THE INDEX OF HARMONIC FOLIATIONS ON SPHERES[1]

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ABSTRACT. For foliations on a compact oriented manifold there is a natural energy functional, defined with respect to a Riemannian metric. Harmonic Riemannian foliations are then the critical foliations for this functional under an appropriate class of special variations. The index of the title is the index of the Hessian of the energy functional at a critical, i.e., harmonic foliation. It is a finite number. In this note it is shown that for a harmonic Riemannian foliation $\mathcal{F}$ of codimension q on the n-sphere ($n > 2$) this index is greater or equal to $q + 1$. Thus $\mathcal{F}$ is unstable. Moreover the given bound is best possible.

Introduction. For foliations on a compact oriented manifold M there is a natural energy functional, defined with respect to a Riemannian metric g_M on M [4, 5]. Harmonic Riemannian foliations are then the critical foliations for this functional under an appropriate class of so-called special variations, defined by sections of the normal bundle Q [4, 5]. For foliations defined by the inverse images of submersions, harmonic foliations correspond to harmonic maps in the sense of Eells and Sampson [2].

The Hessian of the energy functional can be evaluated by the following fundamental formula, which is proved in the appendix to [7].

SECOND VARIATION FORMULA. *Let M be a compact oriented manifold, and $\mathcal{F}$ a Riemannian and harmonic foliation with respect to a bundle-like metric g_M. Consider the 2-parameter family $\mathcal{F}_{s,t}$ of special variations of $\mathcal{F} = \mathcal{F}_{0,0}$ defined by two sections μ, ν of the normal bundle Q. Then for the second derivative of the energy we have*

$$\frac{\partial^2 E(\mathcal{F}_{s,t})}{\partial s \partial t}\bigg|_{s=0, t=0} = \langle (\Delta - \rho_\nabla)\mu, \nu \rangle.$$

The sections $\mu, \nu \in \Gamma Q$ define a 2-parameter special variation $\mathcal{F}_{s,t}$ of $\mathcal{F}_{0,0} = \mathcal{F}$ through Riemannian foliations by patching the local data

$$\Phi_{s,t}^\alpha(x) = \exp_{f^\alpha(x)}(s\mu^\alpha(x) + t\nu^\alpha(x)).$$

Here f^α is a local submersion defining $\mathcal{F}$ in an open set U_α. $\Phi_{s,t}^\alpha$ is then the local submersion defining $\mathcal{F}_{s,t}$ for $|s| \leqslant \varepsilon$, $|t| \leqslant \varepsilon$, $\varepsilon > 0$. The RHS of this formula denotes the endpoint of the geodesic segment starting at $f^\alpha(x)$ and determined by $s\mu^\alpha(x) + t\nu^\alpha(x)$, where $\mu^\alpha = \mu \,|\, U_\alpha$ and $\nu^\alpha = \nu \,|\, U_\alpha$.

To explain the RHS in the second variation formula, observe that the canonical connection ∇ in Q defines an exterior differential d_∇ and codifferential d_∇^* on Q-valued forms on M. Then $\Delta = d_\nabla^* d_\nabla$ is the Laplacian on the space of sections of Q, and the scalar product of sections $\mu, \nu \in \Gamma Q$ is given by

$$\langle \mu, \nu \rangle = \int_M g_Q(\mu, \nu) \cdot \eta_M$$

in terms of the canonical metric g_Q on Q and the volume form η_M associated to the metric g_M on M. The operator $\rho_\nabla \colon Q \to Q$ is the Ricci operator of the foliation, associated to the canonical metric and torsion free connection ∇ in Q (see [7] and §1 below).

This leads to the Jacobi operator of $\mathscr{F}$ [7] defined for $\mu \in \Gamma Q$ by

$$\mathcal{J}_\nabla \mu = (\Delta - \rho_\nabla)\mu.$$

It is selfadjoint and strongly elliptic, thus with real eigenvalues

$$\lambda_1 < \lambda_2 < \cdots < \lambda_i < \cdots \to +\infty \quad \text{for } i \to \infty.$$

The dimension of each eigenspace is finite, and the index of the foliation $\mathscr{F}$ occurring in the title is the sum of the dimensions of the eigenspaces corresponding to negative eigenvalues.

The purpose of this note is to prove the following result.

