DIMENSION OF STRATIFIABLE SPACES

BY

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ABSTRACT. We define a subclass, denoted by EM_3, of the class of stratifiable spaces, and obtain several dimension theoretical results for EM_3 including the coincidence theorem for dim and Ind. The class EM_3 is countably productive, hereditary, preserved under closed maps and, furthermore, the largest subclass of stratifiable spaces in which a harmonious dimension theory can be established.

1. Introduction. Beyond metric spaces, the following line of generalized metric spaces has been established by many authors [S, C, B, H, Ok]:

$$\text{metric} \to \text{Lašnev}^1 \to M_1 \to \text{stratifiable} \to \text{paracompact } \sigma.$$

After Katětov and Morita's work for metric spaces, the first attack to this line in dimension theory was done by Leĭbo [L₁] who proved the equality dim $X = $ Ind X for any Lašnev space X. Nagami extended this result by defining L-spaces [N₃] and free L-spaces [N₄]. Free L-spaces form a countably productive and hereditary class containing every Lašnev space and included in the class of M_1-spaces. It is now desired to develop a satisfactory dimension theory of a still larger class of generalized metric spaces, say, M_1-spaces or stratifiable spaces.

In this direction we define a subclass of stratifiable spaces in terms of a special kind of σ-closure-preserving collection.

DEFINITION 1.1. Let X be a space. A collection $\mathscr{E}$ of subsets of X is called an *encircling net* (or, for short, *E-net*) if for any point x and any open neighborhood U of x, there exists a subcollection $\mathscr{F}$ of $\mathscr{E}$ such that $x \in X - \mathscr{F}^* \subset U$ and $\mathscr{F}^*$ is a closed set of X (where $\mathscr{F}^*$ denotes the union of the members of $\mathscr{F}$).

By EM_3 we denote *the class of stratifiable spaces with σ-closure-preserving E-nets*, and by M_3 *the class of stratifiable spaces*.

The class EM_3 is countably productive, hereditary and preserved under closed maps as well as perfect maps (Corollary 3.9).

Our first main result is a characterization of members of EM_3 as those spaces which are the perfect (closed) images of zero-dimensional stratifiable spaces (Theorem 3.8). This means that EM_3 is just the maximal perfect subclass of M_3 in the sense of Nagami [N₁].

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[1] A space is called a Lašnev space if it is the closed image of a metric space.

The second main results appear in Theorems 4.2 and 4.3 and consist of the following theorems for EM_3:

(a) the equidimensional G_δ-envelope theorem,

(b) the dimension raising theorem,[2]

(c) the decomposition theorem,

(d) the coincidence theorem for dim and Ind.

These theorems for EM_3 extend the corresponding theorems for free L-spaces [**N$_4$**] as well as those for Lašnev spaces [**L$_1$, L$_2$, O$_1$**].

It is an open problem whether the inclusion $EM_3 \subset M_3$ is proper. But the characterization above implies that EM_3 is the largest[3] subclass of M_3 in which the dimension raising theorem holds. We also see in Corollary 4.5 that EM_3 is the largest[3] subclass of M_3 in which the decomposition theorem and the equidimensional G_δ-envelope theorem *simultaneously* hold.

Our arguments are based on Gruenhage and Junnila's result that a stratifiable space is an M_2-space [**G, J**]. Indeed, though we use the word "stratifiable" in view of its significance, what we need is only the existence of a σ-closure-preserving quasi-base.

Conventions. Throughout this paper a space is a *Hausdorff topological space*, and a map means an *onto continuous* one. Let X, Y be spaces and let $f: X \to Y$ be a map. For a collection $\mathcal{F}$ of subsets of X, the symbol $\mathcal{F}^*$ denotes the union of all members of $\mathcal{F}$, and $f(\mathcal{F})$ means the collection of subsets of Y of the form $\{f(F): F \in \mathcal{F}\}$. For a subset Z of X we denote by $\overline{Z}$ (or Cl Z) the closure of Z, by Int Z the interior of Z, and by Bd Z the boundary of Z.

2. Encircling nets and large encircling nets.

Encircling nets are naturally strengthened as follows:

DEFINITION 2.1. Let X be a space. A collection $\mathcal{S}$ of subsets of X is called a *large encircling net* (or, simply, an *LE-net*) if for any disjoint closed sets C and K of X, there exists a subcollection $\mathcal{F}$ of $\mathcal{S}$ such that $C \subset \mathcal{F}^* \subset X - K$ and $\mathcal{F}^*$ is a closed set of X.

REMARKS. Since an LE-net is a net in the usual sense, it follows from Siwiec-Nagata [**SN**] that a space with a σ-closure-preserving LE-net is a σ-space. But a space with a σ-closure-preserving E-net is not necessarily a σ-space as will be seen in Example 2.8. On the other hand it is trivial that a regular σ-space X with ind $X \leq 0$ admits a σ-closure-preserving E-net, and that a normal σ-space X with dim $X \leq 0$ admits a σ-closure-preserving LE-net.

PROPOSITION 2.2. *A metric space admits a σ-locally finite LE-net.*

PROOF. Let M be a metric spaces and $\{\mathcal{S}_i: i = 1, 2, \ldots\}$ a sequence of locally finite closed covers of M such that, for each i, the diameter of each member of $\mathcal{S}_i$ is smaller than $1/i$. Let C, K be disjoint closed sets of M and put

$$\mathcal{F}_i = \{E \in \mathcal{S}_i: E \cap C \neq \varnothing \text{ and } E \cap K = \varnothing\}.$$

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[2] The dimension raising theorem for a topological class $\mathcal{C}$ is: If $X \in \mathcal{C}$ and dim $X \leq n$, then X is the image of a space $X_0 \in \mathcal{C}$ with dim $X_0 \leq 0$ under a perfect map of order not greater than $n + 1$.

[3] When using this word we take no account of infinite-dimensional spaces in the sense of dim.

It is then clear that $\bigcup_{i=1}^{\infty} \mathscr{F}_i^*$ is a closed set of X including C but not meeting K. Hence $\bigcup_{i=1}^{\infty} \mathscr{E}_i$ is a σ-locally finite LE-net on M, which completes the proof.

