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LINKING NUMBERS AND THE ELEMENTARY IDEALS OF LINKS

BY

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ABSTRACT. Let $L = K_1 \cup \cdots \cup K_\mu \subseteq S^3$ be a tame link of $\mu \geq 2$ components, and H the abelianization of $G = \pi_1(S^3 - L)$. Let $\mathfrak{L} = (\mathfrak{L}_{ij})$ be the $\mu \times \mu$ matrix with entries in $\mathbf{Z}H$ given by $\mathfrak{L}_{ii} = \Sigma_{k \neq i} l(K_i, K_k) \cdot (t_k - 1)$ and for $i \neq j$ $\mathfrak{L}_{ij} = l(K_i, K_j) \cdot (1 - t_i)$. Then if $0 < k < \mu$

$$\sum_{i=0}^{k-1} E_{\mu-k+i}(L) \cdot (IH)^{2i} + (IH)^{2k} = \sum_{i=0}^{k-1} E_{\mu-k+i}(\mathfrak{L}) \cdot (IH)^{2i} + (IH)^{2k}.$$

Various consequences of this equality are derived, including its application to the reduced elementary ideals. These results are used to give several different characterizations of links in which all the linking numbers are zero.

1. Introduction. Let $L = K_1 \cup \cdots \cup K_\mu \subseteq S^3$ be a tame link of $\mu \geq 2$ components, that is, a union of μ pairwise disjoint simple closed curves K_i, each of which is carried onto a polygonal curve by some autohomeomorphism of S^3. If

$$G = \pi_1(S^3 - L)$$

is the group of L, and H is its abelianization $H = G/[G, G]$, then H is freely generated (as an abelian group) by certain elements $t_1, \ldots, t_\mu$, the meridians of L, and its integral group ring $\mathbf{Z}H$ consists of polynomials (with integer coefficients) in $t_1, \ldots, t_\mu, t_1^{-1}, \ldots, t_\mu^{-1}$.

In this paper we will consider the relationship between two families of invariants of L, the linking numbers $l(K_i, K_j)$ (defined for $i \neq j \in \{1, \ldots, \mu\}$), and the elementary ideals $E_k(L)$ (defined for every $k \in \mathbf{Z}$). The linking numbers are integers, whose signs depend on a choice of orientations for the K_i. The elementary ideals form an ascending sequence of ideals of the ring $\mathbf{Z}H$, with the property that $E_k(L) = 0$ $\forall k \leq 0$. The first elementary ideal has an especially simple structure: if ε: $\mathbf{Z}H \to \mathbf{Z}$ is the augmentation map (i.e., the homomorphism with $\varepsilon(t_i) = 1$ $\forall i$), and $IH = \ker \varepsilon$ is the augmentation ideal of $\mathbf{Z}H$, then there is an element $\Delta_1(L) \in \mathbf{Z}H$, the *Alexander polynomial of* L, such that $E_1(L) = \Delta_1(L) \cdot IH$. (Though this does not uniquely describe $\Delta_1(L)$, two Alexander polynomials of L will differ only in that one is the product of the other with some unit of $\mathbf{Z}H$.)

The equality $\varepsilon\Delta_1(K_1 \cup K_2) = \pm l(K_1, K_2)$, valid for any tame link $K_1 \cup K_2$ of two components, was discovered by G. Torres in [7]; this is the simplest instance of

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the relationship between the elementary ideals and linking numbers of a link. Subsequent results, notably those of F. Hosokawa [**3**, Theorem 2] and M. E. Kidwell [**4**], have dealt with the *reduced* Alexander polynomial of L (defined in §5). These results will be discussed in detail in §§5 and 6, respectively.

Let $\mathfrak{L} = (\mathfrak{L}_{ij})$ be the $\mu \times \mu$ matrix whose entries are the elements of $\mathbf{Z}H$ given by $\mathfrak{L}_{ii} = \Sigma_{k \neq i} l(K_i, K_k) \cdot (t_k - 1)$ and for $i \neq j$, $\mathfrak{L}_{ij} = l(K_i, K_j) \cdot (1 - t_i)$. For $0 < k < \mu$ let $E_k(\mathfrak{L}) \subseteq \mathbf{Z}H$ be the ideal generated by the determinants of the $(\mu - k) \times (\mu - k)$ submatrices of $\mathfrak{L}$. We will prove the following:

THEOREM 1. *If* $0 < k < \mu$, *then*

$$\sum_{i=0}^{k-1} E_{\mu-k+i}(L) \cdot (IH)^{2i} + (IH)^{2k} = \sum_{i=0}^{k-1} E_{\mu-k+i}(\mathfrak{L}) \cdot (IH)^{2i} + (IH)^{2k}.$$

(*Here* $(IH)^0 = \mathbf{Z}H$.)

