [**4**, p. 405], defines its Reidemeister torsion $\Delta(L)$ in $K_1(\mathbf{Q}R_\pi)/(\pm\pi)$. We shall prove

THEOREM 1.7. $\Delta(\bar{L}) = \operatorname{tr}\Delta(L)$.

PROOF. Recall from [**4**] how Δ is defined. If $C(\tilde{L})$ denotes the cellular chain complex then $C(\tilde{L}) \otimes_{\mathbf{Z}\pi} \mathbf{Q}R_\pi$ (hereafter denoted C_π) is acyclic and based, over $\mathbf{Q}R_\pi$. Similarly for $C(\tilde{L}) \otimes_{\mathbf{Z}\rho} \mathbf{Q}R_\rho$ (written C_ρ), over $\mathbf{Q}R_\rho$. $\Delta(L) = \Pi_i[c_i/c_i']$ where c_i denotes the basis of cells for $(C_\pi)_i$ and c_i' denotes the basis determined by the acyclicity of C_π, and $[c_i/c_i']$ denotes the class of the change of basis matrix (c_i/c_i') in $GL(m_i, \mathbf{Q}R_\pi)$ ($m_i = \operatorname{rank} C_i$, over $\mathbf{Z}\pi$).

Now if $g_1, \ldots, g_h$ are a complete set of coset representatives of ρ in π, each basis c_i for $(C_\pi)_i$ determines a basis for $(C_\rho)_i$ over $\mathbf{Q}R_\rho$—namely: $c_ig_1 \cup c_ig_2 \cup \cdots \cup c_ig_h$ which we write $\bar{c}_i$. Observe that $\bar{c}_i'$, $i = 0, 1, 2, \ldots$, are the bases for $(C_\rho)_i$ determined by its acyclicity. Going back to 1.6, we check that $(\lambda r)_*(c_i/c_i') = (\bar{c}_i/\bar{c}_i')$ in $GL(m_ih, \mathbf{Q}R_\rho)$. Hence $\operatorname{tr}(L) = \Pi_i \operatorname{tr}[c_i/c_i'] = \Pi_i \lambda r_*(c_i/c_i') = \Pi_i[\bar{c}_i/\bar{c}_i'] = \Delta(\bar{L})$ because, by its definition, it is clear that $\bar{c}_i$ is the basis of cells for $(C_\rho)_i$ over $\mathbf{Q}R_\rho$. This completes the proof.

1.8. PROOF OF THEOREM 1. First we show that if $\bar{u}$ is the associated unit of an action $\bar{\mu}\colon \mathbf{Z}/k \times S^{2n+1} \to S^{2n+1}$ which does extend to a free $\mathbf{Z}/hk$ action μ, we have $\bar{u} = \operatorname{tr} u$ for some $u \in (R_\pi)^\times/(\pm\pi)$. In fact u can be taken to be the associated unit of μ. To see this we calculate

$$\Delta(\mu) = (T-1)^{n+1}j(u) \quad \text{and} \quad \Delta(\bar{\mu}) = (T^h - 1)^{n+1}j(\bar{u})$$

by 0.1. By Theorem 1.7, $\operatorname{tr}\Delta(\mu) = \Delta(\bar{\mu})$ and $\operatorname{tr}(T-1)^{n+1} = (T^h-1)^{n+1}$ (since $\Delta(\mu_0) = (T-1)^{n+1}$). Hence we see $j\bar{u} = \operatorname{tr} j(u) = j \operatorname{tr} u$ by 1.6.A. But

$$j\colon R_\rho^\times/(\pm\rho) \to \mathbf{Q}R_\rho^\times/(\pm\rho)$$

is evidently injective. So $\operatorname{tr} u = \bar{u}$ as required.

Conversely, now we suppose $\bar{\mu}$ is an action with associated unit $\bar{u}$ and that $\bar{u} = \operatorname{tr} u$ for some unit in $(R_\pi)^\times/(\pm\pi)$. By 0.1, there is a finite complex L with fundamental group π, universal cover $\simeq S^{2n+1}$, such that $\Delta(L) = (T-1)^{n+1}j(u)$. Let $\bar{L}$ be the h-fold cover of L. By Theorem 1.7, $\Delta(\bar{L}) = (T^h - 1)^{n+1}j(\bar{u}) = \Delta(\bar{M})$, where $\bar{M}$ is the orbit manifold of the action $\bar{\mu}$: $\rho \times S^{2n+1} \to S^{2n+1}$. But this implies that $\bar{M}$ is simple homotopy equivalent to $\bar{L}$ via a map $\bar{f}$: $\bar{M} \to \bar{L}$ say. Hence $(\bar{M}, \bar{f}) \in S^s_{\mathrm{Top}}(\bar{L})$. But by Proposition 1.1, there is an (M, f) in $S^s_{\mathrm{Top}}(L)$ whose cover is $(\bar{M}, \bar{f})$. This means that M is the orbit space of a free action of π on $\tilde{M}$. Its restriction to an action of ρ is just $\bar{\mu}$. Thus $\bar{\mu}$ extends as required. This completes the proof of Theorem 1.