PROPOSITION 2.3. *The property of having a σ-closure-preserving LE-net is preserved under closed maps.*

We thus have

PROPOSITION 2.4. *A Lašnev space admits a σ-closure-preserving LE-net, and hence it is a member of EM_3.*

LEMMA 2.5. *If $\mathscr{E}$ is an E-net (resp. LE-net) on a space, then $\{\overline{E}: E \in \mathscr{E}\}$ is an E-net (resp. LE-net) on the space.*

PROPOSITION 2.6. *The property of having a σ-closure-preserving E-net is countably productive, hereditary and preserved under perfect maps.*

PROOF. Let X_i, $i = 1, 2, \ldots,$ be spaces with σ-closure-preserving E-nets $\mathscr{E}_i$. It is then clear that

$$\left\{ E_j \times \prod_{i=1, i \neq j}^{\infty} X_i : E_j \in \mathscr{E}_j, j = 1, 2, \ldots \right\}$$

is a σ-closure-preserving E-net on $\prod_{i=1}^{\infty} X_i$.

By the preceding lemma it is obvious that the property is hereditary.

Let X be a space with a σ-closure-preserving E-net $\mathscr{E}$ and let $f: X \to Y$ be a perfect map onto a space Y. By Lemma 2.5 we may assume that every finite intersection of members of $\mathscr{E}$ is again a member of $\mathscr{E}$. To show that $f(\mathscr{E})$ is an E-net on Y let $y \in Y$ and let U be an open neighborhood of y. There exist subcollections $\mathscr{E}_i$, $1 \leqslant i \leqslant k$, of $\mathscr{E}$ such that $f^{-1}(y) \subset X - \bigcap_{i=1}^{k} \mathscr{E}_i^* \subset f^{-1}(U)$ and $\mathscr{E}_i^*$ is a closed set of X. It then follows from assumption that $f(\bigcap_{i=1}^{k} \mathscr{E}_i^*)$ is a closed set of Y written as a union of members of $f(\mathscr{E})$ such that $y \in Y - f(\bigcap_{i=1}^{k} \mathscr{E}_i^*) \subset U$. This completes the proof.

PROPOSITION 2.7. *Let X be a space (resp. a semistratifiable space). Then the following statements are equivalent:*
 (1) *X admits a σ-closure-preserving LE-net (resp. E-net).*
 (2) *X admits a σ-locally finite LE-net (resp. E-net).*
 (3) *X admits a σ-discrete LE-net (resp. E-net).*

PROOF. It follows from Lemma 2.5 and a remark above that a space with a σ-closure-preserving LE-net admits a σ-closure-preserving net of closed sets, and therefore it is semistratifiable. Hence the proposition is immediate from Lemma 2.5 and the following fact, which is essentially due to Siwiec and Nagata [SN]: *Let X be a semistratifiable space and $\mathscr{E}$ a σ-closure-preserving collection of closed sets of X. Then there exists a σ-discrete collection $\mathscr{F}$ of closed sets of X such that each member of $\mathscr{E}$ is a union of members of $\mathscr{F}$.*

As for famous pathological spaces, we have the following results which imply particularly that the existence of σ-closure-preserving E-nets does not mean, in general, that of σ-closure-preserving LE-nets (but, for stratifiable spaces, the former means the latter as will be seen in Theorem 3.8).

EXAMPLES 2.8. (1) *The Michael line $I(M)$ has a σ-discrete E-net, but does not have a σ-closure-preserving LE-net.*

(2) *The same is true for the Sorgenfrey line $R(S)$.*

(3) *$[0, \omega_1]$ does not admit a σ-closure-preserving E-net.*

(4) *The quotient space $I(M)/Q$ obtained by identifying the rational points in $I(M)$ does not admit a σ-closure-preserving E-net. In particular the property of having a σ-closure-preserving E-net is not preserved under closed maps.*

PROOF. (1) and (2) (simultaneously). Let $\mathcal{F}$ be a σ-discrete net of closed sets in the unit interval I (resp. the real line R) with the usual topology. It is easy to see that $\mathcal{F}$ is a σ-discrete E-net on $I(M)$ (resp. $R(S)$). But $I(M)$ (resp. $R(S)$) does not admit a σ-closure-preserving LE-net because it is not a σ-space.

(3) For any σ-closure-preserving collection $\mathcal{F}$ of $[0, \omega_1]$, $\mathcal{F}$ fails to be an E-net at ω_1; indeed, $\mathrm{Cl}(\{\bar{F}: F \in \mathcal{F}, \omega_1 \notin \bar{F}\}^*) \cap \{\omega_1\} = \varnothing$.

(4) If $I(M)/Q$ had a σ-closure-preserving E-net, then every point in $I(M)/Q$, in particular the quotient image of Q, would be a G_δ-set of $I(M)/Q$; but this is impossible because Q is not a G_δ-set of $I(M)$.

3. Characterizations of EM_3.

LEMMA 3.1 [O_2, LEMMA 3.1]. *Let X be a submetrizable space (that is, X admits a weaker metric topology), and let $\mathcal{U}$ be a σ-discrete collection of cozero sets of X. Then there exist a metric space M and a one-to-one map $f: X \to M$ such that $f(U)$ is an open set of M for every $U \in \mathcal{U}$.*

The following lemma plays a fundamental role in this paper.

