

## CONVERGENCE ACCELERATION FOR CONTINUED FRACTIONS $K(a_n/1)$

BY

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**ABSTRACT.** A known method for convergence acceleration of limit periodic continued fractions  $K(a_n/1)$ ,  $a_n \rightarrow a$ , is to replace the approximants  $S_n(0)$  by "modified approximants"  $S_n(f^*)$ , where  $f^* = K(a/1)$ . The present paper extends this idea to a larger class of converging continued fractions. The "modified approximants" will then be  $S_n(f^{(n)'})$ , where  $K(a'_n/1)$  is a converging continued fraction whose tails  $f^{(n)'}$  are all known, and where  $a_n - a'_n \rightarrow 0$ .

As a measure for the improvement obtained by this method, upper bounds for the ratio of the two truncation errors are found.

**1. Introduction.** We are going to study continued fractions of the form

$$K\left(\frac{a_n}{1}\right) = \frac{a_1}{1 + \frac{a_2}{1 + \frac{a_3}{1 + \dots}}} = \frac{a_1}{1} + \frac{a_2}{1} + \dots + \frac{a_n}{1} + \dots$$

where  $0 \neq a_n \in \mathbb{C}$ . The  $a_n$ 's are called *elements*, and the numbers

$$(1.1) \quad f_m = \frac{m}{n=1} K(a_n/1) = \frac{a_1}{1} + \frac{a_2}{1} + \dots + \frac{a_m}{1}; \quad m \geq 1, \\ f_0 = 0,$$

are called the *approximants*. All  $f_m$  are well defined in the extended complex plane  $\hat{\mathbb{C}}$ , since all the elements are  $\neq 0$ . When we let  $S_m$  denote the linear fractional transformation

$$(1.2) \quad S_m(w) = \frac{a_1}{1} + \dots + \frac{a_m}{1+w},$$

the  $m$ th approximant is  $f_m = S_m(0)$ . The approximants are often written in the form  $f_m = A_m/B_m$ , where  $A_m$  and  $B_m$ , the  $m$ th numerator and denominator of the continued fraction, are given by

$$(1.3) \quad A_{-1} = 1, \quad A_0 = 0, \quad B_{-1} = 0, \quad B_0 = 1, \\ A_m = A_{m-1} + a_m A_{m-2}, \quad B_m = B_{m-1} + a_m B_{m-2}; \quad m \geq 1.$$

It is easy to prove (see for instance [7, Theorem 2.1, p. 20]) that

$$(1.4) \quad S_m(w) = \frac{A_m + A_{m-1}w}{B_m + B_{m-1}w}.$$

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$K(a_n/1)$  is said to converge when its sequence of approximants  $\{f_m\}_{m=0}^\infty$  converges, possibly to  $\infty$ . The value of  $K(a_n/1)$  then is  $f = K(a_n/1) = \lim_{m \rightarrow \infty} f_m$ . In some special cases this value is easy to determine, for instance in the periodic case or for some continued fraction expansions of known functions. But in most cases one has to compute the value numerically; i.e. to compute  $S_n(0)$  for a sufficiently large  $n$ . (Here truncation error estimates are needed; see for instance [7, Chapter 8, p. 297].)

In 1873, Glaisher [4] found that for a certain continued fraction  $K(a_n/1)$  where  $a_n \rightarrow \infty$ , the sequence  $\{S_n(x_n)\}$  converges much faster to the value  $f$  of  $K(a_n/1)$  than  $\{S_n(0)\}$ , for an appropriate choice of  $\{x_n\}$ . In 1958, Wynn [13] proved that the same applies for all convergent continued fractions  $K(a_n/1)$  where  $a_n \rightarrow \infty$ . Hayden [5] presented in 1965 methods for finding appropriate  $\{x_n\}$  in some of these cases.

Beginning in 1973, Gill [1, 2, 3] studied, among other things, the convergence of limit periodic continued fractions  $K(a_n/1)$ ,  $a_n \rightarrow a$ . He observed that  $\{S_n(x_1)\}$ , where  $x_1 = K(a/1)$ , in some cases converges faster to  $f = K(a_n/1)$  than  $\{S_n(0)\}$ . In 1978, Thron and Waadeland [12] proved that

$$\lim_{n \rightarrow \infty} \left| \frac{f - S_n(x_1)}{f - S_n(0)} \right| = 0$$

for any continued fraction  $K(a_n/1)$  where  $a_n \rightarrow a \in \mathbb{C} \setminus (-\infty, -\frac{1}{4}]$ , and for some  $K(a_n/1)$  where  $a_n \rightarrow -\frac{1}{4}$ . They also quantified the improvement by deriving upper bounds for  $|f - S_n(x_1)|/|f - S_n(0)|$ .

In the present paper we extend this method to other types of continued fractions. To apply the method to a continued fraction  $K(a_n/1)$ , we have to know another convergent continued fraction  $K(a'_n/1)$  such that  $(a_n - a'_n) \rightarrow 0$ , and such that all of the values  $f^{(n)'} = K_{\nu=n+1}^\infty(a'_\nu/1)$ ,  $n = 0, 1, 2, \dots$ , are known.

This is for instance the case when  $K(a'_n/1)$  is  $k$ -periodic or the  $C$ -fraction expansion of the ratio of certain hypergeometric functions.  $K_{\nu=n+1}^\infty(a'_\nu/1)$  is called the  $n$ th tail of a continued fraction  $K(a_n/1)$ , and  $f^{(n)}$  is the value of the  $n$ th tail (when it is converging).

In §2 of this paper, we will find sufficient conditions for

$$(1.5) \quad \lim_{n \rightarrow \infty} \left| \frac{f - S_n(f^{(n)'})}{f - S_n(0)} \right| = 0.$$

In §4, upper bounds for this ratio are found for a given  $n$ .

The following example may serve as an illustration of what to expect by this method.

EXAMPLE 1.1. The 5-periodic continued fraction given by

$$K\left(\frac{a'_n}{1}\right) = \frac{8}{1} + \frac{12}{1} + \frac{8}{1} + \frac{6}{1} + \frac{11}{1} + \dots$$

has the following tail values

$$\begin{aligned} f^{(5n)'} &= (\sqrt{11257} - 35)/33, & f^{(5n+1)'} &= (\sqrt{11257} - 3)/38, \\ f^{(5n+2)'} &= (3\sqrt{11257} - 65)/74, & f^{(5n+3)'} &= (3\sqrt{11257} - 99)/164, \\ f^{(5n+4)'} &= (\sqrt{11257} + 2)/31 & \text{for } n &= 0, 1, 2, 3, \dots \end{aligned}$$

This continued fraction may be used as a tool for improving the convergence of the continued fraction  $K(a_n/1) = K((a'_n + 0.3^n)/1)$ . Table 1.1 shows the improvement obtained by using the modified approximants  $S_n(f^{(n)'})$ .

TABLE 1.1

| $n$ | $S_n(0)$            | $S_n(f^{(n)'})$     |
|-----|---------------------|---------------------|
| 1   | 8.3                 | 2.23                |
| 2   | 0.6                 | 2.223               |
| 5   | 2.8                 | 2.2265              |
| 10  | 2.13                | 2.22649360          |
| 20  | 2.2236              | 2.22649361320389    |
| 24  | 2.225               | 2.22649361320389408 |
| 50  | 2.2264935           |                     |
| 75  | 2.22649361321       |                     |
| 100 | 2.226493613203892   |                     |
| 117 | 2.22649361320389408 |                     |

We find that  $S_n(f^{(n)'})$  gives the value of  $f$  with 8 significant digits when  $n = 10$ , and with 18 significant digits when  $n = 24$ , whereas the same accuracy in  $S_n(0)$  requires  $n > 50$  and  $n = 117$ , respectively.

(This computation was done on UNIVAC 1108, RUNIT, The University of Trondheim.)

The conditions for obtaining (1.5) will turn out to be intimately connected with certain regions related to continued fractions:

DEFINITION 1.1.  $\{E_n\}_{n=1}^\infty$  is a sequence of element regions and  $\{V_n\}_{n=0}^\infty$  is a corresponding sequence of value regions if:

- (i)  $\{0\}, \emptyset \neq E_n \subseteq \mathbb{C}$  for  $n = 1, 2, 3, \dots$ ,
- (ii)  $V_n \subseteq \hat{\mathbb{C}}$  (the extended complex plane) for  $n = 0, 1, 2, \dots$ ,
- (iii)  $E_n \subseteq V_{n-1}$  for  $n = 1, 2, 3, \dots$ ,
- (iv)  $E_n/(1 + V_n) \subseteq V_{n-1}$  for  $n = 1, 2, 3, \dots$ , and
- (v)  $V_n$  is closed for  $n = 0, 1, 2, \dots$

This and the following Definitions 1.2, 1.3 and 1.4 are given in accordance with [7, p. 64] which also points out the fact that the term region is used loosely to mean any subset of  $\mathbb{C}$  or  $\hat{\mathbb{C}}$  respectively. The only difference is that condition (v) in Definition 1.1 is not included in [7].

DEFINITION 1.2. A sequence  $\{E_n\}$  of element regions is called a sequence of convergence regions if every continued fraction  $K(a_n/1)$  with  $0 \neq a_n \in E_n$  for  $n = 1, 2, 3, \dots$  converges.

In particular, if  $E_n = E$  for  $n = 1, 2, 3, \dots$  we say that  $E$  is a simple convergence region; and similarly, if  $E_{2n+1} = E_1$ ,  $E_{2n} = E_2$  for  $n = 1, 2, 3, \dots$ , we say that  $\langle E_1, E_2 \rangle$  is a set of twin convergence regions.

DEFINITION 1.3.  $\{E_n\}_{n=1}^\infty$  is called a uniform sequence of convergence regions (u.s.c.), if there exists a sequence  $\{\lambda_n\}_{n=1}^\infty$  of positive numbers converging to 0, such that if  $0 \neq a_n \in E_n$  for  $n = 1, 2, 3, \dots$  then  $|K(a_n/1) - f_k| \leq \lambda_k$  for  $k = 1, 2, 3, \dots$

The sequence  $\{\lambda_n\}$  will be called a *sequence of positive numbers corresponding to the u.s.c.  $\{E_n\}$* .

DEFINITION 1.4.  $\{V_n\}_{n=0}^\infty$  is called the *best sequence of value regions corresponding to a sequence  $\{E_n\}$  of element regions*, if  $\{V_n\}$  is a sequence of value regions corresponding to  $\{E_n\}$ , and if for every sequence  $\{V'_n\}$  of value regions corresponding to  $\{E_n\}$ ,  $V_n \subseteq V'_n$  for  $n = 0, 1, 2, \dots$ .

