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DYNAMICAL SYSTEMS AND EXTENSIONS OF STATES ON C^*-ALGEBRAS

BY

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ABSTRACT. Let (A, G, τ) be a noncommutative dynamical system, i.e. A is a C^*-algebra, G a topological group and τ an action of G on A by $*$-automorphisms, and let (M_α) be an M-net on G. We characterize the set of a in A such that $M_\alpha a$ converges in norm. We show that this set is intimately related to the problem of extensions of pure states of R. V. Kadison and I. M. Singer: if B is a maximal abelian subalgebra of A, we can associate a dynamical system (A, G, τ) such that $M_\alpha a$ converges in norm if and only if all extensions to A, of a homomorphism of B, coincide on a. This result allows us to construct different examples of a C^*-algebra A with maximal abelian subalgebra B (isomorphic to $C(\mathbf{R}/\mathbf{Z})$ or $L^\infty[0, 1]$) for which the property of unique pure state extension of homomorphisms is or is not verified.

1. Introduction and principal results. Let $\mathcal{Q}$ be a von Neumann algebra, G a locally compact group, τ an action of G on $\mathcal{Q}$ by $*$-automorphisms and ρ a faithful τ-invariant normal state on $\mathcal{Q}$. When $G = \mathbf{Z}$, Lance has proved in **[13]** that for each $a \in \mathcal{Q}$, the average $\frac{1}{n}\Sigma_0^{n-1} \tau_k a$ converges "ρ-almost uniformly" to a τ-invariant element of $\mathcal{Q}$. If we take (commutative case) for $\mathcal{Q}$ the algebra $L^\infty(X, \mu)$ with (X, μ) a probability space, ρ the associated integral, and τ induced by an invertible measure-preserving transformation on X, we obtain the Birkhoff pointwise ergodic theorem (for bounded functions) as a particular case of Lance's theorem; in **[2]**, Lance's theorem has been extended to the case $G = \mathbf{R}^m, \mathbf{Z}^m$ or a connected amenable locally compact group.

In another way, if we consider $\mathcal{Q}_{\text{unif}}$, the set of a in $\mathcal{Q}$ such that $\frac{1}{n}\Sigma_0^{n-1} \tau_k a$ converges *uniformly* (i.e. in operator norm) can be easily seen to be ultraweakly dense in $\mathcal{Q}$ but the problem is to find in $\mathcal{Q}_{\text{unif}}$, a C^*-algebra A, globally invariant by τ and ultraweakly dense in $\mathcal{Q}$ (such a system (A, G, τ) is called a *uniform system*). For a commutative dynamical system $\mathcal{Q} = L^\infty(X, \mu)$ and $G = \mathbf{Z}$ or $\mathbf{R}$ such a C^*-algebra A exists, and one says that such a system admits a uniform realization (cf. **[4, 5, 9, 12]**).

A natural question is whether we have a similar result for noncommutative $\mathcal{Q}$ and for more general groups G. In this paper we show that the above result is true for some particular cases of interest. Furthermore, it turns out that the notion of dynamical systems is intimately related to the problem of extension of pure states of R. V. Kadison and I. M. Singer (cf. **[10]** and also **[1, 14]**): let A be a C^*-algebra, B be a sub-C^*-algebra of A. Following J. Anderson's definition (cf. **[1]**), one says that B

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has the *extension property* relative to A if every pure state on B has a unique pure state extension to A.

When A is finite-dimensional, B a maximal abelian subalgebra, then every homomorphism of B can be extended uniquely to a pure state of A (cf. **[10]**). But when A is infinite-dimensional the situation is quite different; in **[10]** Kadison and Singer showed that if $A = \mathfrak{B}(\mathfrak{H})$, the set of all bounded operators on an infinite-dimensional Hilbert space $\mathfrak{H}$, and if B is isomorphic to $L^\infty([0, 1])$, B does not have the extension property relative to A (for the case $B = l^\infty(\mathbf{N})$, see **[1, 10, 14]**).

We assume that B is a maximal abelian subalgebra of A; we can associate a dynamical system (A, G, τ) with $G \subset \mathfrak{U}_B$ the set of unitaries of B. We show that (Theorem 3.1) the system (A, G, τ) is uniform if and only if B has the extension property relative to A. We also characterize the elements a of A on which all state extensions of the same homomorphism on B take the same value in terms of norm convergence (Proposition 3.4). This allows us to construct different uniform systems of interest. Our results concerning the state extension problem are

THEOREM a. *Let B be a maximal abelian subalgebra of a C^*-algebra A, let G be a subgroup of $\mathfrak{U}_B$, the unitaries of B, such that G generates the C^*-algebra B. Let (M_α) be an M-net on G. The following conditions are equivalent for an element a of A*:

(α) *the net $M_\alpha a$ converges in norm to an element of B*;

(β) $\overline{\mathrm{co}}(\tau_g a,\ g \in G) \cap B = \{\text{one point}\}$, *where $\overline{\mathrm{co}}(\tau_g a,\ g \in G)$ is the norm closure of the convex set generated by $\tau_g a$, $g \in G$*;

