

# CLASSICAL SOLUTIONS OF THE HAMILTON-JACOBI-BELLMAN EQUATION FOR UNIFORMLY ELLIPTIC OPERATORS

BY

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**ABSTRACT.** We prove under appropriate hypotheses that the Hamilton-Jacobi-Bellman dynamic programming equation with uniformly elliptic operators,  $\max_{1 \leq k \leq m} \{L^k u - f^k\} = 0$ , has a classical solution  $u \in C^{2,\beta}$ , for some (small) Hölder exponent  $\beta > 0$ .

**1. Introduction.** This paper extends some methods developed in [4] to prove the existence and uniqueness of a classical solution  $u$  of the *Hamilton-Jacobi-Bellman equation*:

$$(1.1) \quad \begin{cases} \max_{1 \leq k \leq m} \{L^k u(x) - f^k(x)\} = 0, & x \in \Omega, \\ u(x) = 0 & x \in \partial\Omega. \end{cases}$$

Here  $\Omega$  denotes a bounded, smooth domain in  $\mathbf{R}^n$  ( $n \geq 1$ ), the  $f^k$  are given smooth functions, and the  $L^k$  are second order, uniformly elliptic operators of the form

$$(1.2) \quad L^k u \equiv a_{ij}^k(x) u_{x_i x_j} + b_i^k(x) u_{x_i} + c^k(x) u,$$

with smooth coefficients satisfying

$$(1.3) \quad \begin{cases} c^k(x) \leq 0, & x \in \bar{\Omega}, \\ \theta |\xi|^2 \leq a_{ij}^k(x) \xi_i \xi_j \leq \Theta |\xi|^2, & x \in \bar{\Omega}, \xi \in \mathbf{R}^n, \end{cases}$$

for constants  $0 < \theta \leq \Theta < \infty$ ,  $k = 1, \dots, m$ .

Problem (1.1) is the *equation of dynamic programming* arising in the study of the optimal control of certain diffusion processes. For an explanation of its formal derivation from control theory principles see Krylov [9], Bensoussan-J. L. Lions [2], Fleming-Rishel [7], or Kushner [13]. In this interpretation  $L^k$  describes in some sense the response of the stochastic system under control to the constant control  $k$ ,  $f^k$  denotes the running cost of control  $k$ , and  $u(x)$  is the value function, i.e., the (expected) payoff for the optimally controlled system starting in state  $x$ .

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The references just mentioned all provide calculations showing that should the value function  $u$  be smooth enough (say  $C^2$ ), then it satisfies (1.1); and—conversely—if (1.1) has a smooth enough solution it must be the value function. This second fact motivates a direct study of (1.1), in analogy with the indirect methods of the calculus of variations. Many papers in the last few years have made rigorous the formal calculations of dynamic programming theory by demonstrating under assorted hypotheses the existence of a (unique) solution with various regularity properties: see Krylov [9–10], Nisio [19], Brezis-Evans [3], Evans-Friedman [5], P. L. Lions-Menaldi [18], P. L. Lions [16–17], Evans-P. L. Lions [6], Belbas [1], etc. Under the assumption (1.3) of uniform ellipticity the best results to date were in [6], where we proved (1.1) has a unique solution  $u \in W^{2,\infty}(\Omega)$ , solving (1.1) for a.e.  $x \in \Omega$ .

In this work we will show that the solution of (1.1) is in fact a classical solution, lying in the space  $C_{\text{loc}}^{2,\beta}$  for some perhaps very small Hölder exponent  $\beta > 0$ . (This was proved in [3] for the case  $m = 2$  by other, much simpler methods.) The main ideas for the proof are taken from a recent paper [4], where we in fact already obtained this result for the (unrealistic) situation that the operators  $L^k$  have constant coefficients, with no lower order terms. The key observation there was that any pure second derivative  $v \equiv u_{\xi\xi}$  ( $\xi$  a unit vector) is a *supersolution* of a certain linear elliptic p.d.e., to which some recent estimates of Krylov-Safonov [11] apply. In the general situation that the coefficients of the  $L^k$  depend on  $x$ ,  $v = u_{\xi\xi}$  is unfortunately no longer a supersolution.

Happily it turns out, as we will show in this paper, that a relatively small modification of  $v = u_{\xi\xi}$  in fact does act like a supersolution; this technique we borrow from the proof of Ladyženskaja-Ural'ceva [14, p. 340] (see also Gilbarg-Trudinger [8, p. 270]) concerning  $C^{1,\alpha}$  estimates for quasilinear elliptic equations. With these calculations in hand it is then relatively routine to modify the methods of [4] to obtain an interior *a priori* estimate on the Hölder modulus of continuity of  $D^2u$ .