LEMMA 3.2. *Let X be a paracompact σ-space and let $\mathcal{F} = \bigcup_{i=1}^{\infty}\mathcal{F}_i$ be a collection of closed sets of X such that $\mathcal{F}_i$ is closure-preserving for each i. Then there exist a metric space M and a one-to-one map $f: X \to M$ such that $f(F)$ is a closed set of M for every $F \in \mathcal{F}$ and such that $f(\mathcal{F}_i)$ is closure-preserving in M for every i.*

PROOF. Let $\mathcal{B} = \bigcup_{i=1}^{\infty}\mathcal{B}_i$ be a net of X consisting of closed sets such that $\mathcal{B}_i$ is discrete for each i. For each i let $\mathcal{V}_i = \{V_i(B): B \in \mathcal{B}_i\}$ be a discrete collection of open sets of X such that $B \subset V_i(B)$ for each $B \in \mathcal{B}_i$. For $i, j = 1, 2, \ldots$, $B \in \mathcal{B}_i$, put

$$W_i^j(B) = V_i(B) \cap \left(X - \{F: F \in \mathcal{F}_j, F \cap B = \varnothing\}^* \right).$$

Then $W_i^j(B)$ is an open set of X, and $\{W_i^j(B): B \in \mathcal{B}_i\}$ is discrete in X. Hence Lemma 3.1 applies to give a metric space M and a one-to-one map $f: X \to M$ such that $f(W_i^j(B))$ is an open set of M for every $B \in \mathcal{B}_i$, $i, j = 1, 2, \ldots$. It is then obvious that for each i, $f(\mathcal{F}_i)$ is a closure-preserving collection of closed sets of M. This completes the proof.

DEFINITION 3.3. Let $X \in EM_3$ and let $\{\mathcal{F}, \mathcal{V}, \mathcal{E}, \mathcal{S}\}$ be a quartet of collections of subsets of X. The quartet is called an *E-quartet* if we can write $\mathcal{F} = \bigcup_{i=1}^{\infty}\mathcal{F}_i$, $\mathcal{V} = \bigcup_{i=1}^{\infty}\mathcal{V}_i$, $\mathcal{E} = \bigcup_{i=1}^{\infty}\mathcal{E}_i$, $\mathcal{S} = \bigcup_{i=1}^{\infty}\mathcal{S}_i$ and if the following four conditions are satisfied:

(1_q) *$\mathcal{F}$ is a net on X consisting of closed sets.*

(2_q) *For each i, $\mathcal{V}_i$ is a discrete collection of open sets of X written as $\mathcal{V}_i = \{V_i(F):$ $F \in \mathcal{F}_i\}$ in such a manner that $F \subset V_i(F)$ for each $F \in \mathcal{F}_i$.*

(3_q) *$\mathcal{E}$ is an E-net on X consisting of closed sets and $\mathcal{E}_i$ is closure-preserving for each i.*

(4_q) *$\mathcal{S}$ is a quasi-base[4] for X consisting of closed sets and $\mathcal{S}_i$ is closure-preserving for each i.*

By Heath **[H]**, Gruenhage **[G]** and Junnila **[J]**, each member of EM_3 admits an E-quartet.

DEFINITION 3.4. Let X be a member of EM_3 with an E-quartet $\{\mathcal{F}, \mathcal{V}, \mathcal{E}, \mathcal{S}\}$. A map $f: X \to Y$ onto a normal space Y is called an *E-map* with respect to the E-quartet if the following five conditions are satisfied:

(0_f) *f is one-to-one.*

(1_f) *$f(F)$ is a closed set for every $F \in \mathcal{F}$.*

(2_f) *$f(V)$ is an open set for every $V \in \mathcal{V}$.*

(3_f) *$f(E)$ is a closed set for every $E \in \mathcal{E}$, and $f(\mathcal{E}_i)$ is closure-preserving in Y for every i.*

(4_f) *$f(S)$ is a closed set for every $S \in \mathcal{S}$, and $f(\mathcal{S}_i)$ is closure-preserving in Y for every i.*

Noting that $\{X - V: V \in \mathcal{V}\}$ is a σ-closure-preserving collection of closed sets of X, we have the following result by virtue of Lemma 3.2.

PROPOSITION 3.5. *Let X be a member of EM_3. Then for any E-quartet of X there exist a metric space M and an E-map $f: X \to M$ with respect to the E-quartet.*

The following lemma is well known (see, for example, **[E, 2.3.16]**).

LEMMA 3.6. *Let X be a space and let C, K be disjoint closed sets of X. Let $\mathcal{U}$ be a countable open cover of X such that for each $U \in \mathcal{U}$, either $\overline{U} \cap C = \varnothing$ or $\overline{U} \cap K = \varnothing$. Then C and K are separated by a closed set S such that $S \subset \{\mathrm{Bd}\, U: U \in \mathcal{U}\}^*$.*

Now we have the following result frequently used later.

PROPOSITION 3.7. *Let X be a member of EM_3 with an E-quartet $\{\mathcal{F}, \mathcal{V}, \mathcal{E}, \mathcal{S}\}$. Let f: $X \to Y$ be an E-map with respect to the E-quartet onto a normal space Y. Then $\mathrm{Ind}\, X \leqslant \mathrm{Ind}\, Y$.*

PROOF. The proof is by induction on $\mathrm{Ind}\, Y$. If $Y = \varnothing$ then the proposition is trivial. Suppose that the proposition is valid when $\mathrm{Ind}\, Y \leqslant n - 1$ and consider the case of $\mathrm{Ind}\, Y = n$. To show $\mathrm{Ind}\, X \leqslant n$, let C, K be disjoint closed sets of X. For the time being, fix a point x in $X - C$ arbitrarily. We show that there exists an open neighborhood W of x such that $\overline{W} \cap C = \varnothing$ and $\mathrm{Ind}\, \mathrm{Bd}\, W \leqslant n - 1$. Let $\mathcal{E}(x)$ be a subcollection of $\mathcal{E}$ such that $x \in X - \mathcal{E}(x)^* \subset X - C$ and $\mathcal{E}(x)^*$ is a closed set. Write $\mathcal{E}(x) = \bigcup_{i=1}^{\infty} \mathcal{E}_i(x)$ where $\mathcal{E}_i(x) \subset \mathcal{E}_i$. Put $\mathcal{S}_i(x) = \{S \in \mathcal{S}_i: S \cap \mathcal{E}(x)^* = \varnothing\}$

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[4] A collection $\mathcal{S}$ of subsets of a space X is called a quasi-base for X if for any point x and any open neighborhood U of x there exists a member S of $\mathcal{S}$ such that $x \in \mathrm{Int}\, S \subset S \subset U$.