Clearly $E_j(\mathfrak{L}) \subseteq (IH)^{\mu-j}$ whenever $0 < j < \mu$. As observed in [**8**], it is also true that $E_j(L) \subseteq (IH)^{\mu-j}$ whenever $0 < j < \mu$. Combining these observations with Theorem 1, we obtain a simpler (but weaker) statement,

COROLLARY 1. *If* $0 < k < \mu$, *then*

$$E_{\mu-k}(L) + (IH)^{k+1} = E_{\mu-k}(\mathfrak{L}) + (IH)^{k+1}.$$

Corollary 1 stands in contrast to the fact that $E_{\mu+k}(L) + (IH)^j = \mathbf{Z}H$ for any $j, k \geqslant 0$. (See [**8**], where the equivalent statement $\varepsilon E_{\mu+k}(L) = \mathbf{Z}\ \forall k \geqslant 0$ is proven. To show that these two statements are, in fact, equivalent, note that the former trivially implies the latter, which implies that $E_{\mu+k}(L) + IH = \mathbf{Z}H$. If $j \geqslant 1$ and $E_{\mu+k}(L) + (IH)^j = \mathbf{Z}H$, then $IH = IH \cdot \mathbf{Z}H = IH \cdot (E_{\mu+k}(L) + (IH)^j) \subseteq E_{\mu+k}(L) + (IH)^{j+1}$, and hence $\mathbf{Z}H = E_{\mu+k}(L) + IH = E_{\mu+k}(L) + (IH)^{j+1}$.)

For $1 \leqslant i \leqslant \mu$ let $d_i \geqslant 0 \in \mathbf{Z}$ be the g.c.d. of the linking numbers of the other components of L with K_i. Then $E_{\mu-1}(\mathfrak{L})$ is the ideal of $\mathbf{Z}H$ generated by the elements $d_i \cdot (t_i - 1)$, as can easily be seen, so by Corollary 1 $E_{\mu-1}(L) + (IH)^2 = E_{\mu-1}(\mathfrak{L}) + (IH)^2$ is the ideal generated by $(IH)^2$ together with these elements $d_i \cdot (t_i - 1)$; thus the sequence $(d_1, \ldots, d_\mu)$ determines the ideal $E_{\mu-1}(L) + (IH)^2$. Conversely, if $1 \leqslant i \leqslant \mu$ then either $p \cdot (t_i - 1) \notin E_{\mu-1}(L) + (IH)^2\ \forall p \neq 0 \in \mathbf{Z}$ (in which case $d_i = 0$), or else there is a least integer $p > 0$ with $p \cdot (t_i - 1) \in E_{\mu-1}(L) + (IH)^2$ (in which case d_i is this least p); thus, in either case, the ideal $E_{\mu-1}(L) + (IH)^2$ determines d_i. Since this holds for each i, we have

COROLLARY 2. *The sequence* $(d_1, \ldots, d_\mu)$ *is equivalent, as an invariant of* L, *to the ideal* $E_{\mu-1}(L) + (IH)^2$, *that is, the sequence determines the ideal, and vice versa.*

The *lower central series subgroups of* G are defined by: $G_1 = G$, and if $q \geqslant 2$ then G_q is the subgroup of G generated by the set of all commutators $[g, h] = ghg^{-1}h^{-1}$ with $g \in G_{q-1}$ and $h \in G$. Our proof of Theorem 1 is based on the investigation of the quotient G/G_3 by K. T. Chen in [**1**].

In §6 we turn our attention to a family of special links, those in which the linking numbers are all zero. A few different characterizations of this family are set forth in

THEOREM 2. *Any two of the following statements about the tame link* $L = K_1 \cup \cdots \cup K_\mu \subseteq S^3$ *are equivalent*:

(a) $l(K_i, K_j) = 0$ *whenever* $i \neq j \in \{1,\dots,\mu\}$;

(b) *if* Φ *is a free group on* μ *generators then* $G/G_3 \cong \Phi/\Phi_3$;

(c) *there is an epimorphism* $G \twoheadrightarrow \Phi/\Phi_3$;

(d) *the Alexander modules of* G/G_3 *and* Φ/Φ_3 (*defined in* §4) *are isomorphic*;

(e) *there is a* $\mathbf{Z}H$-*epimorphism of the Alexander module of* G *onto that of* Φ/Φ_3;

(f) $E_{\mu-k}(L) \subseteq (IH)^{2k}$ *whenever* $0 < k < \mu$;

(g) $E_{\mu-1}(L) \subseteq (IH)^2$; *and*

(h) $E_{\mu-1}(L) \subseteq J$, *where* J *is the ideal of* $\mathbf{Z}H$ *generated by the products*

$$(t_i - 1)(t_j - 1), \qquad i \neq j.$$

In particular, we may note, without further comment, the obvious analogy between condition (c) and the defining condition of (homology) boundary links.

Before completing this introduction, we must mention our gratitude to Jonathan A. Hillman, whose correspondence, during the time this work was in progress, was both informative and inspirational. Also, the referee is to be thanked for suggesting a significant simplification of the original argument.

