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MONOTONE DECOMPOSITIONS OF θ-CONTINUA[1]

BY

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ABSTRACT. A *θ-continuum* (*θ_n-continuum*) is a compact, connected, metric space that is not separated into infinitely many (more than n) components by any subcontinuum. The following results are among those proved. The first generalizes earlier joint work with E. J. Vought for θ_n-continua, and the second generalizes earlier work by Vought for θ_1-continua.

A θ-continuum X admits a monotone, upper semicontinuous decomposition $\mathfrak{D}$ such that the elements of $\mathfrak{D}$ have void interiors and the quotient space $X/\mathfrak{D}$ is a finite graph, if and only if, for each nowhere dense subcontinuum H of X, the continuum $T(H) = \{x \in X \mid$ if K is a subcontinuum of X and x is in the interior of K, then $K \cap H \neq \varnothing\}$ is nowhere dense. Also, if X satisfies this condition, then X is in fact a θ_n-continuum, for some natural number n, and, for each natural number m, X is a θ_m-continuum, if and only if $X/\mathfrak{D}$ is a θ_m-continuum.

1. Introduction. R. H. Bing, in developing the characterization of simple closed curves as continua that are neither separated by any subcontinuum nor cut weakly by any point, showed [**1**, Theorem 2, p. 499] that aposyndesis [**6**, **9**] is equivalent to local connectedness in continua that are not separated by any subcontinua. In 1966, I became aware that they are equivalent in any continuum that is not separated into infinitely many components by any subcontinuum, and suggested that R. W. Fitzgerald study that class of continua for his (1969) Arizona State University dissertation. That resulted in a quite thorough study of θ-spaces (connected Hausdorff spaces that are not separated into infinitely many components by any closed, connected subspace) published as [**4**]. Independently, and simultaneously, H. S. Davis did a limited study [**2**, pp. 1240–1241] of compact θ-spaces, calling them weakly irreducible.

Some definitions will be helpful in discussing the relationship of the work presented here, on compact, metric θ-spaces, to earlier work.

DEFINITIONS. Here a *continuum* is a nondegenerate, compact, connected, metric space. A *θ-continuum* (*θ_n-continuum*) is a continuum that is not separated into

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infinitely many (more than n) components by any subcontinuum. Note that a θ_n-continuum is a continuum that is not an n-od. A *θ_0-continuum* is a singleton point set (to be used when saying a continuum is a θ_n-continuum but not a θ_{n-1}-continuum, in the case where $n = 1$). If A is a subset of a continuum X, then $T(A) = \{x \in X \mid X$ is not aposyndetic at x with respect to $A\} = \{x \in X \mid$ if x is in the interior of a subcontinuum K of X then $K \cap A \neq \varnothing\}$. $T^0(A) = A$, $T^1(A) = T(A)$ and [3] for $n = 1, 2, \ldots$, $T^{n+1}(A) = T(T^n(A))$. Also $T^\omega(A) = T(\bigcup_{n=1}^{\infty} T^n(A))$. If $x \in X$, then $T^n(x) = T^n(\{x\})$, for $n = 0, 1, 2, \ldots$. See [7, note 1, p. 78] for the history of the aposyndetic set function T.

DEFINITIONS. A subcontinuum K of X is a *continuum of condensation* of X if the interior of K is void. T is *condensation preserving* on X if the interior of $T(K)$ is void whenever K is a continuum of condensation in X. A decomposition $\mathcal{D}$ of X is a *condensation decomposition* of X if each element of $\mathcal{D}$ is a continuum of condensation of X.

In Fitzgerald's dissertation he showed that every locally connected θ-continuum is a θ_n-continuum, for some natural number n [**4**, Corollary 4.8, p. 157]. Last year Jo Ford constructed a θ-continuum that is not a θ_n-continuum for any natural number n [**5**, Example 1].

Here another contribution is made to the solution of the problem of determining which θ-continua are θ_n-continua, for some n. It is shown that a θ-continuum X is a θ_n-continuum, for some n, if T is condensation preserving on X (or X has any one of several other equivalent properties).

E. J. Vought has shown [**10**, Theorem 3, p. 74] that a characterizing condition for a θ_1-continuum X to have an upper semicontinuous, condensation decomposition into a simple closed curve (i.e., with a simple closed curve as the decomposition space) is that T be condensation preserving on X. In joint work with Vought [**8**, Theorem 2, p. 269], it was shown further that this condition characterizes those θ_n-continua, for any natural number n, that have upper semicontinuous, condensation decompositions into a finite graph. Here, that result is generalized to θ-continua.