It is easy to prove [7, Theorem 4.1, p. 65] that, corresponding to a sequence  $\{E_n\}$  of element regions, the best sequence of value regions is given by:

$$(1.6) \quad V_n = c \left( \left\{ f_m^{(n)} = \frac{n+m}{K} (a_\nu/1); m = 1, 2, 3, \dots, a_\nu \in E_\nu \text{ for all } \nu > n \right\} \right) \\ \text{for } n = 0, 1, 2, \dots,$$

where  $c(A)$  means the closure of a set  $A$ .

In §3, certain properties of u.s.c.'s will be investigated. The ratio  $B_n/B_{n-1}$  ( $B_n$  defined by (1.3)) frequently occurs in the theory of continued fractions. It is denoted by  $h_n$  (see for instance [7]), and a simple induction argument gives that

$$(1.7) \quad h_n = \frac{B_n}{B_{n-1}} = \begin{cases} 0 & \text{for } n = \infty, \\ 1 & \text{for } n = 1, \\ 1 + \frac{a_n}{1} + \dots + \frac{a_2}{1} & \text{for } n \geq 2. \end{cases}$$

At the end of this section, some formulas for later reference are stated:

$$(1.8) \quad f_m^{(n)} = \frac{a_{n+1}}{1 + f_{m-1}^{(n+1)}}, \quad f^{(n)} = \frac{a_{n+1}}{1 + f^{(n+1)}} \\ \text{for } n = 0, 1, 2, \dots, m = 1, 2, 3, \dots,$$

where  $f_0^{(n)} = 0$  for  $n = 0, 1, 2, \dots$

$$(1.9) \quad B_n + B_{n-1}f^{(n)} = \prod_{k=1}^n (1 + f^{(k)}) \quad \text{for } n = 0, 1, 2, \dots,$$

$$(1.10) \quad A_n - f^{(0)}B_n = (-1)^{n-1} \prod_{k=0}^n f^{(k)} \quad \text{for } n = 0, 1, 2, \dots$$

Equations (1.8) are trivial, (1.9) and (1.10) are proved easily by induction.

**2. Sufficient conditions for  $\lim_{n \rightarrow \infty} (f - S_n(f^{(n')})) / (f - S_n(0)) = 0$ .** As described in the introduction,  $K(a_n/1)$  is a convergent continued fraction whose value  $f$  we want to compute, whereas  $K(a'_n/1)$  is a convergent continued fraction whose tails  $f^{(n')}$  are all known. In this section we shall establish sufficient conditions for  $\{S_n(f^{(n')})\}$  to converge to  $f$  substantially faster than  $\{S_n(0)\}$  in the sense of (1.5). Since  $f = K(a_n/1) = S_n(f^{(n)})$ , it is no surprise that the condition  $f^{(n)} - f^{(n')} \rightarrow 0$  turns out to be part of the sufficient conditions.

For a sequence  $\{w_n\}$  from  $\hat{\mathbf{C}}$ , straightforward use of (1.4) gives

$$|f - S_n(w_n)| = |S_n(f^{(n)}) - S_n(w_n)| = \frac{|A_{n-1}B_n - A_nB_{n-1}| |f^{(n)} - w_n|}{|B_n + B_{n-1}f^{(n)}| |B_n + B_{n-1}w_n|}.$$

This leads to the following:

$$(2.1) \quad \left| \frac{f - S_n(f^{(n)'})}{f - S_n(0)} \right| = \left| \frac{B_n}{B_n + B_{n-1}f^{(n)'}} \right| \left| \frac{f^{(n)} - f^{(n)'}}{f^{(n)}} \right|.$$

Looking for conditions to ensure (1.5), we see at once, by (2.1), that requiring

$$(2.2) \quad \frac{|B_n|}{|B_n + B_{n-1}f^{(n)'}| |f^{(n)}|} \leq M \quad \text{for } n = 1, 2, 3, \dots \text{ and some } M > 0,$$

and

$$(2.3) \quad \lim_{n \rightarrow \infty} |f^{(n)} - f^{(n)'}| = 0$$

is sufficient. This is exactly the idea that leads to Theorem 2.5. But before stating this theorem, we introduce some new concepts and find conditions for (2.3) (Theorem 2.4).

**DEFINITION 2.1.**  $\{E_n\}_{n=1}^\infty$  is called a *totally uniform sequence of convergence regions* (t.u.s.c.) if

(i)  $\{E_n^{(p)}\}_{n=1}^\infty$  where  $E_n^{(p)} = E_{n+p}$  for  $n = 1, 2, 3, \dots$ , is a u.s.c. for every nonnegative integer  $p$ , and

(ii) there exists a sequence  $\{\lambda_n\}_{n=1}^\infty$  of positive numbers converging to 0, such that for every given  $p \geq 0$ ,  $a_n \in E_n^{(p)}$  for  $n = 1, 2, 3, \dots$  implies that the continued fraction  $K(a_n/1)$  converges in such a way that  $|f - f_n| \leq \lambda_n$  for  $n = 1, 2, 3, \dots$ .

A sequence  $\{\lambda_n\}$  as described in (ii) will be called a *sequence of positive numbers corresponding to the t.u.s.c.  $\{E_n\}$* .

The concept of a t.u.s.c. will be used in Theorem 2.4. A simple uniform convergence region is easily seen to be a t.u.s.c. (with  $E_n = E$  for all  $n$ ). Other examples will be given in the next section as part of a general discussion of the concept u.s.c.

**DEFINITION 2.2.**  $\{E_n\}_{n=1}^\infty$  is called a *sequence of CA-regions* (CA-sequence) if there exists an  $N \in \mathbb{N}$  such that the following two conditions are satisfied:

(i)  $\{E_n^{(N)}\}$  is a t.u.s.c., and  
(ii)  $0 \notin c(\cup_{n=N}^\infty (V_n + W_n))$  where  $\{V_n\}$  is the best sequence of value regions corresponding to  $\{E_n\}$ , and

$$(2.4) \quad W_n = c(\{h_n \in \hat{\mathbb{C}}; a_k \in E_k \text{ for } k = 2, 3, \dots, n\}) \quad \text{for } n = 1, 2, \dots$$

CA stands for convergence acceleration. ( $h_n$  in (2.4) is defined by (1.7).)

In particular, when  $E_n = E_1$  for  $n = 1, 2, 3, \dots$ , we say that  $E_1$  is a *simple CA-region*, and when  $E_{2n-1} = E_1$ ,  $E_{2n} = E_2$  for  $n = 1, 2, 3, \dots$ , we say that  $\langle E_1, E_2 \rangle$  is a *set of twin CA-regions*. CA-sequences play a vital part in the conditions of Theorem 2.5, the main theorem of this section; examples of such sequences will be given in the next section. We mention here that only slight modifications are needed, to turn a simple uniform convergence region into a simple CA-region.

At present we shall need some properties of t.u.s.c.'s. Therefore we state the following proposition, although the proof is postponed to the next section.

**PROPOSITION 2.3.** Let  $\{E_n\}$  be a t.u.s.c. and  $\{V_n\}$  be the best sequence of value regions corresponding to  $\{E_n\}$ . Then

- (i)  $\bigcup_{n=1}^{\infty} E_n$  is bounded if and only if  $\bigcup_{n=0}^{\infty} V_n$  is bounded.
- (ii) If there exists an  $\varepsilon > 0$  such that  $E_n \setminus \{z; |z| < \varepsilon\} \neq \emptyset$  for all  $n$ , then
  - ( $\alpha$ )  $\bigcup_{n=1}^{\infty} E_n$  is bounded, and
  - ( $\beta$ )  $\bigcup_{n=1}^{\infty} V_n$  and  $\bigcup_{n=2}^{\infty} E_n$  have positive distances from  $-1$ .

We shall use this proposition in the next theorem:

**THEOREM 2.4.** Let  $\{E_n\}$  be a t.u.s.c. such that  $E_n \setminus \{z; |z| < \mu\} \neq \emptyset$  for all  $n$  and some  $\mu > 0$ . Furthermore let  $K(a_n/1)$  and  $K(a'_n/1)$  be continued fractions such that  $a_n, a'_n \in E_n$  for  $n = 1, 2, 3, \dots$ . Then  $\lim_{n \rightarrow \infty} (f^{(n)} - f^{(n)'}) = 0$  if and only if  $\lim_{n \rightarrow \infty} (a_n - a'_n) = 0$ , where  $f^{(n)}$  and  $f^{(n)'}$  are the values of the  $n$ th tails of  $K(a_n/1)$  and  $K(a'_n/1)$  respectively.

**PROOF.** We first prove the following statement: If  $\lim_{n \rightarrow \infty} (a_n - a'_n) = 0$ , then  $\lim_{n \rightarrow \infty} (f_m^{(n)} - f_m^{(n)'}) = 0$  for any  $m \in \mathbb{N}$ . ( $f_m^{(n)}$  and  $f_m^{(n)'}$  denote the  $m$ th approximants of the  $n$ th tails of  $K(a_n/1)$  and  $K(a'_n/1)$  respectively.)

If  $m = 1$  then  $f_1^{(n)} - f_1^{(n)'} = a_{n+1} - a'_{n+1}$ .

Hence, the statement is true for  $m = 1$ .

Suppose it is true for  $m = \nu - 1$  where  $\nu \geq 2$ .

Then, since, by use of (1.8),

$$f_\nu^{(n)} - f_\nu^{(n)'} = \frac{a_{n+1} - a'_{n+1} - f_\nu^{(n)'}(f_{\nu-1}^{(n+1)} - f_{\nu-1}^{(n+1)'})}{1 + f_{\nu-1}^{(n+1)}},$$

we get the following inequality:

$$|f_\nu^{(n)} - f_\nu^{(n)'}| \leq \frac{|a_{n+1} - a'_{n+1}| + |f_\nu^{(n)'}| |f_{\nu-1}^{(n+1)} - f_{\nu-1}^{(n+1)'}|}{|1 + f_{\nu-1}^{(n+1)}|}.$$

Since by Proposition 2.3(ii),  $|1 + f_{\nu-1}^{(n+1)}| \geq \delta$  and  $|f_\nu^{(n)'}| \leq K$  for all  $\nu \geq 2, n \geq 0$ , for some  $\delta > 0$  and  $K < \infty$ , this proves the statement for  $m = \nu$ . Induction completes the proof of the statement.