(γ) *we have $\varphi_1(a) = \varphi_2(a)$ whenever φ_1 and φ_2 are G-invariant states possessing the same restriction on B*;

(δ) *all state extensions to A of a homomorphism of B coincide on a.*

THEOREM b. *There exist separable C^*-algebras $B \subset A_1 \subset A$, with B maximal abelian in both A_1, A; B isomorphic to $C(\mathbf{T})$, the algebra of complex valued continuous functions on the one-dimensional torus, such that*:

(a) *the algebra B has the extension property relative to A_1*;

(b) *the algebra B does not have the extension property relative to A*;

(c) *the algebra A_1 does not have the extension property relative to A.*

In Theorem b, the spectrum of B is "reasonable" (it is isomorphic to $\mathbf{T}$). We look now at the case where the spectrum of B is more pathological.

Let $\mathcal{C}$ be a maximal abelian subalgebra of $\mathfrak{B}(\mathfrak{H})$ isomorphic to $L^\infty([0, 1], \mu)$ where μ is the Lebesgue measure on $[0, 1]$. The natural question is whether there exists a "sufficiently big" C^*-algebra A_2 with $\mathcal{C} \subset A_2 \subset \mathfrak{B}(\mathfrak{H})$ such that C has the extension property relative to A_2? We prove

THEOREM c. *There exists a C^*-algebra A_2 ultraweakly dense in $\mathfrak{B}(\mathfrak{H})$ and verifying $\mathcal{C} \subset A_2 \subset \mathfrak{B}(\mathfrak{H})$ such that $\mathcal{C}$ has the extension property relative to A_2.*

We remark that the idea to associate dynamical systems to maximal abelian subalgebras has been used in **[3]** to prove Takesaki's theorem on the characterization of finite von Neumann algebra by the existence of conditional expectations onto its maximal abelian subalgebras (cf. **[16]**) and Tomiyama's theorem on the existence of "completely nonsmooth" maximal abelian subalgebras in properly infinite von Neumann algebras (cf. [**17**]); this idea has also been implicitly developed by J. Anderson in [**1**].

2. Uniform dynamical systems. We shall call a triple (A, G, τ), a *dynamical system* where A is a C^*-algebra, G a locally compact group and τ an action of G on A by *-automorphisms such that the mapping $G \ni g \to \tau_g a \in A$ is norm-continuous for each a in A (for the applications in this paper, we generally take the discrete topology on G, so the continuity is trivially satisfied).

In this paper, we assume that G is *amenable* (cf. [**8**]) or equivalently there exists a family (M_α) of measurable functions on G satisfying the following conditions:

(a) The set $\{\alpha\}$ of indices is ordered and such that given α_1, α_2, there exists $\alpha \geqslant \alpha_1, \alpha_2$.

(b) $M_\alpha \geqslant 0$, $\int_G M_\alpha\, dg = 1$, $\forall \alpha$, where dg is a right Haar measure on G.

(c) $\lim_\alpha \int | M_\alpha(gh) - M_\alpha(g) |\, dg = 0$, $\forall h \in G$.

Such a family (M_α) is called an M-net.

When we have a dynamical system (A, G, τ), for $a \in A$, we denote $M_\alpha a = \int M_\alpha(g)\tau_g a\, dg$ (when G is discrete, $Ma = \sum_{g\in G} M_\alpha(g)\tau_g a$).

When $G = \mathbf{Z}$, we can take $M_n = \frac{1}{n} 1_{\{0,1,\ldots,n-1\}}$. For all this see [**8**].

We say that the system (A, G, τ) is *uniform* if $M_\alpha a$ converges in norm (to a necessarily invariant element of A).

Let B be the set of invariant elements of A, A^* the dual of A. If $a \in A$ we denote $\overline{\text{co}}(\tau_g a, g \in G)$ the norm closure of the convex set generated by $\tau_g a$, $g \in G$.

If φ is a state on A, we denote by ξ_φ the cyclic vector, $\mathcal{H}_\varphi$ the Hilbert space, and π_φ the representation associated to φ by the Gelfand-Naimark-Segal representation.

If φ is a state on A, we denote by $\varphi|_B$ the restriction of φ on B.

If F is a closed linear space of A containing B, stable for the involution, then it is a B-right and -left module (i.e. $\forall f \in F$, $b \in B$, the elements f^*, fb and bf are in F). We call conditional expectation, any positive linear mapping ϕ of F onto B such that $\phi(bfb') = b\phi(f)\phi(b')$, $\phi(b) = b$, $\forall b \in B$.

We denote $[A, B] = \{ab - ba \mid a \in A, b \in B\}$.

LEMMA 2.1. (1) *Let φ be in A^*, φ invariant, and $\varphi = \varphi_1 + i\varphi_2$ be its decomposition into real and imaginary parts, then φ_1 and φ_2 are invariant.*

(2) *Let $\varphi \in A^*$, φ real and invariant, and let $\varphi = \varphi^+ - \varphi^-$ be its decomposition into positive and negative parts; then φ^+ and φ^- are invariant.*

PROOF. Let A^{**} be the enveloping von Neumann algebra of A; the action τ of G is extended to an ultraweakly continuous action on A^{**}. We remark that the invariant continuous linear forms on A are exactly invariant ultraweakly continuous linear forms on A^{**}. The lemma follows then from Proposition 1, p. 193 of [**3**].