THEOREM. *Let $\mathscr{F}$ be a Riemannian and harmonic foliation on the sphere S^n ($n > 2$) for which the standard metric is bundle-like. Then* index$(\mathscr{F}) \geq q + 1$, *where q is the codimension of $\mathscr{F}$. Thus $\mathscr{F}$ is unstable.*

The case $q = 1$ does not occur for S^n, since a Riemannian foliation of codimension 1 is necessarily given by a closed nowhere zero 1-form [5, Proposition 3.14]. Thus $q \geq 2$ in the statement above. The case $q = n$ corresponds to the point foliation of S^n. Its index is the index of the identity map, which by Smith [9] equals $n + 1$. This shows that the estimate above is best possible.

The positivity of the index was suggested by the corresponding result for harmonic maps on the sphere by Xin [10]. The two points to consider for the proof are the evaluation of the Ricci operator ρ_∇ of a harmonic foliation via the Weitzenböck formula (§1), and the calculation of the Laplacian for sections of the normal bundle of $\mathscr{F}$ arising from the tangential projections of constant vector fields of $\mathbf{R}^{n+1}$ to S^n (§2). We thank Luc Lemaire for the suggestion of the precise estimate $q + 1$. In [3] the index of a harmonic map of maximal rank $q \geq 1$ on S^n ($n > 2$) is shown to be $\geq q + 1$.

For the harmonic Hopf foliation $S^1 \to S^{2n+1} \to P^n\mathbf{C}$ the bound under discussion is $2n + 1$. It is of interest to note that Duchamp and Kalka [1] have shown that the space of holomorphic foliations near the Hopf foliation, i.e. foliations given by local submersions into $\mathbf{C}^n$ patching together via local biholomorphic transition functions $\mathbf{C}^n \to \mathbf{C}^n$, is already parametrized by a space of complex dimension $(n + 1)^2$.

1. The Ricci operator of a harmonic foliation.

Let $\mathscr{F}$ be a Riemannian foliation on M given by $L \subset TM$. There is a unique metric and torsion free connection ∇ in the normal bundle $Q = TM/L$ [5, Theorem 1.11]. Its curvature R_∇ is an End(Q)-valued

2-form on M. Since $i(X)R_\nabla = 0$ for $X \in \Gamma L$, it follows that $R_\nabla(\mu, \nu)$: $Q \to Q$ for $\mu, \nu \in \Gamma Q$ is a well-defined operator. Define $P_\nabla(\mu, \nu)$: $Q \to Q$ by

$$(1.1) \qquad P_\nabla(\mu, \nu)s = -R_\nabla(\mu, s)\nu$$

for $s \in \Gamma Q$. The Ricci curvature of $\mathcal{F}$ is then

$$(1.2) \qquad S_\nabla(\mu, \nu) = \operatorname{Trace} P_\nabla(\mu, \nu)$$

which is symmetric and bilinear. The Ricci operator $\rho_\nabla \colon Q \to Q$ is the corresponding selfadjoint operator given by

$$(1.3) \qquad g_Q(\rho_\nabla \mu, \nu) = S_\nabla(\mu, \nu)$$

where g_Q denotes the holonomy invariant metric in Q. In terms of an orthonormal basis $e_{p+1}, \ldots, e_n$ of Q_x at some $x \in M^n$ we have

$$(1.4) \qquad (\rho_\nabla \mu)_x = \sum_{\alpha=p+1}^{n} R_\nabla(\mu, e_\alpha)e_\alpha.$$

For more details see [7]. The underlying principle is that the usual curvature properties of a Riemannian manifold are well defined for a Riemannian foliation $\mathcal{F}$. They can be thought of as the corresponding curvature properties of a Riemannian manifold serving as model space for $\mathcal{F}$.