Now it is easy to prove the theorem. Let  $\{\lambda_n\}$  be a sequence of positive numbers corresponding to the t.u.s.c.  $\{E_n\}$ , and let  $\varepsilon > 0$  be arbitrarily chosen. Since  $\lambda_n \rightarrow 0$ , there exists an  $m_0 \in \mathbb{N}$  such that  $\lambda_m \leq \varepsilon/3$  for every  $m \geq m_0$ . Choose a fixed  $m \geq m_0$ . Then  $|f^{(n)} - f_m^{(n)}| \leq \varepsilon/3$  and  $|f^{(n)'} - f_m^{(n)'}| \leq \varepsilon/3$  for every  $n \geq 0$ , because the tails  $K_{\nu=n+1}^\infty(a_\nu/1)$  and  $K_{\nu=n+1}^\infty(a'_\nu/1)$  may be regarded as continued fractions  $K_{\nu=1}^\infty(C_\nu/1)$  and  $K_{\nu=1}^\infty(C'_\nu/1)$  where the elements

$$C_\nu = a_{\nu+n} \in E_\nu^{(n)} = E_{n+\nu}, \quad C'_\nu = a'_{\nu+n} \in E_\nu^{(n)} = E_{n+\nu} \quad \text{for } \nu = 1, 2, 3, \dots$$

Now suppose  $\lim_{n \rightarrow \infty} (a_n - a'_n) = 0$ . Then we have proved that

$$\lim_{n \rightarrow \infty} |f_m^{(n)} - f_m^{(n)'}| = 0.$$

Hence there exists an  $n_0 \in \mathbb{N}$  such that  $|f_m^{(n)} - f_m^{(n)'}| \leq \varepsilon/3$  for every  $n \geq n_0$ . Since

$$|f^{(n)} - f^{(n)'}| \leq |f^{(n)} - f_m^{(n)}| + |f_m^{(n)} - f_m^{(n)'}| + |f_m^{(n)'} - f^{(n)'}| \leq \varepsilon \quad \text{when } n \geq n_0,$$

we then have  $\lim_{n \rightarrow \infty} (f^{(n)} - f^{(n)'}) = 0$ .

Conversely, suppose  $\lim_{n \rightarrow \infty} (f^{(n)} - f^{(n')}) = 0$ . Since, by (1.8),

$$a_{n+1} - a'_{n+1} = (f^{(n)} - f^{(n')})(1 + f^{(n+1)}) + f^{(n')}(f^{(n+1)} - f^{(n+1)'})$$

where  $\{f^{(n+1)}\}$  and  $\{f^{(n')}\}$  are bounded, we must have  $\lim_{n \rightarrow \infty} (a_n - a'_n) = 0$ .  $\square$

By this theorem, we have conditions for (2.3). Hence, we only have to find conditions for (2.2) to ensure (1.5), the ultimate goal of this section.

**THEOREM 2.5.** *Let  $K(a_n/1)$  and  $K(a'_n/1)$  be two convergent continued fractions such that  $\lim_{n \rightarrow \infty} (a_n - a'_n) = 0$ . Furthermore let  $f$  be the finite value of  $K(a_n/1)$  and  $\{f^{(n')}\}_{n=0}^\infty$  be the values of the tails of  $K(a'_n/1)$ . If  $\{a_n\}$  has no limit point at 0, and there exists a CA-sequence  $\{E_n\}$  such that  $a_n, a'_n \in E_n$  for  $n = 1, 2, 3, \dots$ , then*

$$\lim_{n \rightarrow \infty} \left| \frac{f - S_n(f^{(n')})}{f - S_n(0)} \right| = 0.$$

**PROOF.** By (2.1) and (1.7) we get

$$(2.5) \quad \left| \frac{f - S_n(f^{(n')})}{f - S_n(0)} \right| = \frac{|h_n|}{|h_n + f^{(n')}|} \cdot \frac{|f^{(n)} - f^{(n')}|}{|f^{(n)}|} \quad \text{for } n = 1, 2, 3, \dots$$

Let  $\{E_n\}$  be a CA-sequence such that  $a_n, a'_n \in E_n$  for  $n = 1, 2, 3, \dots$ ,  $\{V_n\}$  be the best sequence of value regions corresponding to  $\{E_n\}$ , and  $\{W_n\}$  be defined by (2.4). Since  $h_n \in W_n$  and  $h_n + f^{(n')} \in W_n + V_n$ , the first factor on the right side of (2.5) is bounded for  $n \geq N$  (this follows from the definition of CA-sequences). Besides, since  $|a_n| \geq \varepsilon$  for all  $n \geq 0$  for some  $\varepsilon > 0$  (since  $\{a_n\}$  has no limit point in 0) and

$$f^{(n)} = a_{n+1} / (1 + f^{(n+1)})$$

where  $|1 + f^{(n+1)}| \leq M$  for all  $n \geq N$  and some positive constant  $M$  (by Proposition 2.3),  $|f^{(n)}| \geq \varepsilon/M > 0$  for all  $n \geq N$ . Hence, the theorem follows by use of Theorem 2.4.  $\square$

With this theorem we have found sufficient conditions for the modified approximants  $\{S_n(f^{(n')})\}$  to yield a substantial convergence acceleration relative to the approximants  $f_n = S_n(0)$  of  $K(a_n/1)$  in the sense of (1.5).

**REMARKS.** (1) It is sufficient to require  $a_n, a'_n \in E_n$  for  $n \geq 2$  in the theorem since  $h_n$  is independent of  $a_1$ . But if that is satisfied, then  $\{E_n^*\}$ , where  $E_1^* = E_1 \cup \{a_1, a'_1\}$  and  $E_n^* = E_n$  for  $n \geq 2$ , will also be a CA-sequence.

(2) The conditions of Theorem 2.5 were obtained by requiring (2.2) and (2.3). But obviously one might allow  $\{f^{(n)}\}$  or  $\{h_n + f^{(n')}\}$  to have limit points at 0, provided that  $(f^{(n)} - f^{(n')})$  converges so fast to 0, that the whole expression still converges to 0.

(3) The applicability of Theorem 2.5 is dependent upon easy ways to prescribe CA-sequences. This subject will be handled in §3.

(4) The usefulness of Theorem 2.5 also depends on how good the improvement will be by this method. §4 gives upper bounds for (2.1) as a measure for this improvement.

**3. Properties of certain sequences of convergence regions.** The purpose of this section is to study the various types of element regions occurring in §2. U.s.c.'s are treated in Proposition 3.1, t.u.s.c.'s were treated in Proposition 2.3 which will be proved in this section, and some CA-sequences are handled in Proposition 3.2. Some examples of t.u.s.c.'s and CA-sequences will also be given.

**PROPOSITION 3.1.** *Let  $\{E_n\}$  be a sequence of element regions and  $\{V_n\}$  be the best sequence of value regions corresponding to  $\{E_n\}$ . Then*

(i) *the following two statements are equivalent for any  $k \in \mathbb{N}$ :*

(A):  $E_k$  is bounded and  $-1 \notin V_k$ .

(B):  $V_{k-1}$  is bounded.

(ii) *If  $V_{k-1}$  is bounded,  $k \in \mathbb{N}$ , then  $E_{k+1}$  has a positive distance from  $-1$ .*

(iii) *If  $\{E_n\}$  is a u.s.c., then  $V_0$  is bounded.*

(iv) *If  $\{E_n\}$  is a u.s.c., then*

( $\alpha$ )  $\{E_n^{(-1)}\}_{n=1}^\infty$  is a u.s.c. if and only if  $-1 \notin V_0$ ,

( $\beta$ )  $\{E_n^{(1)}\}_{n=1}^\infty$  is a u.s.c. if and only if  $V_1$  is bounded.

(Here  $E_n^{(p)} = E_{n+p}$  for  $n = 1, 2, 3, \dots$ ,  $p = \pm 1$ , where  $E_0$  may be any bounded subset of  $\mathbb{C}$  such that  $E_0 \setminus \{0\} \neq \emptyset$ .)

**PROOF.** (i) Let  $k \in \mathbb{N}$  be arbitrarily chosen and  $\{V_n^*\}$  be given by

$$(3.1) \quad V_n^* = \left\{ \sum_{\nu=n+1}^{n+m} K (a_\nu/1); m \geq 1, a_\nu \in E_\nu \text{ for all } \nu > n \right\} \quad \text{for } n = 0, 1, 2, \dots$$

Then, by (1.6),  $V_n = c(V_n^*)$ , and besides

$$(3.2) \quad V_{k-1}^* = E_k \cup E_k / (1 + V_k^*)$$

where  $V_k^*$  is independent of  $E_k$ . Since  $E_k \setminus \{0\} \neq \emptyset$ , it follows that (A)  $\Leftrightarrow$  (B).

(ii) Follows from (i) since  $E_{k+1} \subseteq V_k$  and  $V_k$  is a closed set.

(iii) Suppose  $\{E_n\}$  is a u.s.c. Let  $\{\lambda_n\}$  be a sequence of positive numbers corresponding to  $\{E_n\}$ . Then

$$|f - f_n| = \left| \frac{a_1}{1 + f^{(1)}} - \frac{a_1}{1 + f_{n-1}^{(1)}} \right| = \frac{|a_1| |f^{(1)} - f_{n-1}^{(1)}|}{|1 + f^{(1)}| |1 + f_{n-1}^{(1)}|} \leq \lambda_n \quad \text{for } n \geq 2$$

for any continued fraction  $K(a_n/1)$  where  $a_n \in E_n$ ;  $n = 1, 2, 3, \dots$ . Since  $f^{(1)}$  and  $f_{n-1}^{(1)}$  are independent of  $a_1$ , we have the following two possibilities:

(a)  $|1 + f^{(1)}| = \infty$  for every continued fraction where  $a_n \in E_n$  for all  $n$ . Then  $f = 0$  for every such continued fraction, and  $|f_n| \leq \lambda_n$ , hence,  $V_0$  is bounded.

(b) There exists a continued fraction  $K(a_n/1)$  where  $|1 + f^{(1)}| < \infty$ . Then  $E_1$  must be bounded. (Otherwise  $f^{(1)} - f_{n-1}^{(1)} = 0$  for all  $n \geq 2$  which is impossible because  $a_n \neq 0$  for all  $n \geq 1$ .) To see that  $V_0$  is bounded, we observe that

$$\begin{aligned} |f - f_1| &= |f - a_1| \leq \lambda_1 \Rightarrow |f| \leq \lambda_1 + |a_1| \leq L + M, \\ |f - f_n| &\leq \lambda_n \Rightarrow |f_n| \leq \lambda_n + |f| \leq 2L + M, \end{aligned}$$

where  $L = \sup_{n \geq 1} \lambda_n < \infty$  and  $M = \sup_{z \in E_1} |z| < \infty$ . So, by (1.6)  $V_0$  is bounded.