In [**8**, **10**], and here, the decomposition shown to have the desired properties is $\mathcal{D} = \{T^\omega(x) \mid x \in X\}$ (for the continua considered here $T^\omega(x) = T^n(x)$, for some natural number n). Vought also showed [**10**, Theorem 4, p. 75] that if X is a θ_1-continuum on which T is condensation preserving, then $\mathcal{D}$ is the only upper semicontinuous, condensation decomposition of X, and $X/\mathcal{D}$ is a simple closed curve, i.e., $X/\mathcal{D}$ is a locally connected θ_1-continuum. Here, that result is shown to also apply to θ_n-continua, for $n = 2, 3, \ldots$, as well.

2. Preliminaries. The following lemma [**8**, Lemma 1, p. 262] is used in generalizing Vought's work on θ_1-continua [**10**].

LEMMA 1 (GRACE AND VOUGHT). *If X is a θ-continuum, P is the projection map from X onto an upper semicontinuous, condensation decomposition of X and $[p, q]$ is an arc in $P(X)$ with the quotient topology such that $(p, q) = [p, q] \setminus \{p, q\}$ is open, then any subcontinuum M of X, such that $P(M) \supset (p, q)$, contains $P^{-1}((p, q))$ and, hence,* cl$[P^{-1}((p, q))]$ *is a continuum that is irreducible between $P^{-1}(p)$ and $P^{-1}(q)$.*

LEMMA 2 (FITZGERALD [4, THEOREM 3.4, P. 14]). *A continuum X is a θ-continuum if, and only if, it is not separated into infinitely many components by the union of any finite collection of subcontinua (i.e., if, and only if, it is weakly irreducible* [2, *p.* 1240]).

Notation. The closure of A is $\mathrm{cl}(A)$ or $\mathrm{cl}\, A$, and the interior of A is A°. If $\mathcal{C}$ is a collection of point sets then $\mathcal{C}^* = \{x \mid x \text{ is in some member of } \mathcal{C}\}$.

DEFINITIONS. An open set D is *regular* if $D = [\mathrm{cl}(D)]^\circ$. If X is a continuum, $p \in X$ and $A \subset X$, then X is *aposyndetic at p with respect to A*, if there is a subcontinuum K of X such that $X \setminus A \supset K \supset K^\circ \supset \{p\}$ [9].

LEMMA 3. *Let each of m and n be a nonnegative integer, x and y be points of a θ-continuum X, D' be an open set in X and H be a subcontinuum of X. Then* (1) *if $D' \supset T(H)$, then there is a connected, regular, open set D such that $D' \supset \mathrm{cl}\, D \supset D \supset H$, and* (2) *$x \notin T^{m+n}(y)$ if, and only if, $T^m(x) \cap T^n(y) = \varnothing$.*

PROOF. (1) Since $D' \supset T(H)$, X is aposyndetic at each point of $X \setminus D'$ with respect to H. Hence, $X \setminus D'$, being compact, is contained in the union of the interiors of the members of a finite collection $\{H_1, \ldots, H_i\}$ of continua in $X \setminus H$. By Lemma 2, $X \setminus [\bigcup_{j=1}^{i} H_j]$ has a finite number of components which are, therefore, open. Let D'' be the one that contains H. Then $D = [\mathrm{cl}\, D'']^\circ$ has the desired properties. (2) Note that $T^0(x) \cap T^{m+n}(y) = \varnothing$ is equivalent to $x \notin T^{m+n}(y)$. We prove only the following, which is essentially the inductive step in the (elementary) mathematical induction proofs of the two conditional statements in (2), and leave the rest of the proof to the reader.

Let m and n be nonnegative integers, with $n \neq 0$, and x and y be points of X such that $T^m(x) \cap T^n(y) = \varnothing$. Then $T^{m+1}(x) \cap T^{n-1}(y) = \varnothing$. To prove this, let $H = T^{n-1}(y)$ and $D' = X \setminus T^m(x)$. Then, by part (1), there is a connected open set D such that $X \setminus T^m(x) = D' \supset \mathrm{cl}\, D \supset D \supset H = T^{n-1}(y)$. By the definition of aposyndesis, X is aposyndetic at each point of D with respect to $T^m(x)$ and, hence, at each point of $T^{n-1}(y)$ with respect to $T^m(x)$. Consequently, $T^{m+1}(x) \cap T^{n-1}(y) = \varnothing$.

3. Decompositions. The proof of Lemma 4 is complicated slightly so that a proof of Lemma 5 (to be used in the proof of Theorem 2) can be isolated in it.