PROPOSITION 2.2. *Let E_1 be the norm closure of the linear space generated by B and $a - \tau_g a$, $a \in A$, $g \in G$. Let $E_2 = \{a \in A \mid \overline{\text{co}}(\tau_g a, g \in G) \cap B = \{\text{one point}\}\}$, $E_3 = \{a \in A \mid \varphi(a) = 0$ if $\varphi \in A^*$, φ invariant and $\varphi|_B = 0\}$, $E_4 = \{a \in A \mid \varphi_1(a) = \varphi_2(a)$ if φ_1, φ_2 are invariant states and $\varphi_1|_B = \varphi_2|_B\}$, and $A_{\text{unif}} = \{a \in A \mid M_\alpha a$ converges in norm$\}$.*

Then all these sets are equal, A_{unif} is norm closed, stable by the involution and is a B-right and -left module. There exists a conditional expectation ϕ from A_{unif} onto B such that for every invariant state φ on A we have $\varphi(a) = \varphi(\phi(a))$, $\forall a \in A_{\text{unif}}$.

PROOF. It follows by the definition that $E_1 \subset A_{\text{unif}}$ and by Eberlein's theorem (cf. [7, Theorem 3.1]) we have $A_{\text{unif}} = \overline{E_2}$.

Now let $a \in E_2$ and $\{b\} = \overline{\text{co}}\{\tau_g a, g \in G\} \cap B$. If φ is in A^*, φ invariant and $\varphi|_B = 0$ then $\varphi(a) = \varphi(b) = 0$. Therefore $a \in E_3$ and $E_2 \subset E_3$.

The equality $E_3 = E_4$ follows from Lemma 2.1.

And we remark, finally, that E_1 and E_3 are norm closed and $E_1^\perp = E_3^\perp$ where $E_i^\perp = \{\varphi \in A^* \mid \varphi(a) = 0, \forall a \in E_i\}$.

We conclude that $E_1 = E_2 = E_3 = E_4 = A_{\text{unif}}$. It is clear that if $a \in A_{\text{unif}}$, $b \in B$ then a^*, ab, ba are in A_{unif}.

Let $a \in A_{\text{unif}}$. $M_\alpha a$ converges in norm to an invariant element $\phi(a) \in B$. It is immediate to verify that ϕ is a conditional expectation and $\varphi(a) = \varphi(\phi(a))$, $\forall a \in A_{\text{unif}}$.

THEOREM 2.3. *Let (A, G, τ) be a dynamical system and (M_α) an M-net of G. The following conditions are equivalent*:

(i) *The system (A, G, τ) is uniform, i.e. $M_\alpha a$ converges in norm, $\forall a \in A$.*

(ii) *For every a in A, $\overline{\text{co}}(\tau_g a, g \in G) \cap A^G = \{\text{one point}\}$.*

(iii) *There exists a G-invariant conditional expectation ϕ from A onto B such that every invariant state φ satisfies $\varphi(a) = \varphi(\phi(a))$, $\forall a \in A$.*

(iv) *If φ and φ' are invariant states such that $\varphi|_B = \varphi'|_B$, then $\varphi = \varphi'$.*

PROOF. We remark the following equivalences (with the notations of the proof of Proposition 2.2): (i) $\Leftrightarrow A = A_{\text{unif}}$, (ii) $\Leftrightarrow A = E_2$, (iv) $\Leftrightarrow A = E_4$. Therefore by Proposition 2.2 we have (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iv) $\Rightarrow$ (iii). The implication (iii) $\Rightarrow$ (iv) is immediate.

COROLLARY 2.4. *If the system (A, G, τ) is uniform, the invariant conditional expectation ϕ given by* (iii) *is unique for these properties.*

PROOF. Let ψ be another invariant conditional expectation of A onto B such that every invariant state φ satisfies $\varphi(a) = \varphi(\psi(a))$, $\forall a \in A$. This implies that we have $\varphi(\psi(a)) = \varphi(\phi(a))$ for every state φ on B; therefore $\phi(a) = \psi(a)$, $\forall a \in A$.

COROLLARY 2.5. *If (A, G, τ) is a dynamical system admitting only one invariant state ρ, then it is uniform, i.e.*

$$\left\| \frac{1}{|M_\alpha|} \int_{M_\alpha} \tau_g a \, dg - \rho(a)1 \right\| \underset{\alpha \to \infty}{\to} 0.$$

PROOF. The condition (iv) is trivially satisfied. We also remark that for this system $B = \mathbf{C}1$ and $\phi(a) = \rho(a)1$.

As in the commutative case, a system with a unique invariant state will be called a *uniquely* (*or strictly*) *ergodic* system.

3. Dynamical systems associated to a maximal abelian subalgebra. Let A be a C^*-algebra, B a maximal abelian subalgebra of A, $\mathfrak{U}_B$ the (abelian) group of all unitaries of B, and G a subgroup of $\mathfrak{U}$ such that the C^*-algebra generated by G is B.