And once we have such bounds it is then straightforward to prove:

**THEOREM 1.** *Under the hypotheses described above (1.1) has a unique solution  $u \in C(\bar{\Omega}) \cap C^{2,\beta}(\Omega)$  for some  $\beta > 0$ .*

Note carefully we do not assert  $u \in C^{2,\beta}(\bar{\Omega})$ . Indeed it is an interesting open problem to extend the techniques of [4] to yield boundary estimates for the Hölder continuity of  $D^2u$ . Such bounds would also be useful for studying the Monge-Ampere equation.

The paper is organized so that the proof of Theorem 1 appears in §4, after a derivation in §§2,3 of the key estimate (Lemma 3.3) concerning the oscillation of  $D^2u$  on concentric balls in  $\Omega$ . As noted previously the principal new ideas concern adapting the methods of [4] to cover the variable coefficient case: as such we will devote most of our attention to the new essential difficulties and will continually refer the reader back to [4] for those aspects of the proof which are not much changed. (See the “Note added in proof” at the end of §4.)

*Notation.*

$$\begin{aligned} Du &\equiv (u_{x_1}, \dots, u_{x_n}), \quad D^2u \equiv (u_{x_1x_1}, \dots, u_{x_ix_j}, \dots), \\ D^3u &\equiv (\dots, u_{x_ix_jx_k}, \dots), \\ B(x_0, R) &\equiv \{y \in \mathbf{R}^n \mid |x_0 - y| \leq R\}, \quad B(R) = B(0, R), \\ e_i &\equiv i\text{th unit vector in } \mathbf{R}^n, \\ [f]_{C^\beta(K)} &\equiv \max_{\substack{x, y \in K \\ x \neq y}} \frac{|f(x) - f(y)|}{|x - y|^\beta}. \end{aligned}$$

We employ the summation convention throughout this paper; the letter "C" denotes various constants depending only on known quantities.

**2. Approximation; construction of supersolutions.** The key to everything else in the proof of Theorem 1 is an *a priori* bound on the oscillation of  $D^2u$  in  $\Omega'$ ,  $\Omega' \subset \subset \Omega$ . We present in this section the necessary major modifications of the argument in [4], and will in fact derive the estimates for a sequence of approximating problems defined as follows.

According to [4, Lemma 7.3] there exist smooth functions  $\beta^\varepsilon: \mathbf{R}^m \rightarrow \mathbf{R}$  ( $\varepsilon > 0$ ) satisfying

$$(2.1) \quad \begin{cases} \text{(i)} & \beta^\varepsilon(\cdot) \text{ is smooth and convex } (\varepsilon > 0), \\ \text{(ii)} & 0 \leq \beta_{r_i}^\varepsilon(r) \leq 1, \quad \sum_{i=1}^m \beta_{r_i}^\varepsilon(r) = 1 \quad (\varepsilon > 0, i = 1, \dots, m, r \in \mathbf{R}^m), \\ \text{(iii)} & \beta^\varepsilon(r) = \beta^\varepsilon(r_1, \dots, r_m) \rightarrow \max_{1 \leq k \leq m} r_k \text{ as } \varepsilon \searrow 0, \\ & \text{uniformly on compact subsets of } \mathbf{R}^m, \\ \text{(iv)} & \beta^\varepsilon(r_0, \dots, r_0) = r_0 \text{ for each } r_0 \in \mathbf{R}^1. \end{cases}$$

Let us therefore approximate (1.1) by the problem

$$(2.2)_\varepsilon \quad \begin{cases} \beta^\varepsilon(L^1u^\varepsilon(x) - f^1(x), \dots, L^mu^\varepsilon(x) - f^m(x)) = 0, & x \in \Omega, \\ u^\varepsilon(x) = 0, & x \in \partial\Omega. \end{cases}$$

For the remainder of this section we will suppose that  $(2.2)_\varepsilon$  has a smooth solution  $u^\varepsilon$  and then derive estimates on the oscillation of  $D^2u^\varepsilon$ , provided

$$(2.3) \quad \max_{x \in \Omega} |u^\varepsilon|, |Du^\varepsilon|, |D^2u^\varepsilon| \leq M$$

for some constant  $M$ , independent of  $\varepsilon$ . Later, in §4, we will make use of these bounds to prove the existence of a solution of  $(2.2)_\varepsilon$  and then to pass to limits as  $\varepsilon \searrow 0$  to solve (1.1).