2. Presentations of G **and** G/G_3. A presentation of the group $G = \pi_1(S^3 - L)$ may be obtained from a regular projection of L in the plane, that is, a projection in which the only singularities are double points (or "crossings"), of which there are only finitely many. After removing a short arc on each side of the underpassing point of each crossing, what remains is a collection of pairwise disjoint, simple, tame arcs in the plane. We may denote these arcs by e_{ij} ($1 \leqslant i \leqslant \mu$ and $1 \leqslant j \leqslant j_i$), in such a way that for each i, $e_{i1} \cup \cdots \cup e_{ij_i}$ is the image of K_i in the projection, and $e_{i1},\dots,e_{ij_i}$ appear consecutively around K_i; also, we may orient K_i so that this direction around it (from e_{i1} to e_{i2}, and so on) is preferred. G then has the presentation $\langle x_{ij};\, r_{ij}\rangle$, with a generator x_{ij} whenever $1 \leqslant i \leqslant \mu$ and $1 \leqslant j \leqslant j_i$, and a relator $r_{ij} = x_{pq}^{\delta_{ij}} x_{ij} x_{pq}^{-\delta_{ij}} x_{ij+1}^{-1}$ whenever there is a crossing in which e_{pq} separates e_{ij} from e_{ij+1}; $\delta_{ij} = 1$ or -1 according to whether e_{pq} crosses over e_{ij} moving from left to right or from right to left, as seen by an observer on e_{ij} facing e_{ij+1}. The index j of e_{ij} is to be considered modulo j_i.

If $i \neq p \in \{1,\dots,\mu\}$ the linking number $l(K_i, K_p)$ is the sum $\Sigma\delta_{ij}$, taken over crossings in which K_p passes over K_i. Though it is not immediately apparent from this description, $l(K_i, K_p) = l(K_p, K_i)$ [**6**, p. 132].

Since $\langle x_{ij};\, r_{ij}\rangle$ is a presentation of G, if F is the free group on the set $\{x_{ij} \mid 1 \leqslant i \leqslant \mu, 1 \leqslant j \leqslant j_i\}$ of generators then there is an epimorphism $\eta\colon F \to G$ whose kernel is the least normal subgroup of F containing $\{r_{ij}\}$. If $\alpha\colon G \to H = G/[G, G]$ is the canonical map onto the quotient, then it is clear from the form of the r_{ij} that $\alpha\eta(x_{ij}) = \alpha\eta(x_{ik})$ $\forall j, k \in \{1,\dots,j_i\}$, and no other relations hold between these generators of H, that is, H is the free abelian group with basis $\{t_i = \alpha\eta(x_{i1}) \mid 1 \leqslant i \leqslant \mu\}$.

A finite presentation for G/G_3, namely

$$\langle x_{ij};\, r_{ij},\, r_{(i_1,i_2,i_3)(j_1,j_2,j_3)} = [[x_{i_1j_1}, x_{i_2j_2}], x_{i_3j_3}]\rangle$$

(in which there is a relator $r_{(i_1,i_2,i_3)(j_1,j_2,j_3)}$ whenever $1 \leqslant i_1, i_2, i_3 \leqslant \mu$ and $j_k \in \{1,\ldots,j_{i_k}\}$ for $k = 1, 2$, or 3), can be obtained directly from the presentation $\langle x_{ij};\, r_{ij}\rangle$ of G.

K. T. Chen has shown [**1**] that G/G_3 also has the presentation

$$\langle x_{11},\ldots,x_{\mu 1};\, \rho_i,\, \rho_{ijk}\rangle,$$

in which there are relators $\rho_{ijk} = [[x_{i1}, x_{j1}], x_{k1}]$ whenever $1 \leqslant i, j, k \leqslant \mu$ and

$$\rho_i = \prod_{k \neq i} [x_{k1}, x_{i1}]^{l(K_i, K_k)}$$

whenever $1 \leqslant i \leqslant \mu$. A careful reading of his arguments yields a slightly stronger statement, namely: if $\Phi \subseteq F$ is the subgroup generated by $\{x_{i1} \mid 1 \leqslant i \leqslant \mu\}$, and β: $G \to G/G_3$ is the canonical map onto the quotient, then the restriction $\beta\eta \mid \Phi$: $\Phi \to G/G_3$ of the composite $\beta\eta$ to Φ is surjective, and its kernel is generated (as a normal subgroup of Φ) by the ρ_i and ρ_{ijk}.

3. Elementary ideals. If $p, q \geqslant 1$ and M is a $p \times q$ matrix with entries in a commutative ring R, then the *elementary ideals of* M, denoted $E_k(M)$ and indexed by $k \in \mathbf{Z}$, are ideals of R defined as follows: if $k < 0$ or $k < q - p$ then $E_k(M) = 0$; if $k \geqslant q$ then $E_k(M) = R$; and if $0, q - p \leqslant k < q$ then $E_k(M) \subseteq R$ is the ideal generated by the determinants of the $(q - k) \times (q - k)$ submatrices of M. Clearly $E_k(M) \subseteq E_{k+1}(M)\ \forall k \in \mathbf{Z}$. Also, we note that if f: $R \to S$ is a surjective homomorphism of commutative rings and $f(M)$ is the matrix whose entries are the images under f of the entries of M, then $E_k(f(M)) = f(E_k(M))\ \forall k \in \mathbf{Z}$.