(iv) Suppose  $\{E_n\}$  is a u.s.c. Let  $\{\lambda_n\}$  be a sequence of positive numbers corresponding to this u.s.c., let  $E_0$  be an arbitrarily chosen, bounded subset of  $\mathbb{C}$ , and

let  $V_{-1}$  denote the initial region in the best sequence of value regions corresponding to  $\{E_n^{(-1)}\}$ .

( $\alpha$ ): If  $\{E_n^{(-1)}\}$  is a u.s.c. then  $-1 \notin V_0$  by (i) and (iii). Suppose  $-1 \notin V_0$ . Then for any continued fraction  $K(a_n/1)$  where  $a_n \in E_n^{(-1)}$  for  $n = 1, 2, 3, \dots$ , we must have

$$|f - f_1| \leq |f| + |a_1| \leq C + M,$$

$$|f - f_n| = \frac{|a_1| |f^{(1)} - f_{n-1}^{(1)}|}{|1 + f^{(1)}| |1 + f_{n-1}^{(1)}|} \leq \frac{M \lambda_{n-1}}{\delta^2} \quad \text{for } n = 2, 3, 4, \dots,$$

where  $C = \sup_{z \in V_{-1}} |z| < \infty$  because of (i),

$$M = \sup_{z \in E_0} |z| < \infty \quad \text{and} \quad \delta = \inf_{z \in V_0} |1 + z| > 0.$$

Hence, by defining  $\lambda_1^{(-1)} = C + M$  and  $\lambda_n^{(-1)} = M \lambda_{n-1} / \delta^2$  for  $n = 2, 3, 4, \dots$ , we have found a sequence  $\{\lambda_n^{(-1)}\}$  of positive numbers corresponding to  $\{E_n^{(-1)}\}$ , and therefore  $\{E_n^{(-1)}\}$  is a u.s.c.

( $\beta$ ): If  $\{E_n^{(1)}\}$  is a u.s.c. then by (iii)  $V_1$  is bounded. Suppose  $V_1$  is bounded:  $C = \sup_{z \in V_1} |z| < \infty$ . For any continued fraction  $K(a_n/1)$  where  $a_n \in E_n^{(1)}$  for  $n = 1, 2, 3, \dots$ , and any nonzero  $x \in E_1$ ,  $x/(1 + K(a_n/1))$  will be a continued fraction such that

$$|f - f_n| = \frac{|x| \left| \frac{\infty}{K(a_k/1)} - \frac{n-1}{K(a_k/1)} \right|}{\left| 1 + \frac{\infty}{K(a_k/1)} \right| \left| 1 + \frac{n-1}{K(a_k/1)} \right|} \leq \lambda_n \quad \text{for } n = 2, 3, 4, \dots$$

Thereby

$$\left| \frac{\infty}{K(a_k/1)} - \frac{n-1}{K(a_k/1)} \right| \leq \frac{(1+C)^2 \lambda_n}{|x|} \quad \text{for } n = 2, 3, 4, \dots$$

Since the left side of the last expression is independent of  $x$  and  $E_1$ , we can choose  $\lambda_n^{(1)} = (1+C)^2 \lambda_{n+1} / M$  for  $n = 1, 2, 3, \dots$  where  $M = \sup_{z \in E_1} |z| < \infty$  since  $E_1 \subseteq V_0$  which is bounded, by (iii). Thereby  $\{\lambda_n^{(1)}\}$  is a sequence of positive numbers corresponding to  $\{E_n^{(1)}\}$ .  $\square$

Now the proof of Proposition 2.3 is rather straightforward.

**PROOF OF PROPOSITION 2.3.** Let  $\{\lambda_n\}$  be a sequence of positive numbers corresponding to the t.u.s.c.  $\{E_n\}$  such that  $\lambda_1 \geq \lambda_2$ .

(i) follows from the facts that  $E_n \subseteq V_{n-1}$  for all  $n \geq 1$  (by Definition 1.1), and that for any integer  $p \geq 0$ ,  $|z| \leq 2L + M$  for any  $z \in V_p$ , where  $L = \sup_{n \geq 1} \lambda_n$  and  $M = \sup\{|z|; z \in \bigcup_{n=1}^{\infty} E_n\}$  (by the same argument as in the proof of Proposition 3.1(iii)).

(ii)( $\alpha$ ) Suppose  $\bigcup_{n=1}^{\infty} E_n$  is not bounded. Then, to any  $t' \in \mathbf{R}$  there exists an  $N \in \mathbf{N}$  such that  $\sup\{|z|; z \in E_N\} > t' \cdot \lambda_1$ . Hence, we can choose  $a_n \in E_n \setminus \{z; |z| < \varepsilon\}$  arbitrarily for  $n = 1, 2, 3, \dots$ ,  $n \neq N$ , and  $a_N \in E_N$  such that  $|a_N| = t \lambda_1$  where  $t > t'$ . Since  $\{E_n\}$  is a t.u.s.c., the continued fraction  $K(a_n/1)$  will be such that

$$(3.3) \quad |f^{(N-1)} - a_N| \leq \lambda_1,$$

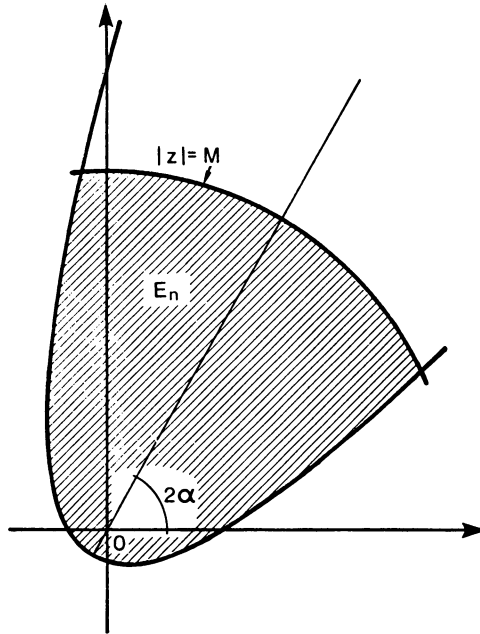


FIGURE 3.1

$$(3.4) \quad \left| f^{(N-1)} - \frac{a_N}{1 + a_{N+1}} \right| \leq \lambda_2 \leq \lambda_1.$$

(3.3) leads to:  $(t-1)\lambda_1 \leq |f^{(N-1)}| \leq (t+1)\lambda_1$ . Therefore

$$(3.5) \quad \begin{aligned} (t-2)\lambda_1 &\leq |f^{(N-1)}| - \lambda_1 \leq \left| \frac{a_N}{1 + a_{N+1}} \right| \\ &\leq |f^{(N-1)}| + \lambda_1 \leq (t+2)\lambda_1, \\ \frac{t}{t+2} &\leq |1 + a_{N+1}| \leq \frac{t}{t-2}, \end{aligned}$$

when  $t' > 2$ .

Besides, (3.3) and (3.4) imply that  $a_N/(1 + a_{N+1})$  is contained in the closed disc centered at  $a_N$  with radius  $2\lambda_1$ . That is

$$\begin{aligned} 2\lambda_1 &\geq \left| \frac{a_N}{1 + a_{N+1}} - a_N \right| = \frac{|a_N| |a_{N+1}|}{|1 + a_{N+1}|} = \frac{t\lambda_1 |a_{N+1}|}{|1 + a_{N+1}|}, \\ |1 + a_{N+1}| &\geq \frac{t}{2} |a_{N+1}| \geq \frac{t}{2} \cdot \varepsilon \rightarrow \infty \quad \text{when } t' \rightarrow \infty. \end{aligned}$$

But this contradicts (3.5). Hence,  $\bigcup_{n=1}^{\infty} E_n$  must be bounded.

(ii)(\beta) That  $\bigcup_{n=1}^{\infty} V_n$  has a positive distance from  $-1$  follows from (ii)(\alpha), (i) and (3.2). That  $\bigcup_{n=2}^{\infty} E_n$  has a positive distance from  $-1$  then follows from the fact that  $\bigcup_{n=2}^{\infty} E_n \subseteq \bigcup_{n=1}^{\infty} V_n$ .  $\square$

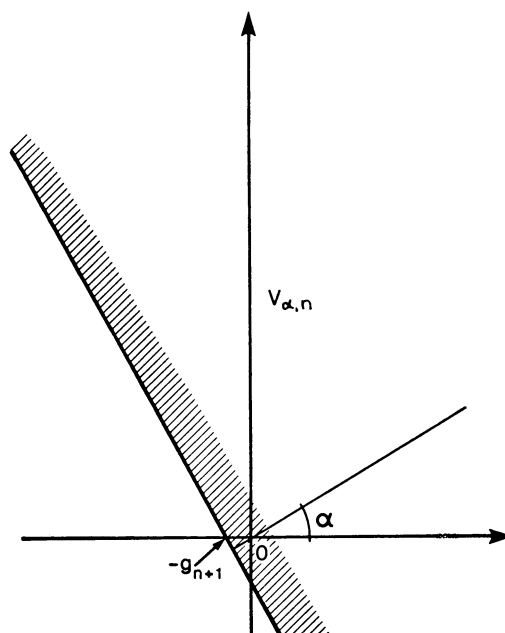


FIGURE 3.2

In the convergence theory for continued fractions  $K(a_n/1)$ , a wide range of sequences of convergence regions are known. (See for instance [6, 7, 10, 11].) Some of these are known to be uniform. Considering Proposition 3.1(i) and (iv), it seems feasible to impose certain boundedness conditions on u.s.c.'s to obtain examples of t.u.s.c.'s. But the existence of a sequence  $\{\lambda_n\}$  satisfying condition (ii) in Definition 2.1 must be handled separately in each case. When  $\{E_n\}$  is a periodic u.s.c., however, boundedness of  $V_n$  for all  $n \geq 1$  ensures that  $\{E_n\}$  is a t.u.s.c. In particular if  $E_{2n+1} = E_1$  and  $E_{2n} = E_2$  for  $n = 1, 2, 3, \dots$ , then bounding  $E_2$  and making sure that  $-1 \notin V_0$ , will be sufficient to make  $\{E_n\}$  a t.u.s.c. This is especially easy to accomplish when  $-1$  is not contained in the interior of  $V_0$  (which will always be the case when  $E_1 = E_2$ ).