So now let  $\xi$  denote some unit vector in  $\mathbf{R}^n$  and set

$$(2.4) \quad v^\varepsilon \equiv u_{\xi\xi}^\varepsilon - \eta \sum_{i,j=1}^n \left( u_{x_ix_j}^\varepsilon - n^2 M \delta_{ij} \right)^2,$$

$\eta$  denoting here some fixed positive number, which will be selected later (in §3).

Define also

$$(2.5) \quad a_{ij}^\varepsilon(x) \equiv \beta_{r_i}^\varepsilon(\cdots) a_{ij}^k(x) \quad (1 \leq i, j \leq n),$$

and notice that according to (2.1)(ii) and (1.3),

$$(2.6) \quad \theta|\xi|^2 \leq a_{ij}^\varepsilon(x) \xi_i \xi_j \leq \Theta|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^n, x \in \bar{\Omega}.$$

LEMMA 2.1. *There exists a constant  $C(\eta)$  such that*

$$-a_{kl}^\varepsilon v_{x_k x_l}^\varepsilon \geq -C(\eta) \quad \text{in } \Omega;$$

$C(\eta)$  depends only on  $\eta$ ,  $M$ ,  $n$ , the coefficients of the  $L^k$ , the  $f^k$ , and not on  $\varepsilon$  or  $\xi$ .

PROOF. To simplify the notation we drop the superscript “ $\varepsilon$ ”. Let us compute

$$(2.7) \quad \begin{aligned} v_{x_k} &= u_{\xi \xi x_k} - 2\eta(u_{x_i x_j} - n^2 M \delta_{ij}) u_{x_i x_j x_k}, \\ v_{x_k x_l} &= u_{\xi \xi x_k x_l} - 2\eta u_{x_i x_j x_l} u_{x_i x_j x_k} - 2\eta(u_{x_i x_j} - n^2 M \delta_{ij}) u_{x_i x_j x_k x_l}. \end{aligned}$$

Next differentiate (2.2) <sub>$\varepsilon$</sub>  with respect to  $x_i$  and  $x_j$ , and also twice with respect to  $\xi$ :

$$(2.8) \quad \begin{aligned} \beta_{r_s}(\cdot)(L^s u - f^s)_{x_i x_j} + \beta_{r_s r_i}(\cdot)(L^s u - f^s)_{x_i}(L^t u - f^t)_{x_j} &= 0, \\ \beta_{r_s}(\cdot)(L^s u - f^s)_{\xi \xi} + \beta_{r_s r_i}(\cdot)(L^s u - f^s)_\xi(L^t u - f^t)_\xi &= 0. \end{aligned}$$

In view of the convexity of  $\beta$ , the last equation implies

$$(2.9) \quad -\beta_{r_s}(\cdot)(L^s u - f^s)_{\xi \xi} \geq 0.$$

Now observe that (2.3), (2.8) and (2.9) yield

$$(2.10) \quad -\beta_{r_s}(\cdot) a_{kl}^s u_{x_i x_j x_k x_l} \geq \beta_{r_s r_i}(\cdot)(L^s u - f^s)_{x_i}(L^t u - f^t)_{x_j} - C - C|D^3 u|,$$

and

$$(2.11) \quad -\beta_{r_s}(\cdot) a_{kl}^s u_{\xi \xi x_k x_l} \geq -C - C|D^3 u|.$$

Finally let us calculate

$$(2.12) \quad \begin{aligned} -a_{kl} v_{x_k x_l} &= -\beta_{r_s}(\cdot) a_{kl}^s v_{x_k x_l} = -\beta_{r_s}(\cdot) a_{kl}^s u_{\xi \xi x_k x_l} + 2\eta \beta_{r_s}(\cdot) a_{kl}^s u_{x_i x_j x_l} u_{x_i x_j x_k} \\ &\quad - 2\eta(u_{x_i x_j} - n^2 M \delta_{ij})(-\beta_{r_s}(\cdot) a_{kl}^s u_{x_i x_j x_k x_l}) \\ &\geq -C - C|D^3 u| + 2\eta \theta |D^3 u|^2; \end{aligned}$$

the last estimate results from (2.11), (2.6) and (2.10), respectively. Notice also that we have used here the inequality

$$-2\eta(u_{x_i x_j} - n^2 M \delta_{ij}) \beta_{r_s r_i}(\cdot)(L^s u - f^s)_{x_i}(L^t u - f^t)_{x_j} \geq 0,$$

resulting from the convexity of  $\beta$  and the matrix inequality

$$\left( (u_{x_i x_j} - n^2 M \delta_{ij}) \right) \leq 0.$$

The lemma is an immediate consequence of (2.12).  $\square$

Lemma 2.1 will be useful because of the following important estimate of Krylov-Safonov [11]. Let us assume now that the ball  $B(R_0)$  lies in  $\Omega$  for some  $R_0 > 0$ .