LEMMA 4. *If X is a θ-continuum, $x \in X$ and $[T^i(x)]^\circ = \varnothing$, for each natural number i, then there is a natural number n such that $T^{n+1}(x) = T^n(x)$.*

PROOF. Assume the lemma is false. Then, for $i = 1, 2, \ldots,$ $T^{7i+1}(x) \setminus T^{7i}(x)$ contains a point p_i'. Let $K_0' = T^0(x) = \{x\}$ and let $K_i' = T^{7i+1}(x)$, for $i = 1, 2, \ldots$. Then $p_i' \notin T^6(p_j')$, for $i > j$, and $p_i' \notin T^6(K_0')$, for $i = 1, 2, \ldots$. Some subsequence of $p_1', p_2', \ldots$ converges to some point p_0 in X. If there is a natural number i such that $p_0 \in T^{7(i-1)}(x)$, then let k be such a number. Otherwise, let $k = 1$. In either case, let $K_0 = T^{7(k-1)}(x)$ and let $p_1, p_2, \ldots$ be a subsequence of $p_k', p_{k+1}', \ldots$ that converges to p_0. Let $K_1, K_2, \ldots$ be the corresponding subsequence of $K_k', K_{k+1}', \ldots$.

For each natural number i, we now have the following. (1) K_i is nowhere dense in $X \setminus K_0$, (2) $K_{i-1} \subset K_i$, (3) $p_i \in K_i \setminus K_{i-1}$, and (4) $T^3(p_i) \cap [K_{i-1} \cup T^3([\{p_1, p_2, \ldots\} \setminus \{p_i\}])] = \varnothing$. These follow from the construction and Lemma

3(2). It is these four properties that will be used in the rest of the proof (thereby yielding a proof of Lemma 5).

Let $P = \{p_1, p_2, \ldots\}$. We wish to define inductively the connected, regular, open sets D_i, D_i', and D_i'' (and, incidentally, the closed set B_i), for each natural number i. Let $B_1 = T^3(K_0 \cup [P \setminus \{p_1\}])$. Then $B_1 \cap T^3(p_1) = \varnothing$ and so, by Lemma 3(1) used thrice ($T^2(p_1)$ and $T(p_1)$ are continua [3, Corollary 1.1, p. 115]), there are strongly nested (i.e., each contains the closure of the next), connected, regular open sets D_1, D_1', D_1'' such that $X \setminus B_1 \supset \text{cl } D_1 \supset D_1' \supset D_1'' \supset \{p_1\}$. Suppose B_i, D_i, D_i', and D_i'' have been defined for each natural number $i \leqslant n$. Let

$$B_{n+1} = K_0 \cup \left[\bigcup_{i=1}^{n} (K_i \cap \text{cl } D_i) \right] \cup T^3(P \setminus \{p_{n+1}\}).$$

Then $B_{n+1} \cap T^3(p_{n+1}) = \varnothing$ and so there are strongly nested, regular, open sets D_{n+1}, D_{n+1}', and D_{n+1}'' such that $X \setminus B_{n+1} \supset \text{cl } D_{n+1} \supset D_{n+1}' \supset D_{n+1}'' \supset \{p_{n+1}\}$. For each natural number i, $D_i \setminus \text{cl } D_i'$ has a finite number of components [4, Corollary 3.6, p. 144]. Hence $\text{cl } D_i \setminus D_i' = \text{cl}(D_i \setminus \text{cl } D_i')$ has a finite number of components. For i and j natural numbers, let $\mathcal{G}_{ij}$ be the collection of all components of $[\text{cl } D_i \setminus D_i'] \cup \text{cl } D_i''$ that intersect K_j. Note that $\text{cl } D_i'' \in \mathcal{G}_{ij}$, for $i \leqslant j$ and $\mathcal{G}_{ij} = \varnothing$, if $i > j$. Let $\mathcal{G}_i = \bigcup_{j=1}^{\infty} \mathcal{G}_{ij}$.

For each natural number i, we wish to define a nowhere dense set F_i, contained in $\text{cl } D_i$, that connects together the elements of $\mathcal{G}_i$ in the same way that $[\bigcup_{j=1}^{\infty} K_j] \cap \text{cl } D_i$ does. First define $F(A, B)$, for each pair $\{A, B\}$ of elements of $\mathcal{G}_{ij}$, as follows. If no component of $K_{j-1} \cap \text{cl } D_i$ intersects both A and B, but some component of $K_j \cap \text{cl } D_i$ does, then let $F(A, b) = F(B, A)$ be some such component. Otherwise let $F(A, B) = \varnothing$. Let $\mathcal{F}_{ij} = \{F(A, B) \mid F(A, B) \neq \varnothing \text{ and } (A, B) \in \mathcal{G}_{ij} \times \mathcal{G}_{ij}\}$. Note that $\mathcal{F}_{ij}$ is a finite collection of components of $K_j \cap \text{cl } D_i$, and that $\bigcup_{j=1}^{\infty} \mathcal{F}_{ij}$ is finite, since $\mathcal{G}_i$ is finite. Hence $F_i = \bigcup_{j=1}^{\infty} \mathcal{F}_{ij}^*$ is nowhere dense.