and $\mathcal{S}(x) = \bigcup_{i=1}^{\infty} \mathcal{S}_i(x)$. Fix i_0 so that $x \in \text{Int } \mathcal{S}_{i_0}(x)^*$. By (3_f) and (4_f) there exist open sets $O_j, j = 1, 2, \ldots,$ of Y such that

$$f\left(\bigcup_{i=1}^{j} \mathcal{S}_i(x)^*\right) \cup f\left(\mathcal{S}_{i_0}(x)^*\right) \subset O_j \subset \overline{O}_j \subset Y - f\left(\mathcal{S}_j(x)^*\right)$$

and

$$\text{Ind Bd } O_j \leqslant n - 1.$$

Define $W = \bigcap_{j=1}^{\infty} f^{-1}(O_j)$. Then

$$x \in W \subset \overline{W} \subset \bigcap_{j=1}^{\infty} f^{-1}(\overline{O}_j) \subset X - \mathcal{S}(x)^* \subset X - C.$$

To show that W is open, let $x' \in W$. Since $x' \in X - \mathcal{S}(x)^*$ and $\mathcal{S}(x)^*$ is a closed set, it follows from (4_q) that $x' \in \text{Int } \mathcal{S}_m(x)^*$ for some m. Then

$$x' \in \bigcap_{j=1}^{m-1} f^{-1}(O_j) \cap \text{Int } \mathcal{S}_m(x)^* \subset W,$$

which implies that W is open. To show $\text{Ind Bd } W \leqslant n - 1$, note that, for any subset Z of X, $f \,|\, Z \colon Z \to f(Z)$ is again an E-map with respect to the E-quartet $\{\mathcal{F} \,|\, Z, \mathcal{V} \,|\, Z, \mathcal{E} \,|\, Z, \mathcal{S} \,|\, Z\}$ on Z. Hence we may apply induction hypothesis to obtain $\text{Ind } f^{-1}(\text{Bd } O_j) \leqslant n - 1$, $j = 1, 2, \ldots,$ which yields

$$\text{Ind Bd } W \leqslant \text{Ind}\left(\bigcup_{j=1}^{\infty} \text{Bd } f^{-1}(O_j)\right)$$

$$= \max\{\text{Ind Bd } f^{-1}(O_j) \colon j = 1, 2, \ldots\}$$

$$\leqslant \max\{\text{Ind } f^{-1}(\text{Bd } O_j) \colon j = 1, 2, \ldots\} \leqslant n - 1.$$

Hence W is a required open neighborhood of x; we have thus finished "local" separation.

Now put

$$\mathcal{F}_i(C) = \{F \in \mathcal{F}_i \colon F \subset W \text{ for some open set } W \text{ with}$$

$$\overline{W} \cap C = \varnothing \text{ and } \text{Ind Bd } W \leqslant n - 1\}.$$

Then by (1_q) and by the "local" separation above, we have $\bigcup_{i=1}^{\infty} \mathcal{F}_i(C)^* = X - C$. For each $F \in \mathcal{F}_i(C)$, fix such a W and denote it by $W_i(C, F)$. On the other hand, by (1_f) and (2_f), there exist open sets $H_i(F)$, $F \in \mathcal{F}_i$, of Y such that $f(F) \subset H_i(F) \subset \text{Cl } H_i(F) \subset f(V_i(F))$ and $\text{Ind Bd } H_i(F) \leqslant n - 1$ (where the set $V_i(F)$ is as in Definition 3.3(2_q)). By induction hypothesis again,

$$\text{Ind Bd } f^{-1}(H_i(F)) \leqslant \text{Ind } f^{-1}(\text{Bd } H_i(F)) \leqslant n - 1.$$

Put for each $F \in \mathcal{F}_i(C)$,

$$D_i(C, F) = W_i(C, F) \cap f^{-1}(H_i(F)).$$

Then

$$\text{Ind Bd } D_i(C, F) \leqslant \max\{\text{Ind Bd } W_i(C, F), \text{Ind Bd } f^{-1}(H_i(F))\} \leqslant n - 1.$$

Put $D_i(C) = \{D_i(C, F): F \in \mathscr{F}_i(C)\}^*$. Since $D_i(C, F) \subset V_i(F)$, (2_q) implies that $\{D_i(C, F): F \in \mathscr{F}_i(C)\}$ is discrete. Thus $\operatorname{Ind} \operatorname{Bd} D_i(C) \leqslant n - 1$, $i = 1, 2, \ldots$. By the same discreteness and by the fact $D_i(C, F) \subset W_i(C, F) \subset \operatorname{Cl} W_i(C, F) \subset X - C$, we have $C \cap \operatorname{Cl} D_i(C) = \varnothing$ for every $i = 1, 2, \ldots$. We also obtain $\cup_{i=1}^{\infty} D_i(C) = X - C$ because $\cup_{i=1}^{\infty} \mathscr{F}_i(C)^* = X - C$.

Quite similarly we can obtain open subsets $D_i(K)$, $i = 1, 2, \ldots$, such that $\operatorname{Ind} \operatorname{Bd} D_i(K) \leqslant n - 1$, $K \cap \operatorname{Cl} D_i(K) = \varnothing$ and $\cup_{i=1}^{\infty} D_i(K) = X - K$. Hence, applying Lemma 3.6, we have a closed set B separating C and K such that

$$B \subset \left(\bigcup_{i=1}^{\infty} \operatorname{Bd} D_i(C) \right) \cup \left(\bigcup_{i=1}^{\infty} \operatorname{Bd} D_i(K) \right).$$

By the countable sum theorem for Ind, we have $\operatorname{Ind} B \leqslant n - 1$. Thus $\operatorname{Ind} X \leqslant n$, which completes the proof of Proposition 3.7.

We can now prove a characterization theorem for EM_3.