Define for  $0 < R < R_0$

$$m^\varepsilon(R) \equiv \min_{B(R)} v^\varepsilon, \quad M^\varepsilon(R) \equiv \max_{B(R)} v^\varepsilon, \quad \text{osc}_{B(R)} v^\varepsilon \equiv M^\varepsilon(R) - m^\varepsilon(R).$$

LEMMA 2.2. *There exist a constant  $C_1(\eta) = C_1(n, \theta, \Theta, M, C(\eta))$  and, for each  $0 < \alpha < 1$ , a constant  $\gamma = \gamma(n, \theta, \Theta, \alpha) > 0$  such that*

(a) *if  $\Gamma \subset B(R)$  is closed and*

$$\text{meas } \Gamma \geq \alpha \text{ meas } B(R),$$

*then*

$$C_1(\eta)R^2 + m^\varepsilon\left(\frac{R}{4}\right) \geq \gamma \min_{\Gamma} v^\varepsilon + (1 - \gamma)m^\varepsilon(R).$$

(b) *In particular,*

$$\text{meas}\{x \in B(R) | v^\varepsilon \geq \lambda m^\varepsilon(R) + (1 - \lambda)M^\varepsilon(R)\} \geq \alpha \text{ meas } B(R)$$

*implies*

$$C_1(\eta)R^2 + m^\varepsilon\left(\frac{R}{4}\right) \geq \gamma(1 - \lambda) \text{osc}_{B(R)} v^\varepsilon + m^\varepsilon(R)$$

*for all  $0 < \lambda < 1$ . The various constants here do not depend on  $\varepsilon, \xi$  or  $R$ , and only  $C_1$  depends on  $\eta$ .*

This lemma follows from Lemma 2.1 and [11]: see [4, §3] for details.

**3. The main estimate.** We are now ready to prove

PROPOSITION 3.1. *Let  $u^\varepsilon$  be a smooth solution of (2.2) <sub>$\varepsilon$</sub> , and assume that the estimate (2.3) holds. Then there exist a constant  $\beta > 0$  and, for each  $\Omega' \subset \subset \Omega$ , a constant  $C(\Omega')$  such that*

$$[D^2 u^\varepsilon]_{C^\beta(\bar{\Omega}')} \leq C(\Omega').$$

*The constant  $\beta$  depends on  $n, \theta, \Theta$ , the coefficients of the operators  $L^k$ , and the  $f^k$ ;  $C(\Omega')$  depends only on these quantities and  $\text{dist}(\Omega', \partial\Omega)$ : neither depends on  $\varepsilon$ .*

The proof will follow several preparatory lemmas:

LEMMA 3.2. *There exist an integer  $N \geq n$ , unit vectors  $\xi_i (i = 1, \dots, N)$  and constants  $0 < \theta' \leq \Theta' < \infty$  such that*

$$(3.1) \quad \beta_\varepsilon(L^1 u^\varepsilon - f^1, \dots, L^m u^\varepsilon - f^m) \equiv G^\varepsilon(u_{\xi_1 \xi_1}^\varepsilon, \dots, u_{\xi_N \xi_N}^\varepsilon, x) = 0$$

*where  $G^\varepsilon: \mathbf{R}^n \times \Omega \rightarrow \mathbf{R}$  is a smooth function, with*

$$(3.2) \quad \theta' \leq G_k^\varepsilon(\cdot) \leq \Theta' \quad (k = 1, 2, \dots, N).$$

This lemma follows from considerations in [4, §5]. In particular we demonstrated there (following Kurtz [12]) that each operator  $L^k$  can be rewritten in the form

$$(3.3) \quad L^k u^\varepsilon \equiv \sum_{i=1}^N a_i^k(x) u_{\xi_i \xi_i}^\varepsilon + \sum_{i=1}^n b_i^k(x) u_{x_i}^\varepsilon + c^k(x) u^\varepsilon(x)$$

where the  $\{\xi_i\}$  are some collection (*independent of  $k$* ) of unit vectors in  $\mathbf{R}^n$ , and the  $a_i^k(x)$  are smooth functions,

$$\theta' \leq a_i^k(x) \leq \Theta' \quad (i = 1, \dots, N; k = 1, \dots, m; x \in \Omega).$$

Now insert the representation (3.3) into the  $\beta_\epsilon$ , recall (2.1)(ii) to check (3.2), and note that the lower order terms in (3.3) are Lipschitz according to (2.3).  $\square$

Let us also note that we may arrange things so that the collection of unit vectors  $\{\xi_i\}$  includes

$$(3.4) \quad f_{ij}^\pm = \frac{e_i \pm e_j}{\sqrt{2}} \quad (1 \leq i, j \leq n).$$

To see this fix  $\delta > 0$  so small that

$$\tilde{L}^k u \equiv L^k u - \delta \sum_{i,j=1}^n (u_{f_{ij}^+, f_{ij}^+} + u_{f_{ij}^-, f_{ij}^-})$$

is uniformly elliptic and then apply the representation (3.3) to  $\tilde{L}^k$  instead of  $L^k$ .