A simple consequence of this definition follows.

LEMMA (3.1). *Suppose $q \geqslant 1$, $p_1 \geqslant p_2 \geqslant 1$, $D \subseteq \mathbf{Z}H$ is an ideal, and $M = (m_{ij})$ and $N = (n_{ij})$ are $p_1 \times q$ and $p_2 \times q$ matrices, respectively, such that $m_{ij} = n_{ij}\ \forall i \in \{1,\ldots,p_2\} \forall j \in \{1,\ldots,q\}$, and $m_{ij} \in D\ \forall i \in \{p_2 + 1,\ldots,p_1\} \forall j \in \{1,\ldots,q\}$. Then for any $k \in \mathbf{Z}$*

$$\sum_{i \geqslant 0} E_{k+i}(M) \cdot D^i = \sum_{i \geqslant 0} E_{k+i}(N) \cdot D^i.$$

PROOF. If $k \geqslant q$, $E_k(M) \cdot D^0 = \mathbf{Z}H = E_k(N) \cdot D^0$, so the equality is trivially true.

Suppose $0, q - p_1 \leqslant j < q$; then $E_j(M) \subseteq \mathbf{Z}H$ is the ideal generated by the determinants of the $(q - j) \times (q - j)$ submatrices of M.

Let P be such a submatrix, involving, say, n of the last $p_1 - p_2$ rows of M. If $n = q - j$, then $\det P \in D^n$; also, $E_{j+n}(N) = E_q(N) = \mathbf{Z}H$, so $\det P \in E_{j+n}(N) \cdot D^n$. If $n = 0$, then P is a submatrix of N, so $\det P \in E_j(N) = E_{j+n}(N) \cdot D^n$. Finally, if $0 < n < q - j$ then expansion of $\det P$ by minors along the last n rows of P expresses $\det P$ as a sum, in which each summand is the product of some element of D^n with the determinant of some $(q - j - n) \times (q - j - n)$ submatrix of N; hence $\det P \in E_{j+n}(N) \cdot D^n$.

Since this argument applies to any $(q-j)\times(q-j)$ submatrix P of M, we conclude that

$$E_j(M) \subseteq \sum_{n\geq 0} E_{j+n}(N)\cdot D^n$$

whenever $0, q-p_1 \leq j < q$. Thus if $0, q-p_1 \leq k < q$

$$\begin{aligned}\sum_{i\geq 0} E_{k+i}(M)\cdot D^i &\subseteq \sum_{i\geq 0}\Big(\sum_{n\geq 0} E_{k+i+n}(N)\cdot D^n\Big)\cdot D^i\\ &= \sum_{i\geq 0}\sum_{n\geq 0} E_{k+i+n}(N)\cdot D^{n+i}\\ &= \sum_{i\geq 0} E_{k+i}(N)\cdot D^i.\end{aligned}$$

Suppose now that $k < q-p_1$ or $k<0$, and let $k_0 = \max\{0, q-p_1\}$. Then since $E_j(M) = 0\ \forall j < k_0$

$$\begin{aligned}\sum_{i\geq 0} E_{k+i}(M)\cdot D^i &= D^{k_0-k}\cdot \sum_{i\geq 0} E_{k_0+i}(M)\cdot D^i\\ &\subseteq D^{k_0-k}\cdot \sum_{i\geq 0} E_{k_0+i}(N)\cdot D^i\\ &\subseteq \sum_{i\geq 0} E_{k+i}(N)\cdot D^i.\end{aligned}$$

Thus for any $k\in\mathbf{Z}$

$$\sum_{i\geq 0} E_{k+i}(M)\cdot D^i \subseteq \sum_{i\geq 0} E_{k+i}(N)\cdot D^i.$$

The opposite inclusion is trivially true, since N can be obtained from M simply by deleting its last p_1-p_2 rows, and so certainly $E_j(M)\supseteq E_j(N)\ \forall j\in\mathbf{Z}$. Q.E.D.

If S and T are free $\mathbf{Z}H$-modules with bases $\{s_1,\ldots,s_p\}$ and $\{t_1,\ldots,t_q\}$, respectively, and $f\colon S\to T$ is a $\mathbf{Z}H$-homomorphism, then the *matrix of* f (with respect to the given bases) is the $p\times q$ matrix $M=(m_{ij})$ with $f(s_i)=\Sigma_j m_{ij}t_j$ for each s_i. If B is a $\mathbf{Z}H$-module and there is an exact sequence $S\xrightarrow{f} T\to B\to 0$ then this sequence is a (finite) *presentation of* B, and M is a *presentation matrix of* B. (Note that since $\mathbf{Z}H$ is noetherian, any finitely generated $\mathbf{Z}H$-module possesses a finite presentation.) In this case we define the *elementary ideals of* B (also known as the Fitting invariants of B) to be $E_k(B)=E_k(M)$; these ideals do not depend on the choice of the presentation matrix M [**5**, p. 58]. In particular, $E_k(B)=\mathbf{Z}H$ whenever $k\geq q$.