2. Proof of Theorem 2. We now turn to free actions of cyclic groups of order p^r with the aim of proving Theorem 2.

PROOF OF THEOREM 2(A). By Theorem 1, $\bar{\mu}$ extends to a free action μ only if $\bar{u} = \operatorname{tr} u$ for some unit u in $(R_\pi)^\times$. But by 1.6.B the diagram below commutes. So $\bar{\mu}$ does not extend unless $\varepsilon_*(\bar{u})$ is in the image of N.

$$\begin{array}{ccc} (R_\pi)^\times & \xrightarrow{\varepsilon} & \mathbf{Z}(\zeta_{r+1})^\times \\ \downarrow \operatorname{tr} & & \downarrow N \\ (R_\rho)^\times & \xrightarrow{\varepsilon} & \mathbf{Z}(\zeta_r)^\times \end{array}$$

PROOF OF THEOREM 2(B). If $\pi = \mathbf{Z}/p^2$, and $\zeta_i =$ the primitive p^ith root of unity, we have the exact sequence,

$$1 \to R_\pi^\times \xrightarrow{\lambda_1 \oplus \lambda_2} Z(\zeta_1)^\times \oplus Z(\zeta_2)^\times \xrightarrow{k} (Z(\zeta_1)/(p))^\times,$$

$\lambda_i f(T) = f(\zeta_i)$, $k(f_1(\zeta_1), f_2(\zeta_2)) = f_1(\zeta_1)^{-1} f_2(\zeta_1) \bmod p$ (obtained by noting $\mathbf{Z}[T]/\Phi_1 \cdot \Phi_2 = R_\pi$, $\mathbf{Z}[T]/(\Phi_1, \Phi_2) = \mathbf{Z}(\zeta_1)/(p)$).

Now $R_\rho = \mathbf{Z}(\zeta_1)$ and it is easy to see from the definition of tr that $\operatorname{tr} = N \circ \lambda_2$: $R_\pi^\times \to \mathbf{Z}(\zeta_2)^\times \to \mathbf{Z}(\zeta_1)^\times$ where N is a norm map as in 1.6.B. Iwasawa's conjecture asserts that N: $\mathbf{Z}(\zeta_2)^\times \to \mathbf{Z}(\zeta_1)^\times$ is surjective. So if Iwasawa's conjecture holds, we need only prove that tr is onto (by Theorem 1). So let $\bar{u} \in \mathbf{Z}(\zeta_1)^\times$. We are given that $\bar{u} = N(f(\zeta_2))$ for some unit $f(\zeta_2)$ in $\mathbf{Z}(\zeta_2)^\times$. Consider $z = (\bar{u}, f(\zeta_2)) \in \mathbf{Z}(\zeta_1)^\times \oplus \mathbf{Z}(\zeta_2)^\times$. It is well known that $N(a) \equiv a^p \bmod p$ if $a \in \mathbf{Z}(\zeta_2)$, so we see that

$$k(z) = \left(Nf(\zeta_2)^{-1} f(\zeta_1)\right) \bmod p \equiv \left(f(\zeta_2)^p\right)^{-1} f(\zeta_2^p) \quad \bmod p$$
$$\equiv \left(f(\zeta_2)^p\right)^{-1} f(\zeta_2)^p \quad \bmod p = 1.$$

So $z \in \operatorname{Ker} k = \operatorname{Im} \lambda_1 \oplus \lambda_2$. This implies there is an element $u \in R_\pi^\times$ such that $\lambda_1(u) = \bar{u}$, $\lambda_2(u) = f(\zeta_2)$ and so $\operatorname{tr} u = N\lambda_2(u) = Nf(\zeta_2) = \bar{u}$. Thus tr is onto and we are done.