In light of all this there is a constant  $C_2 = C_2(n, m, N, M)$  such that if  $B(R_0) \subset \Omega$  for some  $R_0 > 0$ , then

$$(3.5) \quad \text{osc}_{B(R)} \sum_{i,j=1}^n (u_{x_i x_j}^\epsilon - n^2 M \delta_{ij})^2 \leq C_2 \max_{1 \leq i \leq N} \text{osc}_{B(R)} u_{\xi_i}^\epsilon$$

for each  $0 < R < R_0$ .

Then (3.5) implies

$$(3.6) \quad \max_{1 \leq i \leq N} \text{osc}_{B(R)} u_{\xi_i}^\epsilon \leq C_3 \max_{1 \leq i \leq N} \text{osc}_{B(R)} v_i^\epsilon$$

for some  $C_3 = C_3(C_2, n, m, N)$ , provided  $\eta > 0$  is small enough; here

$$(3.7) \quad v_i^\epsilon \equiv u_{\xi_i}^\epsilon - \eta \sum_{i,j=1}^n (u_{x_i x_j}^\epsilon - n^2 M \delta_{ij})^2.$$

Our principal estimate is this (cf. [4, Lemma 4.3]):

LEMMA 3.3. *Assume that the ball  $B(R_0) \subset \Omega$  for some  $R_0 > 0$ . If  $\eta > 0$  is small enough, there exist constants  $\delta_1 < 1$  and  $C_4$  such that if  $0 < R < R_0$ , then*

$$(3.8) \quad \text{osc}_{B(R/4)} v_i^\epsilon \leq \delta_1 \text{osc}_{B(R)} v_i^\epsilon + C_4 R$$

for some index  $i \in \{1, \dots, N\}$ . These constants do not depend on  $R$  or  $\epsilon$ .

PROOF. As before we suppress the superscript “ $\epsilon$ ” for notational simplicity;

$$m_i(R) \equiv \min_{B(R)} v_i, \quad M_i(R) \equiv \max_{B(R)} v_i.$$

There are two possibilities to consider:

Case I. Suppose

$$\text{meas}\{x \in B(R) | v_i \geq \lambda m_i(R) + (1 - \lambda) M_i(R)\} \geq \text{meas } B(R)/2N$$

for some index  $i \in \{1, \dots, N\}$ , say  $i = 1$ , where  $0 < \lambda < 1$  is a constant to be selected later. Then applying Lemma 2.2(b) to  $v_1$  we find

$$C_1(\eta)R^2 + m_1\left(\frac{R}{4}\right) \geq \gamma(1 - \lambda) \operatorname{osc}_{B(R)} v_1 + m_1(R)$$

for some fixed  $\gamma > 0$ . Thus

$$\begin{aligned} \operatorname{osc}_{B(R/4)} v_1 &= M_1\left(\frac{R}{4}\right) - m_1\left(\frac{R}{4}\right) \leq M_1(R) - m_1\left(\frac{R}{4}\right) \\ &\leq (1 - \gamma(1 - \lambda)) \operatorname{osc}_{B(R)} v_1 + C_1(\eta)R^2. \end{aligned}$$

This proves (3.8) with  $\delta_1 = (1 - \gamma(1 - \lambda)) < 1$ .

Case II. Assume now that (3.9) fails for each  $i = 1, 2, \dots, N$ . Then

$$(3.9) \quad \operatorname{meas} \Gamma_1 \geq \frac{1}{2} \operatorname{meas} B(R),$$

where

$$(3.10) \quad \Gamma_1 \equiv \{x \in B(R) | v_i < \lambda m_i(R) + (1 - \lambda)M_i(R), i = 1, \dots, N\}.$$

Let us suppose with no loss of generality

$$(3.11) \quad \operatorname{osc}_{B(R/4)} v_N = \max_{1 \leq i \leq N} \operatorname{osc}_{B(R/4)} v_i.$$