THEOREM 3.8. *The following statements about a space X are equivalent*:
(1) *X is a stratifiable space with a σ-closure-preserving E-net.*
(2) *X is the perfect image of a stratifiable space X_0 with $\dim X_0 \leqslant 0$.*
(3) *X is the closed image of a stratifiable space X_0 with $\operatorname{ind} X_0 \leqslant 0$.*
(4) *X is a stratifiable space with a σ-closure-preserving LE-net.*

PROOF. The implications $(2) \to (3)$ and $(4) \to (1)$ are obvious. To show $(1) \to (2)$ let X be a member of EM_3 with an E-quartet $\{\mathscr{F}, \mathscr{V}, \mathscr{E}, \mathscr{S}\}$. By Proposition 3.5 there exists an E-map $f: X \to M$ onto a metric space M with respect to $\{\mathscr{F}, \mathscr{V}, \mathscr{E}, \mathscr{S}\}$. By Morita [**M**], M is the image of a metric space P with $\dim P \leqslant 0$ under a perfect map g. Now let T be the fiber product of P and X with respect to g and f, that is,

$$T = \{(p, x) \in P \times X: g(p) = f(x)\}$$

with the topology induced from $P \times X$. Let t_P, t_X be the restrictions to T of the projections from $P \times X$ onto P and X, respectively. We thus have the following commutative diagram:

$$
\begin{array}{ccc}
X & \xleftarrow{\; t_X \;} & T \\
f \downarrow & & \downarrow t_P \\
M & \xleftarrow[g]{} & P
\end{array}
$$

It is a well-known property of fiber products that the perfectness of g implies the perfectness of t_X (see [**Pe**, Lemma 7.5.13]). T is stratifiable by [**C**, Theorems 2.3, 2.4]. Hence what should be proved is the zero-dimensionality of T. By Proposition 2.2, P admits an E-quartet $\{\mathscr{F}_P, \mathscr{V}_P, \mathscr{E}_P, \mathscr{S}_P\}$. Now define

$$\mathscr{F}_T = \left\{ t_P^{-1}(F_P) \cap t_X^{-1}(F): F_P \in \mathscr{F}_P, F \in \mathscr{F} \right\},$$

$$\mathscr{V}_T = \left\{ t_P^{-1}(V_P) \cap t_X^{-1}(V): V_P \in \mathscr{V}_P, V \in \mathscr{V} \right\},$$

$$\mathscr{S}_T = \left\{ t_P^{-1}(S_P) \cap t_X^{-1}(S): S_P \in \mathscr{S}_P, S \in \mathscr{S} \right\}, \quad \text{and}$$

$$\mathscr{E}_T = \left\{ t_P^{-1}(E_P): E_P \in \mathscr{E}_P \right\} \cup \left\{ t_X^{-1}(E): E \in \mathscr{E} \right\}.$$

Then it is easy to see that the quartet $\{\mathcal{F}_T, \mathcal{V}_T, \mathcal{E}_T, \mathcal{S}_T\}$ is an E-quartet of T. Furthermore, the map t_P is an E-map with respect to $\{\mathcal{F}_T, \mathcal{V}_T, \mathcal{E}_T, \mathcal{S}_T\}$ because, in general, $t_P(t_P^{-1}(P') \cap t_X^{-1}(X')) = P' \cap g^{-1} \circ f(X')$ for any $P' \subset P$ and $X' \subset X$, and because f is an E-map with respect to $\{\mathcal{F}, \mathcal{V}, \mathcal{E}, \mathcal{S}\}$. Hence, applying Proposition 3.7, we have Ind $T \leqslant 0$. Thus the implication (1) → (2) has been proved.

To show (3) → (4) let X_0 be a stratifiable space with ind $X_0 \leqslant 0$ and let $f: X_0 \to X$ be a closed map. Note that every net on X_0 is an E-net; hence X_0 is a member of EM_3 by Heath [H]. It now follows from the implication (1) → (2) that X_0 is the image of a stratifiable space X_1 with dim $X_1 \leqslant 0$ under a perfect map h. Since every net on X_1 is an LE-net, it follows from Heath [H] again that X_1 admits a σ-closure-preserving LE-net. Hence, applying Proposition 2.3 to the closed map $f \circ h$, we see that X admits a σ-closure-preserving LE-net. On the other hand X is stratifiable by Borges [B, Theorem 3.1]. This completes the proof of Theorem 3.8.

COROLLARY 3.9. *The class EM_3 is countably productive, hereditary and preserved under closed maps.*

PROOF. This is immediate from Theorem 3.8, Proposition 2.6 and the analogous result for M_3 due to Ceder [C] and Borges [B].

A topological class $\mathcal{C}$ is called *perfect* (Nagami [N₁], also see [N₂]) if it is countably productive, hereditary, preserved under perfect maps, included in the class of normal spaces, and every member of $\mathcal{C}$ is the perfect image of a zero-dimensional (in the sense of dim) member of $\mathcal{C}$. Theorem 3.8 and Corollary 3.9 say

COROLLARY 3.10. *The class EM_3 is the maximal perfect subclass of M_3.*

Recently Itō [I] has presented a free L-space, a certain closed image of which is not a free L-space. But we have

COROLLARY 3.11. *Every closed image of a free L-space is a member of EM_3.*

PROOF. By Nagami [N₄, Theorem 2.10] and Theorem 3.8, every free L-space is a member of EM_3 (it is also easy to directly prove that every free L-space admits a σ-closure-preserving E-net). Hence this corollary is immediate from Corollary 3.9.

4. Dimension for EM_3. We begin with the equidimensional G_δ-envelope theorem. To show this, the following lemma is useful.

LEMMA 4.1 (OKA [O₄, LEMMA 3.3]). *Let X be a hereditarily normal space and let $f: X \to L$ be a map onto a metric space L. Then for any subset $Y \subset X$, there exist a G_δ-set Z of X, a metric space M and maps $g: Z \to M$, $h: M \to f(Z)$ such that*
   (i) $Y \subset Z$,
   (ii) dim $g(Y) \leqslant$ dim Y and
   (iii) $f | Z = h \circ g$.