Now we restate and prove

THEOREM 3. *If k is an odd integer, $n > 1$, any free $\mathbf{Z}/k$ action on S^{2n+1} extends to a free $\mathbf{Z}/2k$ action.*

PROOF. According to Theorem 1 we have to prove that tr: $(R_\pi)^\times \to (R_\rho)^\times$ is onto, where $\pi = \mathbf{Z}/2k$, $\rho = \mathbf{Z}/k$. But there is an exact sequence of units (used earlier),

$$1 \to (\mathbf{Z}\rho)^\times \xrightarrow{i_*} (R_\rho)^\times \xrightarrow{\varepsilon_*} (\mathbf{Z}/k)^\times/(\pm 1),$$

where ε_* is induced by the augmentation $\varepsilon\colon R\rho \to \mathbf{Z}/k$. Hence it is enough to prove: (a) $\operatorname{Im} i_* \subset \operatorname{Im} \operatorname{tr}$ where $i_*\colon (\mathbf{Z}\rho)^\times \to (R_\rho)^\times$ is the map induced by projection and (b) $\varepsilon_* \circ \operatorname{tr}$ is onto.

The proof of (b) is elementary: Each unit of $(\mathbf{Z}/k)^\times$ is of the form $2r + 1 \bmod k$ for some integer such that $0 < 2r + 1 < 2k$. We claim that, in $(R\pi)^\times$ the unit $u = (T^{2r+1} - 1)/(T - 1)$ satisfies $\varepsilon_* \operatorname{tr} u = 2r + 1$. For $u = 1 + T + \cdots + T^{2r} = \alpha + T^k\beta$ where $\alpha = 1 + T^2 + \cdots + T^{2r}$, $\beta = (T^{k+1} + T^{k+3} + \cdots + T^{k+2r-1})$. Note α and β lie in the image of $\mathbf{Z}\rho$. Since $\pi = \rho \times \mathbf{Z}/2$, it is an elementary exercise to see $\operatorname{tr} u = \alpha^2 - \beta^2 \bmod \overline{\Sigma}$. Set $\gamma = \alpha - T^{2r}$, and note $\operatorname{tr} u = (\gamma + T^{2r})^2 - T^2(\gamma^2) = (1 - T^2)\gamma^2 + 2\gamma T^{2r} + T^{4r}$. Hence $\varepsilon_* \operatorname{tr} u = 0 \cdot \varepsilon(\gamma^2) + 2\varepsilon(\gamma) + 1 = 2r + 1$ since $\varepsilon(\gamma) = r$. This proves (b).

To prove (a) we observe that the diagram

$$\begin{array}{ccc} \mathbf{Z}(\pi)^\times & \rightarrow & (R\pi)^\times \\ \downarrow N & & \downarrow \operatorname{tr} \\ \mathbf{Z}(\rho)^\times & \xrightarrow{i} & (R_\rho)^\times \end{array}$$

commutes, where N is the norm map, so we can concentrate on proving that N is onto. Now $\mathbf{Z}\pi \approx A(\mathbf{Z}/2)$ where $A = \mathbf{Z}(\rho)$ so we show $N\colon A(\mathbf{Z}/2)^\times \to A^\times$ is onto. Each element of $A(\mathbf{Z}/2)$ can be written $a + bS$, where $a, b \in A$, $S \in \mathbf{Z}/2$. Also N is given by $N(a + bS) = a^2 - b^2$. An element $\nu = a + bS$ is a unit of $A(\mathbf{Z}/2)$ if and only if $a + b = u$ and $a - b = u'$ are units of A; conversely, given units u, u' in $A^\times$ with $u \equiv u' \bmod 2A$ they determine a unit ν uniquely. Note $N(\nu) = uu'$.

Now, the diagram

$$\begin{array}{ccc} A^\times & \xrightarrow{r} & A/2A^\times \\ \downarrow S & & \downarrow F \\ A^\times & \xrightarrow{r} & A/2A^\times \end{array}$$

commutes where r is reduction, $S(x) = x^2$ and F is given by the Frobenius map $F(x) = x^2$. $F\colon A/2A \to A/2A$ is an isomorphism of rings because $A/2A$ is a product of finite fields of characteristic $2(A = \mathbf{Z}_{(\rho)})$. It follows that $\operatorname{Im} r = \operatorname{Im} F \circ r$.

Now we prove $N\colon A(\mathbf{Z}/2)^\times \to A^\times$ is onto. Let $w \in A^\times$. By the last paragraph, $r(w) = r(u^2)$ for some $u \in A^\times$. Write w as $w = uu'$ for some $u' \in A^\times$. Then $r(u'/u) = r(w/u^2) = 1$ so $u'/u \equiv 1 \bmod 2A$. It follows that $u \equiv u' \bmod 2A$ and so, as seen above, these determine a unit $\nu = (u + u')/2 + S(u - u')/2$, in $A(\mathbf{Z}/2)^\times$ with $N\nu = uu' = w$ as required. This completes the proof.

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