Now (3.1) and (3.2) together imply that we can solve for  $u_{\xi_N \xi_N}$  in terms of the other derivatives and  $x$ :

$$u_{\xi_N \xi_N} = g(u_{\xi_1 \xi_1}, \dots, u_{\xi_{N-1} \xi_{N-1}}, x),$$

where  $g: \mathbf{R}^{N-1} \times \Omega \rightarrow \mathbf{R}$  is smooth,

$$-\frac{\Theta'}{\theta'} \leq g_k \equiv -\frac{G_k}{G_n} \leq -\frac{\theta'}{\Theta'} \quad (k = 1, 2, \dots, N-1).$$

The function  $g$  is thus strictly *decreasing* in its first  $N-1$  arguments; and so for  $x \in \Gamma_1$  we may estimate

$$\begin{aligned} u_{\xi_N \xi_N}(x) &= g(u_{\xi_1 \xi_1}(x), \dots, u_{\xi_{N-1} \xi_{N-1}}(x), x) \\ &= g(v_1(x) + \eta w(x), \dots, v_{N-1}(x) + \eta w(x), x), \end{aligned}$$

where

$$(3.12) \quad w \equiv \sum_{i,j=1}^n (u_{x_i x_j} - n^2 M \delta_{ij})^2.$$

Hence, since  $x \in \Gamma_1$ ,

$$\begin{aligned} u_{\xi_N \xi_N}(x) &\geq g(\dots, \lambda m_i(R) + (1 - \lambda)M_i(R) + \eta w(x), \dots, x) \\ &\geq g\left(\dots, \lambda \min_{B(R)} u_{\xi_i \xi_i} + (1 - \lambda) \max_{B(R)} u_{\xi_i \xi_i}, \dots, x\right) - C\eta \operatorname{osc}_{B(R)} w, \end{aligned}$$

because

$$\left| \lambda m_i(R) + (1 - \lambda)M_i(R) + \eta w(x) - \left( \lambda \min_{B(R)} u_{\xi_i \xi_i} + (1 - \lambda) \max_{B(R)} u_{\xi_i \xi_i} \right) \right| \leq \eta \operatorname{osc}_{B(R)} w.$$

Therefore

$$\begin{aligned} u_{\xi_N \xi_N}(x) &\geq g\left(\dots, \min_{B(R)} u_{\xi_i \xi_i}, \dots, x\right) - C(\eta + (1 - \lambda)) \max_{1 \leq i \leq N} \operatorname{osc}_{B(R)} u_{\xi_i \xi_i} \quad (\text{see (3.5)}) \\ &\geq \max_{B(R)} u_{\xi_N \xi_N} - CR - C(\eta + (1 - \lambda)) \max_{1 \leq i \leq N} \operatorname{osc}_{B(R)} v_i \end{aligned}$$

by (3.6).

Hence for  $x \in \Gamma_1$ ,

$$\begin{aligned} v_N(x) &= u_{\xi_N \xi_N}(x) - \eta w(x) \\ &\geq M_N(R) - CR - C(\eta + (1 - \lambda)) \max_{1 \leq i \leq N} \operatorname{osc}_{B(R)} v_i. \end{aligned}$$

Next apply Lemma 2.2(a) to  $v_N$ :

$$\begin{aligned} C(\eta)R^2 + m_N\left(\frac{R}{4}\right) &\geq \gamma\left(M_N(R) - CR - C(\eta + (1 - \lambda)) \max_{1 \leq i \leq N} \operatorname{osc}_{B(R)} v_i\right) \\ &\quad + (1 - \gamma)m_N(R). \end{aligned}$$

From this we calculate

$$\begin{aligned} \operatorname{osc}_{B(R/4)} v_N &= M_N\left(\frac{R}{4}\right) - m_N\left(\frac{R}{4}\right) \\ &\leq (1 - \gamma + C(\eta + (1 - \lambda))) \max_{1 \leq i \leq N} \operatorname{osc}_{B(R)} v_i + CR + C(\eta)R^2 \\ &\leq \delta_1 \max_{1 \leq i \leq N} \operatorname{osc}_{B(R)} v_i + C(\eta)R, \end{aligned}$$

with  $\delta_1 < 1$ , provided  $\lambda < 1$  is now fixed close enough to 1 and  $\eta$  is small enough.

In view of (3.11), (3.8) is now proved for Case II.  $\square$

The next lemma extends Lemma 3.3 by showing that if we pass from  $B(R)$  to a (much) smaller ball, the sum of the oscillations of all the  $v^\epsilon$  diminishes by a fraction less than one, except for a correction term of order  $R$ .