THEOREM 4.2. *Let $X \in EM_3$ and let Y be a subset of X with dim $Y \leqslant n$. Then there exists a G_δ-set G of X such that $Y \subset G$ and dim $G \leqslant n$.*

PROOF. Let $f: X \to L$ be an E-map onto a metric space L with respect to an E-quartet, say $\{\mathcal{F}, \mathcal{V}, \mathcal{E}, \mathcal{S}\}$, on X. By the above lemma there exist a G_δ-set Z of X, a

metric space M and maps $g: Z \to M$, $h: M \to f(Z)$ satisfying (i), (ii), (iii) above. Since $\dim g(Y) \le n$ and M is metrizable, we can find a G_δ-set H of M such that $g(Y) \subset H$ and $\dim H \le n$ (see, for example, [E, 4.1.19]). Define $G = g^{-1}(H)$. Then G is a G_δ-set of Z, and hence of X. To show $\dim G \le n$, note that $g \mid G$ is an E-map with respect to $\{\mathcal{F} \mid G, \ \mathcal{V} \mid G, \ \mathcal{E} \mid G, \ \mathcal{S} \mid G\}$ because $f \mid G$ is so and because $f \mid G = h \circ g \mid G$ by (iii). Hence by Proposition 3.7 we have $\mathrm{Ind}\, G \le \mathrm{Ind}\, H$. Consequently

$$\dim G \le \mathrm{Ind}\, G \le \mathrm{Ind}\, H = \dim H \le n,$$

as required. This completes the proof.

The following theorem occupies the central position in dimension theory of EM_3. The key argument of the proof has already appeared in the proof of Theorem 3.8.

THEOREM 4.3. *The following statements about a space X are equivalent*:

(1) $X \in EM_3$ *and* $\dim X \le n$.

(2) X *is the image of a stratifiable space X_0 with* $\dim X_0 \le 0$ *under a perfect map of order not greater than* $n + 1$.

(3) X *is a stratifiable space which is the union of G_δ-sets X_i, $1 \le i \le n + 1$, with* $\dim X_i \le 0$.

(4) $X \in EM_3$ *and* $\mathrm{Ind}\, X \le n$.

PROOF. (1) $\to$ (2). Let X be a member of EM_3 such that $\dim X \le n$. Let $\{\mathcal{F}, \mathcal{V}, \mathcal{E}, \mathcal{S}\}$ be an E-quartet of X. By Proposition 3.5 there exist a metric space L and an E-map $f: X \to L$ with respect to the E-quartet. By Pasynkov's factorization theorem [P, Theorem 29], there exist a metric space M and maps $g: X \to M$, $h: M \to L$ such that $\dim M \le n$ and $f = h \circ g$. It then follows from Morita [M] that M is the image of a metric space P with $\dim P \le 0$ under a perfect map r such that $\mathrm{ord}\, r \le n + 1$. Let T be the fiber product of P and X with respect to r and g, and let t_p, t_X be the restrictions to T of the projections from $P \times X$ onto P and X, respectively. We thus obtain the following commutative diagram:

$$
\begin{array}{ccccc}
X & = & X & \overset{t_X}{\leftarrow} & T \\
f \downarrow & & g \downarrow & & \downarrow t_P \\
L & \underset{h}{\leftarrow} & M & \underset{r}{\leftarrow} & P
\end{array}
$$

It is obvious that t_X is a perfect map of order not greater than $n + 1$ and that T is a stratifiable space. Note that g is an E-map with respect to $\{\mathcal{F}, \mathcal{V}, \mathcal{E}, \mathcal{S}\}$ because f is so and $f = h \circ g$. Now, as in the proof of Theorem 3.8, t_P is also an E-map with respect to a certain E-quartet of T, and hence $\dim T \le 0$ by Proposition 3.7.

(2) $\to$ (3). Let $t: X_0 \to X$ be a perfect map from a stratifiable space X_0 with $\dim X_0 \le 0$ onto a space X such that $\mathrm{ord}\, t \le n + 1$. Put $Y_i = \{x \in X: |\, t^{-1}(x)\,| = i\}$, $1 \le i \le n + 1$. It then follows from Nagami [N_2, Lemma 4] that $\dim Y_i \le 0$ for each $i = 1, 2, \ldots, n + 1$. Since X is a member of EM_3 by Theorem 3.8, we may apply Theorem 4.2 to obtain G_δ-sets X_i, $1 \le i \le n + 1$, such that $\dim X_i \le 0$ and $Y_i \subset X_i$.

The implication (4) $\to$ (1) is trivial.

Finally the implication (3) → (4) is assured by the following theorem (but the fact Ind $X \leqslant n$ only is direct from (3) as a consequence general for hereditarily normal spaces).

THEOREM 4.4. *Let X be a normal σ-space expressed as the finite union of G_δ-sets X_i, $1 \leqslant i \leqslant k$, such that* dim $X_i \leqslant 0$. *Then X admits a σ-closure-preserving LE-net.*

PROOF. The proof is by induction on k. When $k = 1$, the theorem is trivial. Now suppose that the theorem is valid when $k = m - 1$, and consider the case $k = m$. Put $Y_m = X - X_m$. Then by induction hypothesis and Lemma 2.5, the normal σ-space Y_m admits a σ-closure-preserving LE-net, say $\mathcal{S}$, consisting of closed sets of Y_m. Write $Y_m = \bigcup_{i=1}^\infty C_i$ with closed sets C_i such that $C_i \subset C_{i+1}$, and put $\mathcal{S}_i = \mathcal{S} \mid C_i$. Let $\mathcal{F}$ be a σ-locally finite net of X. Now consider the σ-closure-preserving collection $\bigcup_{i=1}^\infty \mathcal{S}_i \cup \mathcal{F}$ of X. To show that the collection is an LE-net on X, let C, K be disjoint closed sets of X. Since X is hereditarily normal and Ind $X_m \leqslant 0$, there exists a closed set S separating C and K such that $S \cap X_m = \varnothing$. Represent X as the disjoint union $V \cup S \cup W$, where V and W are open sets of X including C and K respectively. Write $V = \bigcup_{i=1}^\infty V_i$ with open sets V_i such that $\overline{V_i} \subset V_{i+1}$ for every i. For each i take a subcollection $\mathcal{F}_i$ of $\mathcal{S}_i$ such that