A metric g_M together with the connection ∇ gives rise to the usual calculus for Q-valued forms on M. In particular there are exterior differentials and codifferentials

$$d_\nabla \colon \Omega^r(M, Q) \to \Omega^{r+1}(M, Q), \quad r \geqslant 0,$$
$$d_\nabla^* \colon \Omega^r(M, Q) \to \Omega^{r-1}(M, Q), \quad r > 0,$$

and the Laplacian $\Delta = d_\nabla d_\nabla^* + d_\nabla^* d_\nabla$. This applies to the canonical projection π: $TM \to Q$ viewed as $\pi \in \Omega^1(M, Q)$. In the following we assume g_M to be bundle-like for $\mathcal{F}$, i.e. inducing g_Q on Q. This defines a canonical metric splitting $\sigma \colon Q \to TM$ of π, so that $\sigma Q \cong L^\perp$. On TM we consider the Riemannian connection ∇^M with curvature R_{∇^M}. At a point $x \in M$ we fix an orthonormal basis $e_1, \ldots, e_n$ of $T_x M$ with $e_1, \ldots, e_p$ in L_x and e_α ($\alpha = p+1, \ldots, n$) in $(\sigma Q)_x \cong Q_x$. Let $E_1, \ldots, E_n$ be a local framing of TM in a neighborhood of x coinciding with $e_1, \ldots, e_n$ at x and satisfying $\nabla_{e_i}^M E_j = 0$. (We neither claim nor require that $(E_i)_y \in L_y$ for $1 \leqslant i \leqslant p$ or $(E_\alpha)_y \in \sigma Q_y$ for $p+1 \leqslant \alpha \leqslant n$ at $y \neq x$. But clearly $(\pi E_i)_x = \pi e_i = 0$ for $1 \leqslant i \leqslant p$ and $(\pi E_\alpha)_x = e_\alpha$ for $p+1 \leqslant \alpha \leqslant n$.) The Weitzenböck formula evaluates the Laplacian $\Delta \pi \in \Omega'(M, Q)$ as follows (see e.g. [8 or 6]):

$$(1.5) \qquad (\Delta \pi)_x = -\sum_{i=1}^{n} \nabla_{e_i} \nabla_{E_i} \pi + S(\pi)_x$$

where for a vector field X

$$(1.6) \qquad S(\pi)_x(X) = \sum_{i=1}^{n} \left(R_\nabla(e_i, X)\pi(e_i) - \pi(R_{\nabla^M}(e_i, X)e_i) \right).$$

Only the value $X_x \in T_x M$ enters into this formula. With our choice of basis the first terms on the RHS disappear for $1 \leqslant i \leqslant p$. What remains is by (1.4) the term

$(-\rho_\nabla \pi(X))_x$. The second terms on the RHS add up to

$$(1.7) \qquad \pi\left(\sum_{i=1}^n R_{\nabla^M}(X, e_i)e_i\right) = \pi(\rho_{\nabla^M}(X))_x$$

where ρ_{∇^M} denotes the Ricci operator of g_M, resp. ∇^M. But for $M^n = S^n$ with its standard metric one has $\rho_{\nabla^S} = (n-1)\mathrm{Id}$: $TS^n \to TS^n$. Thus finally

$$(\Delta\pi)_x = -\sum_{i=1}^n \nabla_{e_i}\nabla_{E_i}\pi - (\rho_\nabla \circ \pi)_x + (n-1)\pi_x$$

as a map $T_xM \to Q_x$. The harmonicity of $\mathcal{F}$ is characterized by $\Delta\pi = 0$ [5, Theorem 3.3]. This yields the following result.

1.8 PROPOSITION. *Let $\mathcal{F}$ be a Riemannian and harmonic foliation on S^n for which the standard metric is bundle-like. Let π: $TM \to Q$ be the canonical projection onto the normal bundle. Let $x \in S^n$ and $E_1, \ldots, E_n$ be a local frame of TS^n on a neighborhood of x such that $(E_i)_x = e_i$ form an orthonormal frame of T_xS^n with $e_1, \ldots, e_p \in L_x$ and $e_{p+1}, \ldots, e_n \in \sigma Q_x \cong \sigma Q_x \cong Q_x$; and further $\nabla_{e_i}^M E_j = (\nabla_{E_i}^M E_j)_x = 0$. For the Ricci operator of $\mathcal{F}$ we have then*

$$(1.9) \qquad (\rho_\nabla \circ \pi)_x = -\sum_{i=1}^n \nabla_{e_i}\nabla_{E_i}\pi + (n-1)\pi_x.$$

2. The Laplacian for special sections of Q. Let $X \in \Gamma TM$ and $\pi(X) \in \Gamma Q$. With the notations of the preceding section

$$(2.1) \qquad (\Delta\pi(X))_x = (d_\nabla^* d_\nabla \pi(X))_x = -\sum_{i=1}^n \nabla_{e_i}(\nabla_{E_i}\pi(X)).$$