LEMMA 3.4. *There exist an integer  $k_N$  and constants  $\delta_N < 1$ ,  $C_5$  such that*

$$(3.13) \quad \sum_{i=1}^N \operatorname{osc}_{B(R/4^{k_N})} v_i^\epsilon \leq \delta_N \sum_{i=1}^N \operatorname{osc}_{B(R)} v_i^\epsilon + C_5 R.$$

*These constants do not depend on  $R$  or  $\epsilon$ .*

PROOF. The argument is very similar to the proofs of [4, Lemmas 4.4, 4.5] and so we only outline here the principal ideas. Once more the superscripts “ $\epsilon$ ” are omitted.

First, owing to Lemma 3.3, the oscillation of at least one of the  $v_i$ , say  $v_1$ , diminishes by a fraction  $\delta_1 < 1$  (and there is a correction term  $CR$ ) if we pass from  $B(R)$  to  $B(R/4)$ :

$$(3.14) \quad \operatorname{osc}_{B(R/4)} v_1 \leq \delta_1 \operatorname{osc}_{B(R)} v_1 + CR.$$



Now consider  $B(R/4)$  and  $B(R/4^2)$ . Again the oscillation of at least one of the  $v_j$  decreases by the fraction  $\delta_1$ , and so there are two possibilities:

(a) If we can take  $j \in \{2, \dots, N\}$ , say  $j = 2$ , then a statement analogous to (3.14) obtains for  $v_2$  and we may add this inequality to (3.14) to obtain an inequality of the type (3.13), but with the summation only from 1 to 2.

(b) If on the other hand the oscillation of only  $v_1$  diminishes in moving from  $B(R/4)$  to  $B(R/4^2)$ , we may combine (3.14) and the same statement for the smaller ball to find

$$(3.15) \quad \operatorname{osc}_{B(R/4^2)} v_1 \leq (\delta_1)^2 \operatorname{osc}_{B(R)} v_1 + CR.$$

Now consider  $B(R/4^2)$  and  $B(R/4^3)$ . If case (a) holds here we have (3.13) for  $v_1$  and  $v_2$ ; and case (b) on the other hand implies

$$(3.16) \quad \operatorname{osc}_{B(R/4^3)} v_1 \leq (\delta_1)^3 \operatorname{osc}_{B(R)} v_1 + CR.$$

Continuing this way, after finitely many steps we reach a ball  $B(R/4^{k_2-1})$  by which either (3.13) obtains for  $N = 2$  or else inequalities like (3.15), (3.16) are valid, the latter possibility implying the oscillation of  $v_1$  over  $B(R/4^{k_2-1})$  is so negligible that an argument like that establishing Lemma 3.3 shows at the next step that the oscillation of some *other*  $v_i$  ( $i \in \{2, \dots, N\}$ ) must diminish. See [4, Lemma 4.4] for details.

Having so far proved (3.13) with 2 in place of  $N$ , we continue the process by induction eventually to arrive at (3.13) for  $N$  itself: again we refer to [4, Lemma 4.5] for more explanation.  $\square$

**PROOF OF PROPOSITION 3.1.** A standard lemma in elliptic theory (cf. Gilbarg-Trudinger [8, p. 191]) states that Lemma 3.4 implies

$$\operatorname{osc}_{B(R)} v_\varepsilon^i \leq C \left( \frac{R}{R_0} \right)^\beta \quad (i = 1, \dots, N)$$

for certain numbers  $C > 0$ ,  $0 < \beta < 1$ ; these constants depend only on known quantities. We apply this calculation to any ball  $B(x_0, R_0)$ ,  $x_0 \in \bar{\Omega}' \subset \subset \Omega$ ,  $R_0 < \operatorname{dist}(x_0, \partial\Omega)$  to complete the proof.  $\square$

**4. Proof of Theorem 1.** It is now fairly easy to exploit the *a priori* estimates from §3 in proving an existence, regularity, and uniqueness theorem for the Hamilton-Jacobi-Bellman equation. First we solve (2.2) $_\varepsilon$  and will then pass to limits as  $\varepsilon \searrow 0$ .

As previously noted we do not have *a priori* Hölder estimates for  $D^2u$  near  $\partial\Omega$ , and so first we must modify the operators  $L^k$  to be identical near the boundary.