LEMMA (3.2). *Let B and C be finitely generated $\mathbf{Z}H$-modules, and suppose there is a $\mathbf{Z}H$-epimorphism $e\colon B\to C$. Then $E_k(B)\subseteq E_k(C)\ \forall k\in\mathbf{Z}$.*

PROOF. Let $S\xrightarrow{f} T\xrightarrow{g} B\to 0$ be a finite presentation of B. Since $\mathbf{Z}H$ is noetherian, $g^{-1}(\ker e)$, a submodule of a finitely generated $\mathbf{Z}H$-module, is itself finitely generated; that is, there is a finitely generated free $\mathbf{Z}H$-module S' and a $\mathbf{Z}H$-epimorphism $f'\colon S'\to g^{-1}(\ker e)$. Let $\hat{f}\colon S\oplus S'\to T$ be the $\mathbf{Z}H$-homomorphism $\hat{f}(s,s')=f(s)+f'(s')\ \forall s\in S\ \forall s'\in S'$; then $\hat{f}(S\oplus S')=\ker g+g^{-1}(\ker e)=g^{-1}(\ker e)=\ker(eg)$.

Hence

$$S \oplus S' \xrightarrow{\hat{f}} T \xrightarrow{eg} C \to 0$$

is a presentation of C. By a felicitous choice of basis for $S \oplus S'$, we may conclude that C has a presentation matrix M', from which a presentation matrix M for B can be obtained simply by deleting certain rows. Then $E_k(C) = E_k(M') \supseteq E_k(M) = E_k(B)\ \forall k \in \mathbf{Z}$. Q.E.D.

4. Proof of Theorem 1. The *free derivatives* are functions $\partial/\partial x_{ij}$: $F \to \mathbf{Z}F$, given by the following rule: if $x = \prod_{k=1}^{p} x_{i_k j_k}^{\varepsilon_k} \in F$, and each ε_k is ± 1, then for $1 \leqslant i \leqslant \mu$ and $1 \leqslant j \leqslant j_i$

$$\frac{\partial}{\partial x_{ij}}(x) = \sum \varepsilon_k \cdot \left(\prod_{q=1}^{k-1} x_{i_q j_q}^{\varepsilon_q} \right) \cdot y_k,$$

the sum being taken over all values of k for which $i_k = i$ and $j_k = j$, where y_k is 1 or $x_{i_k j_k}^{-1}$ according to whether ε_k is 1 or -1. Free derivatives $\partial/\partial x_{i1}$: $\Phi \to \mathbf{Z}\Phi$ are defined in an analogous manner.

The *Alexander matrix* of the presentation $\langle x_{ij};\, r_{ij} \rangle$ of G is a matrix with entries in $\mathbf{Z}H$, which has one row for each relator r_{ij}, and one column for each generator x_{mn}; the common entry of the row corresponding to r_{ij} and the column corresponding to x_{mn} is

$$\alpha\eta\left(\frac{\partial r_{ij}}{\partial x_{mn}} \right).$$

(Here α: $\mathbf{Z}G \to \mathbf{Z}H$ and η: $\mathbf{Z}F \to \mathbf{Z}G$ are the linear extensions of the original α and η from the groups to the group rings.) We will denote this matrix A_G. Alexander matrices of the presentations $\langle x_{ij};\, r_{ij}, r_{(i_1, i_2, i_3)(j_1, j_2, j_3)} \rangle$ and $\langle x_{i1};\, \rho_i, \rho_{ijk} \rangle$ of G/G_3 are defined analogously; we will denote these matrices A_3 and M_3, respectively. We will consider both of these as matrices with entries in $\mathbf{Z}H$ by identifying $H = G/G_2$ with $(G/G_3)/(G/G_3)_2$ in the obvious way. Similarly, we can identify Φ/Φ_2 with $H = G/G_2$ via $\alpha\eta \mid \Phi$: $\Phi \to H$, and then the Alexander matrix A of the presentation $\langle x_{i1};\, \rho_{ijk} \rangle$ of Φ/Φ_3 is also a matrix with entries in $\mathbf{Z}H$. Note that the matrices A and M_3 have only μ columns apiece, one corresponding to each generator appearing in the group presentations from which they are obtained.

If $\mathbf{Z}H$ is considered as a $\mathbf{Z}G$-module via α: $\mathbf{Z}G \to \mathbf{Z}H$, then the tensor product $\mathbf{Z}H \otimes_{\mathbf{Z}G} IG$ is a well-defined abelian group, which may be made into a $\mathbf{Z}H$-module by performing the scalar multiplication in the factor $\mathbf{Z}H$ (i.e., $x \cdot (y \otimes z) = (xy) \otimes z$ $\forall x, y \in \mathbf{Z}H\ \forall z \in IG$). This $\mathbf{Z}H$-module is the *Alexander module of* G. Analogously, the tensor product of $\mathbf{Z}H$ and the augmentation ideal IG/G_3 (over the integral group ring $\mathbf{Z}G/G_3$) is the Alexander module of G/G_3, and the Alexander module of Φ/Φ_3 is the tensor product of $\mathbf{Z}H$ with $I\Phi/\Phi_3$ (over $\mathbf{Z}\Phi/\Phi_3$). (Here we must again identify the abelianizations of G/G_3 and Φ/Φ_3 with H, as above.) We define the elementary ideals of G to coincide with those of its Alexander module, and denote them $E_k(G)$, for $k \in \mathbf{Z}$; similarly, the elementary ideals $E_k(G/G_3)$ and $E_k(\Phi/\Phi_3)$

coincide with the elementary ideals of the Alexander modules of these groups. The ideals $E_k(G)$ are also called the *elementary ideals of the link* L, and denoted $E_k(L)$.