The following is an example of a nonperiodic t.u.s.c.

EXAMPLE 3.1. The sequence  $\{E_n\}$  given by  $E_n = P_{\alpha, n} \cap \mathcal{D}_M$  for  $n = 1, 2, 3, \dots$  (see Figure 3.1) where  $P_{\alpha, n} = \{z \in \mathbb{C}; |z| - \operatorname{Re}(ze^{-i2\alpha}) \leq 2g_n(1 - g_{n+1})\cos^2 \alpha\}$  for  $n = 1, 2, 3, \dots$  and  $\mathcal{D}_M = \{z \in \mathbb{C}; |z| \leq M\}$  is a t.u.s.c. provided that

$$(3.6) \quad |\alpha| < \frac{\pi}{2}, \quad M < \infty, \quad 0 < x < g_n < 1 - x \quad \text{for } n = 1, 2, 3, \dots$$

and the series

$$(3.7) \quad \sum_{k=1}^{\infty} \prod_{n=1}^k \left( \frac{1}{g_{n+p+1}} - 1 \right)$$

diverges for every  $p \geq 0$ , and this divergence is uniform with respect to  $p$ .

If

$$(3.8) \quad V_{\alpha, n} = \{z \in \hat{\mathbb{C}}; \operatorname{Re}(ze^{-i\alpha}) \geq -g_{n+1} \cos \alpha\} \quad \text{for } n = 0, 1, 2, \dots,$$

and  $\{V_n\}$  is the best sequence of value regions corresponding to  $\{E_n\}$ , then  $V_n \subseteq V_{\alpha, n} \cap \mathcal{D}_{M/x \cos \alpha}$  for  $n = 0, 1, 2, \dots$ . The reason for this is as follows: Thron

proved in 1958 [10] that  $\{E_n\}$  is a u.s.c. provided that (3.6) is satisfied and the series (3.7) diverges for  $p = 0$ . Furthermore he proved that  $\{\lambda_n^{(0)}\}$ , where

$$\lambda_n^{(p)} = \frac{M}{2x \cos \alpha \prod_{k=2}^n (1 + d_{k-1}^{(p)} x^2 \cos^2 \alpha / M)} \quad \text{for } n = 1, 2, 3, \dots, p = 0, 1, 2, \dots,$$

$$d_k^{(p)} = \frac{\prod_{n=1}^k (1/g_{n+p+1} - 1)}{\sum_{m=0}^{k-1} \prod_{n=1}^m (1/g_{n+p+1} - 1)} \quad \text{for } k = 1, 2, 3, \dots,$$

is a sequence of positive numbers corresponding to the u.s.c.  $\{E_n\}$ , and that  $\{V_{\alpha,n}\}$  is a corresponding sequence of value regions. That  $V_n \subseteq \mathcal{D}_{M/x \cos \alpha}$  follows from (3.2).

It is easy to see that  $\{E_n^{(p)}\}_{n=1}^\infty$  where  $E_n^{(p)} = E_{n+p}$ , is a u.s.c. with a corresponding sequence  $\{\lambda_n^{(p)}\}$  of positive numbers for any  $p \geq 0$ . So we only have to check the sequence  $\{\lambda_n\}$  where

$$\lambda_n = \sup_{p \geq 0} \lambda_n^{(p)} = \frac{M}{2x \cos \alpha \prod_{k=2}^n (1 + d_{k-1} x^2 \cos^2 \alpha / M)} \quad \text{for } n = 1, 2, 3, \dots,$$

and  $d_k = \inf_{p \geq 0} d_k^{(p)} > 0$  because of (3.6).

The uniform divergence with respect to  $p$  of the series (3.7) ensures the divergence of  $\sum_{k=1}^\infty d_k$ . (It is well known that the series  $\sum_{n=1}^\infty C_n$  and  $\sum_{n=1}^\infty (C_n / \sum_{k=1}^{n-1} C_k)$  converge and diverge together [8, p. 327].) Hence  $\{\lambda_n\}$  is a sequence of positive numbers converging to 0, and  $\{E_n\}$  is a t.u.s.c.

**PROPOSITION 3.2.** *Let  $\langle E_1, E_2 \rangle$  be a uniform set of twin convergence regions, and  $\langle V_0, V_1 \rangle$  be the corresponding best value regions. If  $V_1$  is bounded and  $-1 \notin V_0 + V_1$ , then  $\langle E_1, E_2 \rangle$  is a set of twin CA-regions.*

(If  $E_1 = E_2$ , then by Proposition 3.1 (iii),  $V_1 = V_0$  is always bounded.)

**PROOF.** By Proposition 3.1(iv),  $\{E_n\}$  is a t.u.s.c. when  $V_1$  is bounded. Besides  $W_{2n} \subseteq 1 + V_1$  and  $W_{2n+1} \subseteq 1 + V_0$  for  $n = 1, 2, 3, \dots$ , where  $W_n$  is defined by (2.4). Hence, the proposition follows.  $\square$

This proposition furnishes an easy way to find examples of simple or twin CA-regions.

**EXAMPLE 3.2.** By choosing  $g_n = \frac{1}{2}(1 - \delta)$  for  $n = 1, 2, 3, \dots$ , where  $0 < \delta < 1$ , in Example 3.1, we get a simple uniform convergence region

$$E_\alpha = \{z \in \mathbb{C}; |z| - \operatorname{Re}(ze^{-i2\alpha}) \leq \tfrac{1}{2}(1 - \delta^2) \cos^2 \alpha\} \cap \mathcal{D}_M$$

with a corresponding best value region

$$V_\alpha \subseteq \{z \in \hat{\mathbb{C}}; \operatorname{Re}(ze^{-i\alpha}) \geq -\tfrac{1}{2}(1 - \delta) \cos \alpha\} \cap \mathcal{D}_{M/\frac{1}{2}(1 + \delta) \cos \alpha}$$

when  $|\alpha| < \pi/2$ ,  $0 < \delta < 1$  and  $0 < M < \infty$ .

Since  $\inf\{|1 + x + y|; x, y \in V_\alpha\} \geq \delta > 0$ ,  $E_\alpha$  is a simple  $C_A$ -region.

**EXAMPLE 3.3.** By choosing  $g_{2n-1} = \frac{1}{2}(1 - \delta_1)$ ,  $g_{2n} = \frac{1}{2}(1 - \delta_2)$ ;  $n = 1, 2, 3, \dots$ , where  $-1 < \delta_1 < 1$ ,  $-1 < \delta_2 < 1$  and  $\delta_1 + \delta_2 > 0$  in Example 3.1, we get by the same argument as in Example 3.2, a set of twin CA-regions.

**EXAMPLE 3.4.**

$$E_1 = \{z \in \mathbb{C}; |z| \leq \rho^2 - \epsilon^2\},$$

$$E_2 = \{z \in \mathbb{C}; |z| \geq (1 + \rho + \epsilon)^2\} \cap \mathcal{D}_M,$$

where  $0 < \varepsilon < \rho < \rho + \varepsilon \leq 1$  and  $\mathcal{D}_M$  is defined as in Example 3.1, is a set of twin  $CA$ -regions with a corresponding set of value regions

$$V_0 = \{w; |w| \leq \rho - \varepsilon\}, \quad V_1 = \{w; |w + 1| \geq \rho + \varepsilon\}.$$

The reasons for this are as follows:

(1)  $\langle E_1, E_2 \rangle$  and  $\langle V_0, V_1 \rangle$  are corresponding element and value regions because:

$$\frac{E_1}{1 + V_1} = \left\{ w; |w| \leq \frac{\rho^2 - \varepsilon^2}{\rho + \varepsilon} \right\} = V_0,$$

and if  $z = re^{i\varphi} \in E_2$  and  $w = Re^{i\theta} \in V_0$  then

$$\left| 1 + \frac{z}{1 + w} \right| \geq \frac{r - 1 - R}{R + 1} \geq \frac{(1 + \rho + \varepsilon)^2 - (1 + \rho - \varepsilon)}{1 + \rho - \varepsilon} > \rho + \varepsilon.$$

Hence,  $E_2/(1 + V_0) \subseteq V_1$ . Furthermore,  $E_1 \subseteq V_0$  since  $\rho^2 - \varepsilon^2 = (\rho + \varepsilon)(\rho - \varepsilon) \leq \rho - \varepsilon$ , and  $E_2 \subseteq V_1$  since  $(1 + \rho + \varepsilon)^2 > 1 + \rho + \varepsilon$ .

(2)  $\langle E_1, E_2 \rangle$  is a t.u.s.c. because

$$E_1 \subseteq E_1^* = \{z; z = v^2 \text{ and } |v| \leq \rho\} \quad \text{and}$$

$$E_2 \subseteq E_2^* = \{z; z = v^2 \text{ and } |v \pm i| \geq \rho\}$$

where  $\langle E_1^*, E_2^* \rangle$  is proved by Thron [7, Theorem 4.46, p. 115] to be a set of uniform twin convergence regions. Besides  $-1 \notin V_0$  and  $E_2$  is bounded, so by Proposition 3.1(i), the best value regions corresponding to  $\{E_n\}$  are bounded.

(3)  $-1 \notin V_0 + V_1$  because

$$\inf\{|z + w + 1|; z \in V_0, w \in V_1\} = 2\varepsilon > 0.$$

EXAMPLE 1.1 CONTINUED. All the elements  $a_n$  and  $a'_n$ ,  $n = 1, 2, 3, \dots$ , are contained in the simple  $CA$ -region  $E_\alpha$  in Example 3.2 with  $\alpha = 0$ ,  $M = 13$ , and  $\delta$  arbitrarily chosen from  $(0, 1)$ .

**4. Upper bounds for the ratio  $|f - S_n(f^{(n)})|/|f - S_n(0)|$ .** After having established conditions for obtaining a substantial convergence acceleration, we want to know how much better the convergence is by this new method. The upper bounds for this ratio, given by the following theorem, will serve as a measure for the improvement.