For $j = 1, 2, \ldots$, let

$$L_j = \bigcup_{i=1}^{j} \left[\bigcup_{k=1}^{j} (\mathcal{F}_{ik}^* \cup \mathcal{G}_{ik}^*) \right] \cup \left[K_j \setminus \left(\bigcup_{i=1}^{j} D_i' \right) \right].$$

L_j is a continuum. To see this, assume otherwise, i.e., assume there is a natural number j such that L_j is not connected (it is clearly closed). Let $L_j = R \cup S$, separated, where $K_0 \subset R$. Let $R_{\mathcal{G}} = R \cap [\bigcup_{i=1}^{j} \mathcal{G}_{ij}^*]$ and $S_{\mathcal{G}} = S \cap [\bigcup_{i=1}^{j} \mathcal{G}_{ij}^*]$. Some subcontinuum of K_1 is irreducible from K_0 to $\text{cl } D_1$ and it is contained in $K_j \setminus [\bigcup_{i=1}^{j} D_i']$, so $R_{\mathcal{G}} \neq \varnothing$. Each component of $\bigcup_{i=1}^{j} \mathcal{F}_{ij}^*$ and of $K_j \setminus \bigcup_{i=1}^{j} D_j'$ contains a point $\bigcup_{i=1}^{j} \mathcal{G}_{ij}^*$, by definition in the first instance, and by the "to the boundary" theorem in the second instance, so $S_{\mathcal{G}} \neq \varnothing$ since $S \neq \varnothing$.

Each $\mathcal{G}_{ij}$ is either void or a finite collection of continua that intersect K_j, so $K_j \cup [\bigcup_{i=1}^{j} \mathcal{G}_{ij}^*]$ is a continuum containing L_j. From this, it follows (by the "wire cutting" theorem) that some component N of $K_j \setminus [\bigcup_{i=1}^{j} \mathcal{G}_{ij}^*]$ has a limit point in $R_{\mathcal{G}}$ and in $S_{\mathcal{G}}$. But either (1) $N \subset [K_j \setminus (\bigcup_{i=1}^{j} D_i')] \subset L_j$, and there is no separation of L_j between $R_{\mathcal{G}}$ and $S_{\mathcal{G}}$ (a contradiction), or (2) $N \subset [D_i' \setminus \text{cl } D_i'']$, in which case, for some $i, k \leqslant j$, there is a member of $\mathcal{F}_{ik}$ which is contained in L_j and intersects both $R_{\mathcal{G}}$ and $S_{\mathcal{G}}$ (also a contradiction). Hence L_j is connected.

Let $L = \mathrm{cl}(\bigcup_{j=1}^{\infty} L_j)$. It is the union of connected sets each of which contains K_0, so it is connected. It is obviously closed, so it is a continuum. Also, for each natural number i, $[\bigcup_{j=1}^{\infty} L_j] \cap [D_i' \setminus \mathrm{cl}\, D_i''] \subset \bigcup_{j=1}^{\infty} \mathcal{F}_{ij}^*$, which is the union of fewer than n^2 nowhere dense sets, where n is the number of elements in $\mathcal{G}_i$, which is not more than the (finite) number of components of $[\mathrm{cl}\, D_i \setminus D_i'] \cup \mathrm{cl}\, D_i''$. It follows that $L \cap [D_i' \setminus \mathrm{cl}\, D_i'']$ is nowhere dense. Since X is a θ-continuum, $X \setminus L$ has a finite number of components. Hence $\mathrm{cl}(X \setminus L)$ is the union of a finite collection of continua, so $X \setminus \mathrm{cl}(X \setminus L)$ has a finite number of components [**4**, Corollary 3.6, p. 144]. But $\mathrm{cl}(X \setminus L) \supset (\mathrm{cl}\, D_i' \setminus D_i'')$ and $\mathrm{cl}(X \setminus L) \cap D_i'' = \varnothing$ (since $D_i'' \subset L$), for $i = 1, 2, \ldots$. It follows that $D_1'', D_2'', \ldots$ are all components of $X \setminus \mathrm{cl}(X \setminus L)$. But, again, the complement of the union of a finite collection of continua has a finite number of components [**4**, Corollary 3.6, p. 144]. Since the assumption that the lemma is false leads to a contradiction, the lemma is true.

Lemma 4 yields an alternate proof of the "if" part of the following theorem [**8**, Theorem 1, p. 263].