As discussed in [**2**, pp. 216–220], the matrix A_G is a presentation matrix for the Alexander module of G. Similarly, A is a presentation matrix for the Alexander module of Φ/Φ_3, and A_3 and M_3 are both presentation matrices for the Alexander module of G/G_3. Thus $E_k(L) = E_k(G) = E_k(A_G)$, $E_k(G/G_3) = E_k(A_3) = E_k(M_3)$, and $E_k(\Phi/\Phi_3) = E_k(A)\ \forall k \in \mathbf{Z}$.

LEMMA (4.1). *Let* $d\colon F \to \mathbf{Z}F$ *be a derivation, that is, a function with the property that* $d(xy) = d(x) + xd(y)\ \forall x, y \in F$. *Then* $\alpha\eta(d([[x, y], z])) \in (IH)^2\ \forall\ x, y, z \in F$.

PROOF. Note that $d(1) = d(1 \cdot 1) = d(1) + d(1)$, so $d(1) = 0$. Hence $d(x^{-1}x) = d(x^{-1}) + x^{-1}d(x) = 0\ \forall x \in F$, that is, $d(x^{-1}) = -x^{-1}d(x)\ \forall x \in F$. Suppose $x, y, z \in F$. Then by expanding

$$\alpha\eta(d([[x, y], z])) = \alpha\eta\left(d\left(xyx^{-1}y^{-1}z\left(xyx^{-1}y^{-1}\right)^{-1}z^{-1}\right)\right),$$

we conclude that $\alpha\eta(d([[x, y], z])) = (1 - \alpha\eta(z))\alpha\eta(d([x, y]))$. Also,

$$\begin{aligned}\alpha\eta(d([x, y])) &= \alpha\eta\left(d\left(xyx^{-1}y^{-1}\right)\right)\\ &= (\alpha\eta(x) - 1)\alpha\eta(d(y)) + (1 - \alpha\eta(y))\alpha\eta(d(x)) \in IH.\end{aligned}$$

Hence $\alpha\eta(d([[x, y], z])) = (1 - \alpha\eta(z))\alpha\eta(d([x, y])) \in (1 - \alpha\eta(z)) \cdot IH \subseteq (IH)^2$, as claimed. Q.E.D.

Noting that the free derivatives are derivations, we obtain

PROPOSITION (4.2). *For any* $k \in \mathbf{Z}$

$$\sum_{i \geq 0} E_{k+i}(G) \cdot (IH)^{2i} = \sum_{i \geq 0} E_{k+i}(G/G_3) \cdot (IH)^{2i}.$$

PROOF. Recall that $E_k(G) = E_k(A_G)$, $E_k(G/G_3) = E_k(A_3)\ \forall k \in \mathbf{Z}$. The matrix A_G is, in fact, the submatrix of A_3 consisting of those rows which correspond to the relators r_{ij}. The remaining rows of A_3 are those corresponding to the relators $r_{(i_1, i_2, i_3)(j_1, j_2, j_3)}$. By Lemma (4.1), any entry of one of these rows is an element of $(IH)^2$. Applying Lemma (3.1) (with $D = (IH)^2$) completes the proof. Q.E.D.

Before proceeding, we may note that Proposition (4.2) is valid for *any* finitely presented group G, with no change in the proof. Furthermore, given such a group G

$$\sum_{i \geq 0} E_{k+i}(G) \cdot (IH)^{qi} = \sum_{i \geq 0} E_{k+i}(G/G_{q+1}) \cdot (IH)^{qi}$$

for any $k \in \mathbf{Z}$ and $q \geq 1$. (This statement can be proven with an argument closely paralleling the proof of Proposition (4.2).)

Consider the matrix M_3. If $1 \leq i, j \leq \mu$, then its ijth entry is

$$\begin{aligned}\alpha\eta\left(\frac{\partial \rho_i}{\partial x_{j1}}\right) &= \alpha\eta\left(\frac{\partial}{\partial x_{j1}}\left(\prod_{k \neq i} [x_{k1}, x_{i1}]^{l(K_i, K_k)}\right)\right)\\ &= \sum_{k \neq i} l(K_i, K_k) \cdot \alpha\eta\left(\frac{\partial}{\partial x_{j1}}([x_{k1}, x_{i1}])\right).\end{aligned}$$

Thus if $i \neq j$ the ijth entry of M_3 is $l(K_i, K_j) \cdot (1 - t_i)$, while if $i = j$ the ijth entry of M_3 is $\Sigma_{k \neq i} l(K_i, K_k) \cdot (t_k - 1)$.