Define for  $\delta > 0$

$$\Omega_\delta \equiv \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > \delta\},$$

and next choose a smooth cutoff function  $\zeta = \zeta_\delta$  such that

$$(4.1) \quad 0 \leq \zeta \leq 1, \quad \zeta \equiv 1 \text{ on } \Omega_\delta, \quad \zeta \equiv 0 \text{ near } \partial\Omega.$$

Now define

$$(4.2) \quad L_\delta^k u \equiv (1 - \zeta)\theta\Delta u + \zeta L^k u \quad (k = 1, 2, \dots, m);$$

these operators are uniformly elliptic, with smooth coefficients. We also set

$$f_\delta^k = \zeta f^k \quad (k = 1, 2, \dots, m).$$

In view of (4.1) and (2.1)(iv)

$$\beta^\varepsilon(\dots, L^k u - f_\delta^k, \dots) = \theta \Delta u \quad \text{for } x \text{ near } \partial\Omega.$$

Hence Proposition 3.1 and the standard boundary Schauder estimates provide the *a priori* estimate

$$(4.3) \quad [D^2 u_\delta^\varepsilon]_{C^0(\bar{\Omega})} \leq \text{Const.}$$

(for some  $\beta > 0$ ), where  $u_\delta^\varepsilon$  is a smooth solution of

$$(4.4)_{\varepsilon, \delta} \quad \begin{cases} \beta^\varepsilon(\dots, L^k u_\delta^\varepsilon - f_\delta^k, \dots) = 0 & \text{in } \Omega, \\ u_\delta^\varepsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

The bound in (4.3) requires a previous estimate (cf. (2.3))

$$(4.5) \quad \max_{x \in \Omega} |u_\delta^\varepsilon|, |\dot{D}u_\delta^\varepsilon|, |D^2 u_\delta^\varepsilon| \leq M_\delta,$$

but this follows from fairly straightforward modifications of the techniques in Evans-Lions [6].

It is now routine to prove the existence of a unique, smooth solution of  $(4.4)_{\varepsilon, \delta}$  using a standard continuation argument. The solution belongs to  $C^{2, \beta}(\bar{\Omega})$  and so customary bootstrap-type arguments demonstrate that  $u_\varepsilon^\delta$  is in fact smooth.

Next we send  $\delta \searrow 0$  to solve  $(2.2)_\varepsilon$ . For this it suffices to note that

$$(4.6) \quad \beta^\varepsilon(\dots, L^k u_\varepsilon^\delta - f^k, \dots) = 0 \quad \text{in } \Omega_\delta.$$

Now an estimate like that used for [4, Lemma 2.2] proves

$$(4.7) \quad \max_\Omega |u_\varepsilon^\delta| \leq C_6,$$

the constant independent of  $\varepsilon$  and  $\delta$ . From this bound and interior estimates (cf. Lenhart [15]) of the form

$$(4.8) \quad \max_{\Omega'} |Du_\varepsilon^\delta|, |D^2 u_\varepsilon^\delta| \leq C_7(\Omega')$$

(for  $\Omega' \subset \subset \Omega_\delta \subset \subset \Omega$ ;  $C_7(\Omega')$  independent of  $\varepsilon, \delta$ ), we may recall Proposition 3.1 and extract a subsequence of the  $u_\varepsilon^\delta$  converging in  $C^{2, \alpha}(\Omega)$  ( $0 < \alpha < \beta$ ) as  $\delta \searrow 0$  to a function  $u^\varepsilon$  solving

$$\beta_\varepsilon(\dots, L^k u^\varepsilon - f^k, \dots) = 0 \quad \text{in } \Omega.$$

Furthermore

$$u^\varepsilon \in C(\bar{\Omega}), \quad u^\varepsilon = 0 \quad \text{on } \partial\Omega;$$

this follows since we can find (as in [4, Lemma 2.2]) a smooth barrier function  $\Phi$  such that

$$(4.9) \quad \begin{cases} -\Phi \leq u_\varepsilon^\delta \leq \Phi & \text{in } \Omega, \\ \Phi = 0 & \text{on } \partial\Omega, \end{cases}$$

for all  $\varepsilon, \delta$ . Hence  $u^\varepsilon$  solves  $(2.2)_\varepsilon$  and—from (4.7)–(4.9)—we have

$$\begin{cases} \max_{\Omega} |u^\varepsilon| \leq C_6, \\ \max_{\Omega'} |Du^\varepsilon|, |D^2 u^\varepsilon| \leq C_7(\Omega'), \\ -\Phi \leq u^\varepsilon \leq \Phi \quad \text{in } \Omega. \end{cases}$$

Recalling Proposition 3.1 once more we pass to limits as  $\varepsilon \searrow 0$  to construct a solution  $u \in C(\bar{\Omega}) \cap C^{2,\beta}(\Omega)$  of the Hamilton-Jacobi-Bellman equation. Uniqueness is a consequence of the maximum principle.  $\square$

NOTE ADDED IN PROOF. N. S. Trudinger has recently greatly simplified the proofs in this paper and in [4]: see the forthcoming second edition of [8] for this.

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