THEOREM 1 (GRACE AND VOUGHT). *Let X be a θ_n-continuum. Then X admits an upper semicontinuous, condensation decomposition into a finite graph if, and only if, T is condensation preserving on X.*

PROOF OF "IF" PART. Suppose T is condensation preserving on X. By Lemma 4, the elements of $\mathcal{D} = \{T^{\omega}(x) \mid x \in X\}$ are of the form $T^{m}(x)$, and hence, by mathematical induction, are continua [**3**, Corollary 1.1, p. 115] and nowhere dense. By [**4**, Theorem 6.3, p. 170], any decomposition of X, whose elements are connected, T-closed sets, is upper semicontinuous, so $\mathcal{D}$ is upper semicontinuous. Lemma 3(2) can be used to show that no T-closed set intersects an element of $\mathcal{D}$ in a proper subset of that element. Hence, $\mathcal{D}$ refines any (upper semicontinuous) decomposition of X with T-closed elements. By [**4**, Theorem 6.1, p. 169], such decomposition spaces are finite graphs.

In the process of proving Lemma 4, we have proved the following lemma, which will be used in proving Lemma 6.

LEMMA 5. *If X is a θ-continuum, then X does not contain a continuum K_0, a sequence $K_1, K_2, \ldots$ of continua, and points p_i in $K_i \setminus K_{i-1}$ such that, for each natural number i, (1) K_i is nowhere dense in $X \setminus K_0$, (2) $K_{i-1} \subset K_i$, and (3) $T^3(p_i) \cap [K_{i-1} \cup T^3([\{p_1, p_2, \ldots\} \setminus \{p_i\}])] = \varnothing$.*

LEMMA 6. *If X is a θ-continuum such that $T^n(x)$ is nowhere dense, for each x in X, and for $n = 1, 2, \ldots$, then $\mathcal{D} = \{T^{\omega}(x) \mid x \in X\}$ is upper semicontinuous.*

PROOF. By Lemma 4, the elements of $\mathcal{D}$ are continua of condensation. Assume $\mathcal{D}$ is not upper semicontinuous. Then there exists a sequence $(p_1, p_1'), (p_2, p_2'), \ldots$ of ordered pairs of points of X such that (1) $p_i \in T^{\omega}(p_i')$, for $i = 1, 2, \ldots$, (2) $T^{\omega}(p_i') \cap T^{\omega}(p_j') = \varnothing$, for $i \neq j$, (3) $p_1', p_2', \ldots$ converges to some point p_0' not in $\bigcup_{i=1}^{\infty} T^{\omega}(p_i')$, and (4) $p_1, p_2, \ldots$ converges to some point p_0 not in $T^{\omega}(p_0')$. By Lemmas 4 and 3(2), $T^3(p_0) \cap T^{\omega}(p_0') = \varnothing$, and so, there are strongly nested, connected, regular, open sets D_0, D_0', D_0'' such that $X \setminus T^\omega(p_0') \supset \operatorname{cl} D_0 \supset D_0' \supset D_0'' \supset \{p_0\}$. Since $(\operatorname{cl} D_0) \cap T^\omega(p_0') = \varnothing$, there is a continuum K_0 such that $X \setminus \operatorname{cl} D_0 \supset K_0 \supset K_0^\circ \supset \{p_0'\}$. If $x \in D_0''$, then $T(x) \subset \operatorname{cl} D_0''$, $T^2(x) \subset \operatorname{cl} D_0'$, and $T^3(x) \subset \operatorname{cl} D_0$. Hence $T^3(x) \cap K_0 = \varnothing$. Without loss of generality we can assume that $\{p_1, p_2, \ldots\} \subset D_0''$ and $\{p_1', p_2', \ldots\} \subset K_0^\circ$. For $i = 1, 2, \ldots,$ let $K_i = K_0 \cup [\bigcup_{j=1}^i T^\omega(p_j')]$. By Lemma 4, for each natural number j, there is a natural number $n(j)$ such that $T^\omega(p_j') = T^{n(j)}(p_j')$. Hence K_i is the union of K_0 and i continua of condensation that intersect K_0. Consequently, (1) K_i is nowhere dense in $X \setminus K_0$, (2) $K_{i-1} \subset K_i$, and (3) $p_i \in K_i \setminus K_{i-1}$. Above, it was shown (without loss of generality) that $T^3(p_i) \cap K_0 = \varnothing$. Also (4) $T^3(p_i) \cap [K_{i-1} \cup T^3([\{p_1, p_2, \ldots\} \setminus \{p_i\}])] \subset T^\omega(p_i) \cap [K_0 \cup (\bigcup_{j \neq i} T^\omega(p_j))] = \varnothing$. This contradicts Lemma 5, so $\mathfrak{D}$ is upper semicontinuous.