PROPOSITION (4.3). *If $\mathfrak{L}$ is the matrix mentioned in the introduction, then for any $k \in \mathbf{Z}$*

$$\sum_{i \geq 0} E_{k+i}(G/G_3) \cdot (IH)^{2i} = \sum_{i \geq 0} E_{k+i}(\mathfrak{L}) \cdot (IH)^{2i}.$$

PROOF. Recall that $E_j(G/G_3) = E_j(M_3)$ $\forall j \in \mathbf{Z}$. As we have just noted, if $1 \leq i, j \leq \mu$ the ijth entries of M_3 and $\mathfrak{L}$ coincide. By Lemma (4.1), each entry of any of the remaining rows of M_3 lies in $(IH)^2$. The result now follows from Lemma (3.1). Q.E.D.

Combining Propositions (4.2) and (4.3), and recalling that $E_k(L) = E_k(G)$ $\forall k \in \mathbf{Z}$, we conclude that

$$\sum_{i \geq 0} E_{k+i}(L) \cdot (IH)^{2i} = \sum_{i \geq 0} E_{k+i}(\mathfrak{L}) \cdot (IH)^{2i} \qquad \forall k \in \mathbf{Z}.$$

In particular,

$$\sum_{i \geq 0} E_{\mu+i}(L) \cdot (IH)^{2i} = \sum_{i \geq 0} E_{\mu+i}(\mathfrak{L}) \cdot (IH)^{2i} = \mathbf{Z}H.$$

Thus if $0 < k < \mu$

$$\begin{aligned}\sum_{i=0}^{k-1} E_{\mu-k+i}(L) \cdot (IH)^{2i} + (IH)^{2k} &= \sum_{i \geq 0} E_{\mu-k+i}(L) \cdot (IH)^{2i} \\ &= \sum_{i \geq 0} E_{\mu-k+i}(\mathfrak{L}) \cdot (IH)^{2i} \\ &= \sum_{i=0}^{k-1} E_{\mu-k+i}(\mathfrak{L}) \cdot (IH)^{2i} + (IH)^{2k}.\end{aligned}$$

This completes the proof of Theorem 1.

5. The reduced elementary ideals. Let $\mathbf{Z}[t, t^{-1}]$ be the ring of integral Laurent polynomials in the single variable t, and let π: $\mathbf{Z}H \to \mathbf{Z}[t, t^{-1}]$ be the (unique) ring homomorphism with $\pi(t_i) = t$ $\forall i \in \{1, \ldots, \mu\}$. The ideals $\pi E_k(L) \subseteq \mathbf{Z}[t, t^{-1}]$ are the *reduced elementary ideals of* L, and $\pi\Delta_1(L)$ is the *reduced Alexander polynomial of* L.

Recall that if R is a commutative ring with unity and $D, E \subseteq R$ are ideals then their quotient $D : E$ is the ideal $D : E = \{x \in R \mid xE \subseteq D\}$.

PROPOSITION (5.1). *Let ε: $\mathbf{Z}[t, t^{-1}] \to \mathbf{Z}$ be the homomorphism given by $\varepsilon(t) = 1$. Let l be the $\mu \times \mu$ integral matrix $l = (l_{ij})$ with entries $l_{ii} = -\Sigma_{k \neq i} l(K_i, K_k)$, and for $i \neq j$, $l_{ij} = l(K_i, K_j)$; define elementary ideals $E_k(l) \subseteq \mathbf{Z}$, $k \in \mathbf{Z}$, as in* §3. *Then*

$$\varepsilon\left(\pi E_{\mu-k}(L) : (t-1)^k\right) = E_{\mu-k}(l) \quad \textit{whenever } 0 < k < \mu.$$

PROOF. First, we may note that since $\varepsilon(t-1)=0$

$$\begin{aligned}\varepsilon\left(\pi E_{\mu-k}(L):(t-1)^k\right) &= \varepsilon\left((t-1)+\left(\pi E_{\mu-k}(L):(t-1)^k\right)\right)\\ &= \varepsilon\left(\left((t-1)^{k+1}+\pi E_{\mu-k}(L)\right):(t-1)^k\right)\\ &= \varepsilon\left(\pi\left((IH)^{k+1}+E_{\mu-k}(L)\right):(t-1)^k\right).\end{aligned}$$

By Corollary 1, $(IH)^{k+1}+E_{\mu-k}(L)=(IH)^{k+1}+E_{\mu-k}(\mathfrak{L})$; hence

$$\begin{aligned}\varepsilon\left(\pi E_{\mu-k}(L):(t-1)^k\right) &= \varepsilon\left(\pi\left((IH)^{k+1}+E_{\mu-k}(\mathfrak{L})\right):(t-1)^k\right)\\ &= \varepsilon\left(\left((t-1)^{k+1}+E_{\mu-k}(\pi(\mathfrak{L}))\right):(t-1)^k\right).\end{aligned}$$

Here we have used the equality $\pi E_{\mu-k}(\mathfrak{L})=E_{\mu-k}(\pi(\mathfrak{L}))$ mentioned in §3.