We consider the action τ of G on A by $\tau_g a = gag^*$, $\forall g \in G$, $a \in A$. G is provided with the discret topology. We have then a dynamical system (A, G, τ) with B the set of invariant elements; we shall call such a system a dynamical system *associated* to A and B.

As G is abelian then it is amenable. Let (M_α) be an M-net of G (cf. [**7**, **8**]). We remark that G need not be equal to $\mathfrak{U}_B$.

THEOREM 3.1. *Let A be a C^*-algebra, B a maximal abelian subalgebra, and (A, G, τ) an associated dynamical system. The following conditions are equivalent*:

(i) *The system (A, G, τ) is uniform, i.e. $M_\alpha a$ converges in norm to an element of B.*

(v) *For every homomorphism φ on B, there exists a unique pure state $\tilde{\varphi}$ on A extending φ.*

(vi) *The linear space generated by $B + [A, B]$ is norm dense in A.*

REMARK. We use the notations of Theorem 2.3. In [**1**], J. Anderson has already proven (ii) $\Leftrightarrow$ (v) $\Leftrightarrow$ (vi) under the conditions: B is weakly closed and $G = \mathfrak{U}_B$; for our applications we need the theorem in the general context.

We need some auxiliary results.

LEMMA 3.2. *Let φ be a state on A such that the restriction of φ to B is a homomorphism*; *then*

(a) $\pi_\varphi(u)\xi_\varphi = \varphi(u)\xi_\varphi$, $\forall u \in \mathfrak{U}_B$.

(b) $\varphi(uau^*) = \varphi(a)$, $\forall a \in A$, $\forall u \in \mathfrak{U}_B$.

PROOF (CLASSICAL). (a) As φ is a homomorphism on B, for $u \in \mathfrak{U}_B$, we have $1 = |\varphi(u)| = |\langle \pi_\varphi(u)\xi_\varphi, \xi_\varphi\rangle|$, and by Schwarz's inequality we have $\pi_\varphi(u)\xi_\varphi = \varphi(u)\xi_\varphi$.

(b) is an immediate consequence of (a) since $\varphi(uau^*) = \langle \pi_\varphi(u)\pi_\varphi(a)\pi_\varphi(u^*)\xi_\varphi, \xi_\varphi\rangle$, $\forall a \in A$.

Let I be the convex compact set of G-invariant states on A.

LEMMA 3.3. *Let φ be an extremal point of I*; *then the restriction of φ to B is a homomorphism.*

PROOF. For every $g \in G$, let $\tilde{g}$ be the unique unitary of $\mathcal{H}_\varphi$ verifying

(1) $\tilde{g}\pi_\varphi(a)\xi_\varphi\tilde{g}^* = \pi_\varphi(gag^*)\xi_\varphi$, $\forall a \in A$;

(2) $\tilde{g}\xi_\varphi = \xi_\varphi$.

We have then

(3) $\pi_\varphi(gag^*) = \tilde{g}\pi_\varphi(a)\tilde{g}^*$, $\forall a \in A$, $\forall g \in G$;

(4) $\pi_\varphi(g)\pi_\varphi(a)\pi_\varphi(g)^* = \tilde{g}\pi_\varphi(a)\tilde{g}^*$, $\forall a \in A$, $\forall g \in G$.

Therefore

(5) $\pi_\varphi(g)\tilde{g}^* \in \pi_\varphi(A)'$, $\forall g \in G$.

Taking $a \in B$ in (3) we obtain

(6) $\tilde{g} \in \pi_\varphi(B)'$, $\forall g \in G$.

For every g, h in G, we have

(7) $\tilde{h}(\pi_\varphi(g)\tilde{g}^*)\tilde{h}^* = \pi_\varphi(g)\tilde{g}^*$,

since $\tilde{h}$ commutes with $\tilde{g}^*$ and $\pi_\varphi(g)$ by (6). The relations (5) and (7) imply

(8) $\pi_\varphi(g)\tilde{g}^* \in \pi_\varphi(B)' \cap \{\tilde{g}, g \in G\}'$.

As φ is an extremal point of I, $\pi_\varphi(B)' \cap \{\tilde{g}, g \in G\}' = \{\text{scalars}\}$ (cf. **[15]**), therefore $\pi_\varphi(g) = (\text{scalar})\tilde{g}$, $\forall g \in G$, and the relation (2) implies $\pi_\varphi(g)\xi_\varphi = \varphi(g)\xi_\varphi$, $\forall g \in G$. As G generates the C^*-algebra B, we obtain $\pi_\varphi(b)\xi_\varphi = \varphi(b)\xi_\varphi$, $\forall b \in B$. Hence φ is a homomorphism on B.

We use the notations of Proposition 2.2 in

PROPOSITION 3.4. *Let E_5 be the set of elements a of A such that if φ is a homomorphism of B and φ_1, φ_2 two state extensions of φ we have $\varphi_1(a) = \varphi_2(a)$.*

We have $E_1 = E_2 = E_3 = E_4 = A_{\text{unif}} = E_5$.