By the definition of the induced connections on Q-valued forms we have further

$$(\nabla_{E_j}\pi)(X) = \nabla_{E_j}\pi(X) - \pi(\nabla_{E_j}^M X)$$

so that

$$(\Delta\pi(X))_x = -\sum_{i=1}^n \nabla_{e_i}((\nabla_{E_i}\pi)(X) + \pi(\nabla_{E_i}^M X)).$$

Since

$$(\nabla_{e_i}(\nabla_{E_j}\pi))(x) = \nabla_{e_i}((\nabla_{E_j}\pi)(X)) - (\nabla_{E_j}\pi)(\nabla_{e_i}^M X)$$

we get

$$(2.2) \qquad (\Delta\pi(X))_x = -\sum_{i=1}^n (\nabla_{e_i}\nabla_{E_i}\pi)(X) - \sum_{i=1}^n (\nabla_{E_i}\pi)(\nabla_{e_i}^M X)$$
$$- \sum_{i=1}^n \nabla_{e_i}(\pi(\nabla_{E_i}^M X)).$$

Note that the first sum occurs already in the formula (1.9) for the Ricci operator of $\mathcal{F}$. These terms therefore drop out for the Jacobi operator $\mathcal{J}_\nabla = \Delta - \rho_\nabla$.

The next step is to specialize (2.2) for the vector fields $V^T \in \Gamma TS^n$ arising from constant vector fields V on $\mathbf{R}^{n+1}$ by tangential projection on TS_x^n at each point

$x \in S^n$. This means that for $x \in S^n$

(2.3)
$$V_x^T = V - \langle V, N \rangle_x \cdot N_x$$

where N_x is the normal vector field defined by $N_x = x$.

2.4 PROPOSITION. *Let the situation be as in 1.8. Then for $X = V^T \in \Gamma TS^n$ we have*

(2.5)
$$\left(\Delta \pi (V^T) \right)_x = - \sum_{i=1}^n \left(\nabla_{e_i} \nabla_{E_i} \pi \right) (V^T) + \pi_x (V^T).$$

PROOF. The connections ∇^R in $\mathbf{R}^{n+1}$ and ∇^S in S^n are related by the formula of Gauss

$$\nabla_{e_i}^R V^T = \nabla_{e_i}^S V^T + \alpha_x \left(e_i, V^T \right).$$

The second fundamental form α of $S^n \subset \mathbf{R}^{n+1}$ at x is easily evaluated to be $\alpha = -g_R \cdot N$, so that

$$\alpha_x \left(e_i, V^T \right) = -\langle e_i, V^T \rangle \cdot N_x = -\langle e_i, V \rangle \cdot N_x.$$

Using this and (2.3) one finds

$$\nabla_{e_i}^S V^T = \nabla_{e_i}^R (V - \langle V, N \rangle N) + \langle e_i, V \rangle \cdot N_x$$
$$= -\langle V, N \rangle_x \cdot \nabla_{e_i}^R N - e_i \langle V, N \rangle \cdot N_x + \langle e_i, V \rangle \cdot N_x.$$

Note that for $\langle V, N \rangle_x = f(x)$,

$$e_i \langle V, N \rangle = e_i f = df_x(e_i) = \langle V, e_i \rangle.$$

Further one verifies $\nabla_X^R N = X$ for every $X \in \Gamma TS^n$, which expresses the fact that the Weingarten map of $S^n \subset \mathbf{R}^{n+1}$ equals $-$Id: $TS^n \to TS^n$. In particular $\nabla_{e_i}^R N = e_i$ and thus

(2.6)
$$\nabla_{e_i}^S V^T = -\langle V, N \rangle_x e_i.$$

Since $\nabla_X^S V^T$ depends linearly on $X \in T_x S^n$, it follows more generally that

(2.7)
$$\nabla_X^S V^T = -\langle V, N \rangle X.$$

This formula implies further

$$\left(\nabla_{E_i} \pi \right) \left(\nabla_{E_i}^S V^T \right) = -\langle V, N \rangle \left(\nabla_{E_i} \pi \right) (E_i).$$

But

$$\left(\nabla_{E_i} \pi \right) (E_i) = \nabla_{E_i} \pi (E_i) - \pi \left(\nabla_{E_i}^S E_i \right).$$