$$(W \cup S) \cap C_i \subset \mathcal{F}_i^* \subset C_i - (\overline{V_i} \cup C)$$

and $\mathcal{F}_i^*$ is a closed set of C_i. Now put

$$B = W \cup \left(\bigcup_{i=1}^\infty \mathcal{F}_i^* \right).$$

It is easy to see that B is a closed set of X including K and not meeting C. Since W is the union of some members of $\mathcal{F}$, B is the union of some members of $\bigcup_{i=1}^\infty \mathcal{S}_i \cup \mathcal{F}$. Thus $\bigcup_{i=1}^\infty \mathcal{S}_i \cup \mathcal{F}$ is a σ-closure-preserving LE-net on X. This completes the proof of Theorem 4.4 and, therefore, of Theorem 4.3.

REMARK. Slightly modifying the above proof, we can weaken the condition "X_i is G_δ" in Theorem 4.4 to "X_i is either G_δ or F_σ".

As a trivial version of Theorem 4.4, we have the following result which tells us that the dimension theory does not work well in the remainder $M_3 - EM_3$.

COROLLARY 4.5. *Let X be a normal σ-space not admitting a σ-closure-preserving LE-net. Then either*

(1) *X cannot be decomposed into finitely many zero-dimensional (in the sense of* dim*) subsets, or*

(2) *there exists a zero-dimensional (in the sense of* dim*) subset of X not admitting an equidimensional G_δ-envelope.*

As an immediate consequence of Theorem 4.3, we have

COROLLARY 4.6. *Let X be a stratifiable space with* ind $X \leqslant 0$. *Then* dim $X = $ Ind X.

REMARK. This result, however, is generalized to paracompact σ-spaces in my recent paper [O_5].

We conclude this section with the following result, an immediate consequence of Corollary 3.11 and Theorem 4.3.

COROLLARY 4.7. *Let X be the closed image of a free L-space. Then* $\dim X = \operatorname{Ind} X$.

5. Other spaces admitting σ-closure-preserving E-nets.

Let $\mathcal{C}$ be a topological property. A space is called *peripherally* $\mathcal{C}$ if every point in the space admits an open neighborhood base, the boundary of each member of which is $\mathcal{C}$.

THEOREM 5.1. (1) *A peripherally σ-discrete, paracompact σ-space admits a σ-closure-preserving E-net.*

(2) *A peripherally σ-compact, stratifiable space admits a σ-closure-preserving E-net.*

PROOF. We shall prove (1) and (2) simultaneously. Let $\mathcal{F}$ be a σ-locally finite net (resp. a σ-closure-preserving quasi-base) of X consisting of closed sets. To show that $\mathcal{F}$ itself is an E-net on X let x be a point of X and V an open neighborhood of x. Take an open set U such that $x \in U \subset \overline{U} \subset V$ and $\operatorname{Bd} U$ is σ-discrete (resp. σ-compact). Write $U = \bigcup_{i=1}^{\infty} U_i$ with open sets U_i such that $\overline{U}_i \subset U_{i+1}$ for every i. Write $\operatorname{Bd} U = \bigcup_{i=1}^{\infty} C_i$ with discrete (resp. compact) closed sets C_i, $i = 1, 2, \ldots$. There exists, for each i, a discrete (resp. finite) subcollection $\mathcal{F}_i$ of $\mathcal{F}$ such that $C_i \subset \mathcal{F}_i^* \subset X - (\overline{U}_i \cup \{x\})$. Then $\bigcup_{i=1}^{\infty}\mathcal{F}_i^* \cup (X - \overline{U})$ is a closed set of X including $X - V$, not meeting $\{x\}$ and expressed as a union of members of $\mathcal{F}$. Thus $\mathcal{F}$ is an E-net of X, which completes the proof.

Now we have the following generalization of Corollary 4.6.

COROLLARY 5.2. *Let X be a peripherally σ-compact (or peripherally σ-discrete) stratifiable space. Then* $\dim X = \operatorname{Ind} X$.

We next verify a countable sum theorem for σ-closure-preserving LE-nets.

THEOREM 5.3. *Let X be a normal space expressed as the countable union of closed sets X_i, $i = 1, 2, \ldots$, each of which admits a σ-closure-preserving LE-net. Then X has a σ-closure-preserving LE-net.*

PROOF. Note that X is perfectly normal because each X_i is. Let $\mathcal{S}_i$ be a σ-closure-preserving LE-net of X_i. It is clear that $\bigcup_{i=1}^{\infty}\mathcal{S}_i$ is σ-closure-preserving in X. To show that $\bigcup_{i=1}^{\infty}\mathcal{S}_i$ is an LE-net, let C and K be disjoint closed sets of X. Write $X - C = \bigcup_{i=1}^{\infty} V_i$ with open sets V_i such that $\overline{V}_i \subset V_{i+1}$. For each i let $\mathcal{F}_i$ be a subcollection of $\mathcal{S}_i$ such that $\mathcal{F}_i^*$ is a closed set of X_i and $C \cap X_i \subset \mathcal{F}_i^* \subset X_i - (K \cup \overline{V}_i)$. It is then obvious that $\bigcup_{i=1}^{\infty}\mathcal{F}_i^*$ is a closed set of X and $C \subset \bigcup_{i=1}^{\infty}\mathcal{F}_i^* \subset X - K$. This completes the proof.

The following result is immediate from Theorem 5.3, Proposition 2.2 and Ceder [C, Theorem 8.3].

COROLLARY 5.4. *A chunk complex (and hence a CW-complex) is a member of EM_3.*

We list several unsolved problems below.

Problem 5.5. (1) *Does every stratifiable space admit a σ-closure-preserving E-net?* By virtue of Theorem 3.8, this is equivalent to:

(2) (Nagami [N_1, Problem 4]) *Is every stratifiable space a perfect image of a zero-dimensional* (*in the sense of* dim) *stratifiable space?*

The author also does not know whether the inclusion $EM_3 \subset M_1$ (or $M_1 \subset EM_3$) holds or not.