THEOREM 2. *Let X be a θ-continuum. Then X admits an upper semicontinuous, condensation decomposition into a finite graph, if $[T^n(x)]^\circ = \varnothing$, for all x in X and each natural number n, and only if T is condensation preserving on X.*

PROOF. Suppose $[T^n(x)]^\circ = \varnothing$ for all appropriate x and n. By Lemmas 4 and 6, $\mathfrak{D} = \{T^\omega(x) \mid x \in X\}$ is a condensation decomposition and is upper semicontinuous. By Lemma 3(2), $\mathfrak{D}$ refines any decomposition of X with T closed elements. Consequently, $\mathfrak{D}$ is the minimal decomposition (with respect to refinement) of X (i.e., the core decomposition of X, in the sense of FitzGerald [**4**, Definition 6.0, p. 168]) with respect to being upper semicontinuous with T-closed elements. By [**4**, Theorem 6.1, p. 169], the decomposition space of such a decomposition is a finite graph.

The proof of the "only if" part of [**8**, Theorem 1, p. 263] proves the "only if" part of this theorem, since it is only assumed that X is a θ-continuum, rather than a θ_n-continuum, in that proof.

THEOREM 3. *Suppose X is a θ-continuum and T is condensation preserving on X. Then X is a θ_n-continuum, for some natural number n, and $\mathfrak{D} = \{T^\omega(x) \mid x \in X\}$ is an upper semicontinuous, condensation decomposition of X such that X is a θ_m-continuum if, and only if, $X/\mathfrak{D}$ is a θ_m-continuum. Also $\mathfrak{D} = \{T^{2n}(x) \mid x \in X\}$.*

PROOF. Suppose $[T(H)]^\circ = \varnothing$, for each continuum of condensation H, then, by mathematical induction $[T^n(x)]^\circ = \varnothing$, for all x in X and each natural number n. Then, by the proof of Theorem 2, $X/\mathfrak{D}$ is an upper semicontinuous, condensation decomposition into a finite graph. But a finite graph is a locally connected θ_n-continuum, for some natural number n. By [**8**, Lemma 4, p. 268], $\mathfrak{D} = \{T^{2n}(x) \mid x \in X\}$.

Let P be the projection map from X onto $X/\mathfrak{D}$ and let K be any subcontinuum of X. Then $P(K)$ is a subcontinuum of $X/\mathfrak{D}$ and, hence, separates $X/\mathfrak{D}$ into n, or fewer (perhaps 0) components. By Lemma 1, $P^{-1}(P(K)) \setminus K$ is nowhere dense. Also, for each component R of $(X/\mathfrak{D}) \setminus P(K)$, $P^{-1}(R)$ is connected. Together these imply that $X \setminus K$ has no more components than does $(X/\mathfrak{D}) \setminus P(K)$, i.e., $X \setminus K$ has no more than n components. Hence X is a θ_n-continuum. The proof, in fact, shows that X is a θ_m-continuum if $X/\mathfrak{D}$ is.

All that remains to be shown is that $X/\mathfrak{D}$ is a θ_m-continuum, if X is. That follows from the proof of [**4**, Theorem 3.1, p. 142].

Lemmas 7 and 8, following, are used in the proof of Theorem 4. Lemma 7 is similar to Lemma 1 viewed differently.

LEMMA 7. *If X is a θ-continuum and $\mathfrak{G}$ is an upper semicontinuous, condensation decomposition of X into a finite graph, then any continuum of condensation in X is a subset of some element of $\mathfrak{G}$.*

PROOF. Let P be the projection map from X onto $X/\mathfrak{G}$. Let M be a subcontinuum of X that is not contained in any element of $\mathfrak{G}$, i.e., such that $P(M)$ is not a singleton set in $X/\mathfrak{G}$. Since $X/\mathfrak{G}$ is a finite graph, there is an arc $[p, q]$, contained in $P(M)$, such that $(p, q) = [p, q] \setminus \{p, q\}$, is open in $X/\mathfrak{G}$. By Lemma 1, $M \supset P^{-1}((p, q))$, which is open in X, so M is not a continuum of condensation. Hence each continuum of condensation of X is contained in some element of $\mathfrak{G}$, as was to be proven.

LEMMA 8. *If X is a continuum and $\mathfrak{G}$ is a monotone decomposition of X such that $X/\mathfrak{G}$ is locally connected, then each element of $\mathfrak{G}$ is T-closed in X.*

PROOF. Let P be the projection map from X onto $X/\mathfrak{G}$. Let E be any element of $\mathfrak{G}$ and let $x \in X \setminus E$. Since $X/\mathfrak{G}$ is locally connected at $P(x)$ there is a connected, open set A, in $X/\mathfrak{G}$, such that $(X/\mathfrak{G}) \setminus \{E\} \supset \operatorname{cl} A \supset A \supset \{P(x)\}$. Then $X \setminus E \supset P^{-1}(\operatorname{cl} A) \supset P^{-1}(A) \supset \{x\}$. But $P^{-1}(\operatorname{cl} A)$ is a continuum, since $\mathfrak{G}$ is monotone and $P^{-1}(A)$ is open, by the definition of the quotient topology. Hence X is aposyndetic at x with respect to E. But E is any element of $\mathfrak{G}$ and x is any point in $X \setminus E$. Hence all elements of $\mathfrak{G}$ are T-closed.