**THEOREM 4.1.** *Let  $K(a_n/1)$  be a convergent continued fraction whose value  $f$  one wants to approximate, and  $K(a'_n/1)$  a convergent continued fraction where all the values  $\{f^{(n')}\}_{n=0}^\infty$  of the tails are known. If*

(i) *there exist a t.u.s.c.  $\{E_n\}$  and a  $\mu > 0$  such that  $E_n \setminus \{z; |z| < \mu\} \neq \emptyset$  and  $a_n, a'_n \in E_n$  for all  $n$ ,*

(ii) *there exist a  $D > 0$  and a sequence  $\{r_n\}_{n=0}^\infty$  of numbers  $\geq 1$ , such that*

$$\frac{|1 + f^{(n+1)' }|}{r_{n+1}} - \frac{|f^{(n)' }|}{r_n} \geq D \quad \text{for } n = 0, 1, 2, 3, \dots,$$

and

(iii)  $\lim_{n \rightarrow \infty} (a_n - a'_n) = 0$  and  $|a_n - a'_n| \leq \min\{D^2/4, |a'_n|/2\}$  for  $n = 1, 2, 3, \dots$ ,

then

$$(4.1) \quad \left| \frac{f - S_n(f^{(n)'})}{f - S_n(0)} \right| \leq \left( 1 + \frac{|f^{(n)'}|}{\delta_n(f^{(n)'})} \right) \left( 2 + 4|f^{(n)'}| \frac{r_{n+1}}{D_{n+1}} \right) \frac{d_{n+1}}{|a'_{n+1}|}$$

for  $n = 1, 2, 3, \dots$ , where

$$(4.2) \quad d_n = \sup_{t \geq n} |a_t - a'_t| \text{ for } n = 1, 2, 3, \dots,$$

$$D_n = \inf_{t \geq n} \left[ \frac{|1 + f^{(t+1)'}|}{r_{t+1}} - \frac{|f^{(t)'}|}{r_t} \right] \text{ for } n = 0, 1, 2, \dots,$$

$$\delta_n(f^{(n)'}) = \inf \{ |z + f^{(n)'}| ; z \in W_n \} \text{ for } n = 1, 2, 3, \dots,$$

and  $W_n = c(\{h_n \in \mathbb{C}; a_k \in E_k \text{ for } k = 2, \dots, n\})$  for  $n = 1, 2, \dots$

Before presenting the proof, we shall give some remarks.

(1) We do not know, within the conditions of the theorem, that (4.1) converges to 0. We do not even know that the left-hand side of (4.1) converges unless  $\{E_n\}$  is a  $CA$ -sequence and  $\{a_n\}$  has no limit point at 0 (Theorem 2.5). On the other hand, these conditions ( $\{E_n\}$   $CA$ -sequence and  $\{a_n\}$  no limit point at 0) are not necessary for the convergence of (4.1) to 0, because if  $\{d_n\}$  converges to 0 sufficiently fast, it may compensate for possible limit points at 0 for  $\{\delta_n(f^{(n)'})\}$  or  $\{a'_n\}$ .

(2) Which convergent continued fractions  $K(a'_n/1)$  satisfy condition (ii) in the theorem? We see at once that  $|1 + f^{(n+1)'}| \geq \delta$ ,  $n = 0, 1, 2, \dots$ , for some  $\delta > 0$  is necessary for (ii). This will however always be so when condition (i) is satisfied (by Proposition 2.3).

Furthermore, Proposition 4.3 later in this section gives the following set of sufficient conditions:  $K(a'_n/1)$   $k$ -periodic and  $S'_k(x)$  has two distinct fixed points; or, equivalently,  $K(a'_n/1)$   $k$ -periodic and  $a'_n \in E_n$  for  $n = 1, 2, 3, \dots$ , where  $\{E_n\}$  is a  $CA$ -sequence. An open question is: will  $a'_n \in E_n$  for  $n = 1, 2, 3, \dots$ , where  $\{a'_n\}$  has no limit point at 0 and  $\{E_n\}$  is a  $CA$ -sequence, always imply that  $K(a'_n/1)$  satisfies condition (ii)?

(3) The actual choice of the sequence  $\{r_n\}$ , when it exists, should be made such that  $\{r_n/D_n\}$  is kept bounded. And that is always possible to achieve since  $D_n \geq D > 0$  for all  $n \geq 0$  and  $r_n$  is bounded (since  $\{f^{(n)'}\}$  is bounded by Proposition 2.3 and

$$\frac{|1 + f^{(n+1)'}|}{r_{n+1}} \geq \frac{|1 + f^{(n+1)'}|}{r_{n+1}} - \frac{|f^{(n)'}|}{r_n} \geq D > 0$$

by condition (ii) in the theorem).

(4) The computation of  $\delta_n(f^{(n)'})$  depends on the choice of  $\{E_n\}$ . If  $\{E_n\}$  is a  $CA$ -sequence, we know that  $0 \notin c(\bigcup_{n=1}^{\infty} (V_n + W_n))$ , where  $\{V_n\}$  is the best sequence of value regions corresponding to  $\{E_n\}$ . Since  $f^{(n)' \in V_n}$  for  $n = 1, 2, 3, \dots$ , that means that there exists a  $\delta > 0$  such that

$$\delta \leq \inf \{ |x + y| ; x \in V_k, y \in W_k, k = 1, 2, 3, \dots \} \leq \delta_n(f^{(n)'})$$

for all  $n \geq 1$ . By replacing  $\delta_n(f^{(n)'})$  by  $\delta$  in (4.1), we have a value which is valid for any continued fraction  $K(a'_n/1)$  where  $a'_n \in E_n$  for  $n = 1, 2, \dots$  (profitable if

$E_n \setminus \{a'_n, a_n, 0\} \neq \emptyset$ ), and a value which is known for several  $CA$ -sequences (as for instance in the Examples 3.2, 3.3 and 3.4).

An alternative lower bound for  $\delta_n(f^{(n)'})$  is easy to get if we know a sequence  $\{W_n^*\}$  of regions (or subsets of  $\hat{C}$ ) such that  $W_n \subseteq W_n^*$  for  $n = 1, 2, 3, \dots$ , namely the value we get by replacing  $W_n$  with  $W_n^*$  in the expression for  $\delta_n(f^{(n)'})$ . (Such a sequence  $\{W_n^*\}$  is for instance known in the Examples 3.2–3.4 of  $CA$ -sequences.)

PROOF OF THEOREM 4.1. By (2.5) we have

$$(4.3) \quad \left| \frac{f - S_n(f^{(n)'})}{f - S_n(0)} \right| = \left| 1 - \frac{f^{(n)'}}{h_n + f^{(n)'}} \right| \left| 1 - \frac{f^{(n)'}}{f^{(n)}} \right| \quad \text{for } n = 1, 2, 3, \dots,$$

where  $|1 - f^{(n)'}/(h_n + f^{(n)'})| \leq 1 + |f^{(n)'}/\delta_n(f^{(n)'})|$  for  $n = 1, 2, 3, \dots$

To find upper bounds for  $|1 - f^{(n)'}/f^{(n)}|$ , we first establish upper bounds for  $|f^{(n)} - f^{(n)'}|$ . Let  $\rho_n = f^{(n)} - f^{(n)'}$  and  $\varepsilon_{n+1} = a_{n+1} - a'_{n+1}$  for  $n = 0, 1, 2, \dots$ . Furthermore, let  $n \geq 0$  be arbitrarily chosen, and  $m \geq n$ . By (1.8), we then get

$$\rho_m = \frac{\varepsilon_{m+1} - \rho_{m+1} f^{(m)'}}{1 + f^{(m+1)'} + \rho_{m+1}}.$$

Suppose there exists an  $R_n > 0$  such that  $|\rho_{m+1}| \leq R_n \cdot r_{m+1} \leq \frac{1}{2} |1 + f^{(m+1)'}|$ . If

$$(4.4) \quad R_n = \frac{D_n d_{n+1}}{D_n^2 - 2d_{n+1}}$$

then  $|\rho_m| \leq r_m R_n$  also, because

$$|\rho_m| \leq \frac{d_{m+1} + R_n r_{m+1} |f^{(m)'}|}{|1 + f^{(m+1)'}| - r_{m+1} R_n} \leq r_m R_n$$

when

$$r_m r_{m+1} R_n^2 - [r_m |1 + f^{(m+1)'}| - r_{m+1} |f^{(m)'}|] R_n + d_{n+1} \leq 0,$$

i.e. when

$$R_n^2 - \left[ \frac{|1 + f^{(m+1)'}|}{r_{m+1}} - \frac{|f^{(m)'}|}{r_m} \right] R_n + \frac{d_{n+1}}{r_m r_{m+1}} \leq 0.$$

Since the value (4.4) of  $R_n$  gives

$$\begin{aligned} R_n^2 - \left[ \frac{|1 + f^{(m+1)'}|}{r_{m+1}} - \frac{|f^{(m)'}|}{r_m} \right] R_n + \frac{d_{n+1}}{r_m r_{m+1}} & \\ \leq R_n^2 - D_n R_n + d_{n+1} &= R_n \left( R_n - D_n + \frac{d_{n+1}}{R_n} \right) \\ &= R_n \left( \frac{D_n d_{n+1}}{D_n^2 - 2d_{n+1}} - D_n + \frac{D_n^2 - 2d_{n+1}}{D_n} \right) \\ &= \frac{R_n d_{n+1}}{D_n^2 - 2d_{n+1}} \left( -\frac{D_n^2 - 4d_{n+1}}{D_n} \right) \leq 0 \end{aligned}$$

because of the conditions (ii) and (iii) in the theorem, the statement  $|\rho_m| \leq r_m R_n$  follows. But will  $|\rho_{m+1}| \leq R_n \cdot r_{m+1} \leq \frac{1}{2} |1 + f^{(m+1)' }|$ ? By the conditions (ii) and (iii) in the theorem, the right inequality is easily proved:

$$\begin{aligned} R_n r_{m+1} &= \frac{D_n d_{n+1}}{D_n^2 - 2d_{n+1}} r_{m+1} \leq \frac{D_n d_{n+1}}{2d_{n+1}} r_{m+1} \\ &\leq \frac{1}{2} \left[ \frac{|1 + f^{(m+1)' }|}{r_{m+1}} - \frac{|f^{(m)' }|}{r_m} \right] \cdot r_{m+1} \leq \frac{1}{2} |1 + f^{(m+1)' }|. \end{aligned}$$

The left side of the inequality is surely satisfied if we choose  $m \geq n$  big enough, because  $\{E_n\}$  is a t.u.s.c. where  $E_n \setminus \{z; |z| < \mu\} \neq \emptyset$  for all  $n$ , so by Theorem 2.4,  $\rho_m \rightarrow 0$ .