Problem 5.6. *Let X be a paracompact σ-space admitting a σ-closure-preserving E-net. Then*:

(1) *Does the equality* dim $X = $ Ind X *hold?*

(2) *Is X a perfect image of a zero-dimensional* (*in the sense of* dim) *paracompact σ-space?* More weakly:

(3) *Does X admit a σ-closure-preserving LE-net?*

In the specific case of ind $X \leqslant 0$, (1) admits an affirmative answer by the inequality Ind $X \leqslant$ dim $X + $ ind X for every nonempty paracompact σ-space X [O_5]; (2) is also affirmative, that is, *a paracompact σ-space of* ind $\leqslant 0$ *is the perfect image of a paracompact σ-space of* dim $\leqslant 0$.

To outline the proof, let X be a nonempty paracompact σ-space with ind $X = 0$. Let $\mathcal{F} = \bigcup_{i=1}^{\infty} \mathcal{F}_i$ and $\mathcal{V} = \bigcup_{i=1}^{\infty} \mathcal{V}_i$ be as in Definition 3.3. Let f: $X \to M$ be a one-to-one map onto a metric space M such that $f(\mathcal{V}_i^*)$ is open and $f(\mathcal{F}_i^*)$ is closed for every i. In [O_5, Lemma 5] it is proved that, in general, Ind $X \leqslant$ Ind $M + $ ind X for any such map f: $X \to M$. The metric space M is the image of a metric space L with dim $L = 0$ under a perfect map g. Let T be the fiber product of L and X with respect to g and f. Let t_L, t_X be the restrictions to T of the projections from $L \times X$ onto L and X, respectively. Then, since the map t_L is of the "same type" as f, we have Ind $T \leqslant$ Ind $L + $ ind $T = $ ind T. But, in the present case, ind $T \leqslant$ ind($L \times X$) $ = 0$; hence Ind $T = 0$. It is clear that T is a paracompact σ-space and t_X is a perfect map. This completes the proof.

Problem 5.7. *Let X be a stratifiable space expressed as the union of countably many metrizable* (G_δ-) *subsets. Does the equality* dim $X = $ Ind X *hold? More strongly, does X admit a σ-closure-preserving E-net?* (A space of this type is a natural generalization of a Lašnev space in view of Lašnev's well-known decomposition theorem [La].)

BIBLIOGRAPHY

[B] C. R. Borges, *On stratifiable spaces*, Pacific J. Math. **17** (1966), 1–16.

[C] J. G. Ceder, *Some generalizations of metric spaces*, Pacific J. Math. **11** (1961), 105–125.

[Cr] G. D. Creede, *Concerning semi-stratifiable spaces*, Pacific J. Math. **32** (1970), 47–54.

[E] R. Engelking, *Dimension theory*, North-Holland, Amsterdam, 1978.

[G] G. Gruenhage, *Stratifiable spaces are M_2*, Topology Proc. **1** (1976), 221–226.

[H] R. W. Heath, *Stratifiable spaces are σ-spaces*, Notices Amer. Math. Soc. **16** (1969), 761.

[I] M. Itō, *Weak L-spaces are free L-spaces*, J. Math. Soc. Japan **34** (1982), 507–514.

[J] H. Junnila, *Neighbornets*, Pacific J. Math. **76** (1978), 83–108.

[La] N. S. Lašnev, *Continuous decompositions and closed mappings of metric spaces*, Soviet Math. Dokl. **6** (1965), 1504–1506.

[L_1] I. M. Leĭbo, *On the equality of dimensions for closed images of metric spaces*, Soviet Math. Dokl. **15** (1974), 835–839.

[L_2] _____, *On closed images of metric spaces*, Soviet Math. Dokl. **16** (1975), 1292–1295.

[M] K. Morita, *A condition for the metrizability of topological spaces and for n-dimensionality*, Sci. Rep. Tokyo Kyoiku Daigaku Sect. A **5** (1955), 33–36.

[N_1] K. Nagami, *Normality of products*, Actes Congrès Intern. Math. **2** (1970), 33–37.

[N_2] _____, *Perfect class of spaces*, Proc. Japan Acad. **48** (1972), 21–24.

[N_3] _____, *The equality of dimensions*, Fund. Math. **106** (1980), 239–246.

[N_4] _____, *Dimension of free L-spaces*, Fund. Math. **108** (1980), 211–224.

[O_1] S. Oka, *A note on the covering dimension of Lašnev spaces*, Proc. Japan Acad. Ser. A **54** (1978), 73–75.

[O_2] _____, *Dimension of finite unions of metric spaces*, Math. Japon. **24** (1979), 351–362.

[O_3] _____, *Every Lašnev space is the perfect image of a zero-dimensional one*, Bull. Acad. Polon. Sci. Ser. Sci. Math. **28** (1980), 591–594.

[O_4] _____, *Free patched spaces and fundamental theorems of dimension theory*, Bull. Acad. Polon. Sci. Ser. Sci. Math. **28** (1980), 595–602.

[O_5] _____, *An inequality concerning three fundamental dimensions of paracompact σ-spaces*, Proc. Amer. Math. Soc. **83** (1981), 790–792.

[Ok] A. Okuyama, *Some generalizations of metric spaces, their metrization theorems and product spaces*, Sci. Rep. Tokyo Kyoiku Daigaku Sect. A **9** (1967), 236–254.

[P] B. A. Pasynkov, *On the spectral decomposition of topological spaces*, Amer. Math. Soc. Transl. Ser. (2) **73** (1966), 83–134.

[Pe] A. R. Pears, *Dimension theory of general spaces*, Cambridge Univ. Press, Cambridge, 1975.

[S] F. G. Slaughter, *The closed image of a metrizable space is M_1*, Proc. Amer. Math. Soc. **37** (1973), 309–314.

[SN] F. Siwiec and J. Nagata, *A note on nets and metrization*, Proc. Japan Acad. **44** (1968), 623–627.

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