THEOREM 4. *Let X be a θ-continuum and let $\mathfrak{D} = \{T^\omega(x) \mid x \in X\}$. Then the following are equivalent.*

(1) *$[T^n(x)]^\circ = \varnothing$, for all x in X and any natural number n.*

(2) *T is condensation preserving on X.*

(3) *Each point of X is contained in a nowhere dense, T-closed subset of X.*

(4) *Every continuum of condensation in X is contained in a T-closed continuum of condensation in X.*

(5) *X has an upper semicontinuous, condensation decomposition into a finite graph.*

(6) *$\mathfrak{D}$ is an upper semicontinuous, condensation decomposition of X into a finite graph, and there is a subset A of X such that* (i) *$T^{n+1}(y) = T^n(y)$, for each y in A, where $n = [(m+1)/2]$, for m the Menger order of $X/\mathfrak{D}$, and* (ii) *$\mathfrak{D} = \{T^n(y) \mid y \in A\}$.*

(7) *$\mathfrak{D}$ is the only upper semicontinuous, condensation decomposition of X with T-closed elements.*

(8) *$\mathfrak{D}$ is the only condensation decomposition of X such that $X/\mathfrak{D}$ is aposyndetic (= semi-locally-connected).*

(9) *X has a condensation decomposition whose elements are T-closed, and X is a θ_n-continuum, for some natural number n.*

(10) *For some natural number* n, X *is a* θ_n*-continuum but not a* θ_{n-1}*-continuum, and* $\mathcal{D}$ *is an upper semicontinuous, condensation decomposition such that* $X/\mathcal{D}$ *is a* θ_n*-continuum but not a* θ_{n-1}*-continuum.*

(11) *For some natural number* n, X *is a* θ_n*-continuum, but not a* θ_{n-1}*-continuum,* $T^\omega(x) = T^{2n}(x)$, *for each* x *in* X *(so* $\mathcal{D} = \{T^{2n}(x) \mid x \in X\}$*), and* $\mathcal{D}$ *is an upper semicontinuous, condensation decomposition into a finite graph.*

(12) *For some natural number* n, X *is a* θ_n*-continuum but not a* θ_{n-1}*-continuum, and there is a subset* A *of* X *such that* (i) $T^{n+1}(y) = T^n(y)$, *for each* y *in* A, (ii) $\mathcal{D} = \{T^n(y) \mid y \in A\}$, *and* (iii) $\mathcal{D}$ *is an upper semicontinuous, condensation decomposition of* X *such that* $X/\mathcal{D}$ *is a locally connected* θ_n*-continuum that is not a* θ_{n-1}*-continuum.*

PROOF. The plan of the proof is to prove the conditional statements indicated by the arrows in the following. (1) → (7) → (8) → (9) → (4) → (2) → (10) → (11) → (5) → (6) → (12) → (3) → (1). In the proofs of most of these conditional statements (e.g., If (1) then (7)), the antecedent is assumed, without that being stated explicitly, and the consequent is proved.

(1) → (7). PROOF. By the proof of Theorem 2, $\mathcal{D}$ is the minimal decomposition of X with respect to being upper semicontinuous with T-closed elements, and also $X/\mathcal{D}$ is a finite graph. By Lemma 7, any condensation decomposition of X refines $\mathcal{D}$. Therefore $\mathcal{D}$ is the only upper semicontinuous, condensation decomposition of X with T-closed elements.

(7) → (8). PROOF. By [**4**, Theorem 6.1, p. 169], $\mathcal{D}$ is the minimal decomposition of X with respect to being monotone and having $X/\mathcal{D}$ be semi-locally-connected (= aposyndetic), and $X/\mathcal{D}$ is a finite graph. By Lemma 7, any condensation decomposition of X refines $\mathcal{D}$. Consequently, $\mathcal{D}$ is the only condensation decomposition of X such that $X/\mathcal{D}$ is aposyndetic.

(8) → (9). PROOF. Since any decomposition of X that refines $\mathcal{D}$ is a condensation decomposition of X, $\mathcal{D}$ is the minimal decomposition of X with respect to being monotone and having $X/\mathcal{D}$ be aposyndetic. By [**4**, Theorem 6.1, p. 169], $\mathcal{D}$ is upper semicontinuous and has T-closed elements. Also, for each appropriate x and m, $T^m(x)$ is a subset of some member of $\mathcal{D}$, and hence has void interior. By Theorem 2 then, T is condensation preserving on X and hence X is a θ_n-continuum, for some natural number n, by Theorem 3.