For the framing chosen at x, the second term on the RHS vanishes at x. But this also holds for the value at x of the first term. Namely if $1 \leqslant i \leqslant p$, then by the definition of the connection ∇ in Q [5, 1.3]

$$\nabla_{e_i} \pi (E_i) = \pi [E_i, E_i]_x = 0.$$

If $p + 1 \leqslant \alpha \leqslant n$, then

$$\nabla_{e_\alpha} \pi (E_\alpha) = \pi \left(\nabla_{e_\alpha}^S E_\alpha \right)_x = 0.$$

Thus in any case

(2.8)
$$\left(\nabla_{E_i} \pi \right) \left(\nabla_{e_i}^S V^T \right) = 0.$$

Finally again by (2.7) and the results above

$$\nabla_{e_i}\!\left(\pi\!\left(\nabla^S_{E_i}V^T\right)\right) = \nabla_{e_i}\!\left(-\langle V, N\rangle\pi(E_i)\right)$$
$$= -\langle V, N\rangle_x \cdot \nabla_{e_i}\pi(E_i) - e_i\langle V, N\rangle \cdot \pi(E_i)_x$$
$$= -\langle V, e_i\rangle \cdot \pi(E_i)_x = -\pi(\langle V, e_i\rangle e_i).$$

Since clearly $V_x^T = \Sigma_{i=1}^m \langle V, e_i\rangle e_i$ this shows that

$$(2.9) \qquad\qquad \sum_{i=1}^{n} \nabla_{e_i}\!\left(\pi\!\left(\nabla^S_E V^T\right)\right) = -\pi_x(V^T).$$

Substituting (2.8) and (2.9) in (2.2) yields (2.5). □

3. The estimation of the index. Combining Propositions 1.8 and 2.4 we obtain for the Jacobi operator of a Riemannian and harmonic foliation $\mathcal{F}$ on S^n the formula

$$\mathcal{J}_{\nabla}\pi(V^T) \equiv (\Delta - \rho_{\nabla})\pi(V^T) = (2 - n)\pi(V^T).$$

For $n > 2$ every nontrivial section of Q of the form $\pi(V^T)$ for a constant vector field V of $\mathbf{R}^{n+1}$ thus gives a 1-dimensional contribution to index($\mathcal{F}$). It suffices to show that the space $\mathcal{U} \subset \Gamma Q$ thus obtained satisfies

$$(3.1) \qquad\qquad \dim \mathcal{U} \geqslant q + 1.$$

Let $\mathcal{V} \subset \Gamma TS^n$ be the space of tangent vector fields of the form V^T for $V \in \mathbf{R}^{n+1}$. Since $\mathbf{R}^{n+1} \to \mathcal{V}$ is clearly injective, $\dim \mathcal{V} = n + 1$. Note that every vector field V^T has exactly two zeros. Note further that the vector fields V^T span $T_x S^n$ for any $x \in S^n$. Under the projection $\pi\colon TS^n \to Q$ clearly

$$\begin{array}{ccc} \Gamma TS^n & \to & \Gamma Q \\ \cup & & \cup \\ \mathcal{V} & \to & \mathcal{U} \to 0 \end{array}$$

Since the sections of Q in $\mathcal{U}$ necessarily span Q_x for any $x \in S^n$, we have $\dim \mathcal{U} \geqslant q$.

It remains to show that the case $\dim \mathcal{U} = q$ is not possible. Namely assuming this, there is a basis $X_1, \ldots, X_{n+1}$ of $\mathcal{V}$ such that the vector fields $X_1, \ldots, X_{n+1-q}$ span the kernel of $\mathcal{V} \to \mathcal{U}$ and the remaining q vector fields map onto a basis $Y_1, \ldots, Y_q$ of $\mathcal{U}$. Since the $Y_1, \ldots, Y_q$ generate Q_x in every point $x \in S^n$ and $q = \dim Q_x$, none of the Y_j can ever vanish. But the corresponding X's in $\mathcal{V}$ are then a fortiori nonsingular vector fields on S^n. Since by construction all elements of $\mathcal{V}$ do have zeros, this contradiction eliminates the case $\dim \mathcal{U} = q$. Thus $\dim \mathcal{U} \geqslant q + 1$ which finishes the proof. □

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