PROOF. Let E_B be the state space of B, Ext E_B and Ext I the sets of extremal points of E_B and I respectively.

Every element a of E_5 defines a continuous function $\hat{a}$ on I, the set of invariant states. Let $F\colon I \to E_B$ be the canonical continuous mapping defined by $F(\varphi) = \varphi|_B$, $\forall \varphi \in I$. As $\hat{a}$ is continuous on the compact set $F^{-1}(\text{Ext } E_B)$ and is constant on $F^{-1}(x)$, $\forall x \in \text{Ext}(E_B)$ let $\hat{b}$ be the unique function on $\text{Ext}(E_B)$ such that $\hat{b} \circ F(y) = \hat{a}(y)$, $\forall y \in F^{-1}(\text{Ext } E_B)$. Because of the compactness of Ext E_B and $F^{-1}(\text{Ext } E_B)$, $\hat{b}$ is continuous. Let b be the corresponding element of B.

Consider $\{\varphi \in I \mid \varphi(a) = (F\varphi)(b)\}$; this set is a convex and compact (since closed) subset of I and containing Ext I ($\subset F^{-1}(\text{Ext } E_B)$ by Lemma 3.3); hence it coincides with I. Therefore if φ_1 and φ_2 are invariant states such that $\varphi_1|_B = \varphi_2|_B$ we have $\varphi_1(a) = \varphi_1(b) = \varphi_2(b) = \varphi_2(a)$; hence $a \in E_4$. We have shown that $E_5 \subset E_4$.

It is clear from Lemma 3.3 and the definition of E_4 that $E_4 \subset E_5$. The proposition is proved.

PROPOSITION 3.5. *Let E_6 be the norm closure of the linear space generated by B and $[A, B]$.*

Then $E_1 = E_2 = E_3 = E_4 = E_5 = E_6 = A_{\text{unif}}$.

PROOF. (1) $E_6 \subset A_{\text{unif}}$: Let $g \in G$ and $a \in A$. We have $a - gag^* \in A_{\text{unif}}$; therefore $ag - ga \in A_{\text{unif}}$ since $A_{\text{unif}}\, g \subset A_{\text{unif}}$.

Let $B_0 = \{b \in B \mid ba - ab \in A_{\text{unif}},\ \forall a \in A\}$, where B_0 is a norm closed $*$-subalgebra of B containing G. Therefore $B_0 = B$ and $[A, B] \subset A_{\text{unif}}$. As $B \subset A_{\text{unif}}$ we have $E_6 \subset A_{\text{unif}}$.

(2) $A_{\text{unif}} \subset E_6$: It suffices to show that $E_1 \subset E_6$. For every $a \in A$, $a - gag^* = [ag^*, g] \in E_6$. Therefore by the definition of E_1, we have $E_1 \subset E_6$. We have proved the proposition.

PROOF OF THEOREM 3.1. We remark the following equivalences: (i) $\Leftrightarrow A = A_{\text{unif}}$; (v) $\Leftrightarrow A = E_5$; (vi) $\Leftrightarrow A = E_6$. As $E_5 = E_6 = A_{\text{unif}}$ by Propositions 3.4 and 3.5 we have (i) $\Leftrightarrow$ (v) $\Leftrightarrow$ (vi).

PROOF OF THEOREM a. Theorem a follows from Proposition 3.4.

4. Proof of Theorem b. Consider the one-dimensional torus **T** that we identify to $[0, 1[$ (with 1 identified to 0) provided with the Lebesgue measure μ; let λ be an irrational of $[0, 1]$ and consider the "rotation" T_λ: $Tx = x + \lambda \pmod 1$, $x \in [0, 1[$, T_λ is measurable and preserves μ. Let $C(\mathbf{T})$ be the abelian C^*-algebra of complex valued continuous functions f on $\mathbf{T}$. T_λ acts by $*$-automorphism on $C(\mathbf{T})$ by $T_\lambda(f) = f \circ T_\lambda$, $\forall f \in C(\mathbf{T})$. The dynamical system $(C(\mathbf{T}), T_\lambda^n,\ n \in \mathbf{Z})$ is strictly ergodic (the only invariant probability is the Haar measure μ); therefore it is uniform and

$$\left\| \frac{1}{n} \sum_{k=0}^{n-1} T_\lambda^k f - \mu(f)1 \right\|_{C(\mathbf{T})} \underset{n\to\infty}{\to} 0. \tag{4.1}$$

Consider the C^*-algebra $L^\infty([0,1[,\mu)$. T_λ acts by $*$-automorphism on $L^\infty([0,1[,\mu)$ by $T_\lambda f = f \circ T_\lambda, f \in L^\infty([0,1[,\mu)$.