(9) → (4). PROOF. By [**4**, Theorem 6.3, p. 170], the decomposition is upper semicontinuous. Hence, by [**4**, Theorem 6.1, p. 169], the minimal decomposition with respect to being upper semicontinuous with T-closed elements is a condensation decomposition, and that decomposition space is a finite graph. By Lemma 7, each continuum of condensation in X is contained in an element of the minimal decomposition, i.e., is contained in a T-closed continuum of condensation.

(4) → (2) obviously.

(2) → (10), by Theorem 3.

(10) → (11). PROOF. By a use of Lemma 3(2), as in the proof of the "if" part of Theorem 1, $X/\mathcal{D}$ is a finite graph and consequently is locally connected. Then, by [**8**, Lemma 4, p. 268], $T^\omega(x) = T^{2n}(x)$, for each x in X.

(11) $\to$ (5) obviously.

(5) $\to$ (6). PROOF. By Theorem 2, T is condensation preserving on X. Then, by Theorem 3, X is a θ_l-continuum, for some natural number l, and hence by Theorem 1, $\mathfrak{D}$ is an upper semicontinuous, condensation decomposition of X into a finite graph. The rest of (6) follows from the proof of [**8**, Lemma 4, p. 268].

(6) $\to$ (12). PROOF. By Theorem 2, T is condensation preserving on x and, hence, by Theorem 3, there is a natural number n such that X and $X/\mathfrak{D}$ are θ_n-continua but not θ_{n-1}-continua. By [**4**, Theorem 4.9, p. 158], the Menger order of X is not more than $2n$. Then $n = [(2n+1)/2] \geqslant [(m+1)/2] = l$, where m is the Menger order of X. Let $A = \{y \in X \mid T^{l+1}(y) = T^l(y)\}$. Then $T^{n+1}(y) = T^n(y) = T^l(y)$, for each y in A, and hence $\mathfrak{D} = \{T^n(y) \mid y \in A\}$.

(12) $\to$ (3). PROOF. By Lemma 8, each element of $\mathfrak{D}$ is T-closed. But the elements of $\mathfrak{D}$ are continua of condensation and $\mathfrak{D}^* = X$. Therefore each point of X is contained in a member of $\mathfrak{D}$ and hence is contained in a nowhere dense, T-closed subset of X.

(3) $\to$ (1) obviously, by mathematical induction.

REFERENCES

1. R. H. Bing, *Some characterizations of arcs and simple closed curves*, Amer. J. Math. **70** (1948), 497–506. MR **10**, 55.

2. H. S. Davis, *A note on connectedness im kleinen*, Proc. Amer. Math. Soc. **19** (1968), 1237–1241. MR **40** #8021.

3. H. S. Davis, D. P. Stadtlander and P. M. Swingle, *Properties of the set function T^n*, Portugal. Math. **21** (1962), 114–133. MR **25** #5501.

4. R. W. FitzGerald, *Connected sets with a finite disconnection property*, Studies in Topology (Proc. Conf., Univ. North Carolina, Charlotte, N. C., 1974; dedicated to Math. Sect. Polish Acad. Sci.), Academic Press, New York, 1975, pp. 139–173. MR **51** #1730.

5. Jo Ford, *On n-ods*, preprint.

6. E. E. Grace, *A bibliography on aposyndesis*, General Topology and Modern Analysis (Proc. Conf., Univ. California, Riverside, Calif., 1980; in honor of F. Burton Jones), Academic Press, New York, 1981, pp. 493–513. MR **82f**:54053.

7. ______, *Aposyndesis and weak cutting*, General Topology and Modern Analysis (Proc. Conf., Univ. California, Riverside, Calif., 1980; in honor of F. Burton Jones), Academic Press, New York, 1981, pp. 71–82. MR **82g**:54049.

8. E. E. Grace and E. J. Vought, *Monotone decompositions of θ_n-continua*, Trans. Amer. Math. Soc. **263** (1981), 261–270. MR **81k**:54058.

9. F. B. Jones, *Concerning aposyndetic and nonaposyndetic continua*, Bull. Amer. Math. Soc. **58** (1952), 137–151. MR **14**, 71.

10. E. J. Vought, *Monotone decompositions of continua not separated by any subcontinua*, Trans. Amer. Math. Soc. **192** (1974), 67–78. MR **49** #6189.

11. ______, *Structure of θ and θ_n-continua*, Proceedings: Conference on Metric Spaces, Generalized Metric Spaces and Continua (Guilford College, Greensboro, N. C., 1979, honoring F. B. Jones), Guilford College, Greensboro, N. C., 1980, pp. 121–129.

12. ______, *Monotone decompositions of continua*, General Topology and Modern Analysis (Proc. Conf., Univ. California, Riverside, Calif., 1980; honoring F. Burton Jones), Academic Press, New York, 1981, pp. 105–113.

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