Repeated use of the implication

$$|\rho_{m+1}| \leq R_n r_{m+1} \Rightarrow |\rho_m| \leq R_n r_m \quad \text{for } m \geq n$$

a finite number of times yields the following upper bounds:  $|f^{(m)} - f^{(m)' }| \leq R_n r_m$  for all  $m \geq n$ ,  $n = 0, 1, 2, \dots$ . By using these, we get

$$\begin{aligned} \left| 1 - \frac{f^{(n)' }}{f^{(n)}} \right| &= \left| 1 - \frac{f^{(n)' }(1 + f^{(n+1)' })}{a_{n+1}} \right| = \left| \frac{\epsilon_{n+1} - f^{(n)' } \rho_{n+1}}{a'_{n+1} + \epsilon_{n+1}} \right| \\ &\leq \frac{|f^{(n)' }| R_{n+1} r_{n+1} + d_{n+1}}{|a'_{n+1}| - |\epsilon_{n+1}|} \leq \frac{|f^{(n)' }| R_{n+1} r_{n+1}}{\frac{1}{2} |a'_{n+1}|} + \frac{d_{n+1}}{\frac{1}{2} |a'_{n+1}|} \\ &= \frac{2 |f^{(n)' }| r_{n+1}}{|a'_{n+1}|} \cdot \frac{D_{n+1} d_{n+2}}{D_{n+1}^2 - 2d_{n+2}} + \frac{2d_{n+1}}{|a'_{n+1}|} \leq \left( 4 |f^{(n)' }| \frac{r_{n+1}}{D_{n+1}} + 2 \right) \frac{d_{n+1}}{|a'_{n+1}|} \end{aligned}$$

for  $n = 0, 1, 2, \dots$ .  $\square$

As an example of the use of this theorem, one may look at limit periodic continued fractions such as some of the ones that Thron and Waadeland considered in their paper [12].

**EXAMPLE 4.1.** Let  $K(a_n/1)$  be a limit periodic continued fraction such that  $\lim_{n \rightarrow \infty} a_n = a \in \mathbb{C} \setminus (-\infty, -\frac{1}{4}]$ . Then we can use the continued fraction  $K(a'_n/1)$  where  $a'_n = a$  for all  $n \geq 1$ , as an auxiliary continued fraction. The tails of  $K(a'_n/1)$  all have the value  $x_1$ , where  $x_1$  is the attractive fixed point of the linear fractional transformation  $S'_1(x) = a/(1+x)$ .

Since  $D = |1 + x_1| - |x_1| > 0$  (because  $S'_1$  always has fixed points  $x_1, x_2$  such that  $|x_1| \neq |x_2|$  for these values of  $a$ ), we can choose  $r_n = 1$  for all  $n \geq 0$ . Thron and Waadeland proved that

$$\delta_n(x_1) \geq \frac{D |1 + x_1|/2 - d_n}{|x_1| + D/2} \quad \text{for } n = 1, 2, 3, \dots$$

Theorem 4.1 then yields

$$\begin{aligned} \left| \frac{f - S_n(x_1)}{f - S_n(0)} \right| &\leq \left( 1 + \frac{|x_1| (|x_1| + D/2)}{|1 + x_1| D/2 - d_n} \right) \left( 2 + 4 |x_1| \cdot \frac{1}{D} \right) \frac{d_{n+1}}{|a|} \\ &< \left( 1 + \frac{|x_1| (|x_1| + D/2)}{|1 + x_1| D/2 - d_n} \right) \left( 4 + 4 \frac{|x_1|}{D} \right) \cdot \frac{d_{n+1}}{|a|} \end{aligned}$$

which coincides with the result by Thron and Waadeland. By rearranging this expression, they proved that

$$\left| \frac{f - S_n(x_1)}{f - S_n(0)} \right| \leq \frac{|a| + |\frac{1}{2} + a + \sqrt{\frac{1}{4} + a}|}{\frac{1}{4} + |\frac{1}{4} + a| - |a|} \cdot \frac{2d_n}{|a|}.$$

In this example we could choose  $r_n = 1$  for all  $n \geq 0$ . But it is not difficult to find examples where the flexibility provided by the sequence  $\{r_n\}$  is needed. The auxiliary continued fraction  $K(a'_n/1)$  in Example 1.1 will for instance be of that kind.

In the following two propositions we see that the existence of such a sequence is closely connected to other tail properties of continued fractions.

**PROPOSITION 4.2.** *Let  $K(a_n/1)$  be a convergent continued fraction with tail values  $\{f^{(n)}\}$ .*

(i) *If  $\{h_n + f^{(n)}\}$  is not converging to 0 ( $h_n$  is defined by (1.7)), then*

$$(4.5) \quad \left\{ \prod_{k=0}^n \frac{|1 + f^{(k+1)}|}{|f^{(k)}|} \right\}_{n=0}^{\infty}$$

*has a limit point at  $\infty$ .*

(ii) *If (4.5) has a limit point at  $\infty$ , and  $f^{(n)} \neq 0$  for all  $n \geq 0$ , then there exists a subsequence  $\{k_n\}_{n=1}^{\infty}$  of the natural numbers,  $k_0 = 0$ , such that*

$$\prod_{m=k_{n-1}}^{k_n-1} |1 + f^{(m+1)}| - \prod_{m=k_{n-1}}^{k_n-1} |f^{(m)}| > 0 \quad \text{for all } n \geq 1.$$

(iii) *If  $0 < \mu \leq |f^{(n)}| \leq M < \infty$  for all  $n \geq 0$ , then the following two statements are equivalent:*

(A): *There exist a  $D > 0$  and a sequence  $\{r_n\}_{n=0}^{\infty}$  where  $1 \leq r_n \leq R < \infty$  for all  $n \geq 0$ , such that*

$$\frac{|1 + f^{(n+1)}|}{r_{n+1}} - \frac{|f^{(n)}|}{r_n} \geq D \quad \text{for all } n \geq 0.$$

(B): *There exist a  $C > 0$  and a subsequence  $\{k_n\}_{n=1}^{\infty}$  of the natural numbers,  $k_0 = 0$ , such that  $k_{n+1} - k_n \leq k < \infty$  for all  $n \geq 0$  and*

$$\prod_{m=k_{n-1}}^{k_n-1} |1 + f^{(m+1)}| - \prod_{m=k_{n-1}}^{k_n-1} |f^{(m)}| \geq C \quad \text{for all } n \geq 1.$$

**PROOF.** (i) By (1.9) and (1.10) we have

$$(4.6) \quad \prod_{k=0}^n \frac{|1 + f^{(k+1)}|}{|f^{(k)}|} = \frac{|B_{n+1} + B_n f^{(n+1)}|}{|A_n - B_n f|} = \frac{|h_{n+1} + f^{(n+1)}|}{|f - f_n|}.$$

Since  $K(a_n/1)$  converges,  $f - f_n \rightarrow 0$ , and (i) follows.

(ii) Follows immediately from (i).

(iii) Suppose  $0 < \mu \leq |f^{(n)}| \leq M < \infty$  for all  $n \geq 0$ . To prove (B)  $\Rightarrow$  (A), suppose (B) is true. Then there exist positive numbers  $\{t_m\}_{m=0}^{k_1}$  such that  $t_0 = t_{k_1} = 1$  and

$$(4.7) \quad |1 + f^{(m+1)}| t_{m+1} - |f^{(m)}| t_m = D^{(1)} > 0 \quad \text{for } m = 0, 1, \dots, k_1 - 1,$$

because after elimination of  $D^{(1)}$ , (4.7) is equivalent to a set of  $k_1 - 1$  linear equations in  $k_1 - 1$  unknowns  $t_1, t_2, \dots, t_{k_1-1}$ :  $\mathbf{MT} = \mathbf{F}$ , where the matrix  $\mathbf{M}$  is given in Figure 4.1, where  $u_m = |1 + f^{(m)}|$  and  $l_m = |f^{(m)}|$  for  $m = 1, \dots, k_1 - 1$ ,

$$\mathbf{T} = [t_1, t_2, \dots, t_{k_1-1}]^t \quad \text{and} \\ \mathbf{F} = [|f^{(0)}|, 0, \dots, 0, |1 + f^{(k_1)}|]^t. \\ \mathbf{M} = \begin{bmatrix} u_1 + l_1 & -u_2 & 0 & \cdots & 0 & 0 \\ -l_1 & u_2 + l_2 & -u_3 & \cdots & 0 & 0 \\ 0 & -l_2 & u_3 + l_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & u_{k_1-2} + l_{k_1-2} & -u_{k_1-1} \\ 0 & 0 & 0 & \cdots & -l_{k_1-2} & u_{k_1-1} + l_{k_1-1} \end{bmatrix}$$

FIGURE 4.1

A simple induction gives

$$\det \mathbf{M} = \sum_{j=0}^{k_1-1} \left[ \prod_{m=1}^j |1 + f^{(m)}| \cdot \prod_{m=j+1}^{k_1-1} |f^{(m)}| \right].$$

Since  $\det \mathbf{M} \geq \mu^{k_1-1}$ , the equations have a unique solution, easily found by Cramer's rule:

$$t_m = (-1)^{m-1} \left[ |f^{(0)}| \det \mathbf{M}_{1,m} + (-1)^{k_1-1} |1 + f^{(k_1)}| \det \mathbf{M}_{k_1-1,m} \right] / \det \mathbf{M}$$

where  $\det \mathbf{M}_{p,q}$  denotes the cofactor to the element  $(p, q)$  with respect to  $\mathbf{M}$ .

By a simple induction argument, we get

$$t_m = \left[ \prod_{j=0}^{m-1} |f^{(j)}| \cdot \sum_{j=m}^{k_1-1} \left( \prod_{n=m+1}^j |1 + f^{(n)}| \cdot \prod_{n=j+1}^{k_1-1} |f^{(n)}| \right) \right. \\ \left. + \prod_{j=m+1}^{k_1} |1 + f^{(j)}| \cdot \sum_{j=0}^{m-1} \left( \prod_{n=1}^j |1 + f^{(n)}| \cdot \prod_{n=j+1}^{m-1} |f^{(n)}| \right) \right] / \det \mathbf{M}$$

for  $m = 1, 2, \dots, k_1 - 1$ .