Let $0 < \varepsilon < \frac{1}{8}$ and consider the open set V of $[0,1[$:

$$V = [0,1[\cap \left(\bigcup_{n\in\mathbf{Z}} \left] n\lambda \ (\mathrm{mod}\, 1) - \frac{\varepsilon}{2^{|n|}},\ n\lambda \ (\mathrm{mod}\, 1) + \frac{\varepsilon}{2^{|n|}} \right[\right).$$

We have $0 < \mu(V) < \frac{1}{2}$. Let E be a Borel subset of $[0,1[\setminus V$ with $\mu(E) = \mu(V)$ and let

$$h = 1_V - 1_E, \qquad h \in L^\infty([0,1[,\mu). \tag{4.2}$$

LEMMA 4.1. *We have*

$$\left\| \frac{1}{n} \sum_{k=0}^{n-1} T_\lambda^k(h) - \mu(h) \right\|_{L^\infty([0,1],\mu)} = 1 \tag{4.3}$$

for every $n \geqslant 1$.

PROOF. We remark firstly that $\mu(h) = 0$. Let

$$W_n = \left\{ x \in [0,1[\,\middle|\, \frac{1}{n} \sum_{k=0}^{n-1} h(T_\lambda^k x) = 1 \right\},$$

$$W_n = \bigcap_{k=0}^{n-1} \{ x \in [0,1] \mid h(T_\lambda^k x) = 1 \},$$

$$W_h = \bigcap_{k=0}^{n-1} T_\lambda^{-k} V.$$

Hence W_n is an open set containing $\mathbf{Z}\lambda$ (mod 1); therefore $\mu(W_n) > 0$ and $\|\frac{1}{n}\sum_{k=0}^{n-1} T_\lambda^k(k)\|_{L^\infty} = 1$, $\forall n \geqslant 1$.

Consider $\mathcal{H} = L^2([0,1],\mu)$ with the trigonometrical basis $(e_n)_{n\in\mathbf{Z}}$, $e_n(x) = e^{2\pi i n x}$, $n \in \mathbf{Z}$, $x \in [0,1]$. We identify $L^\infty([0,1[,\mu)$ as a subalgebra $\mathcal{C}$ of $\mathcal{B}(\mathcal{H})$, the algebra of all bounded operators: each element of $L^\infty[0,1[$ acts as multiplication in $\mathcal{H}$. Let u_λ be the unitary in $\mathcal{H}$ defined by $u_\lambda \eta = \eta \circ T_\lambda$, $\forall \eta \in L^2[0,1]$. We have then $u_\lambda e_n = (e^{2\pi i n\lambda})e_n$; therefore the von Neumann algebra generated by $(u_\lambda^n)_{n\in\mathbf{Z}}$ is exactly the diagonal algebra $\mathcal{D}$ relative to the basis $(e_n)_{n\in\mathbf{Z}}$. We have $\mathcal{D} \sim l^\infty(\mathbf{Z})$ and $\mathcal{D}$ is maximal abelian in $\mathcal{B}(\mathcal{H})$.

The unitary u_λ satisfies $u_\lambda^n f \cdot u_\lambda^{n*} = f \circ T_\lambda^n$, $\forall f \in L^\infty[0,1]$, $n \in \mathbf{Z}$. We also identify $C(\mathbf{T})$ as a subalgebra $\mathcal{C}_0$ of $\mathcal{C}$. Let A_1 be the C^*-algebra generated by $C(\mathbf{T})$ and $(u_\lambda^n)_{n\in\mathbf{Z}}$.

LEMMA 4.2. *The vector space generated by the set* $\tilde{E}_1 = \{fu_\lambda^n \mid f \in C(\mathbf{T}), n \in \mathbf{Z}\}$ *is a* $*$*-algebra norm dense in* A_1.

PROOF. It suffices to prove that $(fu_\lambda^n)^*$ and $(fu_\lambda^n)(ku_\lambda^m)$ are in $\tilde{E}_1$ for $f, k \in C(\mathbf{T})$, $n, m \in \mathbf{Z}$: we have

$$(fu_\lambda^n)^* = u_\lambda^{-n}f^* = u_\lambda^{-n}f^*u_\lambda^n u_\lambda^{-n} = (f^* \circ T_\lambda^{-n}) \cdot u_\lambda^{-n} \in \tilde{E}_1,$$
$$(fu_\lambda^n)(ku_\lambda^m) = f(u_\lambda^n k u_\lambda^{-n})u_\lambda^{m+n} = f \cdot (k \circ T_\lambda^n) \cdot u_\lambda^{m+n} \in \tilde{E}_1$$

since $f^* \circ T_\lambda^{-n}$ and $f \cdot (k \circ T_\lambda^n)$ are in $C(\mathbf{T})$.

The group $G = (u_\lambda^n)_{n\in\mathbf{Z}}$ acts by $*$-automorphism on A_1. $\tau_{u_\lambda^n}a = u_\lambda^n a u_\lambda^{n*}$, $\forall a \in A_1$, $\forall n \in \mathbf{Z}$.

LEMMA 4.3. *The dynamical system* $(A_1, (u_\lambda^n)_{n\in\mathbf{Z}})$ *is uniform. Moreover, the* C^**-algebra* B *generated by* $(u_\lambda^n)_{n\in\mathbf{Z}}$ *is maximal abelian in* A_1 *and* $(A_1, (u_\lambda^n)_{n\in\mathbf{Z}})$ *is associated to* B.