Clearly $\pi(\mathfrak{L})=(1-t)l$, so $E_{\mu-k}(\pi(\mathfrak{L}))=(1-t)^k\cdot E_{\mu-k}(l)=(t-1)^k\cdot E_{\mu-k}(l)$. Hence

$$\begin{aligned}\varepsilon\left(\pi E_{\mu-k}(L):(t-1)^k\right) &= \varepsilon\left(\left((t-1)^k\cdot\left((t-1)+E_{\mu-k}(l)\right)\right):(t-1)^k\right)\\ &= \varepsilon E_{\mu-k}(l)=E_{\mu-k}(\varepsilon(l))=E_{\mu-k}(l).\quad \text{Q.E.D.}\end{aligned}$$

COROLLARY (5.2). *Let m be any $(\mu-1)\times(\mu-1)$ submatrix of l. Then*

$$\varepsilon\left(\frac{\pi\Delta_1(L)}{(t-1)^{\mu-2}}\right)=\pm\det m.$$

PROOF. Note that since the sum of the rows of l is zero, and the sum of the columns of l is zero, any two $(\mu-1)\times(\mu-1)$ submatrices of l have (up to sign) the same determinant. Thus $E_1(l)=(\det m)\subseteq\mathbf{Z}$. By Proposition (5.1),

$$\varepsilon(\pi E_1(L):(t-1)^{\mu-1})=E_1(l)=(\det m).$$

Since $E_1(L)=\Delta_1(L)\cdot IH$,

$$\begin{aligned}\pi E_1(L):(t-1)^{\mu-1} &= (\pi\Delta_1(L)\cdot(t-1)):(t-1)^{\mu-1}\\ &= (\pi\Delta_1(L)):(t-1)^{\mu-2}\end{aligned}$$

is the principal ideal of $\mathbf{Z}[t,t^{-1}]$ generated by $\pi\Delta_1(L)/(t-1)^{\mu-2}$. Q.E.D.

Corollary (5.2) was originally proven by F. Hosokawa in [3]; the quotient $\pi\Delta_1(L)/(t-1)^{\mu-2}$ is the *Hosokawa polynomial of* L, often denoted $\nabla(L)$.

6. Theorem 2. The implications (b) ⇒ (c) ⇒ (e) and (b) ⇒ (d) ⇒ (e) of Theorem 2 are clear. That (a) ⇒ (b) follows from the existence of the presentation $\langle x_{i1};\,\rho_i,\,\rho_{ijk}\rangle$ of G/G_3, for if the linking number of every pair of components of L is zero then the ρ_i are trivial relators, and can be deleted from the presentation.

By Lemma (4.1), each entry of the Alexander matrix A of the presentation $\langle x_{i1};\,\rho_{ijk}\rangle$ of Φ/Φ_3 is an element of $(IH)^2$, from which it follows that $E_{\mu-k}(A)=E_{\mu-k}(\Phi/\Phi_3)\subseteq(IH)^{2k}$ whenever $0<k<\mu$. By Lemma (3.2), it follows that (e) ⇒ (f).

That (g) ⇒ (a) follows immediately from Corollary 1, and clearly (f) ⇒ (g). Thus any two of the statements (a), (b), (c), (d), (e), (f), and (g) are equivalent.

Finally, recall the inclusion $E_{\mu-1}(L) \subseteq J + C$ of [**8**], where C is the ideal of $\mathbf{Z}H$ generated by the elements $t_p^{l(K_p, K_q)} - 1$, $p \neq q$. From this we may conclude that (a) $\Rightarrow$ (h); since (h) $\Rightarrow$ (g) trivially, this completes the proof of Theorem 2.

We note the following consequence of Theorem 2.

COROLLARY (6.1). *Suppose* $l(K_i, K_j) = 0$ $\forall i \neq j \in \{1, \ldots, \mu\}$. *Then* $\Delta_1(L) \in (IH)^{2\mu-3}$.

PROOF. For by (f) $E_1(L) = \Delta_1(L) \cdot IH \subseteq (IH)^{2\mu-2}$. Q.E.D.

M. E. Kidwell has shown, in [**4**], that if $l(K_i, K_j) = 0$ whenever $i \neq j$ then $\pi\Delta_1(L) = x(t-1)^{2\mu-3}$ for some $x \in \mathbf{Z}[t, t^{-1}]$. Furthermore, if μ is even then this element x is divisible by $t - 1$ (i.e., $\varepsilon(x) = 0$), while if μ is odd then $|\varepsilon(x)|$ is a perfect square. Whether or not Corollary (6.1) can be strengthened enough to imply this result is, at present, unknown. (*Added in proof*. Kidwell's result can be deduced from the comment following Proposition (4.2) in the text (with $q = 3$), and the theory of the Milnor invariants of links.)

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