Since  $0 < \mu \leq |f^{(n)}| \leq M < \infty$  for all  $n \geq 0$ ,

$$\frac{\mu^{k_1-1}}{(1+M)^{k_1-1} \cdot k_1} \leq t_m \leq \frac{(1+M)^{k_1-1} \cdot k_1}{\mu^{k_1-1}}, \quad m = 1, 2, \dots, k_1 - 1.$$

Besides, by (4.7)

$$D^{(1)} = |1 + f^{(1)}| t_1 - |f^{(0)}| = \left[ \prod_{n=0}^{k_1-1} |1 + f^{(n+1)}| - \prod_{n=0}^{k_1-1} |f^{(n)}| \right] / \det \mathbf{M}$$

so  $D^{(1)} \geq C/k_1(1+M)^{k_1-1}$ . Repeated use of the argument for the indices  $k_1$  to  $k_2$  and so on, and choosing  $r_n = t_n^{-1} \cdot \sup_{m \geq 0} t_m$ ,  $n = 0, 1, 2, \dots$ , proves that the conditions in (A) are satisfied with  $D = \inf_{n \geq 1} D^{(n)} \geq C/k(1+M)^{k-1} > 0$  since  $k_{n+1} - k_n \leq k < \infty$ . To see that (A)  $\Rightarrow$  (B), assume that (A) is true. Then we get

$$\begin{aligned} \frac{\prod_{m=0}^{n-1} |1 + f^{(m+1)}|}{\prod_{m=0}^{n-1} |f^{(m)}|} &= \frac{r_n}{r_0} \frac{\prod_{m=0}^{n-1} |1 + f^{(m+1)}|/r_{m+1}}{\prod_{m=0}^{n-1} |f^{(m)}|/r_m} \\ &= \frac{r_n}{r_0} \prod_{m=0}^{n-1} \left[ 1 + \frac{|1 + f^{(m+1)}|/r_{m+1} - |f^{(m)}|/r_m}{|f^{(m)}|/r_m} \right] \\ &\geq \frac{r_n}{r_0} \prod_{m=0}^{n-1} \left( \frac{D}{|f^{(m)}|/r_m} + 1 \right) \geq \frac{1}{R} \prod_{m=0}^{n-1} \left( 1 + \frac{D}{M} \right) = \frac{1}{R} \left( 1 + \frac{D}{M} \right)^n, \\ \prod_{m=0}^{n-1} |1 + f^{(m+1)}| - \prod_{m=0}^{n-1} |f^{(m)}| &\geq \left[ \frac{1}{R} \left( 1 + \frac{D}{M} \right)^n - 1 \right] \prod_{m=0}^{n-1} |f^{(m)}| \\ &\geq \left[ \frac{1}{R} \left( 1 + \frac{D}{M} \right)^n - 1 \right] \mu^n. \end{aligned}$$

Since  $(1 + D/M)^n \rightarrow \infty$  when  $n \rightarrow \infty$ , there exists to any  $C_1 > 0$  a  $k_1 \in \mathbb{N}$  such that  $(1 + D/M)^{k_1}/R - 1 \geq C_1$ . So, by choosing  $k_n = nk_1$ ,  $n = 1, 2, 3, \dots$ , and  $C = C_1 \mu^{k_1}$ , the conditions in (B) are satisfied.  $\square$

The next proposition shows that such a sequence  $\{r_n\}$  exists in the important case when the auxiliary continued fraction  $K(a'_n/1)$  is  $k$ -periodic, under a mild condition.

**PROPOSITION 4.3.** *Let  $K(a_n/1)$  be a convergent  $k$ -periodic continued fraction with finite tail values  $\{f^{(n)}\}$ , such that  $a_{kn+p} = a_p$  for  $p \in \{1, \dots, k\}$  and  $n = 1, 2, 3, \dots$ . Then*

$$\prod_{n=0}^{k-1} |1 + f^{(n+1)}| - \prod_{n=0}^{k-1} |f^{(n)}| \geq 0$$

where the equality sign holds if and only if the linear fractional transformation

$$S_k(x) = \frac{a_1}{1} + \dots + \frac{a_k}{1+x}$$

has only one fixed point. If  $a_n \in E_n$  for all  $n$ , where  $\{E_n\}$  is a CA-sequence, then  $S_k(x)$  will always have two distinct fixed points.

**PROOF.** The fixed points  $x_1, x_2$  of  $S_k(x)$  are such that  $f^{(0)} = x_1$  and

$$|B_k + B_{k-1}x_2|/|B_k + B_{k-1}x_1| \leq 1$$

(see for instance [7, Theorem 3.1, p. 47]), where the equality sign holds if and only if  $x_1 = x_2$ . Since, by (1.10),

$$B_k + B_{k-1}x_2 = B_k + B_{k-1} \left( \frac{A_{k-1} - B_k}{B_{k-1}} - x_1 \right) = (-1)^k \prod_{n=0}^{k-1} f^{(n)},$$

and by (1.9),  $B_k + B_{k-1}x_1 = \prod_{n=0}^{k-1} (1 + f^{(n+1)})$ , the first part of the proposition follows.

Suppose  $x_1 = x_2$ . Then

$$\prod_{n=0}^{k-1} |1 + f^{(n+1)}| - \prod_{n=0}^{k-1} |f^{(n)}| = 0$$

and therefore, by (4.6),

$$1 = \prod_{n=0}^{mk-1} \frac{|1 + f^{(n+1)}|}{|f^{(n)}|} = \frac{|h_{mk} + f^{(mk)}|}{|f - f_{mk-1}|} \quad \text{for } m = 1, 2, 3, \dots$$

Since  $|f - f_{mk-1}| \rightarrow 0$  when  $m \rightarrow \infty$ , we must have  $\lim_{m \rightarrow \infty} |h_{mk} + f^{(mk)}| = 0$  which is impossible if  $\{E_n\}$  is a CA-sequence.  $\square$

Another problem is to find this sequence  $\{r_n\}$ . In this case, when the auxiliary continued fraction is  $k$ -periodic, the constructive method given in the proof of Proposition 4.2(iii), (B)  $\Rightarrow$  (A), is very useful.

EXAMPLE 1.1 CONTINUED. The choice of  $\{r_n\}$  can be made such that  $r_{kn+p} = t_p^{-1} \cdot \max(t_1, t_2, t_3, t_4, 1)$  for  $p = 0, 1, 2, 3, 4$  and  $n = 0, 1, 2, \dots$ , where  $t_0 = t_5 = 1$  and  $t_p$  is given by

$$|1 + f^{(p+1)' }| t_{p+1} - |f^{(p)' }| t_p = D \quad \text{for } p = 0, 1, 2, 3, 4.$$

Solving this set of linear equations (after eliminating  $D$ ), gives (when the notation  $|1 + f^{(p)' }| = u_p$ ,  $|f^{(p)' }| = l_p$ ,  $p = 0, 1, 2, 3, 4$  is used):

$$\begin{aligned} t_1 &= \frac{l_0(l_2l_3l_4 + u_2l_3l_4 + u_2u_3l_4 + u_2u_3u_4) + u_2u_3u_4u_0}{l_1l_2l_3l_4 + u_1l_2l_3l_4 + u_1u_2l_3l_4 + u_1u_2u_3l_4 + u_1u_2u_3u_4}, \\ t_2 &= \frac{l_0l_1(l_3l_4 + u_3l_4 + u_3u_4) + u_3u_4u_0(l_1 + u_1)}{l_1l_2l_3l_4 + u_1l_2l_3l_4 + u_1u_2l_3l_4 + u_1u_2u_3l_4 + u_1u_2u_3u_4}, \\ t_3 &= \frac{l_0l_1l_2(l_4 + u_4) + u_4u_0(l_1l_2 + u_1l_2 + u_1u_2)}{l_1l_2l_3l_4 + u_1l_2l_3l_4 + u_1u_2l_3l_4 + u_1u_2u_3l_4 + u_1u_2u_3u_4}, \\ t_4 &= \frac{l_0l_1l_2l_3 + u_0(l_1l_2l_3 + u_1l_2l_3 + u_1u_2l_3 + u_1u_2u_3)}{l_1l_2l_3l_4 + u_1l_2l_3l_4 + u_1u_2l_3l_4 + u_1u_2u_3l_4 + u_1u_2u_3u_4}. \end{aligned}$$

Numerical values:

$$t_1 = 0.8300, \quad t_2 = 0.7188, \quad t_3 = 1.4496, \quad t_4 = 0.6387,$$

which gives

$$\begin{aligned} r_{kn} &= 1.4496, \quad r_{kn+1} = 1.7465, \quad r_{kn+2} = 2.0166, \quad r_{kn+3} = 1, \\ r_{kn+4} &= 2.2697. \end{aligned}$$

Besides

$$\frac{|1 + f^{(n+1)' }|}{r_{n+1}} - \frac{|f^{(n)' }|}{r_n} = \frac{|1 + f^{(1)' }|}{r_1} - \frac{|f^{(0)' }|}{r_0} = 0.6398 = D$$

for all  $n \geq 0$ . In view of the comments on this example in §3,  $\delta_n(f^{(n)' }) \geq 1$ . Hence, by Theorem 4.1:

$$\left| \frac{f - S_n(f^{(n)' })}{f - S_n(0)} \right| \leq (1 + |f^{(n)' }|) \left( 2 + 4|f^{(n)' }| \frac{r_{n+1}}{D} \right) \cdot \frac{0.3^{n+1}}{|a'_{n+1}|} = k_n \cdot 0.3^{n+1}$$

where

$$\begin{aligned} k_{5n+1} &= 11.2, & k_{5n+2} &= 12.9, & k_{5n+3} &= 8.17, \\ k_{5n+4} &= 13.7, & k_{5n+5} &= 10.0 & \text{for } n &= 0, 1, 2, \dots \end{aligned}$$

By Table 1.1 we see that the actual values of this ratio, for different values of  $n$ , are as given in Table 4.1.

The computation of  $\{r_n\}$  does not have to be so detailed. We see easily that for instance the following values for  $\{r_n\}$  will work:  $r'_{kn+1} = 1.5$ ,

$$r'_{kn} = r'_{kn+3} = 1, \quad r'_{kn+2} = r'_{kn+4} = 2 \quad \text{for } n = 0, 1, 2, \dots$$

Then we get  $D' \approx 0.32$  and thereby the upper bounds given by Table 4.1.

TABLE 4.1

| $n$   | 1      | 2     | 5       | 10                   | 20                    |
|---|--------|-------|---------|----------------------|-----------------------|
| $\left  \frac{f - S_n(f^{(n)'})}{f - S_n(0)} \right $ | 0.0006 | 0.002 | 0.00001 | $10^{-7}$            | $1.4 \cdot 10^{-12}$  |
| $k_n \cdot 0.3^{n+1}$                                 | 1.01   | 0.35  | 0.007   | $1.8 \cdot 10^{-5}$  | $1.0 \cdot 10^{-10}$  |
| Upper bounds with $\{r'_n\}$                          | 1.94   | 0.67  | 0.012   | $2.95 \cdot 10^{-5}$ | $1.74 \cdot 10^{-10}$ |

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