PROOF. We have to show that for every $a \in A_1$, $\frac{1}{n}\sum_{k=0}^{n-1} \tau_k a$ converges in norm to a $*$-invariant element $\phi(a)$. As the vector space generated by $\tilde{E}_1$ is norm dense in A_1 it suffices to prove that for $a = fu_\lambda^m$, with $f \in C(\mathbf{T})$, $m \in \mathbf{Z}$:

$$\frac{1}{n}\sum_{k=0}^{n-1} \tau_k(fu_\lambda^m) = \left(\frac{1}{n}\sum_{k=0}^{n-1} f \circ T_\lambda^k\right) u_\lambda^m \to \mu(f)u_\lambda^m = \phi(fu_\lambda^m)$$

in norm.

We conclude that the system $(A_1, (u_\lambda^n)_{n\in\mathbf{Z}})$ is uniform. So the associated conditional expectation ϕ maps A_1 onto $B' \cap A_1$. But we have proven that $\phi(fu_\lambda^m) = \mu(f)u_\lambda^m \in B$, $\forall f \in C[0, 1]$, $m \in \mathbf{Z}$, and as the vector space generated by E_1 is norm dense in A_1 we have $\phi(a) \in B$, $\forall a \in A$. Therefore since $B = B' \cap A_1$ and B is maximal abelian, the system $(A_1, (u_\lambda^n)_{n\in\mathbf{Z}})$ is associated to B.

It follows from Theorem 3.1 that B possesses the extension property relative to A_1: we have proven part (a) of Theorem b.

Now let L be the C^*-algebra generated by $C(\mathbf{T})$ and $(h \circ T_\lambda^n)_{n\in\mathbf{Z}}$ where $h = 1_V - 1_E$ defined by (4.2), L is a subalgebra of $L^\infty([0, 1[)$ globally invariant by T. Let A be the C^*-algebra generated by L and $(u_\lambda^n)_{n\in\mathbf{Z}}$. We show, as for A_1, that the vector space generated by the set $\tilde{E} = \{fu_\lambda^n \mid f \in L, n \in \mathbf{Z}\}$ is uniformly dense in A.

LEMMA 4.4. *The* C^**-algebra* B *generated by* $(u_\lambda^n)_{n\in\mathbf{Z}}$ *is maximal abelian in* A. *The system* $(A, (u_\lambda^n)_{n\in\mathbf{Z}})$ *is associated to* B *and is not uniform.*

PROOF. Consider the dynamical system $(B(\mathcal{H}), (u_\lambda^n)_{n\in\mathbf{Z}})$, as the vector state e_0 is invariant. By Kovács and Szűcs' theorem (cf. **[11]**) there exists a normal conditional expectation ϕ from $B(\mathcal{H})$ onto $\mathfrak{D} = \{u_\lambda^n, n \in \mathbf{Z}\}'$, the algebra of invariant elements and for every $x \in B(\mathcal{H})$

$$\frac{1}{n}\sum_{k=0}^{n-1} u_\lambda^k x u_\lambda^{-k} \to \phi(x)$$

ultrastrongly, $n \to \infty$.

Now let $x = f \cdot u_\lambda^m, f \in L, m \in \mathbf{Z}$. We have

$$\frac{1}{n}\sum_{k=0}^{n-1} \tau_{u_\lambda^k}(fu_\lambda^m) = \left(\frac{1}{n}\sum_{k=0}^{n-1} f \circ T_\lambda^k\right)u_\lambda^m \underset{n\to\infty}{\to} \mu(f)u_\lambda^m = \phi(fu_\lambda^m)$$

ultrastrongly.

Therefore we have $\phi(a) \in B$, $\forall a \in E$; as the vector space generated by E is norm dense in A and ϕ is norm continuous we have $\phi(a) \in B$, $\forall a \in A$. This implies that $A \cap \mathfrak{D} = B$; therefore B is maximal abelian in A and the system $(A, (u_\lambda^n)_{n\in\mathbf{Z}})$ is associated to B. This system is not uniform because of Lemma 4.1.

It follows from Theorem 3.1 that B does not possess the extension property relative to A.

Part (c) of Theorem b follows from parts (a) and (b).

We note that the spectrum of the unitary u_λ associated to the rotation T is $\{e^{2\pi in\lambda}\}_{n\in\mathbf{Z}}$. This set is dense in $\mathbf{T}$, the C^*-algebra generated by $(u_\lambda^n)_{n\in\mathbf{Z}}$ is isomorphic to $C(\mathbf{T})$.

REMARK 4.5. We can also consider the C^*-algebra $\tilde{A}$ generated by $L^\infty[0, 1[$ and $(u_\lambda^n)_{n\in\mathbf{Z}}$ and prove exactly as for A that $\mathfrak{B}$ is maximal abelian in $\tilde{A}$ and does not have the extension property relative to $\tilde{A}$, but $\tilde{A}$ is not separable.

If we take $\mathfrak{D}_1$ as the diagonal algebra associated to the Haar orthogonal bases (cf. [**1**, p. 321]), J. Anderson proved that (cf. [**1**, Corollary 7.3]) there exist pure states φ on $\mathfrak{B}(\mathcal{G})$ such that the restrictions of φ on $\mathcal{C}$ and $\mathfrak{D}_1$ are homomorphisms. The situation is quite different with $\mathfrak{D}$.

LEMMA 4.6. *If φ is a state on $\mathfrak{B}(\mathcal{H})$ such that the restriction of φ to $\mathfrak{D}$ is a homomorphism then restriction of φ to $\mathcal{C}_0 \sim C(\mathbf{T})$ coincides with the Lebesgue measure on $\mathbf{T}$ (and cannot be a homomorphism on $\mathcal{C} \sim l^\infty[0, 1]$).*

PROOF. By Lemma 3.2, $\varphi(u_\lambda a u_\lambda^*) = \varphi(a)$, $\forall a \in \mathfrak{B}(\mathcal{H})$; therefore the restriction φ to $\mathcal{C}_0 \sim C(\mathbf{T})$ is a state invariant by the rotation T_λ, and therefore coincides with the Lebesgue measure on $\mathbf{T}$.

5. Proof of Theorem c. We use the above notations $\mathcal{H} = L^2[0, 1]$ and $\mathcal{C} \sim L^\infty[0, 1]$. Let E_5 be a set of elements a of A such that if φ is a homomorphism of B and φ_1, φ_2 are two state extensions of φ we have $\varphi_1(a) = \varphi_2(a)$.

Let $\mathcal{K}$ be the set of compact operators of $\mathcal{H}$.

LEMMA 5.1. *We have $\mathcal{K} \subset E_5$.*

PROOF. Let $k \in \mathcal{K}$, φ be a homomorphism of B and $\tilde{\varphi}_1$ be a pure state extension of φ to $\mathfrak{B}(\mathcal{H})$. As $\mathcal{C}$ is isomorphic to $L^\infty[0, 1]$, φ and $\tilde{\varphi}_1$ are not normal; therefore $\tilde{\varphi}_1$ is singular (i.e. $\tilde{\varphi}_{1|\mathcal{K}} = 0$) and $\tilde{\varphi}_1(k) = 0$. Consider the convex compact set

$$S_\varphi = \left\{\tilde{\varphi} \text{ state on } \mathfrak{B}(\mathcal{H}) \mid \tilde{\varphi}_{|\mathcal{C}} = \varphi\right\}$$

and let

$$S'_\varphi = \left\{\tilde{\varphi} \in S_\varphi \mid \tilde{\varphi}(k) = 0\right\};$$

S'_φ is a convex compact subset of S_φ containing all the extremal points (pure states) of S_φ. Therefore $S'_\varphi = S_\varphi$ and we have proved that $k \in E_5$, $\forall k \in \mathcal{K}$.

For each r rational, let T_r be the associated rotation $[0, 1[\to x \to x + r \pmod 1) \in [0, 1[$. T_r is periodic with period k and there exist Borel subsets $F_1, F_2 = T_rF_1, \ldots, F_k = T_r^{k-1}F_1$ forming a partition of $\mathbf{T}$. Let $P_i = 1_{F_i}$ be the indicating function of F_i that we identify, as multiplication operator, as a projection of $\mathcal{C}$; let u_r be the unitary canonically associated to T_r. We consider the following operation used by Kadison and Singer: $u_r^{|P_1} = (1 - P_1)u_r(1 - P_1) + P_1u_rP_1$; it is clear that $u_r^{|P_1|P_2|\cdots|P_k} = 0$. Therefore we conclude with [**10**, Lemma 5, p. 395] that

$$u_r \in E_5, \quad \forall r \text{ rational.} \tag{5.1}$$

We show then as in the proof of Theorem b that the linear space generated by the set $F = \{k + fu_r, k \in \mathcal{K}, f \in \mathcal{C}, r \text{ rational}\}$ is a *-algebra contained in E_5. Let A_2 be its norm closure; then A_2 is a C^*-algebra contained in E_5. Therefore $\mathcal{C}$ has the extension property relative to A_2.

We remark that the above proof can be applied to the case $\mathfrak{D} \sim l^\infty(\mathbf{Z})$: let E_5' be the set of $a \in A$ such that $\varphi_1(a) = \varphi_2(a)$ for φ_1, φ_2 pure states on $\mathfrak{B}(\mathcal{G})$ such that $\varphi_{1|\mathfrak{D}} = \varphi_{2|\mathfrak{D}}$ is a homomorphism. We have then $\mathcal{K} \subset E_5'$; if S is a permutation of $\mathbf{Z}$, let u_S be the associated unitary on. Kadison and Singer (cf. [**10**, Theorem 3]) has shown that $u_S \in E_5'$; the linear space generated by $\{k + du_S, k \in \mathcal{K}, d \in \mathfrak{D}, S$ permutation of $\mathbf{Z}\}$ can be seen to be a *-algebra contained in E_5'; the C^*-algebra A_3 obtained by norm closure is in E_5'. We remark in particular that $\mathcal{C}_0 \subset A_3$, and we obtain

LEMMA 5.2. *There exists a C^*-algebra A_3 containing $\mathfrak{D}$, $\mathcal{K}$ and $\mathcal{C}_0$ such that $\mathfrak{D}$ has the extension property relative to A_3.*

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